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On the structure of the quasiconvex hull in planar elasticity

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Abstract

Let K_1 and K_2 be compact sets of real 2×2 matrices with positive determinant. Suppose that both sets are frame invariant, meaning invariant under the left action of the special orthogonal group. Then we give an algebraic characterization for K_1 and K_2 to be incompatible for homogeneous gradient Young measures. This result permits a simplified characterization of the quasiconvex hull and the rank-one convex hull in planar elasticity.

1 Introduction

We study quasiconvexity in the calculus of variations. Morrey [Mor52] proved that this is the essential property for functions in the context of sequentially weakly lower-semicontinuity for multiple integrals. Unfortunately, his definition of quasiconvexity is very hard to test. Kristensen [Kri99] even showed that there cannot be any "local" characterization which is equivalent to quasiconvexity. Kristensen's proof makes use of Šverák's counterexample of a rank-one convex function that fails to be quasiconvex [Šve92]. The difference between quasiconvexity and rank-one convexity is also visible on the level of sets. Milton [Mil04] showed that there is a rank-one convex set which fails to be quasiconvex. However, Milton's as well as Šverák's counterexample work only in the case of an underlying space $\mathbb{M}^{m\times n}$ with $m \geq 3$ and $n \geq 2$.

In contrast to that, the situation in $\mathbb{M}^{2\times 2}$ seems to be fundamentally different. Whether rank-one convexity and quasiconvexity are the same over $\mathbb{M}^{2\times 2}$ remains an open question. Nevertheless, we would like to recall a few of the results for the 2×2 case. Müller [Mül99b] showed that rank-one convexity implies quasiconvexity on diagonal matrices. Dolzmann [Dol03] proved that rank-one convexity and polyconvexity are equivalent for frame-invariant sets with constant determinant. In this paper, we heavily rely on results by Faraco and Székelyhidi [FS08] on the localization of the quasiconvex hull. One of the key tools that we are going to use is the following, see [FS08, Corollary 3].

Theorem 1.1. *Let ν be a compactly supported homogeneous gradient Young measure over $\mathbb{M}^{2\times 2}$. Then the set $\operatorname{supp}(\nu)^{\mathrm{qc}}$ is connected.*

In [Hei11], we applied this to the case of isotropic sets. Now we study quasiconvexity in the context of frame-invariant sets in $\mathbb{M}^{2\times 2}$. We prove the following (see Theorem 6.3).

Theorem (Incompatible sets)**.** *Let $L_1, L_2 \subseteq \mathbb{M}^{2\times 2}$ be compact and frame-invariant sets of matrices with positive determinant. Then the following properties are equivalent:*

(a) L_1 and L_2 are incompatible for homogeneous gradient Young measures.

(b) L_1 *and* L_2 *are incompatible for first-order laminates. In addition,* $\{A^tA \mid A{\in}L_1\}$ *and* $\{C^tC \mid C{\in}L_2\}$ *are incompatible for T4 configurations.*

(c) L_1 *and* L_2 *can be separated by a polyconvex set.*

This theorem gives an algebraic characterization for two frame-invariant sets to be incompatible for homogeneous gradient Young measures. In view of elasticity theory, a $2{\times}2$ matrix A might represent a deformation gradient. Then the key observation of the theorem is that essential conditions can be nicely written down for the so-called right Cauchy-Green tensor A^tA. In addition, we will construct a set Δ which provides information about the structure of the quasiconvex hull. Our second result reads (see Theorem 6.1)

Theorem (Structure)**.** *Let* $K \subseteq \mathbb{M}^{2\times 2}$ *be a compact and frame-invariant set of matrices with positive determinant and* $A, B \in K$*. Then the following properties are equivalent:*

(i) A *and* B *lie in the same connected component of* K^{rc}*.*

(ii) A *and* B *lie in the same connected component of* K^{qc}*.*

(iii) A *and* B *lie in the same connected component of* K^{pc}*.*

(iv) $\det(A)$ *and* $\det(B)$ *lie in the same connected component of* Δ*.*

The paper is organized as follows:
In Section 2, we recall definitions of the convexity notions that are used later on. Then we give a short introduction to T4 configurations in Section 3. Preliminaries can be found in Section 4. Section 5 provides the most important tool and Section 6 collects the main results of this paper.

