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A central limit theorem for the effective conductance: I. Linear boundary data and small ellipticity contrasts

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ABSTRACT. We consider resistor networks on $\mathbb{Z}^d$ where each nearest-neighbor edge is assigned a non-negative random conductance. Given a finite set with a prescribed boundary condition, the effective conductance is the minimum of the Dirichlet energy over functions that agree with the boundary values. For shift-ergodic conductances, linear (Dirichlet) boundary conditions and square boxes, the effective conductance scaled by the volume of the box is known to converge to a deterministic limit as the box-size tends to infinity. Here we prove that, for i.i.d. conductances with a small ellipticity contrast, also a (non-degenerate) central limit theorem holds. The proof is based on the corrector method and the Martingale Central Limit Theorem; a key integrability condition is furnished by the Meyers estimate. More general domains, boundary conditions and arbitrary ellipticity contrasts are to be addressed in a subsequent paper.

1. INTRODUCTION AND MAIN RESULT

As is well known, most materials, regardless how pure they may seem at the macroscopic level, have a rather complicated microscopic structure. It may then come as a surprise that physical phenomena such as heat or electric conduction are described so well using differential equations with smooth, sometimes even constant, coefficients. An explanation for this has been offered by homogenization theory: rapid oscillations at the microscopic level are smoothened out (i.e., homogenized) at the macroscopic scale; that is, in the limit when the separation between these scales tends to infinity. Note, however, that this does not mean that the microscopic structure is simply washed out. Indeed, while it disappears from the structure of the equations, it remains embedded in the values of effective material constants, e.g., the coefficients.

An illustrative example of a homogenization problem is that of effective conductance. We will formulate an instance of this problem directly in the setting of resistor networks. Consider the d-dimensional hypercubic lattice $\mathbb{Z}^d$ and suppose that each unordered nearest-neighbor edge $\langle x,y\rangle$ is assigned a value $a_{xy} = a_{yx} \in (0,\infty)$ — called the *conductance* of $\langle x,y\rangle$. For a finite set $\Lambda \subset \mathbb{Z}^d$, let $\mathbb{B}(\Lambda)$ be those edges with at least one endpoint in Λ. Given a function $f\colon \mathbb{Z}^d \to \mathbb{R}$, let

$$Q_\Lambda(f) := \sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda)} a_{xy}\big[f(y)-f(x)\big]^2, \tag{1.1}$$

where each pair (x,y) is counted only once. This is the electrostatic (Dirichlet) energy for the potential f with Dirichlet boundary condition on the boundary vertices of Λ.

Consider now the square box $\Lambda_L := [0,L)^d \cap \mathbb{Z}^d$. A quantity of prime interest for us is the *effective conductance*,

$$C_L^{\mathrm{eff}}(t) := \inf\big\{Q_{\Lambda_L}(f)\colon\, f(x) = t\cdot x,\ \forall x \in \partial\Lambda_L\big\}, \tag{1.2}$$

where $t \in \mathbb{R}^d$ and where $\partial\Lambda$ are those vertices outside Λ that have an edge into Λ. By Kirchhoff's and Ohm's laws (see, e.g., Doyle and Snell [5]), this represents the total electric current flowing through the network when the boundary vertices are kept at voltage $t\cdot x$.

For homogeneous resistor networks, i.e., when $a_{xy} := a$ for all $\langle x,y\rangle$, the infimum (1.2) is achieved by $f(x) := t\cdot x$ and so $C_L^{\mathrm{eff}}(t) = a|t|^2L^d(d+o(1))$. A question of (reasonably) practical interest is then what happens when the conductances a_{xy} are no longer constant, but remain close to a constant. In particular, we may assume that they are uniformly elliptic, i.e.,

$$\exists\lambda \in (0,1),\ \forall\langle x,y\rangle\colon \qquad \lambda \le a_{xy} \le \frac{1}{\lambda}. \tag{1.3}$$

A comparison of Q_Λ with these a_{xy}'s and the homogeneous case shows that $C_L^{\mathrm{eff}}(t)$ is still of the order of $|t|^2L^d$. Moreover, thanks to the choice of the linear boundary condition, by subadditivity arguments the limit

$$c_{\text{eff}}(t) := \lim_{L\to\infty} \frac{1}{L^d} C_L^{\text{eff}}(t) \tag{1.4}$$

exists almost surely for any ergodic distribution of the conductances. The problem left to resolve is thus a characterization of the limit value.

Interestingly, $c_{\text{eff}}(t)$ can be characterized in large generality: Suppose that $a_{xy} = a_{xy}(\omega)$ is a sample from a shift-ergodic law $\mathbb{P}$ on the product space indexed by edges of $\mathbb{Z}^d$. Formally, we denote by $\mathbb{B}(\mathbb{Z}^d)$ the set of all edges in the lattice, write $\Omega := \bigotimes_{\mathbb{B}(\mathbb{Z}^d)}[\lambda, 1/\lambda]$ for the set of configurations satisfying (1.3) and interpret $a_{xy}(\omega) = a_{yx}(\omega)$ as the coordinate projection on edge $\langle x, y\rangle$. As is well known (e.g., Jikov, Kozlov and Oleinik [7] with ideas going back to Papanicolaou and Varadhan [14], Kozlov [8] and Künnermann [9]),

$$c_{\text{eff}}(t) = \inf_{g\in L^\infty(\mathbb{P})} \mathbb{E}\biggl(\sum_{x=\hat{e}_1,\dots,\hat{e}_d} a_{0,x}(\omega)\bigl|t\cdot x + \nabla_x g(\omega)\bigr|^2\biggr). \tag{1.5}$$

Here $\mathbb{E}$ is expectation with respect to $\mathbb{P}$, the objects $\hat{e}_1,\dots,\hat{e}_d$ are the unit coordinate vectors in $\mathbb{R}^d$ and $\nabla_x g(\omega) := g\circ\tau_x(\omega) - g(\omega)$ is the gradient of g in direction of $x\in\mathbb{Z}^d$ with τ_x denoting the shift by x; i.e., the map on the probability space such that $a_{yz}(\tau_x\omega) := a_{x+y,x+z}(\omega)$. The quantity on the right-hand side of (1.5) can be interpreted as the Dirichlet energy density — with the spatial average naturally replaced by the ensemble average.

Once the (deterministic) leading-order of $C_L^{\text{eff}}(t)$ has been identified, the next natural question is that of fluctuations. It is obvious — e.g., by checking the explicitly computable $d = 1$ case — that no universal limit law can be expected for general conductance distributions, but progress could perhaps be made for the (physically most appealing) i.i.d. case. However, even here establishing just the order of magnitude of the fluctuations turned out to be an arduous task. Indeed, more than a decade ago Wehr [18] showed that $\text{Var}(C_L^{\text{eff}}) \ge \Theta(L^d)$ but a corresponding upper bound has been furnished only recently by Gloria and Otto [6]. Both of these results contain important technical caveats: Wehr requires continuously distributed a_{xy}'s while Gloria and Otto express their results under a "massive" cutoff.

Gloria and Otto [6] drew important ideas from an earlier unpublished note by Naddaf and Spencer [12] where (optimal) upper bounds on the variance have been derived for certain correlated conductance laws. The main tool of [12] is the *Meyers estimate* (cf Meyers [11]), to be used heavily in the present note as well. From earlier derivations of (suboptimal) variance upper bounds we find worthy of mentioning an old paper by Yurinskii [17], cf [6] for a thorough discussion of this work, and a more recent paper by Benjamini and Rossignol [1]. Indidentally, the Meyers estimate was also invoked in the analysis of finite-volume approximations to $c_{\text{eff}}(t)$ by Caputo and Ioffe [4].

The goal of the present note is to prove that, for i.i.d. conductances which are (deterministically) not too far from a constant, the asymptotic law of $C_L^{\text{eff}}(t)$ is in fact Gaussian. Explicitly, let $\mathscr{N}(\mu,\sigma^2)$ denote the normal random variable with mean μ and variance σ^2. Then we have:

Theorem 1.1 *Suppose the conductances a_{xy} are i.i.d. For each $d\ge 1$, there is $\lambda = \lambda(d)\in(0,1)$ such that the following holds: If* (1.3) *is satisfied $\mathbb{P}$-a.s. with this λ, then for each $t\in\mathbb{R}^d$ there is $\sigma_t^2\in[0,\infty)$ such that*

$$\frac{C_L^{\text{eff}}(t) - \mathbb{E}C_L^{\text{eff}}(t)}{|\Lambda_L|^{1/2}} \;\underset{L\to\infty}{\overset{\text{law}}{\longrightarrow}}\; \mathscr{N}(0,\sigma_t^2). \tag{1.6}$$

Moreover, $\sigma_t^2 > 0$ whenever $t\neq 0$ and the conductance law is non-degenerate.

A few remarks are in order:

Remarks 1.2 (1) Notice that the above does not give us much information on the "order expansion" of $C_L^{\text{eff}}(t)$. Indeed, we know that $\mathbb{E}C_L^{\text{eff}}(t)$ is to the leading order equal to $c_{\text{eff}}(t)|\Lambda_L|$ but when this order is subtracted, the next-order term is (presumably) of boundary size. In $d \geq 3$, this is still larger than the typical size of the fluctuations. What the above does tell us is the character of the leading order *random* term.

(2) There is in fact a formula for σ_t^2, see Theorem 2.7 below, which also shows that $t \mapsto \sigma_t^2$ is of a bi-quadratic (and thus smooth) form. However, the formula involves complicated conditioning and does not seem very useful for practical computations.

(3) There is no restriction on the single-conductance law other than (1.3). In particular, $\mathbb{P}$ can have a non-absolutely continuous part including atoms; the support need not be an interval. Certain technical problems do arise at this level of generality; these are discussed in Section 2.5 which, we believe, is of independent interest.

We prove Theorem 1.1 by a reduction to the Martingale Central Limit Theorem. There are two main technical ingredients: homogenization theory (which enables a stationary martingale approximation of $C_L^{\text{eff}}(t)$) and analytical estimates for finite-volume harmonic coordinates (by which we control the errors in the martingale approximation). The restrictions to rectangular boxes, linear boundary conditions and small ellipticity contrasts permit us to encapsulate the analytical input into a single step, the Meyers estimate, cf Proposition 2.4 and Theorem 4.4. These restrictions can be relaxed but not without lengthy additional arguments not all of which have been handled satisfactorily at this time. These are deferred to a follow-up paper.

We remark that two recent preprints have been brought to our attention at the time this work was first announced in conference talks. First, Nolen [13] has established a normal approximation to the effective conductance defined over a periodic environment, in the limit when the period tends to infinity. Second, in a preprint that was posted at the time of writing the present note, Rossignol [15] formulates and proves a central limit law for the *effective resistance* for the corresponding problem on a torus. Rossignol's setting is based on minimizing the electrostatic energy over currents (rather than potentials) subject to a restriction on the total current flowing around the torus. By a well known reciprocity relation between effective conductance and resistance, these papers appear to address very similar problems.

The present paper differs from both Nolen [13] and Rossignol [15] mainly in its emphasis on fixed (Dirichlet), as opposed to periodic, boundary conditions. Indeed, a majority of our technical work is aimed at controlling the resulting boundary effects. Also the way a Gaussian limit law is established is quite different: Nolen appeals to a second-order Poincaré inequality, Rossignol uses concentration-of-measure techniques while we invoke the Martingale Central Limit Theorem. A slight deficiency of the present work compared to [13] and [15] is the limitation on ellipticity contrast. Nolen overcomes this by adapting lengthy analytical estimates from Gloria and Otto [6], for Rossignol this seems to come naturally through the concentration-of-measure approach. While we believe that the Gloria-Otto machinery applies in our situation as well, in the present paper we decided to sacrifice on generality somewhat and solve only the simplest non-trivial (yet still physically appealing) case.

2. Key ingredients

Here we discuss the strategy of the proof of Theorem 1.1 and state its principal ingredients in the form of suitable propositions. The actual proofs begin in Section 3.

2.1 Martingale approximation.

A standard way to control fluctuations of a function of i.i.d. random variables is by way of a *martingale approximation*. Let us order the random variables $\{a_{xy} \colon \langle x,y\rangle \in \mathbb{B}(\Lambda_L)\}$ in any (for now) convenient way and let $\mathscr{F}_k$ to be the σ-algebra generated by the first k of them. (Since we only aim at a distributional convergence, the σ-algebras may depend on L.) Then

$$C_L^{\mathrm{eff}}(t) - \mathbb{E}C_L^{\mathrm{eff}}(t) = \sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} Z_k, \tag{2.1}$$

where

$$Z_k := \mathbb{E}\big(C_L^{\mathrm{eff}}(t)\big|\mathscr{F}_k\big) - \mathbb{E}\big(C_L^{\mathrm{eff}}(t)\big|\mathscr{F}_{k-1}\big). \tag{2.2}$$

Obviously, the quantity Z_k is a martingale increment. In order to show distributional convergence to $\mathscr{N}(0,\sigma^2)$, it suffices to verify the (Lindenberg-Feller-type of) conditions of the Martingale Central Limit Theorem due to Brown [3]:

(1) There exists $\sigma^2 \in [0,\infty)$ such that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}(Z_k^2|\mathscr{F}_{k-1}) \underset{L\to\infty}{\longrightarrow} \sigma^2 \tag{2.3}$$

in probability, and

(2) for each $\varepsilon > 0$,

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}\big(Z_k^2 1_{\{|Z_k|>\varepsilon|\Lambda_L|^{1/2}\}}\big|\mathscr{F}_{k-1}\big) \underset{L\to\infty}{\longrightarrow} 0 \tag{2.4}$$

in probability.

