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Continuous dependence for a nonstandard Cahn–Hilliard system with nonlinear atom mobility

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Abstract

This note is concerned with a nonlinear diffusion problem of phase-field type, consisting of a parabolic system of two partial differential equations, complemented by Neumann homogeneous boundary conditions and initial conditions. The system arises from a model of two-species phase segregation on an atomic lattice [22]; it consists of the balance equations of microforces and microenergy; the two unknowns are the order parameter ρ and the chemical potential μ. Some recent results obtained for this class of problems is reviewed and, in the case of a nonconstant and nonlinear atom mobility, uniqueness and continuous dependence on the initial data are shown with the help of a new line of argumentation developed in [12].

1 About the model and the mathematical problem

This paper deals with a phase field system that is addressed and investigated in a rather general framework. A special situation was already considered and mathematically studied in [9, 10] from the viewpoint of well-posedness and long time behavior. The two papers [11] and [14] are devoted to the optimal control problems for distributed and boundary controls, respectively. The recent contributions [12] and [13] are related to what we are going to discuss and review in this note. As to modeling issues, two directly relevant antecedents have been the papers by Fried & Gurtin [17] and Gurtin [19], while [22], the paper that inspired our research cooperation, led us to begin by studying a system of Allen-Cahn type for phase segregation processes without diffusion [7, 8].

1.1 The nonstandard phase-field system in a simplified form

The initial and boundary value problem introduced in [9] consists in looking for two *fields*, the *chemical potential* μ and the *order parameter* ρ, that solve

$$\varepsilon\,\partial_t\mu + 2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0 \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\delta\,\partial_t\rho - \Delta\rho + f'(\rho) = \mu \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}}\rho = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1.3}$$

$$\mu(\,\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,,0) = \rho_0 \qquad \text{in } \Omega, \tag{1.4}$$

where Ω denotes a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ, $T > 0$, and where ε and δ stand for two positive parameters. Moreover, the nonlinearity f is a double-well potential defined in $(0,1)$, whose derivative f' is singular at the endpoints $\rho=0$ and $\rho=1$: a relevant example is

$$f(\rho)=\alpha\{\rho\ln(\rho)+(1-\rho)\ln(1-\rho)\}+\beta\,\rho\,(1-\rho), \tag{1.5}$$

with some positive constants α and β; according to whether or not $\alpha\geq\beta/2$, it turns out that f is convex in the whole of $[0,1]$ or exhibits two wells with a local maximum at $\rho=1/2$.

The nonstandard phase field model (1.1)–(1.4) can be regarded as a variant of the classic Cahn-Hilliard system for diffusion-driven phase segregation by atom rearrangement:

$$\partial_t\rho-\kappa\Delta\mu=0\,,\qquad \mu=-\Delta\rho+f'(\rho). \tag{1.6}$$

As to differences between (1.1) and $(1.6)_1$, we point out that the former equation, in which the *mobility* coefficient $\kappa>0$ has been taken equal to 1, contains a group of terms involving time derivatives, with two unpleasant nonlinearities. Moreover, (1.2) differs from $(1.6)_2$ due to the presence of the viscous contribution $\delta\,\partial_t\rho$, and keeping the coefficient δ positive is crucial for our analysis. Equations (1.1)–(1.2) actually have the structure of a *phase field system* [5, 21], in which the chemical potential μ appears in the place of the more usual temperature variable. Note that, in general, those equations cannot be combined into one higher-order equation, as is instead customarily done with the equations in (1.6) so as to obtain the well-known *Cahn-Hilliard equation*

$$\partial_t\rho=\kappa\Delta(-\Delta\rho+f'(\rho)). \tag{1.7}$$

1.2 Generalization of Cahn-Hilliard equation according to Fried and Gurtin

In [17, 19] a broad generalization of (1.7) was devised, along three directions:

(i) to regard the second of (1.6) as a *balance of microforces:*

$$\operatorname{div}\boldsymbol{\xi}+\pi+\gamma=0, \tag{1.8}$$

where the distance microforce per unit volume is split into an internal part π and an external part γ, and the contact microforce per unit area of a surface oriented by its normal $\boldsymbol{n}$ is measured by $\boldsymbol{\xi}\cdot\boldsymbol{n}$ in terms of the *microstress* vector $\boldsymbol{\xi}$;[3]

(ii) to regard the first equation of (1.6) as a *balance law for the order parameter*:

$$\partial_t\rho=-\operatorname{div}\boldsymbol{h}+\sigma, \tag{1.9}$$

where the pair $(\boldsymbol{h},\sigma)$ is the *inflow* of ρ;

(iii) to demand that the constitutive choices for $\pi,\boldsymbol{\xi},\boldsymbol{h}$, and the *free energy density* ψ, be consistent in the sense of Coleman and Noll [6] with an *ad hoc* version of the Second Law of Continuum Thermodynamics:

$$\partial_t\psi+(\pi-\mu)\partial_t\rho-\boldsymbol{\xi}\cdot\nabla(\partial_t\rho)+\boldsymbol{h}\cdot\nabla\mu\leq0, \tag{1.10}$$

[3] Let us point out that in [16] the balance of microforces is stated in the form of a principle of virtual powers for microscopic motions.

that is, a postulated "dissipation inequality that accommodates diffusion" (cf. equation (3.6) in [19]).

In [19], the following list of constitutive prescriptions was shown to be consistent with (iii):

$$\psi = \widehat{\psi}(\rho, \nabla\rho), \;\; \widehat{\pi}(\rho, \nabla\rho, \mu) = \mu - \partial_\rho\widehat{\psi}(\rho, \nabla\rho), \;\; \widehat{\boldsymbol{\xi}}(\rho, \nabla\rho) = \partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho). \tag{1.11}$$

Within this framework, let also

$$\boldsymbol{h} = -\boldsymbol{M}\nabla\mu, \quad \text{with } \boldsymbol{M} = \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu), \tag{1.12}$$

where the tensor-valued *mobility mapping* $\widehat{\boldsymbol{M}}$ should satisfy the *residual dissipation inequality*

$$\nabla\mu \cdot \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

With the help of (1.8), (1.9), (1.11), and on account of $\boldsymbol{h} = -\boldsymbol{M}\nabla\mu$, one recovers a general equation for diffusive phase segregation processes:

$$\partial_t\rho = \operatorname{div}\Big(\boldsymbol{M}\nabla\Big(\partial_\rho\widehat{\psi}(\rho, \nabla\rho) - \operatorname{div}\big(\partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho)\big) - \gamma\Big)\Big) + \sigma.$$

Then, the Cahn-Hilliard equation (1.7) is obtained by taking

$$\widehat{\psi}(\rho, \nabla\rho) = f(\rho) + \frac{1}{2}|\nabla\rho|^2, \qquad \boldsymbol{M} = \kappa\boldsymbol{1}, \tag{1.13}$$

and letting the external distance microforce γ and the order parameter source term σ be identically null.

1.3 An alternative generalization of Cahn-Hilliard equation

In [22], a modifed derivation, with respect to Fried-Gurtin's approach to phase-segregation modeling, was proposed. While the crucial step (i) was retained, both the order parameter balance (1.9) and the dissipation inequality (1.10) were dropped and replaced, respectively, by the *microenergy balance*

$$\partial_t\varepsilon = e + w, \quad e := -\operatorname{div}\overline{\boldsymbol{h}} + \overline{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{1.14}$$

and the *microentropy imbalance*

$$\partial_t\eta \geq -\operatorname{div}\boldsymbol{h} + \sigma, \quad \boldsymbol{h} := \mu\overline{\boldsymbol{h}}, \quad \sigma := \mu\,\overline{\sigma}. \tag{1.15}$$

As a new feature in this approach, the *microentropy inflow* $(\boldsymbol{h}, \sigma)$ was deemed proportional to the *microenergy inflow* $(\overline{\boldsymbol{h}}, \overline{\sigma})$ through the *chemical potential* μ, a positive field; consistently, the free energy was defined to be

$$\psi := \varepsilon - \mu^{-1}\eta, \tag{1.16}$$

with the chemical potential playing the same role as the *coldness* in the deduction of the heat equation.[4]

Combining (1.14)-(1.16) yields

$$\partial_t\psi \leq -\eta\partial_t(\mu^{-1}) + \mu^{-1}\,\overline{\boldsymbol{h}}\cdot\nabla\mu - \pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{1.17}$$

an inequality that replaces (1.10) in restricting *à la* Coleman-Noll the possible constitutive choices. On taking all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\overline{\boldsymbol{h}}$, dependent in principle on $\rho, \nabla\rho, \mu, \nabla\mu$, and on choosing

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\,\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{1.18}$$

compatibility with (1.17) implies that we must have:

$$\left\{\begin{array}{c} \widehat{\pi}(\rho,\nabla\rho,\mu) = -\partial_\rho\widehat{\psi}(\rho,\nabla\rho,\mu) = \mu - f'(\rho), \\ \widehat{\boldsymbol{\xi}}(\rho,\nabla\rho,\mu) = \partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho,\mu) = \nabla\rho, \\ \widehat{\eta}(\rho,\nabla\rho,\mu) = \mu^2\partial_\mu\widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu^2\rho \end{array}\right\} \tag{1.19}$$

together with

$$\widehat{\overline{\boldsymbol{h}}}(\rho,\nabla\rho,\mu,\nabla\mu) = \widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu, \quad \nabla\mu\cdot\widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu \geq 0.$$

We now choose for $\widehat{\boldsymbol{H}}$ the simplest expression $\boldsymbol{H} = \kappa\boldsymbol{1}$, implying a constant and isotropic mobility, and once again we assume that the external distance microforce γ and the source $\overline{\sigma}$ are null. Then, with the use of (1.19) and (1.16), the microforce balance (1.8) and the energy balance (1.14) become, respectively,

$$\Delta\rho + \mu - f'(\rho) = 0 \tag{1.20}$$

and

$$2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \mathrm{div}(\kappa\nabla\mu) = 0, \tag{1.21}$$

a nonlinear system for the unknowns ρ and μ.