2 Convexity notions

We denote by $\mathbb{M}^{2\times 2}$ the vector space of all real $2{\times}2$ matrices equipped with the Euclidean structure of $\mathbb{R}^4$. We are going to recall some convexity notions in $\mathbb{M}^{2\times 2}$. A detailed discussion, also for higher dimensions, can be found in [Bal77], [Dac89, §4.1], [Mül99a] and [Dol03].

Let $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a given continuous function. Then f is *convex* if for every $A, B \in \mathbb{M}^{2\times 2}$ we have

$$\forall \lambda \in [0,1] \quad f(\lambda A + (1{-}\lambda)B) \le \lambda f(A) + (1{-}\lambda) f(B). \tag{1}$$

The function f is *polyconvex* if there exists a convex function $g\colon \mathbb{R}^5 \to \mathbb{R}$ such that for every $A \in \mathbb{M}^{2\times 2}$ we have $f(A) = g(A, \det(A))$, where $\det(A)$ denotes the determinant of A. The function f is *quasiconvex* (in Morrey's sense [Mor52]), if for every $A \in \mathbb{M}^{2\times 2}$ and every smooth function $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with compact support we have

$$0 \le \int_{\mathbb{R}^2} \big(f(A + \mathrm{D}\phi(x)) - f(A)\big)\mathrm{d}x.$$

The function f is *rank-one convex* if (1) holds for every $A, B \in \mathbb{M}^{2\times 2}$ that are *rank-one connected*, meaning $A - B$ equals the tensor product $a \otimes b$ for some vectors $a, b \in \mathbb{R}^2$. Polyconvexity and rank-one convexity were introduced by Ball [Bal77].

Now we consider sets of matrices. Therefore fix a compact set $K \subset \mathbb{M}^{2\times 2}$. We denote by $\mathcal{M}(K)$ the set of all probability measures over the Borel sets of K. Let $\nu \in \mathcal{M}(K)$ be a given element. We write $\bar{\nu}$ for its mean value and $\operatorname{supp}(\nu)$ for its support. We define the sets $\mathcal{M}^{\mathrm{pc}}(K)$, $\mathcal{M}^{\mathrm{qc}}(K)$ and $\mathcal{M}^{\mathrm{rc}}(K)$ as follows. A probability measure $\nu \in \mathcal{M}(K)$ lies in $\mathcal{M}^{\mathrm{pc}}(K)$ ($\mathcal{M}^{\mathrm{qc}}(K)$ or $\mathcal{M}^{\mathrm{rc}}(K)$) if and only if Jensen's inequality

$$f(\bar{\nu}) \leq \int_{\mathbb{M}^{2\times 2}} f(A) \mathrm{d}\nu(A) \tag{2}$$

is fulfilled for every continuous function $f \colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ which is polyconvex (quasiconvex or rank-one convex). Kinderlehrer and Pedregal [KP91] show that every $\nu \in \mathcal{M}^{\mathrm{qc}}(K)$ is a *homogeneous gradient Young measure*. Whereas every $\nu \in \mathcal{M}^{\mathrm{rc}}(K)$ is a *laminate*, see [Ped93]. The *polyconvex hull* of K is given by

$$K^{\mathrm{pc}} = \{\bar{\nu} \mid \nu \in \mathcal{M}^{\mathrm{pc}}(K)\}. \tag{3}$$