The sums on the left suggest invoking the Spatial Ergodic Theorem, but for that we would need to ensure that the individual terms in the sum are (at least approximated by) functions that are stationary with respect to shifts of $\mathbb{Z}^d$. This necessitates the following additional input:

(i) a specific choice of the ordering of the edges, and
(ii) a more explicit representation for Z_k.

We will now discuss various aspects of these in more detail.

2.2 Stationary edge ordering.

Recall that $\mathbb{B}(\mathbb{Z}^d)$ denotes the set of all (unordered) edges in $\mathbb{Z}^d$. We will order $\mathbb{B}(\mathbb{Z}^d)$ as follows: Let $\preceq$ denote the lexicographic ordering of the vertices of $\mathbb{Z}^d$. Explicitly, for $x=(x_1,\dots,x_d)$ and $y=(y_1,\dots,y_d)$ we have $x \preceq y$ if either $x=y$ or $x\neq y$ and there exists $i\in\{1,\dots,d\}$ such that $x_j=y_j$ for all $j<i$ and $x_i<y_i$. We will write $x\prec y$ if $x\neq y$ and $x\preceq y$.

For the purpose of defining a stationary ordering of the edges, and also easier notation in some calculations that are to follow, we now identify $\mathbb{B}(\mathbb{Z}^d)$ with the set of pairs (x,i), where $x\in\mathbb{Z}^d$ and $i\in\{1,\dots,d\}$, so that (x,i) corresponds to the edge between the vertices x and $x+\hat{\mathrm{e}}_i$. We will then write

$$(x,i)\preceq(y,j) \quad \text{if} \quad \begin{cases} \text{either } x\prec y \\ \text{or} \quad x=y \text{ and } i\le j. \end{cases} \tag{2.5}$$

Again, $(x,i)\prec(y,j)$ if $(x,i)\preceq(y,j)$ but $(x,i)\neq(y,j)$. It is easy to check that $\preceq$ is a complete order on $\mathbb{B}(\mathbb{Z}^d)$. A key fact about this ordering is its stationarity with respect to shifts:

Lemma 2.1 *If $(x,i) \preceq (y,j)$ then also $(x+z,i) \preceq (y+z,j)$ for all $z \in \mathbb{Z}^d$.*

Proof. This is a trivial consequence of the definition. □

Now we proceed to identify the sigma algebras $\{\mathscr{F}_k\}$ in the martingale representation above. Recall that $\Omega := \bigotimes_{\mathbb{B}(\mathbb{Z}^d)}[\lambda, {}^1\!/\!_\lambda]$ denotes the set of conductance configurations satisfying (1.3). Writing ω for elements of Ω we use $a_{xy} = a_{xy}(\omega)$, for $\langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)$, to denote the coordinate projection corresponding to edge $\langle x,y\rangle$. Given $L \geq 1$, set $N := |\mathbb{B}(\Lambda_L)|$ and let $b_1,\dots,b_N$ be the enumeration of $\mathbb{B}(\Lambda_L)$ induced by the ordering of edges $\preceq$ defined above. Then we set

(2.6) $$\mathscr{F}_k := \sigma(\omega_b\colon b \preceq b_k), \qquad k = 1,\dots,N,$$

with

(2.7) $$\mathscr{F}_0 := \sigma(\omega_b\colon b \prec b_1).$$

By definition $\mathscr{F}_0$ is independent of the edges in $\mathbb{B}(\Lambda_L)$ while $\mathscr{F}_N$ determines the entire configuration in $\mathbb{B}(\Lambda_L)$. Note also that $\mathscr{F}_k$ includes information about edges that are not in $\mathbb{B}(\Lambda_L)$. This will be of importance once we replace Z_k by a random variable that depends on all of ω.

2.3 An explicit form of martingale increment.

Having addressed the ordering of the edges, and thus the definition of the σ-algebras $\mathscr{F}_k$, we now proceed to derive a more explicit form of the quantity Z_k from (2.2). Given $\omega \in \Omega$, define the operator $\mathscr{L}_\omega$ on ($\mathbb{R}$ or $\mathbb{R}^d$-valued) functions on the lattice via

(2.8) $$(\mathscr{L}_\omega f)(x) := \sum_{y\colon \langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)} a_{xy}(\omega)\,\bigl[f(y) - f(x)\bigr].$$

This is an elliptic finite-difference operator — a random Laplacian — that shows up as the generator of the random walk among random conductances (see, e.g., Biskup [2] for a review of these connections). The existence/uniqueness for the associated Dirichlet problem implies that for any finite $\Lambda \subset \mathbb{Z}^d$ there is a unique $\Psi_\Lambda\colon \Omega \times (\Lambda \cup \partial\Lambda) \to \mathbb{R}^d$ such that $x \mapsto \Psi_\Lambda(\omega,x)$ obeys

(2.9) $$\begin{cases} \mathscr{L}_\omega \Psi_\Lambda(\omega,x) = 0, & x \in \Lambda,\\ \Psi(\omega,x) = x, & x \in \partial\Lambda. \end{cases}$$

It is then easily checked that $f(x) := t\cdot\Psi_\Lambda(\omega,x)$ is the unique minimizer of $f \mapsto Q_\Lambda(f)$ over all functions f with the boundary values $f(x) = t\cdot x$ for $x \in \partial\Lambda$. In particular, we have

(2.10) $$C_L^{\mathrm{eff}}(t) = Q_{\Lambda_L}\bigl(t\cdot\Psi_{\Lambda_L}\bigr)$$

for all $t \in \mathbb{R}^d$. The function $x \mapsto \Psi_\Lambda(\omega,x)$ will sometimes be referred to as a *finite-volume harmonic coordinate*. (The first line in (2.9) justifies this term.)

The minimum value $Q_\Lambda(t\cdot\Psi_\Lambda)$ is a differentiable and concave function of $\{a_{xy}\colon \langle x,y\rangle \in \mathbb{B}(\Lambda)\}$. As is readily checked,

(2.11) $$\frac{\partial}{\partial a_{xy}} Q_\Lambda(t\cdot\Psi_\Lambda) = \bigl[t\cdot\Psi_\Lambda(\omega,y) - t\cdot\Psi_\Lambda(\omega,x)\bigr]^2, \qquad \langle x,y\rangle \in \mathbb{B}(\Lambda).$$

This relation is of fundamental importance for what is to come.

Abusing the notation slightly, let $\omega_1,\dots,\omega_N$, with $N := |\mathbb{B}(\Lambda)|$, denote the components of the configuration ω over $\mathbb{B}(\Lambda)$ labeled in the order induced by $\preceq$ defined above. Let

(2.12) $$q(\omega_1,\dots,\omega_N) := Q_\Lambda(t\cdot\Psi_\Lambda)$$

mark explicitly the dependence of the right-hand side on these variables. The product structure of the underlying probability measure then allows us to give a more explicit expression for the increment $Z_k = Z_k(\omega_1, \dots, \omega_k)$:

$$
\begin{aligned}
Z_k = \int \mathbb{P}(\mathrm{d}\omega_k') \dots \mathbb{P}(\mathrm{d}\omega_N') \bigl[& q(\omega_1, \dots, \omega_k, \omega_{k+1}', \dots, \omega_N') \\
& - q(\omega_1, \dots, \omega_{k-1}, \omega_k', \dots, \omega_N') \bigr] \\
= \int \mathbb{P}(\mathrm{d}\omega_k') \dots \mathbb{P}(\mathrm{d}\omega_N') & \int_{\omega_k'}^{\omega_k} \mathrm{d}\tilde{\omega}_k \frac{\partial}{\partial \tilde{\omega}_k} q(\omega_1, \dots, \omega_{k-1}, \tilde{\omega}_k, \omega_{k+1}', \dots, \omega_N').
\end{aligned}
\tag{2.13}
$$

A key point is that the last partial derivative is (modulo notational changes) given by (2.11). In words, Z_k is equal to the modulus-squared of the gradient of $t \cdot \Psi_\Lambda$ over the k-th edge in $\mathbb{B}(\Lambda)$, integrated over part of the variables.

2.4 Input from homogenization theory.

In order to apply the Spatial Ergodic Theorem to the sums on the left of (2.3–2.4), we will substitute for Z_k a quantity that is stationary with respect to the shifts of $\mathbb{Z}^d$. This will be achieved by replacing the discrete gradient of Ψ_Λ — which by (2.11) enters as the partial derivative of q in the formula for Z_k — by the gradient of its stationary infinite-volume counterpart, to be denoted by ψ. The existence and properties of the latter object is quite standard:

Proposition 2.2 (Infinite-volume harmonic coordinate) *Suppose the law of the conductances is (jointly) ergodic with respect to the shifts of $\mathbb{Z}^d$ and assume* (1.3) *for some $\lambda \in (0,1)$. Then there is a function $\psi \colon \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ such that*

(1) *(ψ is $\mathscr{L}_\omega$-harmonic) $\mathscr{L}_\omega \psi(\omega, x) = 0$ for all x and $\mathbb{P}$-a.e. ω.*
(2) *(ψ is shift covariant) For $\mathbb{P}$-a.e. ω we have $\psi(\omega, 0) := 0$ and*

$$
\psi(\omega, y) - \psi(\omega, x) = \psi(\tau_x \omega, y - x), \qquad x, y \in \mathbb{Z}^d.
\tag{2.14}
$$

(3) *(ψ is square integrable)*

$$
\mathbb{E}\biggl(\sum_{x = \hat{e}_1, \dots, \hat{e}_d} a_{0,x}(\omega) \bigl| \psi(\omega, x) \bigr|^2 \biggr) < \infty.
\tag{2.15}
$$

(4) *(ψ is approximately linear) The corrector $\chi(\omega, x) := \psi(\omega, x) - x$ satisfies*

$$
\lim_{|x| \to \infty} \frac{\mathbb{E}\bigl(|\chi(\omega, x)|^2 \bigr)}{|x|^2} = 0.
\tag{2.16}
$$

Proof. Properties (1-3) are standard and follow directly from the construction of ψ (which is done, essentially, by showing that a minimizing sequence in (1.5) converges in a suitable L^2-sense; see, e.g., Biskup [2, Section 3.2] for a recent account of this). As to (4), a moment's thought reveals that it suffices to show this for x of the form $n\hat{e}_i$, where $n \to \pm\infty$. This follows from the Mean Ergodic Theorem, similarly as in [2, Lemma 4.8]. $\square$

The replacement of (the gradients of) Ψ_Λ by ψ necessitates developing means to quantify the resulting error. For this we introduce an L^p-norm on functions $f \colon \Omega \times (\Lambda \cup \partial\Lambda) \to \mathbb{R}^d$ by the usual formula

$$
\|\nabla f\|_{\Lambda, p} := \biggl(\frac{1}{|\Lambda|} \sum_{\langle x, y \rangle \in \mathbb{B}(\Lambda)} \mathbb{E} \bigl| f(\omega, y) - f(\omega, x) \bigr|^p \biggr)^{1/p}.
\tag{2.17}
$$

Analogously, we also introduce a norm on functions $\varphi\colon \Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ by

(2.18) $$\|\nabla\varphi\|_p := \Bigl(\sum_{x=\hat{\mathrm{e}}_1,\dots,\hat{\mathrm{e}}_d} \mathbb{E}\bigl|\varphi(\omega,x)-\varphi(\omega,0)\bigr|^p\Bigr)^{1/p}.$$

Here we introduced the symbol ∇f for an $\mathbb{R}^d$-valued functions the i-th component of which at x is given by $\nabla_i f(x) := f(x+\hat{\mathrm{e}}_i)-f(x)$ — abusing our earlier use of this notation. It is reasonably well known, albeit perhaps not written down explicitly anywhere, that the gradients of Ψ_Λ and ψ are close in $\|\cdot\|_{\Lambda,2}$-norm:

Proposition 2.3 *Suppose the law $\mathbb{P}$ on conductances $\{a_{xy}\}$ is ergodic with respect to shifts of $\mathbb{Z}^d$ and obeys* (1.3) *for some $\lambda\in(0,1)$. Then*

(2.19) $$\bigl\|\nabla(\Psi_{\Lambda_L}-\psi)\bigr\|_{\Lambda_L,2} \underset{L\to\infty}{\longrightarrow} 0.$$

As we will elaborate on later (see Remark 4.3), this is exactly what is needed to establish the representation (1.5) for the limit value $c_{\mathrm{eff}}(t)$ of the sequence $L^{-d}C_L^{\mathrm{eff}}(t)$. However, in order to validate the conditions (2.3–2.4) of the Martingale Central Limit Theorem, more than just square integrability is required. For this we state and prove:

Proposition 2.4 (Meyers estimate) *Suppose $\mathbb{P}$ is ergodic with respect to shifts. For each $d\ge 1$, there is $\lambda=\lambda(d)\in(0,1)$ such that if* (1.3) *holds $\mathbb{P}$-a.s. with this λ, then for some $p>4$,*

(2.20) $$\|\nabla\psi\|_p<\infty$$

and

(2.21) $$\sup_{L\ge 1}\bigl\|\nabla(\Psi_{\Lambda_L}-\psi)\bigr\|_{\Lambda_L,p}<\infty.$$

Proposition 2.4 is the sole reason for our restriction on ellipticity contrast. We believe that, on the basis of the technology put forward in Gloria and Otto [6], no such restriction should be needed. To attest this we note that versions of the above bounds actually hold uniformly for a.e. $\omega\in\Omega$ satisfying (1.3); i.e., for norms without the expectation $\mathbb{E}$. In addition, from [6, Proposition 2.1] we in fact know (2.20) for all $p\in(1,\infty)$ when $d\ge 3$.