1.4 Insertion of the two parameters ε and δ

Compare now the systems (1.20)–(1.21) and (1.6): needless to say, (1.20) is the same 'static' relation between μ and ρ as $(1.6)_2$. However, (1.21) is rather different from $(1.6)_1$, for more than one reason:

(R1) (1.21) is a nonlinear equation, while $\partial_t\rho - \kappa\Delta\mu = 0$ is linear;

[4] As much as absolute temperature is a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*; likewise, as argued in [22], the chemical potential can be seen as a macroscopic measure of microscopic *organization*.

(R2) the time derivatives of both ρ and μ are present in (1.21);

(R3) in front of both $\partial_t\mu$ and $\partial_t\rho$ there are *nonconstant factors* that *should remain nonnegative* during the evolution.

Thus, the system (1.20)–(1.21) deserves a careful analysis. We must confess that at the beginning we boldly attacked this problem as it was, prompted to optimism by the previous successful outcome of the joint cooperation for the papers [7, 8]. Actually, in [7, 8] we tackled the system of Allen-Cahn type derived via the approach in [22] for no-diffusion phase segregation processes. By the way, the evolution problem ruled by (1.20)–(1.21) turned out to be too difficult for us. Therefore, we decided to study its regularized version (1.1)–(1.4) (note that κ has been taken equal to unity in (1.1)): in fact, this initial-boundary value problem is arrived at by introducing the extra terms $\varepsilon\,\partial_t\mu$ in (1.21) and $\delta\,\partial_t\rho$ in (1.20), then by supplementing the obtained equations (1.1) and (1.2) with homogeneous Neumann conditions (1.3) at the body's boundary (where $\partial_{\boldsymbol{n}}$ denotes the outward normal derivative), and with the initial conditions (1.4).

Of course, the positive coefficients ε and δ are intended to be small. The introduction of the $\varepsilon-$term is motivated by the desire to have a strictly positive coefficient as a factor of $\partial_t\mu$ in (1.21), in order to guarantee the parabolic structure of equation (1.1). As to the $\delta-$term, we can say that it transforms (1.20) into an Allen-Cahn equation with source μ; in fact, it is a sort of regularization already employed in various procedures involving the so-called *viscous Cahn-Hilliard equation* (examples can be found in [2, 3, 18, 20, 23] and references therein).

On the one hand, the presence of the term $\delta\,\partial_t\rho$ with a positive δ is very important for our analysis; on the other hand, nonuniqueness may occur if $\delta = 0$. For instance, take $\rho_0 = 1/2$, μ_0 constant, and look for a space-independent solution (which is in agreement with homogeneous Neumann boundary conditions (1.3)). Then, we have that

$$\frac{d}{dt}\left((\varepsilon + 2\rho)^{1/2}\,\mu\right) = 0$$

and simply $f'(\rho) = \mu$. Hence, the solution has the form

$$\mu = z_0\,(\varepsilon + 2\rho)^{-1/2} \quad \text{and} \quad f'(\rho) = z_0\,(\varepsilon + 2\rho)^{-1/2},$$

for some given constant z_0. Now, choose the potential f such that

$$f'(r) = z_0\,(\varepsilon + 2\,r)^{-1/2} \quad \text{for} \quad r \in [1/3, 2/3],$$

and pick any smooth/irregular $\rho : [0, T] \to [1/3, 2/3]$ with $\rho(0) = 1/2$. We then get infinitely many smooth/irregular solutions! This of course means that uniqueness is out of question; and that, moreover, there is no control on the regularity of solutions in time.

We point out that such a modified system, with positive ε and δ, turns out to be a phase field model with a nonstandard equation (1.1) for the chemical potential μ, while quite often phase field systems use temperature (in place of chemical potential) and order parameter as variables.

Concerning a physical interpretation of the regularizing perturbations we introduced, to motivate the presence of $\delta\,\partial_t\rho$ is relatively easy. All we need to do in order to let this term appear in the microforce balance is to add $\partial_t \rho$ to the list of state variables we considered to analyze the constitutive consequences of (1.17). This measure brings in the typical dissipation mechanism of Allen-Cahn nondiffusional segregation processes, where dissipation depends essentially on $(\partial_t \rho)^2$, in addition to Cahn-Hilliard's $|\nabla \mu|^2-$ dissipation (cf. [22]), thus opening the way to *split the distance microforce additively* into an equilibrium and a nonequilibrium part, with $\pi^{eq} = -\partial_\rho \widehat{\psi}(\rho, \nabla\rho, \mu) = \mu - f'(\rho)$ the equilibrium part, just as in $(1.19)_1$, and with $\pi^{neq} = -\delta\, \partial_t \rho$ the nonequilibrium part.

As far as the the introduction of $\varepsilon\, \partial_t \mu$ is concerned, we can say that (formally) the desired term can be made to appear in (1.1) by modifying the choice of the free energy in (1.18) as follows:

$$\psi = -\mu\Big(\rho + \frac{\varepsilon}{2}\Big) + f(\rho) + \frac{1}{2}|\nabla \rho|^2. \tag{1.22}$$

By the way, in [9] we could prove existence and uniqueness of the solution to the initial boundary value problem (1.1)–(1.4) with $\varepsilon > 0$ and in [10] we discussed the asymptotic behavior of such solutions as $\varepsilon \searrow 0$ by showing a suitable convergence to a (weaker) solution of the limiting problem with $\varepsilon = 0$. Thus, in some respect, we can avoid the use of the parameter ε, an issue we expand and make precise in the following subsection.

1.5 Various generalizations

In the first place, we are interested in the generalization of the free energy (1.22). We work in two directions. We extend $f(\rho)$ by allowing f to be the sum of a convex and lower semicontinuous function, with proper domain $D(f_1) \subset \mathbb{R}$, and of a smooth function f_2 with no convexity properties (to allow for a double or multi-well potential f). We point out that in this case f_1 need not be differentiable in its domain and, in place of f_1', one should take the subdifferential $\beta := \partial f_1$ in the order parameter equation. In general, $\beta := \partial f_1$ is only a graph, not necessarily a function, and may include vertical (and horizontal) lines as in the example $\beta = \partial I_{[0,1]}$, i.e.,

$$\eta \in \partial I_{[0,1]}(u) \quad \text{if and only if} \quad \eta \begin{cases} \leq 0 & \text{if } u = 0 \\ = 0 & \text{if } 0 < u < 1 \\ \geq 0 & \text{if } u = 1 \end{cases}, \tag{1.23}$$

which corresponds to the potential

$$f_1(u) = I_{[0,1]}(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1 \\ +\infty & \text{elsewhere} \end{cases}. \tag{1.24}$$

Therefore, f_1 is not required to be smooth so that its subdifferential β might be multi-valued. The other important modification we make in the free energy (1.22) is that of allowing in the first coupling term a general smooth function, say $h(\rho)$, as factor of $-\mu$ in (1.22), with the only restriction that $h(\rho)$ be bounded from below by a positive constant. Then, it could be

$$h(\rho) \geq \frac{\varepsilon}{2} \tag{1.25}$$

to maintain the same notation, and this lower bound should hold at least for the significant values of ρ belonging to the domain of f_1; actually, this was the case for $h(\rho) = \rho + \frac{\varepsilon}{2}$ in the interval $[0, 1]$, which is the effective domain of the potential f in (1.5) (the same domain as in (1.24)). An interesting remark of Alexander Mielke, when one of us was lecturing on our results, was that the behavior of

$$h(\rho) = \rho + \text{ small parameter}$$

in a right neighbourhood of 0 ($h(\rho) \approx 0$) differs from that in a left neighbourhood of 1 ($h(\rho) \approx 1$). Instead, assuming only a boundedness from below for h allows many other instances like, e.g., a specular behavior around the extremal points of the domain of f. On the other hand, we stress the fact that f_1 is just supposed to be proper, convex and lower semicontinuous; hence, any form of double-well or multi-well potential, possibly defined on the whole of $\mathbb{R}$, may result from the free energy

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\, h(\rho) + f_1(\rho) + f_2(\rho) + \frac{1}{2}|\nabla\rho|^2. \tag{1.26}$$

In this respect, we also cover the case of a free energy ψ which is convex or not with respect to ρ according to whether or not the chemical potential μ is greter or less than a critical value μ_c; e.g., this is the case with f_1 given as in (1.24) and

$$h(\rho) = \rho(1-\rho), \quad f_2(\rho) = +\mu_c\, \rho(1-\rho).$$

There is also a third novelty in our approach. Indeed, the mobility factor κ appearing in (1.21) (cf. also the choice for $\widehat{\boldsymbol{H}}$ prior to (1.21)) is no longer assumed to be constant, but rather a nonnegative, continuous and bounded, nonlinear function of μ. In particular, to prove existence of solutions we may let $\kappa(\mu)$ degenerate at $\mu = 0$: indeed, in our model the chemical potential μ is required to take nonnegative values, so that 0 remains critical for μ. The details of such an existence proof are developed in [12], a paper to which we refer frequently in the present note. Let us also mention that in the recent paper [13] an existence theory is presented for a variation of the problem (1.27)–(1.30) below, where the conductivity κ in (1.27) may depend on both variables μ and ρ.