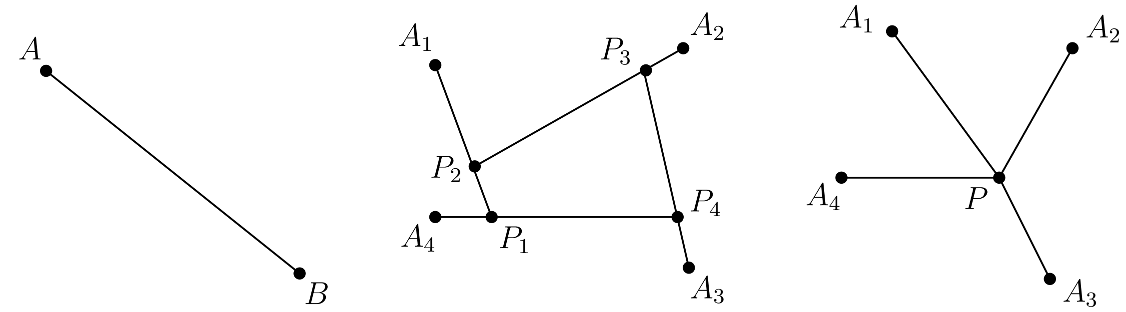
We get the *quasiconvex hull* K^{qc}, the *rank-one convex hull* K^{rc} and the *convex hull* K^{c} if, in (3), we replace $\mathcal{M}^{\mathrm{pc}}(K)$ by $\mathcal{M}^{\mathrm{qc}}(K)$, $\mathcal{M}^{\mathrm{rc}}(K)$ and $\mathcal{M}(K)$, respectively. Finally, we call K *polyconvex* (*quasiconvex*, *rank-one convex* or *convex*) whenever $K = K^{\mathrm{pc}}$ ($K = K^{\mathrm{qc}}$, $K = K^{\mathrm{rc}}$ or $K = K^{\mathrm{c}}$) holds. The previous definitions together with the hierarchy of convexity notions imply that $K \subseteq K^{\mathrm{rc}} \subseteq K^{\mathrm{qc}} \subseteq K^{\mathrm{pc}} \subseteq K^{\mathrm{c}}$. In particular, every laminate is a homogeneous gradient Young measure.

Note that all these convexity notions, both for functions and for sets, are stable against transformations of the underlying space $\mathbb{M}^{2\times 2}$ which are given by $A \mapsto SAT$ for some invertible matrices $S, T \in \mathbb{M}^{2\times 2}$. We will make use of this fact several times.

3 Laminates

Two classes of laminates play an important role in our paper. First, consider the case of two rank-one connected matrices $A, B \in \mathbb{M}^{2\times 2}$, where $A = B$ is possible. Then for every real number $\lambda \in (0, 1)$ the measure $\lambda\delta_A + (1-\lambda)\delta_B$ with support $\{A, B\}$ is a laminate, a so-called *first-order laminate*. In fact, it fulfills Jensen's inequality (see (2)) for rank-one convex functions.

Second, consider the case of four matrices $A_1, \ldots, A_4 \in \mathbb{M}^{2\times 2}$ without rank-one connections. Tartar [Tar93] showed that $\{A_1, \ldots, A_4\}$ can be the support of a laminate. Consider the following definition, compare [Szé05]. A 4-tuple (A_1, A_2, A_3, A_4) over $\mathbb{M}^{2\times 2}$ without rank-one connections forms a *T4 configuration* whenever there exist matrices $C_1, \ldots, C_4 \in \mathbb{M}^{2\times 2}$ of rank equal to 1, a matrix $P \in \mathbb{M}^{2\times 2}$ and real numbers $\kappa_1, \ldots, \kappa_4 > 1$ such that (T4) or (T4*)



(a) Rank-one connected matrices (b) T4 configuration, (T4) (c) T4 configuration, (T4*)

Figure 1: Three matrix configurations in $\mathbb{M}^{2\times 2}$. Solid lines follow rank-one directions.

is fulfilled:

$$\text{(T4)}\begin{cases} A_1 = P + \kappa_1 C_1 \\ A_2 = P + C_1 + \kappa_2 C_2 \\ A_3 = P + C_1 + C_2 + \kappa_3 C_3 \\ A_4 = P + C_1 + C_2 + C_3 + \kappa_4 C_4 \\ \ \ 0 = C_1 + \ldots + C_4 \end{cases}, \quad \text{(T4*)}\begin{cases} A_1 = P + C_1 \\ A_2 = P + C_2 \\ A_3 = P + C_3 \\ A_4 = P + C_4 \\ \ \ 0 \in \{C_1, \ldots, C_4\}^{\mathrm{c}} \end{cases}. \tag{4}$$

The lemma collects well-known properties of T4 configurations.