2.5 Perturbed corrector and variance formula.

Unfortunately, a direct attempt at the substitution of (the gradients of) Ψ_Λ by ψ in (2.13) reveals another technical obstacle: As (2.13) relies on the Fundamental Theorem of Calculus, the replacement of Ψ_Λ by ψ requires the latter function to be defined for ω that may lie outside of the support of $\mathbb{P}$. This is a problem because ψ is generally determined by conditions (1-4) in Proposition 2.2 only on a set of full $\mathbb{P}$-measure. Imposing additional assumptions on $\mathbb{P}$ — namely, that the single-conductance distribution is supported on an interval with a bounded and non-vanishing density — would allow us to replace the Lebesgue integral in (2.13) by an integral with respect to $\mathbb{P}(\mathrm{d}\tilde{\omega}_k)$ and thus eliminate this problem. Notwithstanding, we can do much better by invoking a rank-one perturbation argument which we describe next.

Fix an index $i\in\{1,\dots,d\}$ and recall the notation $\nabla_i f(x) := f(x+\hat{\mathrm{e}}_i)-f(x)$. For a vertex $x\in\mathbb{Z}^d$ and a finite set $\Lambda\subset\mathbb{Z}^d$ satisfying $x\in\Lambda$ or $x+\hat{\mathrm{e}}_i\in\Lambda$, let $\mathfrak{g}_\Lambda^{(i)}(\omega,x)$ be defined by

(2.22) $$\mathfrak{g}_\Lambda^{(i)}(\omega,x)^{-1} := \inf\bigl\{Q_\Lambda(f)\colon f(x+\hat{\mathrm{e}}_i)-f(x)=1,\, f_{\partial\Lambda}=0\bigr\},$$

where $0^{-1} := \infty$. Note that (2.13) and (2.11) ask us to understand how $\nabla_i \Psi_\Lambda(\omega,x)$ changes when the coordinate of ω over $\langle x, x+\hat{\mathrm{e}}_i\rangle$ is perturbed. Somewhat surprisingly, this change takes a purely multiplicative form:

Proposition 2.5 (Rank-one perturbation) *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $x,y \in \Lambda$ be nearest neighbors; $y = x+\hat{\mathrm{e}}_i$ for some $i \in \{1,\dots,d\}$. For any ω,ω' that agree everywhere except at edge $b := \langle x,y\rangle$,*

$$\nabla_i \Psi_\Lambda(\omega',x) = \bigl[1-(\omega'_b-\omega_b)\mathfrak{g}^{(i)}_\Lambda(\omega',x)\bigr]\nabla_i\Psi_\Lambda(\omega,x). \tag{2.23}$$

For the prefactor we alternatively get

$$1-(\omega'_b-\omega_b)\mathfrak{g}^{(i)}_\Lambda(\omega',x) = \exp\Bigl\{-\int_{\omega_b}^{\omega'_b}\mathrm{d}\tilde{\omega}_b\,\mathfrak{g}^{(i)}_\Lambda(\tilde{\omega},x)\Bigr\}, \tag{2.24}$$

where $\tilde{\omega}$ coincides with ω except at b, where it equals $\tilde{\omega}_b$. In particular, $1-(\omega'_b-\omega_b)\mathfrak{g}^{(i)}_\Lambda(\omega',x)$ is bounded away from 0 and ∞ uniformly in $\omega \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$.

It is worthy of noting that (2.23) is a special case of a more general rank-one perturbation formula; cf Lemma 5.1, which may be of independent interest. Incidentally, such formulas have proved extremely useful in the analysis of random Schrödinger operators. The $\Lambda \uparrow \mathbb{Z}^d$-limit of the right-hand side can now be controlled uniformly in $\omega \in \Omega$:

Proposition 2.6 *Suppose* (1.3) *holds for some $\lambda \in (0,1)$. Then $\Lambda \mapsto \mathfrak{g}^{(i)}_\Lambda(\omega,x)$ is non-decreasing and bounded away from zero and infinity uniformly in $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \Omega$. In particular, for all $\omega \in \Omega$ and all $x \in \mathbb{Z}^d$ the limit*

$$\mathfrak{g}^{(i)}(\omega,x) := \lim_{\Lambda\uparrow\mathbb{Z}^d}\mathfrak{g}^{(i)}_\Lambda(\omega,x) \tag{2.25}$$

exists and satisfies

$$\mathfrak{g}^{(i)}(\omega,x)^{-1} = \inf\bigl\{Q_{\mathbb{Z}^d}(f)\colon f(x+\hat{\mathrm{e}}_i)-f(x)=1,\,|\mathrm{supp}(f)|<\infty\bigr\}, \tag{2.26}$$

where $\mathrm{supp}(f) := \{x\in\mathbb{Z}^d\colon f(x)\neq 0\}$. In particular, $(\omega,x)\mapsto\mathfrak{g}^{(i)}(\omega,x)$ is stationary in the sense that $\mathfrak{g}^{(i)}(\tau_z\omega,x+z)=\mathfrak{g}^{(i)}(\omega,x)$ holds for all $\omega\in\Omega$ and all $x,z\in\mathbb{Z}^d$.

Before we wrap up the outline of the proof of Theorem 1.1, let us formulate a representation for the limiting variance σ_t^2 from Theorem 1.1: For $x\in\mathbb{Z}^d$ and $i\in\{1,\dots,d\}$, let b denote the edge corresponding to the pair (x,i) and let

$$h(\omega,x,i) := \int\mathbb{P}(\mathrm{d}\omega'_b)\int_{\omega'_b}^{\omega_b}\mathrm{d}\tilde{\omega}_b\,\bigl[1-(\tilde{\omega}_b-\omega_b)\mathfrak{g}^{(i)}(\tilde{\omega},x)\bigr]^2, \tag{2.27}$$

where $\tilde{\omega}$ is the configuration equal to ω except at b, where it equals $\tilde{\omega}_b$. Define the matrix $\hat{Z}(x,i) := \{\hat{Z}_{jk}(x,i)\}_{j,k=1,\dots,d}$ by the quadratic form

$$\bigl(t,\hat{Z}(x,i)t\bigr) := \mathbb{E}\Bigl(h(\cdot,x,i)\bigl|\nabla_i(t\cdot\psi)(\cdot,x)\bigr|^2\Bigm|\sigma\bigl(\omega_{b'}\colon b'\preceq(x,i)\bigr)\Bigr), \tag{2.28}$$

where (x,i) represents the edge $\langle x,x+\hat{\mathrm{e}}_i\rangle$ and $t\in\mathbb{R}^d$. Then we have:

Theorem 2.7 (Limiting variance) *Under the assumptions of Theorem 1.1, the matrix elements of $\hat{Z}(x,i)$ are square integrable. In particular, σ_t^2 from Theorem 1.1 is given by*

$$\sigma_t^2 = \sum_{i=1}^d\mathbb{E}\Bigl(\bigl(t,\hat{Z}(0,i)t\bigr)^2\Bigr),\qquad t\in\mathbb{R}^d. \tag{2.29}$$

As an inspection of (2.28) reveals, the limiting variance is thus a bi-quadratic form in t. Although concisely written, the expression is not very useful from the practical point of view; particularly, due to the unwieldy conditioning in (2.28). The representation using the h-function also adds to this; it is no longer obvious, albeit still true, that

$$\mathbb{E}\Big((t,\hat{Z}(x,i)t)\,\Big|\,\sigma\big(\omega_{b'}\colon b' \prec (x,i)\big)\Big) = 0, \tag{2.30}$$

i.e., that $(t,\hat{Z}(x,i)t)$ is a martingale increment. A question of interest is whether an expression can be found for σ_t^2 that is more amenable to computations.

2.6 Outline.

The proofs (and the rest of the paper) are organized as follows. In Section 3 we assemble the ingredients — following the steps outlined in the present section — into the proof of Theorems 1.1 and 2.7. In Section 4 we then show that the finite-volume harmonic coordinate approximates its full lattice counterpart in an L^2-sense as stated in Proposition 2.3 and establish the Meyers estimate from Proposition 2.4. A key technical tool is the Calderón-Zygmund regularity theory and a uniform bound on the triple gradient of the Green's function of the simple random walk in finite boxes. Finally, in Section 5, we prove Propositions 2.5 and 2.6 dealing with the harmonic coordinate over environments perturbed at a single edge.

3. Proof of the CLT

In this section we verify the conditions (2.3–2.4) of the Martingale Central Limit Theorem and thus prove Theorems 1.1 and 2.7. All derivations are conditional on Propositions 2.3–2.6 the proofs of which are postponed to later sections. Throughout we will make use of the following simple but useful consequence of Hölder's inequality:

Lemma 3.1 *For any $p' > p > 2$, $\alpha := \frac{2}{p}\frac{p'-p}{p'-2}$ and $\beta := \frac{p'}{p}\frac{p-2}{p'-2}$,*

$$\big\|\nabla(\Psi_{\Lambda_L}-\psi)\big\|_{\Lambda_L,p} \le \big\|\nabla(\Psi_{\Lambda_L}-\psi)\big\|_{\Lambda_L,2}^{\alpha}\,\big\|\nabla(\Psi_{\Lambda_L}-\psi)\big\|_{\Lambda_L,p'}^{\beta}.$$

Proof. Apply Hölder's inequality to the function $f := |\nabla(\Psi_{\Lambda_L}-\psi)|$. □

Assume now the setting developed in Section 2; in particular, the ordering of edges and sigma-algebras $\mathscr{F}_k$ from Section 2.2 and the martingale increment Z_k from (2.2) and its representation (2.13) from Section 2.3. In analogy with equation (2.27), we also define

$$h_\Lambda(\omega,x,i) := \int \mathbb{P}(\mathrm{d}\omega_b')\int_{\omega_b'}^{\omega_b}\mathrm{d}\tilde{\omega}_b\big[1-(\tilde{\omega}_b-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\tilde{\omega},x)\big]^2, \tag{3.1}$$

where $b := \langle x, x+\hat{\mathrm{e}}_i\rangle$ and $\tilde{\omega}$ is the configuration equal to ω except at b, where it equals $\tilde{\omega}_b$. By Proposition 2.5, we may write the martingale increment Z_k as

$$Z_k = \mathbb{E}\Big(h_\Lambda(\cdot,x_k,i_k)\big|\nabla_{i_k}(t\cdot\Psi_\Lambda)(\cdot,x_k)\big|^2\,\Big|\,\mathscr{F}_k\Big), \tag{3.2}$$

where x_k and i_k are the vertex and the edge direction corresponding to b_k, i.e., $b_k = \langle x_k, x_k+\hat{\mathrm{e}}_{i_k}\rangle$. Recall also the notation for $\hat{Z}(x,i)$ from (2.28) and note that this is well defined and finite $\mathbb{P}$-a.s. thanks to the estimate (2.20) and boundedness of h.

Proposition 3.2 (Martingale CLT — first condition) *Assume that the premises (and thus conclusions) of Propositions 2.3–2.6 hold. Then*

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}(Z_k^2|\mathscr{F}_{k-1}) \underset{L\to\infty}{\longrightarrow} \sum_{i=1}^{d}\mathbb{E}\Big(\big(t,\hat{Z}(0,i)t\big)^2\Big) \tag{3.3}$$

in $\mathbb{P}$-probability.