1.6 Aim of this contribution

In this paper, we recall the existence result of [12] and sketch the basic steps of the proof; moreover, in the case when the function $\mu \mapsto \kappa(\mu)$ is Lipschitz continuous and bounded from below by a positive constant, we prove uniqueness and continuous dependence on initial data. This result is new and follows the line of argumentation devised in [12] for the case of κ constant.

We set (cf. (1.25))

$$g(u) := h(u) - \frac{\varepsilon}{2} \geq 0 \quad \text{for all } u \in D(f_1),$$

and take $\varepsilon = \delta = 1$ for the sake of simplicity. The problem we deal with is:

$$\big(1 + 2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \qquad \text{in } \Omega \times (0,T), \tag{1.27}$$

$$\partial_t\rho - \Delta\rho + \xi + f_2'(\rho) = \mu g'(\rho)\,, \quad \text{with } \xi \in \beta(\rho), \qquad \text{in } \Omega \times (0,T), \tag{1.28}$$

$$(\kappa(\mu)\nabla\mu)\cdot \boldsymbol{n}|_\Gamma = \partial_{\boldsymbol{n}}\rho|_\Gamma = 0 \qquad \text{on } \Gamma \times (0,T), \tag{1.29}$$

$$\mu(\,\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,,0) = \rho_0 \qquad \text{in } \Omega. \tag{1.30}$$

Clearly, how to select ξ in $\beta(\rho)$ is part of the problem. For (1.27)–(1.30) we can prove a well-posedness result. In particular, we think that our continuous dependence proof is a nice piece of work, since it can handle the presence of a multivalued graph β (with vertical segments as, e.g., in (1.23)) and only exploits the monotonicity property of β. This was not the case for the uniqueness technique used in [9], since there the difference of two equations (1.1) was tested by the time derivative of the difference of the two ρ components, a procedure that strongly conflicts with nonsmooth potentials.

The longtime behavior of the system (1.1)–(1.4) and the structure of the ω-limit set have been analyzed in [9] and in [10]; the latter paper also deals with the $\varepsilon = 0$ problem, as already mentioned. The two papers [11] and [14] are concerned with the study of two optimal control problems for systems similar to (1.1)–(1.4); precisely, in [11] a distributed control problem is investigated, while [14] focuses on a boundary control problem.

In this paper, we concentrate on existence and uniqueness. In the next section, we state our assumptions and our results. The existence of a solution to problem (1.27)–(1.30) is proved in the Section 3. In Section 4, we show some regularity properties of the solutions. The last section is devoted to the proof of the continuous dependence of the solution on the initial data.

2 Main results

Let Ω be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ (lower-dimensional cases can be treated with minor changes). We introduce a final time $T \in (0,+\infty)$ and set $Q := \Omega \times (0,T)$. Moreover, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma\}, \tag{2.1}$$

and endow these spaces with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$. For $p \in [1,+\infty]$, we write $\|\cdot\|_p$ both for the usual norm in $L^p(\Omega)$ and for the norm in $L^p(Q)$, since no confusion can arise. Moreover, any of the above symbols for norms is used even for any power of these spaces. We remark that the embeddings $W \subset V \subset H$ are compact, since Ω is bounded and smooth. As V is dense in H, we can identify H with a subspace of V^* in the usual way.

We now introduce the structural assumptions on our system. Firstly, since the chemical potential is expected to be at least nonnegative, we assume that the function κ is defined just for nonnegative arguments; moreover, we require that

$$\kappa : [0, +\infty) \to \mathbb{R} \text{ is locally Lipschitz continuous,} \tag{2.2}$$

$$\kappa_*, \kappa^* \in (0, +\infty) \quad \text{and} \quad \mu_* \in [0, +\infty), \tag{2.3}$$

$$\kappa(r) \le \kappa^* \quad \text{for every } r \ge 0 \quad \text{and} \quad \kappa(r) \ge \kappa_* \quad \text{for every } r \ge \mu_*, \tag{2.4}$$

$$K(r) := \int_0^r \kappa(s)\, ds \quad \text{for } r \ge 0 \tag{2.5}$$

(note that, under the above assumptions, K is strictly increasing). As to the other data, we assume that $f = f_1 + f_2$ and that

$$f_1 : \mathbb{R} \to [0, +\infty] \text{ is convex, proper, lower semicontinuous,} \tag{2.6}$$

$$f_2 : \mathbb{R} \to \mathbb{R} \text{ and } g : \mathbb{R} \to [0, +\infty) \text{ are } C^2 \text{ functions,} \tag{2.7}$$

$$f_2',\ g, \text{ and } g' \text{ are Lipschitz continuous,} \tag{2.8}$$

$$\beta := \partial f_1 \quad \text{and} \quad \pi := f_2', \tag{2.9}$$

$$\mu_0 \in V, \quad \rho_0 \in W, \quad \mu_0 \ge 0 \quad \text{and} \quad \rho_0 \in D(\beta) \quad \text{a.e. in } \Omega, \tag{2.10}$$

$$\text{there exists some } \xi_0 \in H \text{ such that } \xi_0 \in \beta(\rho_0) \text{ a.e. in } \Omega, \tag{2.11}$$

where $D(f_1)$ and $D(\beta)$ $(\subseteq D(f_1))$ denote the effective domains of f_1 and β, respectively. It is known that any proper, convex and lower semicontinuous function is bounded from below by an affine function (see, e.g., [1, Prop. 2.1, p. 51]). Hence, assuming $f_1 \ge 0$ looks reasonable, because one can suitably modify the smooth perturbation f_2 by adding a straight line to it. Another positivity condition, $g \ge 0$, is needed on the set $D(\beta)$, while g can take negative values outside of $D(\beta)$. Finally, since f_1 obeys (2.6) and f_2 is smooth, assumptions (2.10)–(2.11) imply that $f(\rho_0) \in L^1(\Omega)$.

Let us discuss the a priori regularity we ask for any solution (μ, ρ, ξ) to our problem. As (1.28) reduces for any given μ to a rather standard phase-field equation, it is natural to look for pairs (ρ, ξ) that satisfy

$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.12}$$

$$\xi \in L^\infty(0,T;H), \tag{2.13}$$

and solve the strong form of the relative subproblem, namely,

$$\partial_t \rho - \Delta\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \qquad \text{a.e. in } Q, \tag{2.14}$$

$$\rho(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{2.15}$$

We note that (2.12) also incorporates the Neumann boundary condition for ρ (see (2.1) for the definition of W).

The situation is different for the component μ : in case of uniform parabolicity, i.e., if $\mu_* = 0$, the coefficient $\kappa(\mu)$ is bounded away from zero, and we require that

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad \mu \ge 0 \quad \text{a.e. in } Q, \tag{2.16}$$

$$\operatorname{div}\bigl(\kappa(\mu)\nabla\mu\bigr) \in L^2(0,T;H), \tag{2.17}$$

so that μ satisfies

$$\int_\Omega \bigl(1+2g(\rho(t))\bigr)\,\partial_t\mu(t)\,v + \int_\Omega \mu(t)\,g'(\rho(t))\,\partial_t\rho(t)\,v \tag{2.18}$$
$$+ \int_\Omega \kappa(\mu(t))\nabla\mu(t)\cdot\nabla v = 0 \qquad \text{for every } v\in V \text{ and for a.a. } t\in(0,T),$$

$$\mu(0)=\mu_0 \quad \text{a.e. in } \Omega. \tag{2.19}$$

Thus, equation (1.27) holds in a strong sense:

$$\bigl(1+2g(\rho)\bigr)\,\partial_t\mu + \mu\,g'(\rho)\,\partial_t\rho - \operatorname{div}\bigl(\kappa(\mu)\nabla\mu\bigr) = 0 \quad \text{a.e. in } Q, \tag{2.20}$$

whereas the related Neumann boundary condition in (1.29) continues to be understood in the usual weak sense. Furthermore, we observe that (2.16)–(2.18) imply further regularity for μ whenever κ is smoother, thanks to the regularity theory of quasilinear elliptic equations.

Such a formulation is too strong when μ_* is allowed to be positive, because sufficient information cannot be obtained on the gradient $\nabla\mu$ and the time derivative $\partial_t\mu$. In this case, we rewrite equation (2.20) as

$$\partial_t\bigl(1+2g(\rho)\mu\bigr) - \mu g'(\mu)\partial_t\rho - \Delta K(\mu) = 0, \tag{2.21}$$

and require lower regularity:

$$\mu\in L^\infty(0,T;H), \quad \mu\ge 0 \quad \text{a.e. in } Q, \quad K(\mu)\in H^1(0,T;H)\cap L^\infty(0,T;V), \tag{2.22}$$

$$(1+2g(\rho))\mu \in H^1(0,T;V^*). \tag{2.23}$$

On accounting for the initial and Neumann boundary conditions, we replace (2.18)–(2.19) by

$$\langle\partial_t\bigl((1+2g(\rho))\mu\bigr)(t),v\rangle - \int_\Omega \bigl(\mu g'(\rho)\partial_t\rho\bigr)(t)\,v + \int_\Omega \nabla K(\mu(t))\cdot\nabla v = 0$$
$$\text{for every } v\in V \text{ and for a.a. } t\in(0,T), \tag{2.24}$$

$$\bigl((1+2g(\rho))\mu\bigr)(0) = \bigl(1+2g(\rho_0)\bigr)\,\mu_0. \tag{2.25}$$

Note that the middle term of (2.24) is meaningful: let us explain why. First, we have that $g'(\rho)\in C^0(\overline{Q})$, because the continuity of ρ,

$$\rho\in C^0([0,T];C^0(\overline{\Omega})) = C^0(\overline{Q}), \tag{2.26}$$

follows directly from (2.12) and the compact embedding $W\subset C^0(\overline{\Omega})$ (see, e.g., [24, Sect. 8, Cor. 4]). Next, (2.22) and the embedding $V\subset L^4(\Omega)$ imply that $K(\mu)\in L^\infty(0,T;L^4(\Omega))$; consequently, $\mu\in L^\infty(0,T;L^4(\Omega))$, as $K(r)$ behaves like r for large $|r|$ (see (2.4)). Finally, (2.12) ensures that $\partial_t\rho\in L^\infty(0,T;H)$, whence $\mu g'(\rho)\partial_t\rho\in L^\infty(0,T;L^{4/3}(\Omega))$, and v is in $L^4(\Omega)$ whenever $v\in V$. We remark that in this framework (2.21) is only satisfied in a distributional sense.