Lemma 3.1. *Let $A_1, \ldots, A_4 \in \mathbb{M}^{2\times 2}$ be matrices such that (A_1, A_2, A_3, A_4) forms a T4 configuration. Then, for (T4*), the matrix P and, for (T4), the matrices $P_1 = P$, $P_2 = P + C_1$, $P_3 = P + C_1 + C_2$ as well as $P_4 = P + C_1 + C_2 + C_3$ lie in the rank-one convex hull $\{A_1, \ldots, A_4\}^{\mathrm{rc}}$.*

Proof. Compare with Figure 1. There exists a laminate ν so that $\bar{\nu} = P$ and $\operatorname{supp}(\nu) = \{A_1, \ldots, A_4\}$. A way how to construct such a ν for (T4) can be found, for example, in [Mül99a, §2.5]. Similar constructions can be used when P is replaced by P_2, P_3 or P_4. Condition (T4*), which can be seen as a limit case of (T4), is handled by Kirchheim [Kir03, Corollary 4.19]. □

4 Incompatibility and frame invariance

Let $K_1, K_2 \subseteq \mathbb{M}^{2\times 2}$ be compact sets. We call K_1 and K_2 *incompatible for homogeneous gradient Young measures* if for every $\nu \in \mathcal{M}^{\mathrm{qc}}(K_1 \cup K_2)$ we have either $\operatorname{supp}(\nu) \subseteq K_1$ or $\operatorname{supp}(\nu) \subseteq K_2$. In the same spirit, we call K_1 and K_2 *incompatible for first-order laminates* if there are no rank-one connected matrices A, B with $A \in K_1$ and $B \in K_2$. We call K_1 and K_2 *incompatible for T4 configurations* if for every T4 configuration $(A_1, \ldots, A_4)$ over $K_1 \cup K_2$ we have either $\{A_1, \ldots, A_4\} \subseteq K_1$ or $\{A_1, \ldots, A_4\} \subseteq K_2$. Furthermore, we say that K_1 and K_2 *can be separated by a polyconvex set* whenever there exist disjoint open sets $U_1, U_2 \subseteq \mathbb{M}^{2\times 2}$ such that $K_1 \subset U_1$, $K_2 \subset U_2$ as well as $(K_1 \cup K_2)^{\mathrm{pc}} \subseteq U_1 \cup U_2$ holds. Then the set polyconvex set $(K_1 \cup K_2)^{\mathrm{pc}}$ "separates" K_1 and K_2.

In view of [FS08, Theorem 2], we obtain a sufficient condition for K_1 and K_2 to be incompatible for homogeneous gradient Young measures:

Theorem 4.1. *Let $K_1, K_2 \subseteq \mathbb{M}^{2\times 2}$ be compact and assume that K_1 and K_2 can be separated by a polyconvex set. Then K_1 and K_2 are incompatible for homogenous gradient Young measures.*

We want to apply Theorem 4.1 to a special situation. Therefore, we introduce additional notation. Given a column vector $x \in \mathbb{R}^2$ and a matrix $A \in \mathbb{M}^{2\times 2}$, we denote by $|x|$ and $|A|$ the corresponding Euclidean norms and by x^t and A^t the transposed objects. The letter I is used for the identity matrix in $\mathbb{M}^{2\times 2}$. Let $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ be the subspace of all symmetric matrices and $A, B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$. Then we write $A \prec B$ whenever $A-B$ is negative definite and $A \preceq B$ whenever $A-B$ is negative semi-definite. If $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is positive semi-definite, then we denote by $\sqrt{A}$ the only positive semi-definite matrix which solves $X^2 = A$ in $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$. Now we define the disjoint open sets

$$P_1 = \{A \in \mathbb{M}^{2\times 2} \mid A^tA \prec I\}, \qquad P_2 = \{A \in \mathbb{M}^{2\times 2} \mid I \prec A^tA \wedge \det(A) > 0\}. \tag{5}$$

We show that the union $P_1 \cup P_2$ is a lower level-set of a polyconvex function. In order to do so, we consider the following notation, which has been used before by many authors in the context of isotropic sets. Let $A \in \mathbb{M}^{2\times 2}$ be a given matrix, then we define $\lambda_1(A), \lambda_2(A) \in \mathbb{R}^2$ as the only real numbers such that $\{|\lambda_1(A)|, \lambda_2(A)\}$ is the set of singular values of A (the eigenvalues of $\sqrt{A^tA}$) and, in addition, $|\lambda_1(A)| \leq \lambda_2(A)$ as well as $\det(A) = \lambda_1(A)\lambda_2(A)$ holds.