Proof. Fix $t\in\mathbb{R}^d$. Thanks to Lemma 2.1 and Proposition 2.2(2), for each $i\in\{1,\dots,d\}$, the collection of conditional expectations

$$\Big\{\mathbb{E}\Big(\big(t,\hat{Z}(x,i)t\big)^2\Big|\,\sigma\big(\omega_b\colon b\prec(x,i)\big)\Big)\colon x\in\mathbb{Z}^d\Big\} \tag{3.4}$$

is stationary with respect to the shifts on $\mathbb{Z}^d$ and, by Proposition 2.4, uniformly in $L^1(\mathbb{P})$. Labeling the edges in $\mathbb{B}(\Lambda_L)$ according to the complete order $\preceq$, the Spatial Ergodic Theorem yields

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big((t,\hat{Z}(x_k,i_k)t)^2\big|\mathscr{F}_{k-1}\big) \underset{L\to\infty}{\longrightarrow} \sum_{i=1}^{d}\mathbb{E}\Big(\big(t,\hat{Z}(0,i)t\big)^2\Big) \tag{3.5}$$

with the limit $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. To see how this relates to our claim, abbreviate

$$A_k := h_{\Lambda_L}(\cdot,x_k,i_k)\big|\nabla_{i_k}(t\cdot\Psi_{\Lambda_L})(\cdot,x_k)\big|^2, \tag{3.6}$$

$$B_k := h(\cdot,x_k,i_k)\big|\nabla_{i_k}(t\cdot\psi)(\cdot,x_k)\big|^2, \tag{3.7}$$

and denote

$$R_{L,k} := \mathbb{E}\Big[\mathbb{E}\big[A_k\,\big|\mathscr{F}_k\big]^2-\mathbb{E}\big[B_k\,\big|\mathscr{F}_k\big]^2\Big|\mathscr{F}_{k-1}\Big]. \tag{3.8}$$

By (3.2) we have $Z_k=\mathbb{E}(A_k\,\big|\mathscr{F}_k)$, while (2.28) reads $(t,\hat{Z}(x_k,i_k)t)=\mathbb{E}(B_k\,\big|\mathscr{F}_k)$. Hence, as soon as we show that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big(|R_{L,k}|\big) \underset{L\to\infty}{\longrightarrow} 0, \tag{3.9}$$

in $\mathbb{P}$-probability, the claim (3.3) will follow.

The proof of (3.9) will proceed by estimating $\mathbb{E}|R_{L,k}|$ which will involve applications of the Cauchy-Schwarz inequality (in order to separate terms) and Jensen's inequality (in order to eliminate conditional expectations). First we note

$$\mathbb{E}|R_{L,k}|\le\big(\mathbb{E}\big[(A_k-B_k)^2\big]\big)^{1/2}\big(\mathbb{E}\big[(A_k+B_k)^2\big]\big)^{1/2}. \tag{3.10}$$

Writing $A_k=B_k+(A_k-B_k)$ and noting $(a+b)^2\le 2a^2+2b^2$ tells us

$$\mathbb{E}\big[(A_k+B_k)^2\big]\le 2\mathbb{E}\big[(A_k-B_k)^2\big]+8\mathbb{E}\big(B_k^2\big). \tag{3.11}$$

Summing over k and applying Cauchy-Schwarz, we find that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big(|R_{L,k}|\big)\le\sqrt{\alpha\big(2\alpha+8\beta\big)}, \tag{3.12}$$

where

$$\alpha:=\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big[(A_k-B_k)^2\big]\quad\text{and}\quad\beta:=\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}(B_k^2). \tag{3.13}$$

By inspection of (3.12) we now observe that it suffices to show that β stays bounded while α tends to zero in the limit $L\to\infty$.

The boundedness of β follows from (2.20) and the fact that $h(\cdot,x,i)$ is bounded; indeed, these yield $\mathbb{E}(|B_k|^2) \le \|h\|_\infty^2 |t|^4 \|\nabla\psi\|_4^4$ uniformly in k and L. Concerning the terms constituting α, using $(a+b)^2 \le 2a^2+2b^2$ we first separate terms as

$$\mathbb{E}\big[(A_k-B_k)^2\big] \le 2\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot,x_k,i_k)\big|^2\big|\,|\nabla_{i_k}(t\cdot\Psi_{\Lambda_L})(\cdot,x_k)|^2 - |\nabla_{i_k}(t\cdot\psi)(\cdot,x_k))|^2\big|^2\Big) + 2\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot,x_k,i_k)-h(\cdot,x_k,i_k)\big|^2\big|\nabla_{i_k}(t\cdot\psi)(\cdot,x_k)\big|^4\Big). \tag{3.14}$$

Since h_Λ is uniformly bounded, the average over k of the first term is bounded by a constant times the product of $(\|\nabla\Psi_{\Lambda_L}\|_{\Lambda_L,4}+\|\nabla\psi\|_{\Lambda_L,4})^2$ and $\|\nabla(\Psi_{\Lambda_L}-\psi)\|_{\Lambda_L,4}^2$. The latter tends to zero as $L\to\infty$ by Proposition 2.4, Proposition 2.3 and Lemma 3.1 (with the choices $p:=4$ and $p'>4$ but sufficiently close to 4).

For the second term in (3.14) we pick $p>4$ and use Hölder's inequality to get

$$\begin{aligned}\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}&\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot,x_k,i_k)-h(\cdot,x_k,i_k)\big|^2\big|\nabla_{i_k}(t\cdot\psi)(\cdot,x_k)\big|^4\Big)\\ &\le |t|^4\,\|\nabla\psi\|_{\Lambda_L,p}^4\Bigg(\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot,x_k,i_k)-h(\cdot,x_k,i_k)\big|^{2q}\Big)\Bigg)^{1/q},\end{aligned} \tag{3.15}$$

where q satisfies $4/p+1/q=1$. The norm of $\|\nabla\psi\|_{\Lambda_L,p}$ is again bounded by Proposition 2.4 as long as p is sufficiently close to 4; to apply (2.20), we need to invoke the stationarity of $\nabla\psi$ to bound $\|\nabla\psi\|_{\Lambda_L,p}\le C\|\nabla\psi\|_p$.

For the second term in (3.15) we first need to show that for each $\varepsilon>0$ there is $N\ge1$ so that for all $\omega\in\Omega$,

$$\operatorname{dist}_{\ell^1(\mathbb{Z}^d)}(x,\Lambda_L^{\mathrm{c}})\ge N \quad\Rightarrow\quad \big|h_{\Lambda_L}(\omega,x,i)-h(\omega,x,i)\big|<\varepsilon. \tag{3.16}$$

For this we use that

$$\big|h_\Lambda(\omega,x,i)-h(\omega,x,i)\big|\le C\int_\lambda^{1/\lambda}\mathrm{d}\tilde\omega_b\,\big|\mathfrak{g}_\Lambda^{(i)}(\tilde\omega,x)-\mathfrak{g}^{(i)}(\tilde\omega,x)\big| \tag{3.17}$$

for some constant $C=C(\lambda)<\infty$. To estimate the right-hand side, by the monotonicity of $\Lambda\mapsto\mathfrak{g}_\Lambda^{(i)}(\tilde\omega,x)$ and its stationarity with respect to shifts, we have

$$\big|\mathfrak{g}_\Lambda^{(i)}(\omega,x)-\mathfrak{g}^{(i)}(\omega,x)\big|\le\big|\mathfrak{g}_{\Lambda_N}^{(i)}(\tau_x\omega,0)-\mathfrak{g}^{(i)}(\tau_x\omega,0)\big|,\qquad \omega\in\Omega, \tag{3.18}$$

as soon as the box $x+\Lambda_N\subset\Lambda$. The implication (3.16) then follows via (3.17) by the fact that the difference on the right-hand side of (3.18) converges to zero uniformly in $\omega\in\Omega$.

We now bound the second term in (3.15) as follows. The terms for which x_k is at least N steps away from Λ_L are bounded by ε thanks to (3.17); the sum over the remaining terms is of order NL^{d-1} thanks to the uniform boundedness of $h_\Lambda-h$. Hence, in the limit $L\to\infty$, the second term in (3.15) is of order $\varepsilon^{1/q}$; taking $\varepsilon\downarrow0$ shows that α tends to zero as $L\to\infty$. This finishes the proof of (3.9) and the whole claim. □

Proposition 3.3 (Martingale CLT — second condition) *Assume that the premises (and thus conclusions) of Propositions 2.3–2.6 hold. Then for each $\varepsilon>0$,*

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big(Z_k^2 1_{\{|Z_k|>\varepsilon|\Lambda_L|^{1/2}}\big|\mathscr{F}_{k-1}\big)\underset{L\to\infty}{\longrightarrow}0, \tag{3.19}$$

in $\mathbb{P}$-probability.

Proof. This could be proved by strengthening a bit the statement of Proposition 3.2 (from squares of the Z's to a slightly higher power), but a direct argument is actually easier.

First we note that it suffices to show convergence in expectation. Let $p > 4$ be such that the statements in Proposition 2.4 hold. By Chebyshev we have

$$\mathbb{E}\big(Z_k^2 1_{\{|Z_k|>\varepsilon|\Lambda_L|^{1/2}\}}\big) \le \Big(\frac{1}{\varepsilon|\Lambda_L|^{1/2}}\Big)^{\frac{p-4}{2}} \mathbb{E}\big(Z_k^{p/2}\big). \tag{3.20}$$

Since h_{Λ_L} is bounded, Jensen's inequality yields

$$\mathbb{E}\big(Z_k^{p/2}\big) \le C\mathbb{E}\bigg(\Big[\mathbb{E}\Big(\big|\nabla_{i_k}(t\cdot\Psi_\Lambda)(\cdot,x_k)\big|^2\Big|\mathscr{F}_k\Big)\Big]^{p/2}\bigg) \le C\mathbb{E}\Big(\big|\nabla_{i_k}(t\cdot\Psi_\Lambda)(\cdot,x_k)\big|^p\Big). \tag{3.21}$$

It follows that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big(Z_k^{p/2}\big) \le C|t|^p\|\nabla\Psi_{\Lambda_L}\|_{\Lambda_L,p}^p. \tag{3.22}$$

The right-hand side is bounded uniformly in L. Using this in (3.20), the claim follows. □

We can now finish the proof of our main results:

Proof of Theorems 1.1 and 2.7 from Propositions 2.3–2.6. The distributional convergence in (1.6) is a direct consequence of the Martingale Central Limit Theorem whose conditions (2.3–2.4) are established in Propositions 3.2 and 3.3. The limiting variance σ_t^2 is given by the right-hand side of (3.3), in agreement with (2.29). It remains to prove that $\sigma_t^2 > 0$ whenever $t \neq 0$ and the law $\mathbb{P}$ is non-degenerate.

Suppose on the contrary that $\sigma_t^2 = 0$. Then for each i we would have $\mathbb{E}((t,\hat{Z}(0,i)t)^2) = 0$ and thus $(t,\hat{Z}(0,i)t) = 0$ $\mathbb{P}$-a.s. Denoting $b := \langle 0,\hat{\mathrm{e}}_i\rangle$, (2.27–2.28) imply that, for $\mathbb{P}$-a.e. ω_b,

$$\int\mathbb{P}(\mathrm{d}\omega_b')\int_{\omega_b'}^{\omega_b}\mathrm{d}\tilde{\omega}_b\,\mathbb{E}\Big(\big[1-(\tilde{\omega}_b-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\tilde{\omega},0)\big]\big|\nabla_i(t\cdot\psi)(\omega,0)\big|^2\Big|\mathscr{F}_{(0,i)}\Big) = 0, \tag{3.23}$$

where $\mathscr{F}_{(0,i)} := \sigma(\omega_b)$. Let $\Omega_1 \subset [\lambda, 1/\lambda]$ be the set of ω_b where this holds. The expectation in (3.23) is independent of ω_b'; subtracting the expression for two (generic) choices of ω_b in Ω_1 then shows that the inner integral must vanish for all $\omega_b,\omega_b' \in \Omega_1$. But (2.24) tells us that the prefactor in square brackets, and thus the conditional expectation, is non-negative. In light of $\mathbb{P}(\Omega_1) = 1$ and the fact that Ω_1 contains at least two points, this can only happen when

$$\nabla_i(t\cdot\psi)(\cdot,0) = 0, \qquad \mathbb{P}\text{-a.s. for all } i = 1,\dots,d. \tag{3.24}$$

But then $c_{\mathrm{eff}}(t) = 0$, which cannot hold for $t \neq 0$ when (1.3) is in force. □

4. The Meyers estimate

The goal of this section is to give proofs of Propositions 2.3 and 2.4. The former is a simple consequence of the Hilbert-space structure underlying the definition of a harmonic coordinate; the latter (to which this section owes its name) is a far less immediate consequence of the Calderón-Zygmund regularity theory for singular integral operators.