Here are our results. The first establishes the existence of a weak solution in the general case and the equivalence of strong and weak formulations in the case $\mu_*=0$; the proof will be outlined in Section 3.

Theorem 2.1. *Assume* (2.2)–(2.9) *and* (2.10)–(2.11)*. Then, there exists at least one triplet* (μ, ρ, ξ) *such that*

$$(\mu, \rho, \xi) \text{ satisfies (2.12)–(2.13), (2.22)–(2.23)} \\ \text{and solves problem (2.14)–(2.15), (2.24)–(2.25).} \tag{2.27}$$

Moreover, if $\mu_* = 0$ *then any triplet*(μ, ρ, ξ) *as in* (2.27) *fulfills also* (2.16)–(2.19)*.*

Notice that, due to (2.26), no further assumption is needed to ensure the boundedness of ρ. As to the first component, we have the following boundedness result.

Theorem 2.2. *Assume* (2.2)–(2.9)*,* (2.10)–(2.11)*, and let*

$$\mu_0 \in L^\infty(\Omega). \tag{2.28}$$

Then, the component μ *of any triplet* (μ, ρ, ξ) *complying with* (2.27) *is essentially bounded.*

The next result holds if we assume that $\mu_* = 0$.

Theorem 2.3. *Assume* (2.2)–(2.9)*,* (2.10)–(2.11)*,* $\mu_* = 0$*, and*

$$K(\mu_0) \in W. \tag{2.29}$$

Then,

$$K(\mu) \in W^{1,p}(0, T; H) \cap L^p(0, T; W) \quad \text{for every } p \in [1, +\infty), \tag{2.30}$$

where μ *is the first component of any triplet* (μ, ρ, ξ) *being as in* (2.27)*.*

Observe that (2.29) implies (2.28), due to the three-dimensional embedding $W \subset L^\infty(\Omega)$ and the strict monotonicity of K^{-1} (see (2.5)). Uniqueness is a consequence of the following continuous dependence result.

Theorem 2.4. *Assume* (2.2)–(2.9) *and* $\mu_* = 0$*. Let* $(\mu_{0,i}, \rho_{0,i})$*,* $i = 1, 2$*, be two sets of initial data satisfying* (2.10)–(2.11) *and* (2.29)*, and let* (μ_i, ρ_i, ξ_i)*,* $i = 1, 2$*, be two corresponding triplets fulfilling* (2.27) *(with the obvious modifications of initial conditions). Then, there exists a constant* C*, depending on the data through the structural assumptions, such that*

$$\|\mu_1 - \mu_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\rho_1 - \rho_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ \leq C \left\{ \|\mu_{0,1} - \mu_{0,2}\|_H + \|\rho_{0,1} - \rho_{0,2}\|_H \right\}. \tag{2.31}$$

Henceforth, we make repeated use of the notation

$$Q_t := \Omega \times (0, t) \quad \text{for } t \in [0, T]. \tag{2.32}$$

Moreover, we account for the well-known embedding $V \subset L^p(\Omega)$ for $1 \leq p \leq 6$ and the related Sobolev inequality:

$$\|v\|_p \leq C_\Omega \|v\|_V \quad \text{for every } v \in V \text{ and } 1 \leq p \leq 6, \tag{2.33}$$

where C depends on Ω only; Hölder inequality and the elementary Young inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}\, b^2 \quad \text{for every } a, b \ge 0 \text{ and } \varepsilon > 0 \tag{2.34}$$

are also frequently employed. Finally, throughout the paper we use a small-case italic c for different constants that may only depend on Ω, the final time T, the shape of the nonlinearities f and g, and the properties of the data involved in the statements; the symbol c_ε denotes a constant that depends also on the parameter ε. The meaning of c and c_ε might change from line to line and even in the same chain of inequalities, whereas those constants that we need to refer to are always denoted by capital letters, just like C_Ω in (2.33).

3 Existence

In this section we sketch the proof of Theorem 2.1, referring to [12] for details.

Approximation. The approximating problem is based on a time delay in the right-hand side of equation (2.14). A translation operator $\mathcal{T}_\tau : L^1(0,T;H) \to L^1(0,T;H)$ is considered, depending on a time step $\tau > 0$: for $v \in L^1(0,T;H)$ and for a.a. $t \in (0,T)$, we set:

$$(\mathcal{T}_\tau v)(t) := v(t-\tau) \quad \text{if } t > \tau \quad \text{and} \quad (\mathcal{T}_\tau v)(t) := \mu_0 \quad \text{if } t < \tau; \tag{3.1}$$

and we replace μ by $\mathcal{T}_\tau \mu$ in (2.14). At the same time, we modify the equation for μ. Precisely, we force uniform parabolicity and allow the solution to take negative values. Accordingly, we define $\kappa_\tau : \mathbb{R} \to \mathbb{R}$ and the related function K_τ to be

$$\kappa_\tau(r) := \kappa(|r|) + \tau \quad \text{and} \quad K_\tau(r) := \int_0^r \kappa_\tau(s)\, ds \quad \text{for } r \in \mathbb{R}. \tag{3.2}$$

Then, the approximating problem involves the following equations:

$$\bigl(1 + 2g(\rho_\tau)\bigr)\, \partial_t \mu_\tau + \mu_\tau\, g'(\rho_\tau)\, \partial_t \rho_\tau - \operatorname{div}\bigl(\kappa_\tau(\mu_\tau) \nabla \mu_\tau\bigr) = 0 \qquad \text{a.e. in } Q, \tag{3.3}$$

$$\partial_t \rho_\tau - \Delta \rho_\tau + \xi_\tau + \pi(\rho_\tau) = (\mathcal{T}_\tau \mu_\tau)\, g'(\rho_\tau) \quad \text{and} \quad \xi_\tau \in \beta(\rho_\tau) \qquad \text{a.e. in } Q, \tag{3.4}$$

supplemented by homogeneous Neumann boundary conditions for both μ_τ and ρ_τ, and by the initial conditions $\mu_\tau(0) = \mu_0$ and $\rho_\tau(0) = \rho_0$. It can be easily shown (cf. [12, Lemma 3.1]) that such an initial and boundary value problem has a unique solution $(\mu_\tau, \rho_\tau, \xi_\tau)$, which satisfies the analogues of (2.12)–(2.13) and (2.16)–(2.17).

Our aim is now to let τ tend to zero in order to obtain a limit triplet (μ, ρ, ξ) complying with (2.27). Our proof uses compactness arguments and relies on a number of uniform-in-τ a priori estimates. In performing the estimates, τ can be taken as small as desired; it will be convenient to assume $\tau \le \kappa^*$. In order to make the formulas more readable, we omit the index τ in the calculations, and write μ_τ and ρ_τ only when each estimate is established.

First a priori estimate. We test (3.3) by μ and observe that

$$\bigl[\bigl(1 + 2g(\rho)\bigr)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho\bigr] \mu = \frac{1}{2}\, \partial_t \bigl[(1 + 2g(\rho)) \mu^2\bigr].$$

Thus, by integrating over $(0,t)$, where $t \in [0,T]$ is arbitrary, we obtain:

$$\int_\Omega \bigl(1+2g(\rho(t))\bigr)|\mu(t)|^2 + \int_{Q_t} \kappa_\tau(\mu)|\nabla\mu|^2 = \int_\Omega (1+2g(\rho_0))\mu_0^2 \,.$$

Hence, on recalling that $g \ge 0$ and that, in view of (2.4), $\kappa_\tau^2(r) \le 2\kappa^*\kappa_\tau(r)$ for every $r \in \mathbb{R}$, we are led to

$$\|\mu_\tau\|_{L^\infty(0,T;H)} + \|K_\tau(\mu_\tau)\|_{L^2(0,T;V)} \le c. \tag{3.5}$$

An analogous test by $-\mu^- = \min\{\mu, 0\}$, and the nonnegativity of μ_0, allow us to deduce that $\mu^- = 0$, whence

$$\mu_\tau \ge 0 \quad \text{a.e. in } Q.$$

Moreover, as K has a linear growth and thanks to (3.1) and (2.10), it follows from (3.5) that

$$\|K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)} + \|\mathfrak{T}_\tau\mu_\tau\|_{L^\infty(0,T;H)} + \|\mathfrak{T}_\tau K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{3.6}$$

The Sobolev inequality (2.33) and estimate (3.5) entail $\|K_\tau(\mu_\tau)\|_{L^2(0,T;L^6(\Omega))} \le c$; consequently,

$$\|\mu_\tau\|_{L^2(0,T;L^6(\Omega))} \le c, \tag{3.7}$$

for (2.4) implies that $K_\tau(r) \ge \kappa_* r - c$ for every $r \ge 0$.