Following Conti et al. [CDLMR03, Lemma 2.2], we know that the function $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ given by $f(A) = \lambda_1(A) + \lambda_2(A) - \det(A) - 1$ is polyconvex. We write the function f in factorized form $f = (1-\lambda_1)(\lambda_2 - 1)$ and realize that f is negative if and only if either $\lambda_2 < 1$ holds or $1 < \lambda_1$. By simple computations, we get that $\lambda_2(A) < 1$ holds whenever A lies in P_1 and $\lambda_1(A) > 1$ holds whenever A lies in P_2. This directly implies that we can write $P_1 \cup P_2$ as a lower level-set of a polyconvex function. We would like to point out that a key idea of this paper is to exploit the fact that P_1 and P_2, as defined in (5), are incompatible for homogeneous gradient Young measures. A first step is done in the next lemma.

Lemma 4.2. *Let $K_1, K_2 \subseteq \mathbb{M}^{2\times 2}$ be compact sets with positive determinant. Moreover, we assume that there exists a matrix $B \in \mathbb{M}^{2\times 2}$ with positive determinant such that for every $A \in K_1$ and $C \in K_2$ we have $A^tA \prec B^tB \prec C^tC$. Then K_1 and K_2 can be separated by a polyconvex set.*

Proof. Take $U_1 = P_1B$ and $U_2 = P_2B$ where P_1 and P_2 are given by (5). Note that the determinant is positive for all elements in U_2 and K_2 by assumption. A simple computation shows that U_1 and U_2 contain K_1 and K_2, respectively. The set $P_1 \cup P_2$ is the lower level-set of a polyconvex function and so is $U_1 \cup U_2$. In fact, if $f\colon \mathbb{M}^{2\times 2} \to R$ is polyconvex, the function given by $A \mapsto f(AB^{-1})$ is also polyconvex, by definition. Here B^{-1} stands for the inverse of the matrix B. In view of (2) and (3), it is not hard to see that the polyconvex hull $(K_1 \cup K_2)^{\mathrm{pc}}$ must be a subset of $U_1 \cup U_2$. □

Now we come to frame invariance. Let $K \subset \mathbb{M}^{2\times 2}$ be a given set. We call K *frame invariant* whenever it is invariant under the left action of the special orthogonal group $\mathrm{SO}(2)$, meaning, for every $A \in K$ the whole orbit $\mathrm{SO}(2)A$ is contained in K. Here we consider $\mathrm{SO}(2)$ as a subset of $\mathbb{M}^{2\times 2}$ so that the group action becomes just matrix multiplication. In the context of frame invariance, we will need the following lemma.

Lemma 4.3. *Let $D \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ be positive definite and $Y \in \mathbb{M}^{2\times 2}$ a matrix with positive determinant. Assume that Y^tY and D are rank-one connected. Then there is a positive real number $\gamma > 0$ and a rotation $R \in \mathrm{SO}(2)$ such that $Y^tY - D = \gamma\sqrt{D}(RY - \sqrt{D})$. In particular, RY and $\sqrt{D}$ are rank-one connected.*

Proof. If Y^tY and D are rank-one connected, so are the matrices $\widetilde{D} = \sqrt{D}^{-1} Y^tY \sqrt{D}^{-1}$ and I. Fix a real number $\alpha \in \mathbb{R}$ and a vector $\widetilde{x} \in \mathbb{R}^2$ with $|\widetilde{x}| = 1$ such that $\widetilde{D} - I = \alpha\widetilde{x}\otimes\widetilde{x}$. Since $\widetilde{D}$ is positive definite, we must have $\alpha > -1$. Then there is a real number $\beta > -1$ with the same sign as α so that

$$(I + \beta\widetilde{x}\otimes\widetilde{x})^t(I + \beta\widetilde{x}\otimes\widetilde{x}) - I = (2\beta + \beta^2)\widetilde{x}\otimes\widetilde{x} = \alpha\widetilde{x}\otimes\widetilde{x}.$$

Note that the matrix $I + \beta\widetilde{x}\otimes\widetilde{x}$ has positive determinant. Multiplication from the right and left by $\sqrt{D}$ gives

$$\left(\sqrt{D} + \beta\sqrt{D}^{-1} x\otimes x\right)^2 - D = \alpha x\otimes x = Y^tY - D$$

where we set $x = \sqrt{D}\widetilde{x}$. This implies that the matrices Y and $\sqrt{D} + \beta\sqrt{D}^{-1} x\otimes x$ have the same symmetric part in the polar decomposition. Since both matrices have positive determinant, there must be a rotation $R \in \mathrm{SO}(2)$ such that $RY - \sqrt{D}$ and $\beta\sqrt{D}^{-1} x\otimes x$ are the same. □

The next lemma states well-known facts, compare [Dol03, §A.2] and the references therein. For the convenience of the reader, we give a proof here.