4.1 L^2 bounds and convergence.

Recall our notation $\mathscr{L}_\omega$ for the operator in (2.8). We begin by noting an explicit representation of the minimum of $f \mapsto Q_\lambda(f)$ as a function of the (Dirichlet) boundary condition:

Lemma 4.1 *Let $\Lambda \subset \mathbb{Z}^d$ be finite and fix an $\omega \in \Omega$. Then there is $K\colon \partial\Lambda \times \partial\Lambda \to [0,\infty)$, depending on Λ and ω, such that for any h that obeys $\mathscr{L}_\omega h(x) = 0$ for $x \in \Lambda$,*

$$Q_\Lambda(h) = \frac{1}{2} \sum_{x,y\in\partial\Lambda} K(x,y)\big[h(y)-h(x)\big]^2. \tag{4.1}$$

Moreover, $K(x,y) = K(y,x)$ for all $x,y \in \partial\Lambda$ and

$$\sum_{y\in\partial\Lambda} K(x,y) = \sum_{\substack{z\in\Lambda \\ \langle x,z\rangle\in\mathbb{B}(\Lambda)}} a_{xz} \tag{4.2}$$

for all $x \in \partial\Lambda$.

Proof. "Integrating" by parts we obtain

$$Q_\Lambda(h) = -\sum_{y\in\Lambda} h(y)(\mathscr{L}_\omega h)(y) + \sum_{\substack{y\in\partial\Lambda, x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy}\big[h(y)-h(x)\big]h(y). \tag{4.3}$$

Employing the fact that h is $\mathscr{L}_\omega$-harmonic, the first sum drops out. For the second sum we recall that $h(x) = \sum_{z\in\partial\Lambda} p_\Lambda(x,z)h(z)$, where $p_\Lambda(x,z)$ is the discrete Poisson kernel which can be defined by $p_\Lambda(x,z) := P^x_\omega(X_{\tau_{\partial\Lambda}} = z)$ for $\tau_{\partial\Lambda}$ denoting the first exit time from Λ of the random walk in conductances ω. Now set

$$K(y,z) := \sum_{\substack{x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy} p_\Lambda(x,z) \tag{4.4}$$

and note that $\sum_{z\in\partial\Lambda} K(y,z) = \sum_{x\in\Lambda,\, \langle x,y\rangle\in\mathbb{B}(\Lambda)} a_{xy}$. It follows that

$$\sum_{\substack{y\in\partial\Lambda, x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy}\big[h(y)-h(x)\big]h(y) = \sum_{y,z\in\partial\Lambda} K(y,z)\big[h(y)-h(z)\big]h(y). \tag{4.5}$$

The representation using the random walk and its reversiblity now imply that K is symmetric. Symmetrizing the last sum then yields the result. ☐

Remark 4.2 We note that Lemma 4.1 holds even for vector valued functions; just replace $[h(y)-h(x)]^2$ by the norm square of $h(y)-h(x)$.This applies to several derivations that are to follow; a point that we will leave without further comment.

We can now prove Proposition 2.3 dealing with the convergence of $\nabla\Psi_\Lambda$ to $\nabla\psi$ in $\|\cdot\|_{\Lambda,2}$-norm, as $\Lambda := \Lambda_L$ fills up all of $\mathbb{Z}^d$.

Proof of Proposition 2.3. Abbreviate $h(x) := \psi(\omega,x) - \Psi_{\Lambda_L}(\omega,x)$. The bound (1.3) implies

$$\big\|\nabla(\Psi_{\Lambda_L} - \psi)\big\|^2_{\Lambda_L,2} \le \frac{1}{\lambda}\frac{1}{|\Lambda_L|}\,\mathbb{E}\bigg(\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|h(y)-h(x)\big|^2\bigg). \tag{4.6}$$

Let $f\colon \Lambda\cup\partial\Lambda \to \mathbb{R}^d$ be the minimizer of

$$\inf\bigg\{\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} \big|f(y)-f(x)\big|^2,\, f(z) = \chi(z) \text{ for all } z\in\partial\Lambda_L\bigg\}. \tag{4.7}$$

Since h is the minimizer of the corresponding Dirichlet energy with conductances $\{a_{xy}\}$ and boundary condition χ, we get using (1.3)

$$
\begin{aligned}
\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|h(y)-h(x)\big|^2 &\le \sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|f(y)-f(x)\big|^2 \\
&\le \frac{1}{\lambda}\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} \big|f(y)-f(x)\big|^2.
\end{aligned}
\tag{4.8}
$$

Writing the last sum coordinate-wise and applying Lemma 4.1, we thus get

$$
\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|h(y)-h(x)\big|^2 \le \frac{1}{2\lambda}\sum_{x,y\in\partial\Lambda_L} K(x,y)\big|\chi(\omega,y)-\chi(\omega,x)\big|^2,
\tag{4.9}
$$

where the kernel $K(x,y)$ pertains to the homogeneous problem, i.e., the simple random walk. Note that these bounds hold for all configurations satisfying (1.3).

By shift covariance and sublinearity of the corrector (cf Proposition 2.2(2,4)), for each $\varepsilon>0$ there is $A=A(\varepsilon)$ such that

$$
\mathbb{E}\big(\big|\chi(\cdot,x)-\chi(\cdot,y)\big|^2\big)\le A+\varepsilon|x-y|^2.
\tag{4.10}
$$

Using this and (4.9) in (4.6) yields

$$
\big\|\nabla(\Psi_{\Lambda_L}-\psi)\big\|_{\Lambda_L,2}^2 \le \frac{1}{2\lambda^2}\frac{1}{|\Lambda_L|}\sum_{x,y\in\Lambda_L} K(x,y)\big(A+\varepsilon|x-y|^2\big).
\tag{4.11}
$$

But $\sum_{y\in\partial\Lambda_L}K(x,y)\le 1$ for each $x\in\partial\Lambda_L$ while $\sum_{x,y\in\partial\Lambda_L}K(x,y)|x-y|^2$ is, by Lemma 4.1, the Dirichlet energy of the function $x\mapsto x$ for conductances all equal to 1. Hence, the last sum in (4.11) is bounded by $A|\partial\Lambda_L|+\varepsilon|\mathbb{B}(\Lambda_L)|$. Taking $L\to\infty$ and $\varepsilon\downarrow 0$ finishes the proof. ☐

Remark 4.3 As alluded to in the introduction, the L^2-convergence $\nabla\Psi_{\Lambda_L}\to\nabla\psi$ permits us to prove the formula (1.5) for $c_{\text{eff}}(t)$. The argument is similar to (albeit much easier than) what we used in the proof of Proposition 3.2. Indeed, we trivially decompose

$$
C_L^{\text{eff}}(t)=Q_{\Lambda_L}\big(t\cdot\Psi_{\Lambda_L}\big)=Q_{\Lambda_L}(t\cdot\psi)+\big(Q_{\Lambda_L}\big(t\cdot\Psi_{\Lambda_L}\big)-Q_{\Lambda_L}(t\cdot\psi)\big).
\tag{4.12}
$$

The stationarity of the gradients of ψ and the Spatial Ergodic Theorem imply

$$
\frac{1}{|\Lambda_L|}Q_{\Lambda_L}(t\cdot\psi)\underset{L\to\infty}{\longrightarrow}\mathbb{E}\bigg(\sum_{x=\hat{\text{e}}_1,\dots,\hat{\text{e}}_d} a_{0,x}(\omega)\big|t\cdot\psi(\omega,x)\big|^2\bigg),
\tag{4.13}
$$

for any ergodic law $\mathbb{P}$ on conductances. The expression on the right coincides with the infimum in (1.5). (There is no gradient on the right-hand side of (4.13) because $\psi(\omega,0):=0$.) It remains to control the difference on the extreme right of (4.12). Using Cauchy-Schwarz,

$$
\frac{\mathbb{E}\big|Q_\Lambda\big(t\cdot\Psi_\Lambda\big)-Q_\Lambda(t\cdot\psi)\big|}{|\Lambda|}\le|t|^2\big\|\nabla(\Psi_\Lambda-\psi)\big\|_{\Lambda,2}^2+2|t|^2\|\nabla\psi\|_2\big\|\nabla(\Psi_\Lambda-\psi)\big\|_{\Lambda,2}.
\tag{4.14}
$$

By Proposition 2.3, the right-hand side tends to zero as $\Lambda:=\Lambda_L$ increases to $\mathbb{Z}^d$. Since we know that $|\Lambda_L|^{-1}C_L^{\text{eff}}(t)$ is bounded and converges almost surely (e.g., by the Subadditive Ergodic Theorem), the limit value $c_{\text{eff}}(t)$ thus satisfies (1.5). Note that Proposition 2.3, and thus all the above, holds for any shift-ergodic (elliptic) law on conductances.

4.2 The Meyers estimate in finite volume.

Key to the proof of Proposition 2.4 is the Meyers estimate. The term owes its name to Norman Meyers [11] who discovered a bound on L^p-continuity (in the right-hand side) of the solutions of Poisson equation with second-order elliptic differential operators in divergence from, provided the associated coefficients are close to a constant. The technical ingredient underpinning this observation is the Calderón-Zygmund regularity theory for certain singular integral operators in $\mathbb{R}^d$. (Incidentally, as noted in [11], Meyers' argument is a generalization of earlier work of Boyarskii, cf [11, ref. 2 and 3] for systems of first-order PDEs and a version of his result was also derived, though not published, by Calderón himself; cf [11, page 190]).

To ease the notation, in addition to (2.18), we will use the notation $\|f\|_p$ also for the canonical norm in $\ell^p(\Lambda)$,

$$\|f\|_p := \Big(\sum_{x\in\Lambda} \big|f(x)\big|^p\Big)^{1/p}, \tag{4.15}$$

throughout the rest of this section.

Let us review the gist of Meyers' argument for functions on $\mathbb{Z}^d$. Our notation is inspired by that used in Naddaf and Spencer [12] and Gloria and Otto [6]. A general form of the second order difference operator $\mathscr{L}$ in divergence form is

$$\mathscr{L} := \nabla^\star \cdot A \cdot \nabla, \tag{4.16}$$

where $A = \{A_{ij}(x) \colon i,j = 1,\dots,d,\, x \in \mathbb{Z}^d\}$ are x-dependent matrix coefficients, $\nabla f(x)$ is a vector whose i-th component is $\nabla_i f(x) := f(x+\hat{\mathrm{e}}_i) - f(x)$ and $\nabla^\star$ is its conjugate acting as $\nabla_i^\star f(x) := f(x) - f(x-\hat{\mathrm{e}}_i)$. The above $\mathscr{L}$ is explicitly given by

$$(\mathscr{L} f)(x) = \sum_{i,j=1}^d \Big(A_{i,j}(x)\big[f(x+\hat{\mathrm{e}}_i) - f(x)\big] - A_{i,j}(x-\hat{\mathrm{e}}_j)\big[f(x+\hat{\mathrm{e}}_i-\hat{\mathrm{e}}_j) - f(x-\hat{\mathrm{e}}_j)\big]\Big). \tag{4.17}$$

Now, if A is close to identity, it makes sense to write

$$\mathscr{L} = \Delta + \nabla^\star \cdot (A - \mathrm{id}) \cdot \nabla, \tag{4.18}$$

where we noted that the standard lattice Laplacian Δ corresponds to $\nabla^\star \cdot \mathrm{id} \cdot \nabla$. This formula can be used as a starting point of perturbative arguments.

Consider a finite set $\Lambda \subset \mathbb{Z}^d$ and let $g \colon \Lambda \cup \partial\Lambda \to \mathbb{R}^d$. Let f be a solution to the Poisson equation

$$-\mathscr{L} f = \nabla^\star \cdot g, \qquad \text{in } \Lambda, \tag{4.19}$$

with $f := 0$ on $\partial\Lambda$. Employing (4.18), we can rewrite this as

$$-\Delta f = \nabla^\star \cdot \big[g + (A - \mathrm{id}) \cdot \nabla f\big]. \tag{4.20}$$

The function on the right has vanishing total sum over Λ and hence it lies in the domain of the inverse $(\Delta)_\Lambda^{-1}$ of Δ with zero boundary conditions. Taking this inverse followed by one more gradient, and denoting

$$\mathscr{K}_\Lambda := \nabla(-\Delta)_\Lambda^{-1}\nabla^\star, \tag{4.21}$$

this equation translates to

$$\nabla f = \mathscr{K}_\Lambda \cdot \big[g + (A - \mathrm{id})\nabla f\big]. \tag{4.22}$$

A first noteworthy point is that this is now an autonomous equation for ∇f. A second point is that, if $\|\mathscr{K}_\Lambda\|_p$ is the norm of $\mathscr{K}_\Lambda$ as a map (on vector valued functions) $\ell^p(\Lambda) \to \ell^p(\Lambda)$, we get

$$\|\nabla f\|_p \le \|\mathscr{K}_\Lambda\|_p \|A - \mathrm{id}\|_\infty \|\nabla f\|_p + \|\mathscr{K}_\Lambda\|_p \|g\|_p. \tag{4.23}$$

Assuming $\|\mathscr{K}_\Lambda\|_p\|A-\mathrm{id}\|_\infty<1$ this yields

$$\|\nabla f\|_p \le \frac{\|\mathscr{K}_\Lambda\|_p\|g\|_p}{1-\|\mathscr{K}_\Lambda\|_p\|A-\mathrm{id}\|_\infty}. \tag{4.24}$$

The perturbative nature of the condition $\|\mathscr{K}_\Lambda\|_p\|A-\mathrm{id}\|_\infty<1$ is further highlighted by the fact that it also ensures the very existence of a unique solution ∇f to (4.22) via a contraction argument; (4.24) then implies the continuity of $g\mapsto\nabla f$ in $\ell^p(\Lambda)$.