Second a priori estimate. Add ρ to both sides of (3.4) and test by $\partial_t\rho$, so as to obtain that

$$\begin{aligned}
&\int_{Q_t} |\partial_t\rho|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t)) \\
&= \frac{1}{2}\|\rho_0\|_V^2 + \int_\Omega f(\rho_0) + \frac{1}{2}\int_\Omega \bigl(\rho^2(t) - 2f_2(\rho(t))\bigr) + \int_{Q_t} g'(\rho)(\mathfrak{T}_\tau\mu)\partial_t\rho \\
&\le c + c\int_\Omega |\rho(t)|^2 + \frac{1}{4}\int_{Q_t} |\partial_t\rho|^2 + c\|\mathfrak{T}_\tau\mu\|^2_{L^\infty(0,T;H)},
\end{aligned}$$

for every $t \in [0,T]$. In view of the chain rule and Young's inequality (2.34), we have that

$$c\int_\Omega |\rho(t)|^2 \le c\int_\Omega |\rho_0|^2 + \frac{1}{4}\int_{Q_t} |\partial_t\rho|^2 + c\int_0^t \|\rho(s)\|_H^2\,ds.$$

Hence, as f_1 is nonnegative, from (3.6) and the Gronwall lemma we infer that

$$\|\rho_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|f_1(\rho_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{3.8}$$

Third a priori estimate. Rewrite (3.4) as

$$-\Delta\rho + \beta(\rho) \ni -\partial_t\rho - \pi(\rho) + (\mathfrak{T}_\tau\mu)g'(\rho)$$

and note that the right-hand side is bounded in $L^2(0,T;H)$, thanks to (2.8)–(2.9) and to the previous estimates. By a standard argument, that consists in testing formally by either $-\Delta\rho$ or $\beta(\rho)$ and using the regularity theory for elliptic equations, we first recover that

$$\|\Delta\rho(s)\|_H^2 + \|\xi(s)\|_H^2 \le 2\|-\partial_t\rho(s) - \pi(\rho(s)) + ((\mathfrak{T}_\tau\mu)g'(\rho))(s)\|_H^2 \tag{3.9}$$

for a.a. $s \in (0,T)$; finally, we conclude that

$$\|\rho_\tau\|_{L^2(0,T;W)} \le c \quad \text{and} \quad \|\xi_\tau\|_{L^2(0,T;H)} \le c. \tag{3.10}$$

Fourth a priori estimate. As this estimate is rather long and technical, let us just describe how it can be obtained, referring to [12, Section 4] for details. The aim is improving estimates (3.8) and (3.10). By proceeding formally, in particular, by writing $\beta(\rho)$ in place of ξ and treating β like a smooth function, one can differentiate (3.4) with respect to time and test the resulting equation by $\partial_t\rho$:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{Q_t} \beta'(\rho)|\partial_t\rho|^2 \\
&= \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 - \int_{Q_t} \left(\pi'(\rho) - g''(\rho)(\mathfrak{T}_\tau\mu)\right)|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho \\
&\le \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 + c\int_{Q_t} (1+\mathfrak{T}_\tau\mu)|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho.
\end{aligned} \tag{3.11}$$

Now, the term difficult to control is the last one on the right-hand side. We compute $\partial_t\mu$ from (3.3), then integrate by parts and repeatedly use the Hölder, Sobolev, and Young inequalities, so as to obtain:

$$\begin{aligned}
&\int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho = \int_0^{t-\tau}\!\!\int_\Omega \partial_t\mu(s)\, g'(\rho(s+\tau))\partial_t\rho(s+\tau)\,ds \\
&= \int_0^{t-\tau}\!\!\int_\Omega \kappa_\tau(\mu)(s)\nabla\mu(s)\cdot\nabla\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds \\
&\quad - \int_0^{t-\tau}\!\!\int_\Omega \frac{g'(\rho(s))\,g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds;
\end{aligned} \tag{3.12}$$

the last two integrals are treated separately, taking the structural assumptions into account. In the subsequent computations, one takes advantage of the compact embedding $V \subset L^4(\Omega)$ and of the regularity theory for linear elliptic equations. In particular, exploiting (3.9) entails that

$$\|\nabla\rho(s)\|_V^2 \le c\big(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\big) \le c\big(\|\partial_t\rho(s)\|_H^2 + 1\big),$$

an inequality that turns out to be helpful in the control of one of the terms. At the end, we arrive at

$$\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 \le c\int_0^t \phi(s)\|\partial_t\rho(s)\|_H^2\,ds + c,$$
$$\text{where} \quad \phi(s) := \|\mu(s)\|_4^2 + \|\nabla K_\tau(\mu)(s)\|_H^2 + \|\nabla(\mathfrak{T}_\tau K_\tau(\mu))(s)\|_H^2\,;$$

hence, as $\phi \in L^1(0,T)$ by (3.5)–(3.7), we can apply the Gronwall lemma and conclude that

$$\|\partial_t\rho_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{3.13}$$

As a consequence, note that $-\Delta\rho+\xi = -\partial_t\rho - \pi(\rho) + g'(\rho)\mathcal{T}_\tau\mu$ belongs to $L^\infty(0,T;H)$ due to (3.5) and (3.13). Therefore, by (3.9), both $-\Delta\rho$ and ξ are in $L^\infty(0,T;H)$. Thanks to elliptic regularity, we conclude that

$$\|\rho_\tau\|_{L^\infty(0,T;W)} \le c \quad \text{and} \quad \|\xi_\tau\|_{L^\infty(0,T;H)} \le c. \tag{3.14}$$

Fifth a priori estimate. Test (3.3) by $\partial_t K_\tau(\mu) = \kappa_\tau(\mu)\partial_t\mu$ and obtain

$$\begin{aligned}
&\int_{Q_t} (1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2 \\
&= \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu_0)|^2 - \int_{Q_t} g'(\rho)\partial_t\rho\,\mu\,\partial_t K_\tau(\mu)
\end{aligned} \tag{3.15}$$

for every $t \in (0,T)$. The first term on the left-hand side can be estimated from below, as follows:

$$\int_{Q_t} (1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 \ge \int_{Q_t} \frac{\kappa_\tau^2(\mu)}{2\kappa^*}\,|\partial_t\mu|^2 = \frac{1}{2\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2. \tag{3.16}$$

On the right-hand side, the first term is trivial due to $(2.10)_1$; as to the second one, by the Young, Hölder, and Sobolev inequalities we have that

$$\begin{aligned}
-\int_{Q_t} g'(\rho)\partial_t\rho\,\mu\,\partial_t K_\tau(\mu) &\le \frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2 \|\partial_t\rho(s)\|_4^2\,ds \\
&\le \frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2 \|\partial_t\rho(s)\|_V^2\,ds.
\end{aligned} \tag{3.17}$$

Next, observe that (2.4) yields $K_\tau(r) \ge \kappa_* r - c_*$ for every $r \ge 0$, where c_* depends only on the structural assumptions. Hence, on recalling (3.6), we deduce that

$$\begin{aligned}
&\|\mu(s)\|_4^2 \le c\big(\|K_\tau(\mu)(s)\|_4^2 + 1\big) \le c\big(\|K_\tau(\mu)(s)\|_V^2 + 1\big) \\
&\le c\|\nabla K_\tau(\mu)(s)\|_H^2 + c\|K_\tau(\mu)(s)\|_H^2 + c \le c\|\nabla K_\tau(\mu)(s)\|_H^2 + c
\end{aligned} \tag{3.18}$$

for a.a. $s \in (0,T)$. By combining (3.16)–(3.18) with (3.15), we obtain that

$$\frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2 \le c + c\int_0^t \|\partial_t\rho(s)\|_V^2 \big(\|\nabla K_\tau(\mu)(s)\|_H^2 + 1\big)\,ds.$$

In view of (3.13), we can apply the Gronwall lemma and conclude that

$$\|K_\tau(\mu_\tau)\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c. \tag{3.19}$$

Moreover, arguing as for (3.7) and (3.6), we derive that

$$\|\mu_\tau\|_{L^\infty(0,T;L^6(\Omega))} + \|\mathcal{T}_\tau\mu_\tau\|_{L^\infty(0,T;L^6(\Omega))} \le c. \tag{3.20}$$

Sixth a priori estimate. Writing (3.3) as $\partial_t\big((1+2g(\rho))\mu\big) = \Delta K_\tau(\mu) + g'(\rho)\mu\,\partial_t\rho$ and then testing by $v \in L^1(0,T;V)$ leads to

$$\left|\int_Q \partial_t\big((1+2g(\rho))\mu\big)\,v\right| = \left|-\int_Q \nabla K_\tau(\mu)\cdot\nabla v + \int_Q g'(\rho)\mu\,\partial_t\rho\,v\right|$$
$$\le \|K_\tau(\mu)\|_{L^\infty(0,T;V)}\|v\|_{L^1(0,T;V)} + \|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^\infty(0,T;L^4(\Omega))}\|v\|_{L^1(0,T;L^4(\Omega))}$$
$$\le \big(\|K_\tau(\mu)\|_{L^\infty(0,T;V)} + c\|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^\infty(0,T;L^4(\Omega))}\big)\|v\|_{L^1(0,T;V)}.$$

Hence, (3.13) and (3.19)–(3.20) enable us to infer that

$$\|\partial_t\big((1+2g(\rho_\tau))\mu_\tau\big)\|_{L^\infty(0,T;V^*)} \le c. \tag{3.21}$$