Lemma 4.4. *Let $A, C \in \mathbb{M}^{2\times 2}$ be given matrices such that $0 < \det(A) \leq \det(C)$. Then $\det(A^tA - C^tC) \geq 0$ holds if and only if $A^tA \preceq C^tC$. In addition, the following properties are equivalent:*

(i) $\mathrm{SO}(2)A$ *and* $\mathrm{SO}(2)C$ *are incompatible for first-order laminates.*

(ii) $\det(A^tA - C^tC) > 0$.

(iii) $A^tA \prec C^tC$.

Proof. Condition (ii) holds if and only if $A^tA - C^tC$ is either negative or positive definite. In view of $\det(A) \leq \det(C)$, it must be negative definite and, hence, we have (ii) ⇔ (iii). The first part of the lemma follows by a similar argument. Condition ¬(ii) holds if and only if there is a vector $x \in \mathbb{R}^2\setminus\{0\}$ which fulfills one of these equivalent properties: $x^tA^tAx = x^tC^tCx$, $|Ax| = |Cx|$ or $QAx = RCx$ for some rotations $Q, R \in \mathrm{SO}(2)$. We conclude that ¬(ii) holds if and only if there are rotations $Q, R \in \mathrm{SO}(2)$ such that the rank of the matrix $QA - RC$ is at most 1. This implies that ¬(ii) ⇔ ¬(i). □

5 A necessary condition for incompatibility

We are going to prove that, in the context of frame invariance, separability by a polyconvex set is also necessary for incompatibility. Here we look at a special case first. In order to prove the proposition, we will use Helly's theorem, see, for example, [DGK63].

Theorem 5.1 (Helly's theorem)**.** *Let d be a positive integer, $\mathcal{I}$ an index set, possibly uncountable, and $\{\mathcal{D}_\alpha\}_{\alpha\in\mathcal{I}}$ a family of compact convex sets in a d-dimensional Euclidean space. If for every $\alpha_1,\ldots,\alpha_{d+1}\in\mathcal{I}$ the intersection $\mathcal{D}_{\alpha_1}\cap\cdots\cap\mathcal{D}_{\alpha_{d+1}}$ is non-empty, then the whole intersection $\bigcap\{\mathcal{D}_\alpha \mid \alpha{\in}\mathcal{I}\}$ is non-empty.*

Proposition 5.2. *Let $K_1, K_2 \subseteq \mathbb{M}^{2\times 2}$ be compact and frame-invariant sets of matrices with positive determinant such that*

$$\max\{\det(A) \mid A{\in}K_1\} < \min\{\det(C) \mid C{\in}K_2\}.$$

Then the following properties are equivalent:

- *(a) K_1 and K_2 are incompatible for homogeneous gradient Young measures.*
- *(b) K_1 and K_2 are incompatible for first-order laminates. In addition, $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ are incompatible for T4 configurations where $\mathcal{C}_{K_i}=\{X^tX \mid X{\in}K_i\}$ for $i=1,2$.*
- *(c) K_1 and K_2 can be separated by a polyconvex set.*

Proof. We assume that K_1 and K_2 are non-empty. Otherwise the proof is trivial. Theorem 4.1 implies that (c) $\Rightarrow$ (a). We show that (b) $\Rightarrow$ (c). Assume that K_1 and K_2 are incompatible for first-order laminates and, in addition, $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ are incompatible for T4 configurations. Then, in particular, K_1 and K_2 are disjoint. Let $\Lambda\in\mathbb{R}$ be a positive real number such that for every $Y\in K_1{\cup}K_2$ we have $Y^tY\preceq\Lambda I$. Such a Λ exists, since K_1 and K_2 are compact. Given $Y\in K_1{\cup}K_2$ we define the set $\mathcal{D}_Y$ via

$$\mathcal{D}_Y=\begin{cases}\{D{\in}\mathbb{M}^{2\times2}_{\mathrm{sym}} \mid Y^tY\prec D\preceq\Lambda I\} & \text{if } Y\in K_1,\\ \{D{\in}\mathbb{M}^{2\times2}_{\mathrm{sym}} \mid 0\preceq D\prec Y^tY\} & \text{if } Y\in K_2.\end{cases}$$