The aforementioned general facts are relevant for us because $\mathscr{L}_\omega$ is of the form (4.16). Indeed, set $A_{ij}(x):=\delta_{ij}a_{x,x+\hat{\mathrm{e}}_i}$ and note that (4.17) reduces to (2.8). The finite-volume corrector

$$\chi_\Lambda(\omega,x):=\Psi_\Lambda(\omega,x)-x \tag{4.25}$$

then solves the Poisson equation

$$-\mathscr{L}_\omega\chi_\Lambda=\nabla^\star\cdot g,\quad\text{where}\quad g(x):=(a_{x,x+\hat{\mathrm{e}}_1},\dots,a_{x+\hat{\mathrm{e}}_d}). \tag{4.26}$$

This is bounded uniformly so, in order to have (4.24) for all finite boxes, our main concern is the following claim:

Theorem 4.4 *For each $p\in(1,\infty)$, the operator $\mathscr{K}_{\Lambda_L}$ is bounded in $\ell^p(\Lambda_L)$, uniformly in $L\ge1$.*

Proof of Proposition 2.4 from Theorem 4.4. Let $p^*>4$. Since (in our setting) $\|A-\mathrm{id}\|_\infty\le\lambda^{-1}-1$, we may choose $\lambda\in(0,1)$ close enough to one so that $\sup_{L\ge1}\|\mathscr{K}_{\Lambda_L}\|_{p^*}\|A-\mathrm{id}\|_\infty<1$. From the above derivation it follows

$$\sup_{L\ge1}\|\nabla\chi_{\Lambda_L}\|_{\Lambda_L,p^*}<\infty. \tag{4.27}$$

We claim that this implies

$$\|\nabla\chi\|_p<\infty,\qquad p<p^*. \tag{4.28}$$

Indeed, pick $\alpha>0$ and note that, for any $\varepsilon\in(0,\alpha)$,

$$\sum_{x\in\Lambda_L}1_{\{|\nabla\chi(\cdot,x)|>\alpha\}}\le\sum_{x\in\Lambda_L}1_{\{|\nabla\chi_{\Lambda_L}(\cdot,x)|>\alpha-\varepsilon\}}+\sum_{x\in\Lambda_L}1_{\{|\nabla\chi_{\Lambda_L}(\cdot,x)-\nabla\chi(\cdot,x)|>\varepsilon\}}. \tag{4.29}$$

Taking expectations and dividing by $|\Lambda_L|$, the left hand side becomes $\mathbb{P}(|\nabla\chi(\cdot,0)|>\alpha)$, while the second sum on the right can be bounded by $\varepsilon^{-2}\|\nabla\chi_{\Lambda_L}-\nabla\chi\|^2_{\Lambda_L,2}$, which tends to zero as $L\to\infty$ by Proposition 2.3. Applying Chebyshev's inequality to the first sum on the right and taking $L\to\infty$ followed by $\varepsilon\downarrow0$ yields

$$\mathbb{P}\big(|\nabla\chi(\cdot,0)|>\alpha\big)\le\frac{1}{\alpha^{p^*}}\sup_{L\ge1}\|\nabla\chi_{\Lambda_L}\|^{p^*}_{\Lambda_L,p^*}. \tag{4.30}$$

Multiplying by α^{p-1} and integrating over $\alpha>0$ then proves (4.28).

Returning to the claims in Proposition 2.4, inequality (4.28) is a restatement of (2.20). Since (4.27–4.28) imply the uniform boundedness of $\|\nabla(\chi_{\Lambda_L}-\chi)\|_{\Lambda_L,p}$, for each $p<p^*$, Lemma 3.1 then shows $\|\nabla(\chi_{\Lambda_L}-\chi)\|_{\Lambda_L,p}\to0$, as $L\to\infty$ for all $p<p^*$. This proves (2.21) as well. □

4.3 Interpolation.

In the proof of Theorem 4.4 we will follow the classical argument spelled out in Chapter 2 (specifically, proof of Theorem 1 in Section 2.2) of Stein's book [16]. The reasoning requires only straightforward adaptations due to discrete setting and finite volume, but we still prefer to give a full argument to keep the present paper self-contained. A key idea is the use of interpolation between the strong ℓ^2-type estimate (Lemma 4.5) and the weak ℓ^1-type estimate for $\mathscr{K}_{\Lambda_L}$ (Lemma 4.6). Both of these of course need to hold uniformly in $L \geq 1$.

Lemma 4.5 *For any finite $\Lambda \subset \mathbb{Z}^d$, the $\ell^2(\Lambda)$-norm of $\mathscr{K}_\Lambda$ satisfies $\|\mathscr{K}_\Lambda\|_2 \leq 1$.*

Proof. Let $\mathscr{H}$ be a Hilbert space and T a positive self-adjoint, bounded and invertible operator. Then for all $h \in \mathscr{H}$,

$$(h, T^{-1}h) = \sup_{g \in \mathscr{H}} \{2(g,h) - (g,Tg)\}. \tag{4.31}$$

We will apply this to $\mathscr{H}$ given by the space (of $\mathbb{R}$-valued functions) $\ell^2(\Lambda)$, $T := \varepsilon - \Delta$ and $h := \nabla^\star \cdot f$ for some $f\colon \Lambda \to \mathbb{R}^d$ with zero boundary conditions outside Λ. Then

$$\begin{aligned}
\big(\nabla^\star \cdot f, (\varepsilon - \Delta)^{-1}\nabla^\star \cdot f\big) &= \sup_{g \in \ell^2(\Lambda)} \big\{2(g, \nabla^\star \cdot f) - \varepsilon(g,g) + (g, \Delta g)\big\} \\
&= \sup_{g \in \ell^2(\Lambda)} \big\{2(\nabla g, f) - \varepsilon(g,g) - (\nabla g, \nabla g) - (f,f)\big\} + (f,f) \\
&= \sup_{g \in \ell^2(\Lambda)} \big\{-(\nabla g - f, \nabla g - f)\big\} + (f,f) \\
&\leq (f,f),
\end{aligned} \tag{4.32}$$

where we used that $\nabla^\star$ is the adjoint of ∇ in the space of $\mathbb{R}^d$-valued functions $\ell^2(\Lambda)$ and where the various inner products have to be interpreted either for $\mathbb{R}$-valued or $\mathbb{R}^d$-valued functions accordingly. Taking $\varepsilon \downarrow 0$, the left-hand side becomes $(f, \mathscr{K}_\Lambda \cdot f)$. The claim follows. □

The second ingredient turns out to be technically more involved.

Lemma 4.6 *$\mathscr{K}_{\Lambda_L}$ is of weak-type (1-1), uniformly in $L > 1$. That is, there exists $\widehat{K}_1$ such that, for all $L > 1$, $f \in \ell^1(\Lambda_L)$ and $\alpha > 0$,*

$$\big|\{z \in \Lambda_L\colon |\mathscr{K}_{\Lambda_L} f(z)| > \alpha\}\big| \leq \widehat{K}_1 \frac{\|f\|_1}{\alpha}. \tag{4.33}$$

Deferring the proof of this lemma to the end of this section, we now show how this enters into the proof of Theorem 4.4.

Proof of Theorem 4.4 from Lemma 4.6. We follow the proof in Stein [16, Theorem 5, page 21]. We begin with the case $1 < p < 2$. Let $f \in \ell^p(\Lambda_L)$ and pick $\alpha > 0$. Let $f_1 := f 1_{\{|f| > \alpha\}}$ and $f_2 := f 1_{\{|f| \leq \alpha\}}$. Then

$$\begin{aligned}
\big|\{z \in \Lambda_L\colon |\mathscr{K}_{\Lambda_L} f(z)| > 2\alpha\}\big| \leq \big|\{z \in \Lambda_L\colon |\mathscr{K}_{\Lambda_L} f_1| > \alpha\}\big| \\
+ \big|\{z \in \Lambda_L\colon |\mathscr{K}_{\Lambda_L} f_2| > \alpha\}\big|.
\end{aligned} \tag{4.34}$$

Lemmas 4.5 and 4.6 then yield

$$\big|\{z \in \Lambda_L\colon |\mathscr{K}_{\Lambda_L} f(z)| > \alpha\}\big| \leq \widehat{K}_1 \frac{\|f_1\|_1}{\alpha} + \widehat{K}_2 \frac{\|f_2\|_2}{\alpha^2}, \tag{4.35}$$

with $\widehat{K}_1$ and $\widehat{K}_2$ independent of L. Multiplying by α^{p-1} and integrating, we infer

$$
\begin{aligned}
\|\mathscr{K}_{\Lambda_L} f\|_p^p &= p\int_0^\infty \alpha^{p-1}\big|\{z\in\Lambda_L\colon |\mathscr{K}_{\Lambda_L} f(z)|>\alpha\}\big|\,\mathrm{d}\alpha \\
&\le p\sum_z \int_0^\infty \Big(\widehat{K}_1\alpha^{p-2}|f(z)|\,1_{\{|f|>\alpha\}}+\widehat{K}_2\alpha^{p-3}|f(z)|^2\,1_{\{|f|\le\alpha\}}\Big)\,\mathrm{d}\alpha \\
&= p\widehat{K}_1\sum_z |f(z)|\int_0^{|f(z)|}\alpha^{p-2}\,\mathrm{d}\alpha + p\widehat{K}_2\sum_z |f(z)|^2\int_{|f(z)|}^\infty \alpha^{p-3}\,\mathrm{d}\alpha \\
&= \frac{p\widehat{K}_1}{p-1}\sum_z |f(z)|^p+\frac{p\widehat{K}_2}{2-p}\sum_z |f(z)|^p,
\end{aligned}
\tag{4.36}
$$

proving the assertion in the case $1<p<2$.

For $p\in(2,\infty)$, the fact that $\mathscr{K}_\Lambda$ is obviously symmetric implies that $\|\mathscr{K}_\Lambda\|_p=\|\mathscr{K}_\Lambda\|_q$, where q is the index dual to p. Hence $\sup_{L\ge1}\|\mathscr{K}_{\Lambda_L}\|_p<\infty$ for all $p\in(1,\infty)$. $\square$

It remains to prove Lemma 4.6. The strategy is to represent the operator using a singular kernel that has a "nearly ℓ^1-integrable" decay. Let $G_\Lambda(x,y)$ be the Green's function (i.e., inverse) of the Laplacian on Λ with zero boundary condition.

Lemma 4.7 *The operator $\mathscr{K}_\Lambda$ admits the representation*

$$
\hat{\mathrm{e}}_i\cdot\big[\mathscr{K}_\Lambda\cdot f(x)\big]=\sum_{y\in\Lambda}\sum_{j=1}^d\big[\nabla_i^{(1)}\nabla_j^{(2)}G_\Lambda(x,y)\big]f_j(y),
\tag{4.37}
$$

where the superscripts on the ∇'s indicate which of the two variables the operator is acting on.

Proof. Since both G_Λ and f vanish outside Λ, we have

$$
\begin{aligned}
\hat{\mathrm{e}}_i\cdot\big[\mathscr{K}_\Lambda\cdot f(x)\big] &= \nabla_i\Big(\sum_{y\in\Lambda}G_\Lambda(\cdot,y)\big(\nabla^\star\cdot f\big)(y)\Big)(x) \\
&= \sum_{y\in\mathbb{Z}^d}\Big(\big(G_\Lambda(x+\hat{\mathrm{e}}_i,y)-G_\Lambda(x,y)\big)\sum_{j=1}^d[f_j(y-\hat{\mathrm{e}}_k)-f_j(y)]\Big) \\
&= \sum_{j=1}^d\sum_{y\in\mathbb{Z}^d}\big(G_\Lambda(x+\hat{\mathrm{e}}_i,y+\hat{\mathrm{e}}_j)-G_\Lambda(x,y+\hat{\mathrm{e}}_j)\big)f_j(y) \\
&\qquad -\sum_{j=1}^d\sum_{y\in\mathbb{Z}^d}\big(G_\Lambda(x+\hat{\mathrm{e}}_i,y)-G_\Lambda(x,y)\big)f_j(y).
\end{aligned}
\tag{4.38}
$$

This is exactly the claimed expression. $\square$

Crucial for the proof of the weak-(1,1) type in Lemma 4.6 is an integrable decay estimate on the gradient of the kernel of the operator $\mathscr{K}_\Lambda$:

Proposition 4.8 *There exists $C>0$ independent of L such that*

$$
\big|\nabla_i^{(2)}\nabla_j^{(1)}\nabla_k^{(2)}G_{\Lambda_L}(x,y)\big|\le\frac{C}{|x-y|^{d+1}}
\tag{4.39}
$$

for all $x,y\in\Lambda_L$ and $i,j,k\in\{1,\dots,d\}$.