Passage to the limit. On setting $\zeta_\tau := (1+2g(\rho_\tau))\mu_\tau$ and recalling the a priori estimates, it turns out that there exist a triplet (μ,ρ,ξ), with $\mu \ge 0$ a.e. in Q, and functions k and ζ such that

$$\mu_\tau \to \mu \qquad \text{weakly star in } L^\infty(0,T;L^6(\Omega)), \tag{3.22}$$
$$\rho_\tau \to \rho \qquad \text{weakly star in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W), \tag{3.23}$$
$$\xi_\tau \to \xi \qquad \text{weakly star in } L^\infty(0,T;H), \tag{3.24}$$
$$K_\tau(\mu_\tau) \to k \qquad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V), \tag{3.25}$$
$$\zeta_\tau \to \zeta \qquad \text{weakly star in } W^{1,\infty}(0,T;V^*)\cap L^\infty(0,T;L^6(\Omega)), \tag{3.26}$$

at least for a susequence $\tau = \tau_i \searrow 0$. By (3.23), (3.25), and the compact embeddings $W \subset C^0(\overline{\Omega})$ and $V \subset H$, we can apply well-known results (see, e.g., [24, Sect. 8, Cor. 4]) and infer that

$$\rho_\tau \to \rho \qquad \text{strongly in } C^0(\overline{Q}), \tag{3.27}$$
$$K_\tau(\mu_\tau) \to k \qquad \text{strongly in } C^0([0,T];H) \text{ and a.e. in } Q. \tag{3.28}$$

Now, convergences (3.24) and (3.27) imply that $\xi \in \beta(\rho)$ a.e. in Q, as is well known (see, e.g., [4, Prop. 2.5, p. 27]). By (3.27), we also recover the Cauchy condition (2.15) and the fact that $\phi(\rho_\tau) \to \phi(\rho)$ strongly in $C^0(\overline{Q})$ for every continuous function $\phi : \mathbb{R} \to \mathbb{R}$; of course, this property can be applied to g, g', and π (see (2.8)). From (3.22) and (3.20), it is not difficult to check that $\mathcal{T}_\tau\mu_\tau \to \mu$ weakly star in $L^\infty(0,T;L^6(\Omega))$; hence, the product $\mathcal{T}_\tau\mu_\tau g'(\rho_\tau)$ has the weak star limit $\mu g'(\rho)$ in $L^\infty(0,T;L^6(\Omega))$ and (2.14) can follow from (3.4).

Next, we check that μ_τ converges to μ a.e. in Q. Note that K_τ^{-1} converges to K^{-1} uniformly on $[0,R]$ for every $R > 0$. Hence, (3.28) implies $\mu_\tau \to K^{-1}(k)$ a.e. in Q, and a comparison with (3.22) enables us to deduce that $K^{-1}(k) = \mu$ (whence $k = K(\mu)$) and

$$\mu_\tau \to \mu \quad \text{strongly in } L^p(0,T;L^q(\Omega)), \text{ for every } p < +\infty \text{ and } q < 6, \tag{3.29}$$

and a.e. in Q (the Egorov theorem is used here). Then, we can also infer that ζ_τ converges to $(1+2g(\rho))\mu$ a.e. in Q, whence $\zeta = (1+2g(\rho))\mu$ by comparing with (3.26). On the other hand, (3.26) implies that $\zeta_\tau \to \zeta$ strongly in $C^0([0,T];V^*)$, thus, $\zeta_\tau(0) \to \zeta(0)$ strongly in V^*, so that the Cauchy condition (2.25) is verified.

It remains for us to identify the limit of $\mu_\tau g'(\rho_\tau)\partial_t\rho_\tau$: we show that it weakly converges to $\mu g'(\rho)\partial_t\rho$ in some L^p-type space. By choosing, e.g., $p = 2,\ q = 4$ in (3.29) and exploiting the weak star convergence of $\partial_t\rho_\tau$ in $L^\infty(0,T;H)$ (see (3.23)) and the uniform convergence of $g'(\rho_\tau)$, we deduce that $\mu_\tau g'(\rho_\tau)\partial_t\rho_\tau \to \mu g'(\rho)\partial_t\rho$ weakly in $L^2(0,T;L^{4/3}(\Omega))$. At this point, it is straightforward to derive (2.24) in an integrated form, namely,

$$\int_0^T \langle \partial_t\big((1+2g(\rho))\mu\big)(t), v(t)\rangle\, dt - \int_Q \mu g'(\rho)\partial_t\rho\, v + \int_Q \nabla K(\mu)\cdot\nabla v = 0 \tag{3.30}$$

for any $v \in L^2(0,T;V) \subset L^2(0,T;L^4(\Omega))$, whence the time-pointwise version (2.24).

End of the proof of Theorem 2.1. Here, we check the last part of the statement of Theorem 2.1. In the case $\mu_* = 0$, we have $\kappa(r) \geq \kappa_*$ for every $r \geq 0$. This implies that the inverse function $K^{-1} : [0,+\infty) \to [0,+\infty)$ is Lipschitz continuous. Hence, (2.22) yields

$$\mu = K^{-1}(K(\mu)) \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

i.e., (2.16) holds. In particular, we can write

$$\nabla K(\mu) = \kappa(\mu)\nabla\mu \quad \text{and} \quad \partial_t\big((1+2g(\rho))\mu\big) = \mu\,\partial_t(1+2g(\rho)) + (1+2g(\rho))\partial_t\mu$$

and thus replace the weak formulation by the strong one. Next, we point out that (2.18) implies that (2.20) holds in the sense of distributions, whence (2.17) follows by comparison. Finally, (2.19) is a consequence of (2.25) and the continuity of μ from $[0,T]$ to H.

4 Regularity properties

In this section, we prove Theorems 2.2 and 2.3 and make some remarks on the regularity of solutions. To achieve the first result, we adapt the arguments used in [9, 12].

Proof of Theorem 2.2. Set $\mu_0^* := \max\{1, \|u_0\|_\infty\}$. We would like to test (2.24) by $(\mu-k)^+$, for some constant k greater than μ_0^*. We have to check that $(\mu-k)^+$ is an admissible test function, which is not obvious since $\nabla\mu$ might not exist in the usual sense.

Now, thanks to (2.4)–(2.5), K is a strictly increasing mapping from $[0,+\infty)$ onto itself and K^{-1} is Lipschitz continuous on the interval $[s_*,+\infty)$, where $s_* := K(\mu_*)$. Therefore, we can choose a strictly increasing map $K_* : [0,+\infty) \to [0,+\infty)$ that is globally Lipschitz continuous and coincides with K^{-1} on $[s_*,+\infty)$. Hence, we have $K_*(K(r)) = r$ for every $r \geq \mu_*$ and $K_*(K(r)) < \mu_*$ for $r < \mu_*$. It follows that $(r-k)^+ = (K_*(K(r))-k)^+$ for every $r \geq 0$ if $k \geq \mu_*$. On the other hand, $K_*(K(\mu)) \in H^1(0,T;H)\cap L^2(0,T;V)$ by (2.22). Hence, $(\mu-k)^+$ enjoys the same regularity and is an admissible test function in (2.24) provided that $k \geq \mu_*$. Thus, from now on we assume $k \geq \max\{\mu_0^*, \mu_*\}$. We have from (2.24) that

$$\int_0^t \langle \partial_t\big[(1+2g(\rho))\mu\big](s), (\mu(s)-k)^+\rangle\, ds + \int_{Q_t} \nabla K(\mu)\cdot\nabla(\mu-k)^+$$
$$= \int_{Q_t} \mu\,\partial_t g(\rho)\,(\mu-k)^+$$

for every $t \in [0,T]$. A simple rearrangement yields:

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+\rangle \, ds + \int_{Q_t} \nabla K(\mu)\cdot \nabla(\mu-k)^+ \tag{4.1}$$
$$= \int_{Q_t} \partial_t g(\rho)\, |(\mu-k)^+|^2 - k\int_{Q_t} \partial_t g(\rho)\, (\mu-k)^+ .$$

Note that $1/(1+2g(\rho)) \in H^1(0,T;V)\cap L^\infty(0,T;W)$, in view of (2.12) and our assumptions on g (cf. (2.7)–(2.8)). Then, we can apply the 'chain-rule' Lemma 5.1 in [12] to deduce that

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+\rangle \, ds = \int_{Q_t} (\mu-k)\partial_t\big[(1+2g(\rho))(\mu-k)^+\big]$$
$$= \int_{Q_t} 2\partial_t g(\rho)\, |(\mu-k)^+|^2 + \int_{Q_t} (\mu-k)(1+2g(\rho))\, \partial_t(\mu-k)^+$$
$$= \frac{1}{2}\int_{Q_t} \partial_t\big[(1+2g(\rho))|(\mu-k)^+|^2\big] + \int_{Q_t} \partial_t g(\rho)\, |(\mu-k)^+|^2 .$$

On the other hand, we have that

$$\nabla(\mu-k)^+ = \nabla\mu = \nabla K^{-1}(K(\mu)) = (K^{-1})'(K(\mu))\nabla K(\mu) = \frac{1}{\kappa(\mu)}\,\nabla K(\mu)$$

almost everywhere in the set where $\mu \geq k$. Furthermore, we observe that $(\mu(0)-k)^+ = 0$ a.e. in Ω on account of $k \geq \mu_0^*$. Hence, (4.1) yields

$$\frac{1}{2}\int_\Omega (1+2g(\rho(t)))|(\mu(t)-k)^+|^2 + \int_{Q_t} \kappa(\mu)|\nabla(\mu-k)^+|^2 = -k\int_{Q_t} \partial_t g(\rho)\, (\mu-k)^+ .$$

As g is nonnegative and $\kappa(r) \geq \kappa_*$ for $r \geq k$ (because $k \geq \kappa_*$), it follows that

$$\frac{1}{2}\int_\Omega |(\mu(t)-k)^+|^2 + \kappa_*\int_{Q_t} |\nabla(\mu-k)^+|^2 \leq k\int_{Q_t} |\partial_t g(\rho)|\, (\mu-k)^+ .$$

At this point, we can repeat the argument used in [9]: indeed, the analog of (2.14) is never used there, and the whole proof is based just on the regularity $\partial_t\rho \in L^\infty(0,T;H)\cap L^2(0,T;V)$. In the present case, we have to exploit the same regularity for $\partial_t g(\rho)$, an easy consequence of (2.12) and (2.8).