Fix a subset $\{Y_1,\ldots,Y_4\}\subseteq K_1{\cup}K_2$. We show that the intersection $\mathcal{D}=\mathcal{D}_{Y_1}\cap\cdots\cap\mathcal{D}_{Y_4}$ is non-empty. In order to do that, we distinguish between different cases. Nothing is to show if $Y_1,\ldots,Y_4\in K_1$ or $Y_1,\ldots,Y_4\in K_2$. To shorten notation, set $X_i=Y_i^tY_i$ for $i=1,\ldots,4$. If $Y_1\in K_1$ and $Y_2,Y_3,Y_4\in K_2$, then, by Lemma 4.4, we must have $X_1\prec X_i$ for $i=1,2,3$. Hence, there is a positive real number $\epsilon>0$ such that $X_1{+}\epsilon I$ lies in $\mathcal{D}$. Similarly, if $Y_1,Y_2,Y_3\in K_1$ and $Y_4\in K_2$, then $X_4{-}\epsilon I$ lies in $\mathcal{D}$ for some $\epsilon>0$. Up to a permutation, there is only one case left: $Y_1,Y_4\in K_1$ and $Y_2,Y_3\in K_2$. If $X_1\preceq X_4$ holds, then $X_4{+}\epsilon I$ lies in $\mathcal{D}$ for some $\epsilon>0$. Similar arguments can be used to treat $X_4\preceq X_1$, $X_2\preceq X_3$ and $X_3\preceq X_2$. Thus, in view of Lemma 4.4, it remains to deal with matrices $Y_1,Y_4\in K_1$ and $Y_2,Y_3\in K_2$ which fulfill the following condition. For indices $i,j\in\{1,2,3,4\}$ with $i<j$ we have

$$\det(X_i{-}X_j)\begin{cases}<0 & \text{if } (i,j){\in}\{(1,4),(2,3)\},\\ >0 & \text{else.}\end{cases} \tag{6}$$

Now we apply a result by Székelyhidi [Szé05, Theorem 2]. Condition (6) is equivalent to what he calls *sign-configuration (B)*. Hence, exactly one of the following three holds:

(i) There is a $O \in \{X_1, \dots, X_4\}^{\mathrm{c}}$ such that $\det(X_i - O) > 0$ for every $i = 1, \dots, 4$.

(ii) There is a $O \in \{X_1, \dots, X_4\}^{\mathrm{c}}$ such that $\det(X_i - O) = 0$ for every $i = 1, \dots, 4$ and then $(X_1, \dots, X_4)$ forms a T4 configuration, (T4*) in (4).

(iii) There is a $O \in \{X_1, \dots, X_4\}^{\mathrm{c}}$ such that $\det(X_i - O) < 0$ for every $i = 1, \dots, 4$ and then $(X_1, \dots, X_4)$ forms a T4 configuration, (T4) in (4).

Note that, by definition, $X_1, \dots, X_4$ are symmetric as well as positive definite and so is O. We have that $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ are incompatible for T4 configurations. Hence, (i) must hold. This can only happen when O lies in $\mathcal{D}$, see Lemma 4.4. This shows that $\mathcal{D}$ is non-empty.