Although (4.39) is certainly not unexpected, and perhaps even well-known, we could not find an exact reference and therefore provide an independent proof in Section 4.4. With this estimate at hand, we can now turn to the proof of Lemma 4.33.

Proof of Lemma 4.6 from Proposition 4.8. To ease the notation, we will write $\Lambda := \Lambda_L$ (note that all bounds will be uniform in L) and, resorting to components, write $\mathscr{K}_\Lambda$ for the scalar-to-scalar operator with kernel $\mathscr{K}_\Lambda^{(i,j)}(x,y) := \nabla_i^{(1)}\nabla_j^{(2)}G_\Lambda(x,y)$ for some fixed $i,j \in \{1,\dots,d\}$. For the most part, we adapt the arguments in Stein [16, pages 30-33].

Take some function $f\colon \Lambda \to \mathbb{R}$, extended to vanish outside Λ, and pick $\alpha > 0$. Consider a partition of $\mathbb{Z}^d$ into cubes of side 3^r, where r is chosen so large that $3^{-rd}\|f\|_1 \le \alpha$. Naturally, each of the cubes in the partition further divides into 3^d equal-sized sub-cubes of side 3^{r-1}, which subdivide further into sub-cubes of side 3^{r-2}, etc. We will now designate these to be either *good cubes* or *bad cubes* according to the following recipe. All cubes of side 3^r are *ex definitio* good. With Q being one of these sub-cubes of side 3^{r-1}, we call Q good if

$$\frac{1}{|Q|}\sum_{z\in Q}\big|f(z)\big| \le \alpha, \tag{4.40}$$

and bad otherwise. For each good cube, we repeat the process of partitioning it into 3^d equal-size sub-cubes and designating each of them to be either good or bad depending on whether (4.40) holds or not, respectively. The bad cubes are not subdivided further.

Iterating this process, we obtain a finite set $\mathscr{B}$ of bad cubes which covers the (bounded) region $B := \bigcup_{Q\in\mathscr{B}} Q$. We define $G := \mathbb{Z}^d \setminus B$, the good region, and note that

$$\big|f(z)\big| \le \alpha, \qquad z \in G, \tag{4.41}$$

and

$$\alpha < \frac{1}{|Q|}\sum_{z\in Q}\big|f(z)\big| \le 3^d\alpha, \qquad Q \in \mathscr{B}, \tag{4.42}$$

where the last inequality is due to the fact that the parent cube of a bad cube is good. Next we define the "good" function

$$g(z) := \begin{cases} f(z), & z \in G \\ \frac{1}{|Q|}\sum_{z\in Q} f(z), & z \in Q \in \mathscr{B}. \end{cases} \tag{4.43}$$

The "bad" function, defined by $b := f - g$, then satisfies

$$\begin{aligned} b(z) &= 0, \qquad && z \in G, \\ \sum_{z\in Q} b(z) &= 0, \qquad && Q \in \mathscr{B}. \end{aligned} \tag{4.44}$$

Since $\mathscr{K}_\Lambda f = \mathscr{K}_\Lambda g + \mathscr{K}_\Lambda b$, as soon as

$$\big|\{z : |\mathscr{K}_\Lambda g(z)| > \alpha/2\}\big| \le \frac{\widehat{K}_1\|f\|_1}{2\alpha} \quad \text{AND} \quad \big|\{z : |\mathscr{K}_\Lambda b(z)| > \alpha/2\}\big| \le \frac{\widehat{K}_1\|f\|_1}{2\alpha}, \tag{4.45}$$

the desired bound (4.33) will hold. We will now show these bounds in separate arguments.

Considering g first, we note that $\|g\|_2^2$ is bounded by a constant times $\alpha\|f\|_1$. Indeed, for $z \in B$ let Q_z denote the bad cube containing z. Then

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^d} g(z)^2 &= \sum_{z\in G} f(z)^2 + \sum_{z\in B} g(z)^2 \\
&\le \alpha \sum_{z\in G} \bigl|f(z)\bigr| + \sum_{z\in B}\Bigl(\frac{1}{|Q_z|}\sum_{y\in Q_z} f(z)\Bigr)^2 \\
&\le \alpha\|f\|_1 + 3^d\alpha\sum_{z\in B}\frac{1}{|Q_z|}\sum_{y\in Q_z}\bigl|f(z)\bigr| \\
&\le (3^d+1)\alpha\|f\|_1
\end{aligned}
\tag{4.46}
$$

by using (4.41) on G and (4.42) on B. By Chebychev's inequality and Lemma 4.5,

$$
\bigl|\{z : |\mathscr{K}_\Lambda g(z)| > \alpha\}\bigr| \le \frac{\|\mathscr{K}_\Lambda g\|_2^2}{\alpha^2} \le \frac{(3^d+1)\|\mathscr{K}_\Lambda\|_2^2\,\|f\|_1}{\alpha}.
\tag{4.47}
$$

Note that this yields an estimate that is uniform in $\Lambda := \Lambda_L$ because $\|\mathscr{K}_\Lambda\|_2 \le 1$ by Lemma 4.5.

Let us turn to the estimate in (4.45) concerning b. Let $\{Q_k\colon k = 1,\dots,|\mathscr{B}|\}$ be an enumeration of the bad cubes and let $b_k := b\,1_{Q_k}$ be the restriction of b onto Q_k. Abusing the notation to the point where we write $\mathscr{K}_\Lambda(x,y)$ for the kernel governing $\mathscr{K}_\Lambda$, from (4.44) we then have

$$
\mathscr{K}_\Lambda b_k(z) = \sum_{y\in Q_k}\bigl[\mathscr{K}_\Lambda(z,y) - \mathscr{K}_\Lambda(z,y_k)\bigr]b(y),
\tag{4.48}
$$

where y_k is the center of Q_k (remember that all cubes are odd-sized). Let $\tilde{Q}_k$ denote the cube centered at y_k but of three-times the size — i.e., $\tilde{Q}_k$ is the union of Q_k with the adjacent $3^d - 1$ cubes of the same side. The bound now proceeds depending on whether $z \in \tilde{Q}_k$ or not.

For $z \notin \tilde{Q}_k$, the distance between z and any $y \in Q_k$ is proportional to the distance between z and y_k. Proposition 4.8 thus implies

$$
\bigl|\mathscr{K}_\Lambda(z,y) - \mathscr{K}_\Lambda(z,y_k)\bigr| \le C\,\frac{\mathrm{diam}(Q_k)}{|z-y_k|^{d+1}}, \qquad z \notin \tilde{Q}_k.
\tag{4.49}
$$

Moreover, thanks to (4.43),

$$
\sum_{y\in Q_k}|b(y)| \le \sum_{y\in Q_k}\bigl(|f(y)| + |g(y)|\bigr) \le 2\sum_{y\in Q_k}|f(y)|.
\tag{4.50}
$$

Using these in (4.48) yields

$$
|\mathscr{K}_\Lambda b_k(z)| \le C\,\frac{\mathrm{diam}(Q_k)}{|z-y_k|^{d+1}}\sum_{y\in Q_k}|f(y)|.
\tag{4.51}
$$

Summing over all $z \notin \tilde{Q}_k$ and taking into account that $|z-y_k| \ge \mathrm{diam}(Q_k)$ for $z \in \tilde{Q}_k$, we conclude

$$
\begin{aligned}
\sum_{z\in\Lambda\setminus\tilde{Q}_k}|\mathscr{K}_\Lambda b_k(z)| &\le C\,\mathrm{diam}(Q_k)\sum_{y\in Q_k}|f(y)|\sum_{z\colon\, |z-y_k|\ge\mathrm{diam}(Q_k)}\frac{1}{|z-y_k|^{d+1}} \\
&\le \tilde{C}\sum_{y\in Q_k}|f(y)|
\end{aligned}
\tag{4.52}
$$

for some constant $\tilde{C}$. Setting $\tilde{B} := \bigcup_k \tilde{Q}_k$ and summing over k, we obtain

$$
\sum_{z\in\Lambda\setminus\tilde{B}}|\mathscr{K}_\Lambda b(z)| \le \tilde{C}\sum_{y\in B}|f(y)| \le \tilde{C}\|f\|_1,
\tag{4.53}
$$

which by an application of Chebychev's inequality yields

$$\big|\{z\in\Lambda\setminus\tilde B\colon\, |\mathscr{K}_\Lambda b(z)|\ge\alpha\}\big|\le\frac{\tilde C\|f\|_1}{\alpha}. \tag{4.54}$$

i.e., a bound of the desired form.

To finish the proof, we still need to take care of $z\in\tilde B$. Here we get (and this is the only step where we are forced to settle on *weak*-type estimates),

$$\begin{aligned}\big|\{z\in\tilde B\colon\, |\mathscr{K}_\Lambda b(z)|\ge\alpha\}\big|&\le|\tilde B|\le 3^d\sum_k|Q_k|\\&\le 3^d\sum_k\frac{1}{\alpha}\sum_{z\in Q_k}\big|f(z)\big|\le\frac{3^d\|f\|_1}{\alpha}.\end{aligned} \tag{4.55}$$

The bound (4.33) then follows by combining (4.47), (4.54) and (4.55). □

4.4 Triple gradient of finite-volume Green's function.

In order to finish the proof of Theorem 4.4, we still need to establish the decay estimate in Proposition 4.8. This will be done by invoking a corresponding bound in the full lattice and reducing it onto a box by reflection arguments. (This is the sole reason why we restrict to rectangular boxes; more general domains require considerably more sophisticated methods.)

For $\varepsilon>0$, let G^ε denote the Green's function associated with the discrete Laplacian Δ on $\mathbb{Z}^d$ with killing rate $\varepsilon>0$, i.e., $G^\varepsilon(\cdot,\cdot)$ is the kernel of the bounded operator $(\varepsilon-\Delta)^{-1}$ on $\ell^2(\mathbb{Z}^d)$. This function admits the probabilistic representation

$$G^\varepsilon(x,y)=\sum_{k=0}^\infty\frac{P^x\big(X_k=y\big)}{(1+\varepsilon)^{k+1}}, \tag{4.56}$$

where X is the simple random walk and P^x is the law of X started at x. This function depends only on the difference of its arguments, so we will interchangeably write $G^\varepsilon(x,y)=G^\varepsilon(x-y)$. We now claim:

Lemma 4.9 *There exists $\widehat C>0$ such that, for all $\varepsilon>0$, all $i,j,k\in\{1,\dots d\}$ and all $x\ne0$,*

$$\big|\nabla_i\nabla_j\nabla_k G^\varepsilon(x)\big|\le\frac{\widehat C}{|x|^{d+1}}. \tag{4.57}$$

Sketch of proof. This is a mere extension (by adding one more gradient) of the estimates from in Lawler [10, Theorem 1.5.5]. (Strictly speaking, this theorem is only for the transient dimensions but, thanks to $\varepsilon>0$, the same proofs would apply here.) The main idea is to use translation invariance of the simple random walk to write $G^\varepsilon(x)$ as a Fourier integral and then control the gradients thereof under the integral sign. We leave the details as an exercise to the reader. □

We now state and prove a stronger form of Proposition 4.8.

Lemma 4.10 *There exists $C>0$ such that, for all $L>1$, $\varepsilon>0$ and arbitrary $i,j,k\in\{1,\dots d\}$,*

$$|\nabla_i^{(2)}\nabla_j^{(1)}\nabla_k^{(2)}G_\Lambda^\varepsilon(x,y)|\le\frac{C}{|x-y|^{d+1}} \tag{4.58}$$

for all $x,y\in\Lambda$ and all $i,j,k\in\{1,\dots,d\}$. Here, the superscripts on the operators indicate the variable the operator is acting on.