Remark 4.1. The property $\mu \in L^\infty(Q)$ may lead to additional regularitity for ρ, of course under suitable assumptions on the initial data. Indeed, note that (2.14) yields:

$$\partial_t\rho - \Delta\rho + \xi = \mu g'(\rho) - \pi(\rho) \in L^\infty(Q).$$

So, if we let $\inf\rho_0$ and $\sup\rho_0$ belong to the interior of $D(\beta)$ (assuming that it is not empty, the significant case), one can easily derive that $\xi \in L^\infty(Q)$. Indeed, one can formally multiply by $|\xi|^{p-1}\operatorname{sign}\xi$ and estimate $\|\xi\|_p$ uniformly with respect to p, if the assumption on ρ_0 is satisfied. This implies that

$$\rho \in W^{1,p}(0,T;L^p(\Omega))\cap L^p(0,T;W^{2,p}(\Omega)) \quad \text{for every } p < +\infty,$$

provided that ρ_0 is smooth enough. However, no further regularity can be proved, in general, since (2.14) cannot be differentiated.[5]

Proof of Theorem 2.3. By virtue of (2.26), it turns out that $g(\rho)$ is continuous and $g'(\rho)$ is bounded. Moreover, thanks to Theorem 2.2, μ is bounded too, because $\mu_0 = K^{-1}(K(\mu_0))$ fulfills (2.28) by virtue of (2.29).

We point out that (2.20) can be seen as a uniformly parabolic linear equation for $w = K(\mu)$, with continuous coefficients and a right-hand side belonging to $L^\infty(0,T;H)$. Indeed, as $\partial_t\mu = (\kappa(\mu))^{-1}\partial_t(K(\mu))$, we have that

$$\partial_t w - \frac{\kappa(\mu)}{1+2g(\rho)}\Delta w = -\frac{\mu\, g'(\rho)\,\partial_t\rho}{1+2g(\rho)}.$$

Therefore, recalling that $w_0 := K(\mu_0) \in W$ and applying the optimal L^p-L^q-regularity results (see, e.g., [15, Thm. 2.3]), we infer (2.30) and Theorem 2.3 is proved.

Let us remark that (2.30) holds under an assumption on w_0 that is actually weaker than $w_0 \in W$. The optimal condition involves a proper Besov space and gives a similar result for a fixed p. We are going to exploit (2.30) just with $p = 4$ in the proof of our continuous dependence result. As $\mu_* = 0$ and

$$\kappa_*|\nabla\mu| \le \kappa(\mu)|\nabla\mu| = |\nabla K(\mu)| \quad \text{a.e. in } Q,$$

(2.30) implies that

$$|\nabla\mu| \in L^4(0,T;L^6(\Omega)); \tag{4.2}$$

this regularity is used for $|\nabla\mu_i|$, $i = 1,2$ in the proof of Theorem 2.4 here below.

5 Continuous dependence

In this section, we prove Theorem 2.4. We point out that, under the assumptions of this theorem, both solutions (μ_1,ρ_1,ξ_1) and (μ_2,ρ_2,ξ_2) satisfy the regularity properties stated in Theorems 2.2 and 2.3. In particular, the following estimate holds true (cf. (4.2) and (2.12)):

$$\sum_{i=1}^{2}\left\{\|\mu_i\|_{L^4(0,T;W^{1,6}(\Omega))} + \|\rho_i\|_{L^4(0,T;W^{1,6}(\Omega))}\right\} \le c, \tag{5.1}$$

for some constant c depending only on the data of the problems, including the initial values $(\mu_{0,i},\rho_{0,i})$, $i = 1,2$.

As a general strategy for both solutions, we rewrite equation (2.20) in the form

$$\partial_t\big(\mu/\alpha(\rho)\big) - \alpha(\rho)\,\mathrm{div}\big(\kappa(\mu)\nabla\mu\big) = 0, \tag{5.2}$$

[5] Unless β has a special form, for instance (compute the derivative of the convex part of (1.5)) ,

$$\beta(\rho) = \ln\frac{\rho}{1-\rho},$$

like in [9]. By the way, in this case the condition $\xi \in L^\infty(Q)$ is equivalent to $\inf\,\rho > 0$ and $\sup\,\rho < 1$.

where the function $\alpha : \mathbb{R} \to (0, +\infty)$ is defined by

$$\alpha(r) := \bigl(1 + 2g(r)\bigr)^{-1/2} \quad \text{for } r \in \mathbb{R}. \tag{5.3}$$

More precisely, let us consider the variational formulation of (5.2) that accounts for the homogeneous Neumann boundary condition and involves a related unknown function, namely,

$$z := \frac{\mu}{\alpha(\rho)}\,,$$

with

$$\int_\Omega \partial_t z(t)\, v + \int_\Omega \kappa\bigl(\alpha(\rho(t))\, z(t)\bigr)\nabla\bigl(\alpha(\rho(t))\, z(t)\bigr) \cdot \nabla\bigl(\alpha(\rho(t))v\bigr) = 0$$
$$\text{for a.a. } t \in (0, T) \text{ and for every } v \in V. \tag{5.4}$$

We point out that, for $i = 1, 2$, the functions $z_i := \mu_i/\alpha(\rho_i)$ are bounded, since both μ_i and ρ_i are. Indeed, (2.26) holds and Theorem 2.2 can be applied (recall (2.29) and (2.28)). Moreover, from (5.1) we can easily deduce that

$$\sum_{i=1}^{2} \|z_i\|_{L^4(0,T;W^{1,6}(\Omega))} \le c \tag{5.5}$$

as well. For the sake of convenience, for $i = 1, 2$ we set

$$a_i := \alpha(\rho_i), \quad \kappa_i := \kappa(\mu_i)$$

and observe that (z_i, ρ_i) satisfy (5.4). In order to simplify formulas and make the proof more readable, let us adopt the notation:

$$\mu := \mu_1 - \mu_2, \quad \rho := \rho_1 - \rho_2, \quad \xi := \xi_1 - \xi_2, \quad z := z_1 - z_2\,, \quad a := a_1 - a_2,$$
$$\mu_0 := \mu_{0,1} - \mu_{0,2}, \quad \rho_0 := \rho_{0,1} - \rho_{0,2}, \quad \text{and} \quad z_0 := \frac{\mu_{0,1}}{\alpha(\rho_{0,1})} - \frac{\mu_{0,2}}{\alpha(\rho_{0,2})}\,.$$

Note that z_0 is the initial value of the difference $z_1 - z_2$.

It is our intention to prove the preliminary estimate

$$\int_\Omega |z(t)|^2 + \int_{Q_t} |\nabla(a_1 z)|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 \le c\left\{\int_\Omega |z_0|^2 + \int_\Omega |\rho_0|^2\right\}, \tag{5.6}$$

and then to explain how to derive (2.31) from (5.6). Here, c depends on the structure and on an upper bound of the norms of the initial data involved in our assumptions. Indeed, in the subsequent estimates, the (varying) value of the constant c depends on some norms of the considered solutions, e.g., through $\|z_i\|_\infty$. However, such norms can be estimated in terms of an upper bound of the quantities that appear in (2.10)–(2.11), (2.28) and (2.29).

We proceed as follows. Having written (5.4) for both solutions and chosen $v = z_1 - z_2$ in the difference, we integrate over $(0, t)$, for an arbitrary $t \in (0, T)$. At the same time, we consider (2.14) for both solutions and test the difference by $\rho_1 - \rho_2$, integrating over Q_t. Finally, we take a suitable linear combination of the resulting equalities and perform a number of estimates that lead us to apply the Gronwall lemma. However, before starting, we recall a list of inequalities that follow from the boundedness and the Lipschitz continuity of α, α', and $1/\alpha$ (cf. (2.7)–(2.8)) and from the Lipschitz continuity of κ. Indeed, in spite of (2.2), here we may assume κ *globally* Lipschitz continuous, as both μ_1 and μ_2 are bounded. We easily infer that

$$
\begin{aligned}
&|a| = |\alpha(\rho_1) - \alpha(\rho_2)| \le c|\rho|, \\
&|\nabla a| = |\alpha'(\rho_1)\nabla\rho + \big(\alpha'(\rho_1) - \alpha'(\rho_2)\big)\nabla\rho_2| \le c|\nabla\rho| + c|\nabla\rho_2|\,|\rho|, \\
&|\nabla a_i| + |\nabla a_i^{-1}| \le c|\nabla\rho_i|, \\
&|\kappa_1 - \kappa_2| \le c|\mu|, \\
&|\mu| \le |a|\,|z_1| + a_2|z| \le c|a| + c|z| \le c|\rho| + c|z|, \\
&|\nabla z| = |\nabla\big(a_1 z\,/a_1\big)| \le c|\nabla(a_1 z)| + c|\nabla\rho_1|\,|z|.
\end{aligned}
$$

In what follows, we will repeatedly use these inequalities without alerting the reader. Let us state a lemma that we proved in [12, Section 6].

Lemma 5.1. *For each $\varphi \in L^4(0,T;L^6(\Omega))$, we have that*

$$
\begin{aligned}
\int_{Q_t} \varphi^2(|z|^2 + |\rho|^2) &\le \varepsilon \int_{Q_t} \big(|\nabla(a_1 z)|^2 + |\nabla\rho|^2\big) \\
&\quad + c_\varepsilon \int_0^t \big(1 + \|\nabla\rho_1(s)\|_6^4 + \|\varphi(s)\|_6^4\big)\big(\|z(s)\|_2^2 + \|\rho(s)\|_2^2\big)\,ds,
\end{aligned}
\tag{5.7}
$$

for every $\varepsilon > 0$ and every $t \in [0,T]$.

Let us start our program and, in order to make the argument more transparent, let us deal just with the first equation, if only for a while. We have that

$$
\frac{1}{2}\int_\Omega |z(t)|^2 + \int_{Q_t} \big(\kappa_1\nabla(a_1 z_1)\cdot\nabla(a_1 z) - \kappa_2\nabla(a_2 z_2)\cdot\nabla(a_2 z)\big) = \frac{1}{2}\int_\Omega |z_0|^2.
$$

It is convenient to transform the last integrand on the left-hand side as follows:

$$
\begin{aligned}
&\kappa_1\nabla(a_1 z_1)\cdot\nabla(a_1 z) - \kappa_2\nabla(a_2 z_2)\cdot\nabla(a_2 z) \\
&= \kappa_1|\nabla(a_1 z)|^2 + \kappa_1\nabla(a_1 z_2)\cdot\nabla(a_1 z) - \kappa_2\nabla(a_2 z_2)\cdot\nabla(a_2 z) \\
&= \kappa_1|\nabla(a_1 z)|^2 + \kappa_1\nabla(a z_2)\cdot\nabla(a_1 z) + \kappa_1\nabla\mu_2\cdot\nabla(a_1 z) - \kappa_2\nabla\mu_2\cdot\nabla(a_2 z). \\
&= \kappa_1|\nabla(a_1 z)|^2 + \kappa_1\nabla(a z_2)\cdot\nabla(a_1 z) + \kappa_1\nabla\mu_2\cdot\nabla(a z) + (\kappa_1 - \kappa_2)\nabla\mu_2\cdot\nabla(a_2 z).
\end{aligned}
$$

Then, thanks to assumption (2.4) with $\mu_* = 0$, the above equality yields:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |z(t)|^2 + \kappa_*\int_{Q_t} |\nabla(a_1 z)|^2 - \frac{1}{2}\int_\Omega |z_0|^2 \\
&\le -\int_{Q_t} \kappa_1\nabla(a z_2)\cdot\nabla(a_1 z) - \int_{Q_t} \kappa_1\nabla\mu_2\cdot\nabla(a z) \\
&\quad - \int_{Q_t} (\kappa_1 - \kappa_2)\nabla\mu_2\cdot\nabla(a_2 z),
\end{aligned}
\tag{5.8}
$$

where each term on the right-hand side has to be estimated separately. First, it is straightforward to obtain

$$
\begin{aligned}
&-\int_{Q_t} \kappa_1 \nabla(az_2)\cdot\nabla(a_1 z) \le \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + c\int_{Q_t}\bigl(z_2^2|\nabla a|^2 + a^2|\nabla z_2|^2\bigr)\\
&\le \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + c\int_{Q_t}\bigl(|\nabla\rho|^2 + |\nabla\rho_2|^2\,|\rho|^2\bigr) + c\int_{Q_t}|\nabla z_2|^2\,|\rho|^2\\
&\le \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + C_1\int_{Q_t}|\nabla\rho|^2 + c\int_{Q_t}\bigl(|\nabla\rho_2|^2 + |\nabla z_2|^2\bigr)\,|\rho|^2,
\end{aligned}
\tag{5.9}
$$

where we have denoted by C_1 the constant we want to refer to. As to the second term, we deduce that

$$
\begin{aligned}
&-\int_{Q_t} \kappa_1 \nabla\mu_2\cdot\nabla(az) \le \kappa^*\int_{Q_t}|\nabla\mu_2|\bigl(|a|\,|\nabla z| + |z|\,|\nabla a|\bigr)\\
&\le c\int_{Q_t}|\nabla\mu_2|\bigl(|\nabla(a_1 z)|\,|\rho| + |z|\,|\nabla\rho_1|\,|\rho| + |z|\,|\nabla\rho| + |z|\,|\nabla\rho_2|\,|\rho|\bigr)\\
&\le \frac{\kappa_*}{4}\int_{Q_t}|\nabla(a_1 z)|^2 + c\int_{Q_t}|\nabla\mu_2|^2\,|\rho|^2 + \int_{Q_t}|\nabla\rho|^2\\
&\quad + c\int_{Q_t}|\nabla\mu_2|^2\,|z|^2 + c\int_{Q_t}\bigl(|\nabla\rho_1|^2 + |\nabla\rho_2|^2\bigr)\,|\rho|^2.
\end{aligned}
\tag{5.10}
$$

For the third and last term, we argue as follows:

$$
\begin{aligned}
&-\int_{Q_t}(\kappa_1-\kappa_2)\nabla\mu_2\cdot\nabla(a_2 z)\\
&\le c\int_{Q_t}|\mu|\,|\nabla\mu_2|\,|\nabla(a_2 z)| \le c\int_{Q_t}\bigl(|\rho|+|z|\bigr)\,|\nabla\mu_2|\,\bigl(|a_2|\,|\nabla z| + |z|\,|\nabla\rho_2|\bigr)\\
&\le c\int_{Q_t}\bigl(|\rho|+|z|\bigr)\,|\nabla\mu_2|\,\bigl(|\nabla(a_1 z)| + |\nabla\rho_1|\,|z| + |z|\,|\nabla\rho_2|\bigr)\\
&\le \frac{\kappa_*}{4}\int_{Q_t}|\nabla(a_1 z)|^2 + c\int_{Q_t}|\nabla\mu_2|^2\bigl(|\rho|^2+|z|^2\bigr)\\
&\quad + c\int_{Q_t}|\nabla\mu_2|\bigl(|\nabla\rho_1| + |\nabla\rho_2|\bigr)|z|\,\bigl(|\rho|+|z|\bigr)\\
&\le \frac{\kappa_*}{4}\int_{Q_t}|\nabla(a_1 z)|^2 + c\int_{Q_t}\bigl(|\nabla\mu_2|^2 + |\nabla\rho_1|^2 + |\nabla\rho_2|^2\bigr)\bigl(|\rho|^2+|z|^2\bigr).
\end{aligned}
$$

Next, we deal with the second equation. Testing the difference of (2.14) by ρ yields:

$$
\begin{aligned}
&\frac12\int_\Omega|\rho(t)|^2 + \int_{Q_t}|\nabla\rho|^2 + \int_{Q_t}\xi\rho\\
&= \int_{Q_t}\bigl(\mu_1 g'(\rho_1) - \mu_2 g'(\rho_2) - \pi(\rho_1) + \pi(\rho_2)\bigr)\rho + \frac12\int_\Omega|\rho_0|^2.
\end{aligned}
\tag{5.11}
$$

We note that the product $\xi\rho$ in the left-hand side is nonnegative by monotonicity, while the first integrand on the right-hand side can be estimated as follows:

$$\begin{aligned}&\bigl(\mu_1 g'(\rho_1) - \mu_2 g'(\rho_2) - \pi(\rho_1) + \pi(\rho_2)\bigr)\rho \\ &\leq \bigl(|\mu|\,|g'(\rho_1)| + |\mu_2|\,|g'(\rho_1) - g'(\rho_2)| + |\pi(\rho_1) - \pi(\rho_2)|\bigr)|\rho| \\ &\leq |g'(\rho_1)|\,|\mu|\,|\rho| + c|\mu_2|\,|\rho|^2 + c|\rho|^2 \leq c\bigl(|\mu|^2 + |\rho|^2\bigr) \leq c\bigl(|z|^2 + |\rho|^2\bigr).\end{aligned}$$

Now, on inspecting the coefficients of the integral $\int_{Q_t} |\nabla\rho|^2$ in the right-hand sides of (5.9) and (5.10), it appears convenient to multiply (5.11) by $C_1 + 2$ and then add it to (5.8). Having done this, it is straightforward to deduce that

$$\begin{aligned}&\int_\Omega |z(t)|^2 + \int_{Q_t} |\nabla(a_1 z)|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 \\ &\leq c\int_{Q_t} \bigl(|\nabla\mu_2| + |\nabla\rho_1| + |\nabla\rho_2| + |\nabla z_2| + 1\bigr)^2 \bigl(|z|^2 + |\rho|^2\bigr) + c\int_\Omega |z_0|^2 + c\int_\Omega |\rho_0|^2.\end{aligned}$$

At this point, we observe that (5.1) and (5.5) allow us to apply Lemma 5.1 with $\varphi = |\nabla\mu_2| + |\nabla\rho_1| + |\nabla\rho_2| + |\nabla z_2| + 1$. After such an application, we choose $\varepsilon > 0$ small enough and use the Gronwall lemma. Thus, we obtain (5.6). Now, we easily check that

$$\begin{aligned}&|\mu| \leq c|\rho| + c|z|, \\ &|\nabla\mu| = |\nabla\left(a_1\, z + z_2\, a\right)| \leq c|\nabla(a_1 z)| + c|\nabla z_2|\,|\rho| + z_2|\nabla a| \\ &\qquad \leq c|\nabla(a_1 z)| + c\left(|\nabla z_2| + |\nabla\rho_2|\right)|\rho| + c|\nabla\rho|\end{aligned}$$

almost everywhere in Q and

$$|z_0| \leq c(1/a_1)|\mu_0| + c\, z_{0,2}|a| \leq c\left(|\mu_0| + |\rho_0|\right)$$

in Ω. By combining these inequalities with (5.6), we obtain the estimate

$$\begin{aligned}&\int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 \\ &\leq c\int_{Q_t} \bigl(|\nabla z_2| + |\nabla\rho_2|\bigr)^2 |\rho|^2 + c\int_\Omega |\mu_0|^2 + c\int_\Omega |\rho_0|^2.\end{aligned}$$

Hence, we can apply once more the Gronwall lemma and plainly conclude that (2.31) holds true.

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