Proof. Throughout, we fix $L \in \mathbb{N}$ and denote $\Lambda := \Lambda_L$. The proof is based on the Reflection Principle for the simple random walk on $\mathbb{Z}^d$. To start, denote $\Lambda_0 := \mathbb{Z}^{d-1} \times \mathbb{N}$ (abusing our earlier notation), write $X^{(d)}$ for the d-th component of X and let $\tau_0 := \inf\{k \geq 0 \colon X_k^{(d)} = 0\}$. For $y \in \Lambda_0$ with components $y = (y_1, \dots, y_d)$, put $r_0(y) := (y_1, \dots, -y_d)$. The Green's function $G^\varepsilon_{\Lambda_0}$ on Λ_0 with zero boundary condition is given by

$$G^\varepsilon_{\Lambda_0}(x,y) = \sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} P^x\big(X_k = y, \tau_0 > k\big). \tag{4.59}$$

The Reflection Principle tells us that, for $x, y \in \Lambda_0$,

$$P^x\big(X_k = y, \tau_0 \leq k\big) = P^x\big(X_k = r_0(y), \tau_0 \leq k\big) = P^x\big(X_k = r_0(y)\big) \tag{4.60}$$

and so

$$\begin{aligned} G^\varepsilon_{\Lambda_0}(x,y) &= G^\varepsilon(x,y) - \sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} P^x\big(X_k = y, \tau_0 \leq k\big) \\ &= G^\varepsilon(x,y) - \sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} P^x\big(X_k = r_0(y)\big) \\ &= G^\varepsilon(x,y) - G^\varepsilon(x, r_0(y)). \end{aligned} \tag{4.61}$$

This holds for all $x, y \in \Lambda_0$ and extends even to $y \in \Lambda_0 \cup \partial\Lambda_0$, as is easy to check.

Next, consider $\Lambda_1 := \mathbb{Z}^{d-1} \times \{0, \dots, L\}$ and set $\tau_L := \inf\{k \geq 0 \colon X_k^{(d)} = L\}$. In analogy with $r_0(y)$ we define $r_L(y) := (y_1, \dots, 2L - y_d)$. The Reflection Principle again yields

$$P^x\big(X_k = y, \tau_0 > k, \tau_L \leq k\big) = P^x\big(X_k = r_L(y), \tau_0 > k\big) \tag{4.62}$$

and so

$$\begin{aligned} G^\varepsilon_{\Lambda_1}(x,y) &= \sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} P^x\big(X_k = y, \tau_0 > k, \tau_L > k\big) \\ &= G^\varepsilon_{\Lambda_0}(x,y) - \sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} P^x\big(X_k = y, \tau_0 > k, \tau_L \leq k\big) \\ &= G^\varepsilon_{\Lambda_0}(x,y) - G^\varepsilon_{\Lambda_0}\big(x, r_L(y)\big). \end{aligned} \tag{4.63}$$

In conjunction with (4.61), we thus obtain

$$\begin{aligned} G^\varepsilon_{\Lambda_1}(x,y) &= G^\varepsilon_{\Lambda_0}(x,y) - G^\varepsilon_{\Lambda_0}\big(x, r_L(y)\big) \\ &= G^\varepsilon(x,y) - G^\varepsilon\big(x, r_0(y)\big) - G^\varepsilon\big(x, r_L(y)\big) + G^\varepsilon\big(x, r_0 \circ r_L(y)\big) \end{aligned} \tag{4.64}$$

for all $x \in \Lambda_1$ and $y \in \Lambda_1 \cup \partial\Lambda_1$.

Proceeding by induction along coordinate directions, we may progressively confine all coordinates of the random walk and obtain a representation for the corresponding Green's function in Λ in terms of its full-lattice counterpart. Indeed,

$$\begin{aligned} &\Lambda_d := \{0, \dots, L\}^d, \\ &\Lambda_i := \mathbb{Z}^{d-i} \times \{0, \dots, L\}^i, \qquad i = 1, \dots, d-1, \\ &\Lambda_0 := \mathbb{Z}^d, \end{aligned} \tag{4.65}$$

and denoting by $r_0^{(i)}$, resp., $r_L^{(i)}$, $i=1,\dots,d$, the reflections in the planes with the i-th coordinate equal to zero, resp., L, we get in analogy with (4.64) the relation

$$\begin{aligned}G^{\varepsilon}_{\Lambda_i}(x,y) =& G^{\varepsilon}_{\Lambda_{i-1}}(x,y) - G^{\varepsilon}_{\Lambda_{i-1}}\big(x,r_0^{(i)}(y)\big)\\ &- G^{\varepsilon}_{\Lambda_{i-1}}\big(x,r_L^{(i)}(y)\big) + G^{\varepsilon}_{\Lambda_{i-1}}\big(x,r_0^{(i)}\circ r_L^{(i)}(y)\big).\end{aligned} \tag{4.66}$$

Solving the recursion, this leads to

$$G^{\varepsilon}_{\Lambda}(x,y) = \sum_{R\in\mathfrak{R}} (-1)^{|R|} G^{\varepsilon}\big(x, R_1\circ\cdots\circ R_d(y)\big), \tag{4.67}$$

where $\mathfrak{R} := \bigotimes_{i=1}^d \big\{\mathrm{id}, r_0^{(i)}, r_L^{(i)}, r_0^{(i)}\circ r_L^{(i)}\big\}$ is the set of all possible reflections that may occur and $|R| := \sum_{i=1}^d (1_{R_i=r_0^{(i)}} + 1_{R_i=r_L^{(i)}})$ has the same parity as the number of reflections involved.

From (4.67) and Lemma 4.9, we thus obtain

$$\big|\nabla_i^{(2)}\nabla_j^{(1)}\nabla_k^{(2)} G^{\varepsilon}_{\Lambda}(x,y)\big| \le \sum_{R\in\mathfrak{R}} \big|\nabla_i^{\star}\nabla_j\nabla_j^{\star} G^{\varepsilon}(x-R(y))\big| \le \sum_{R\in\mathfrak{R}} \frac{\widehat{C}}{|x-R(y)|^{d+1}}. \tag{4.68}$$

Since, $|x-R(y)| \ge |x-y|$ when $x,y\in\Lambda$, and $|\mathfrak{R}| = 2^d$, the result follows. $\square$

We are now ready to complete the proof of Theorem 4.4:

Proof of Proposition 4.8. Although the $\varepsilon\downarrow 0$ limit of G^{ε} exists only in $d\ge 3$, for gradients we have $\nabla G(x,y) = \lim_{\varepsilon\downarrow 0}\nabla G^{\varepsilon}(x,y)$ in all $d\ge 1$. Since the bound in Lemma 4.10 holds uniformly in $\varepsilon>0$, we get the claim in all $d\ge 1$. $\square$

5. Perturbed harmonic coordinate

In this section we will prove Propositions 2.5 and 2.6. Abusing our earlier notation, let

$$G_{\Lambda}(x,y;\omega) = (-\mathscr{L}_{\omega})^{-1}(x,y) \tag{5.1}$$

denote the Green's function in Λ with Dirichlet boundary condition for conductance configuration ω. (Thus, the simple-random walk Green's function from Section 4 corresponds to $\omega := 1$.) The Green's function is the fundamental solution to the Poisson equation, i.e.,

$$\begin{cases} -\mathscr{L}_{\omega} G_{\Lambda}(x,z,\omega) = \delta_x(z) & \text{if } z\in\Lambda,\\ G_{\Lambda}(x,z,\omega) = 0, & \text{if } z\in\partial\Lambda,\end{cases} \tag{5.2}$$

where $\delta_x(z)$ is the Kronecker delta. Note that G_{Λ} is defined for all $\omega\in\Omega$. The solution to (5.2) is naturally symmetric,

$$G_{\Lambda}(x,y;\omega) = G_{\Lambda}(y,x;\omega), \qquad x,y\in\Lambda, \tag{5.3}$$

and so we can extend it to a function on $\Lambda\cup\partial\Lambda$ by setting $G_{\Lambda}(x,\cdot;\omega) = 0$ whenever $x\in\partial\Lambda$. Here is a generalized form of the representation (2.23):

Lemma 5.1 (Rank-one perturbation) *For a finite $\Lambda\subset\mathbb{Z}^d$ let $x,y\in\Lambda$ be nearest neighbors. For any ω,ω' such that $\omega'_b = \omega_b$ except at $b:=\langle x,y\rangle$, and any $z\in\Lambda\cup\partial\Lambda$,*

$$\begin{aligned}&\Psi_{\Lambda}(\omega',z) - \Psi_{\Lambda}(\omega,z)\\ &\qquad = -(\omega'_{xy}-\omega_{xy})\big[G_{\Lambda}(z,y;\omega') - G_{\Lambda}(z,x;\omega')\big]\big[\Psi_{\Lambda}(\omega,y) - \Psi_{\Lambda}(\omega,x)\big].\end{aligned} \tag{5.4}$$

Proof. Suppose $\omega, \omega' \in \Omega$ are such that ω' equals ω except at the edge $b := \langle x,y\rangle$, where $\omega'_b := \omega_b + \varepsilon$. Define the function $\Phi_\Lambda\colon \Lambda \cup \partial\Lambda \to \mathbb{R}^d$ by

$$\Phi_\Lambda(z) := \Psi_\Lambda(\omega,z) - \varepsilon\big[G_\Lambda(z,y;\omega') - G_\Lambda(z,x;\omega')\big]\big[\Psi_\Lambda(\omega,y) - \Psi_\Lambda(\omega,x)\big]. \tag{5.5}$$

We claim that

$$\mathscr{L}_{\omega'}\Phi_\Lambda = 0 \qquad \text{in } \Lambda. \tag{5.6}$$

Since for $z \in \partial\Lambda$ we have $\Phi_\Lambda(z) = \Psi_\Lambda(\omega,z) = z$, this will imply $\Phi_\Lambda(\cdot) = \Psi_\Lambda(\omega',\cdot)$ thanks to the uniqueness of the solution of the Dirichlet problem.

In order to show (5.6), we first use (5.2–5.3) to get

$$\mathscr{L}_{\omega'}\Phi_\Lambda(z) = \mathscr{L}_{\omega'}\Psi_\Lambda(\omega,z) - \varepsilon\big[\delta_y(z) - \delta_x(z)\big]\big[\Psi_\Lambda(\omega,y) - \Psi_\Lambda(\omega,x)\big]. \tag{5.7}$$

To deal with the term $\mathscr{L}_{\omega'}\Psi_\Lambda(\omega,z)$, we think of of $\mathscr{L}_{\omega'}$ as a matrix of dimension $|\Lambda|$. For its coefficients $\mathscr{L}_\omega(z,z') := \langle \delta_z, \mathscr{L}_\omega \delta_{z'}\rangle_{\ell^2(\Lambda)}$ we obtain

$$\mathscr{L}_{\omega'}(z,z') = \mathscr{L}_\omega(z,z') + \varepsilon\big[\delta_y(z) - \delta_x(z)\big]\big[\delta_y(z') - \delta_x(z')\big]. \tag{5.8}$$

Using that $\mathscr{L}_\omega\Psi_\Lambda(\omega,z) = 0$ for $z \in \Lambda$, we now readily confirm (5.6). ☐

Proof of Proposition 2.5. Set $y := x + \hat{\mathrm{e}}_i$ and denote $\nabla_i f(z) := f(z+\hat{\mathrm{e}}_i) - f(z)$. Lemma 5.1 shows

$$\nabla_i\Psi_\Lambda(\omega',x) = \Big[1 - (\omega'_b - \omega_b)\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x,x,\omega')\Big]\nabla_i\Psi_\Lambda(\omega,x), \tag{5.9}$$

where the superindices on ∇ indicate which variable is the operator acting on. The prove the claim we need to show

$$\big[\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x,x,\omega)\big]^{-1} = \inf\big\{Q_\Lambda(f)\colon f(y) - f(x) = 1,\, f_{\partial\Lambda} = 0\big\}, \tag{5.10}$$

where the conductances in Q_Λ correspond to ω. For this, let f be the minimizer of the right-hand side. The method of Largrange multiplies shows

$$-\mathscr{L}_\omega f(z) = \alpha\big[\delta_y(z) - \delta_x(z)\big]. \tag{5.11}$$

Thanks to (5.2), this is solved by

$$f(z) = \alpha\big[G_\Lambda(y,z;\omega) - G_\Lambda(x,z;\omega)\big] = \alpha\nabla_i^{(1)}G_\Lambda(x,z;\omega) \tag{5.12}$$

which in light of the constraint $f(y) - f(x) = 1$ gives $\alpha = [\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x,x,\omega)]^{-1}$. Since also $Q_\Lambda(f) = \langle f, -\mathscr{L}_\omega f\rangle_{\ell^2(\Lambda)}$, (5.11) gives $Q_\Lambda(f) = \alpha$ and so (5.10) holds. The correspondence (2.23) then follows from (5.9–5.10); the identity (2.24) results by differentiation of the left-hand side with respect to ω'_b. ☐

Finally, it remains to establish the limit (2.25), including all of its stated properties:

Proof of Proposition 2.6. Thanks to ellipticity restriction(1.3), we have a bound on this quantity in terms of the lattice Laplacian. This shows that, for some $c = c(\lambda) \in (0,1)$,

$$c < \nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x,x,\omega') < 1/c \tag{5.13}$$

uniformly in Λ. Moreover, $\Lambda \mapsto \nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x,x,\omega')$ is obviously non-decreasing in Λ and so the limit exists. The formula (2.26) and the claimed stationarity then follow as well. ☐

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