Since K_1 and K_2 are compact, there is an integer $n > 0$ and matrices $O_1, \dots, O_n \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ such that the matrix O in (i) can always be taken from $\{O_1, \dots, O_n\}$. We can fix a positive real number $\epsilon > 0$ such that for every subset $\{Y_1 \dots, Y_4\} \subseteq K_1 \cup K_2$ the intersection $\mathcal{D}^\epsilon_{Y_1} \cap \cdots \cap \mathcal{D}^\epsilon_{Y_4}$ is non-empty. Here we set

$$\mathcal{D}^\epsilon_Y = \begin{cases} \{D \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \mid Y^t Y + \epsilon I \preceq D \preceq \Lambda I\} & \text{if } Y \in K_1, \\ \{D \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \mid 0 \preceq D \preceq Y^t Y - \epsilon I\} & \text{if } Y \in K_2. \end{cases}$$

The set $\mathcal{D}^\epsilon = \bigcap \{\mathcal{D}^\epsilon_Y \mid Y \in K_1 \cup K_2\}$ is the intersection of compact convex sets in the 3-dimensional space $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$. What we have just shown together with Theorem 5.1 implies that $\mathcal{D}^\epsilon$ is non-empty. Fix an element $D \in \mathcal{D}^\epsilon$ and set $B = \sqrt{D}$. Then we obtain (c) as a consequence of Lemma 4.2.

Finally, it remains to prove $\neg$(b) $\Rightarrow$ $\neg$(a). All first-order laminates are homogenous gradient Young measures. Hence, we have to show $\neg$(a) given that the following is true: K_1 and K_2 are incompatible for first-order laminates, but $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ are not incompatible for T4 configurations. Recall that we have set $X_i = Y_i^t Y_i$ for $i = 1, \dots, 4$. In view of what we have done above, there are matrices $Y_1, Y_4 \in K_1$ and $Y_2, Y_3 \in K_2$ such that (6) holds together with (ii) or (iii).

Case (ii). Lemma 4.3 implies two things. First, for every $i \in \{1, \dots, 4\}$ there must be a rotation $R_i \in \mathrm{SO}(2)$ such that $R_i Y_i$ and $\sqrt{O}$ are rank-one connected. Second, we must have that

$$0 \in \{X_1 - O, \dots, X_4 - O\}^{\mathrm{c}} = \sqrt{O}\{\gamma_1 R_1 Y_1 - \sqrt{O}, \dots, \gamma_4 R_4 Y_4 - \sqrt{O}\}^{\mathrm{c}}$$

for some positive real numbers $\gamma_1, \dots, \gamma_4 > 0$. In particular, $\sqrt{O}$ must lie in the convex hull $\{R_1 Y_1, \dots, R_4 Y_4\}^{\mathrm{c}}$. Hence, $(R_1 Y_1, \dots, R_4 Y_4)$ forms a T4 configuration (of type (T4*) in (4)) with $P = \sqrt{O}$. We have $\neg$(a).

Case (iii). Assume that $(X_1, \dots, X_4)$ forms a T4 configuration and fix matrices $C_1, \dots, C_4, P$ such that (T4) in (4) is fulfilled. Since the matrices $X_1, \dots, X_4$ are symmetric and positive definite, so are $P_1, \dots, P_4$, see Lemma 3.1. Then, in view of Lemma 4.3, we find matrices $\widetilde{C}_1, \widetilde{C}_2, \dots \in \mathbb{M}^{2\times 2}$ of rank equal to 1, real numbers $\widetilde{\kappa}_1, \widetilde{\kappa}_2, \dots > 1$ and rotations $Q_0, R_1,$

$Q_1, R_2, Q_2, \ldots \in \mathrm{SO}(2)$ such that

$$
\begin{aligned}
R_1Y_1 - Q_0\sqrt{P_1} &= \widetilde{\kappa}_1(Q_1\sqrt{P_2} - Q_0\sqrt{P_1}) = \widetilde{C}_1,\\
R_2Y_2 - Q_1\sqrt{P_2} &= \widetilde{\kappa}_2(Q_2\sqrt{P_3} - Q_1\sqrt{P_2}) = \widetilde{C}_2,\\
R_3Y_3 - Q_2\sqrt{P_3} &= \widetilde{\kappa}_3(Q_3\sqrt{P_4} - Q_2\sqrt{P_3}) = \widetilde{C}_3,\\
R_4Y_4 - Q_3\sqrt{P_4} &= \widetilde{\kappa}_4(Q_4\sqrt{P_1} - Q_3\sqrt{P_4}) = \widetilde{C}_4,\\
R_5Y_1 - Q_4\sqrt{P_1} &= \widetilde{\kappa}_5(Q_5\sqrt{P_2} - Q_4\sqrt{P_1}) = \widetilde{C}_5,\\
R_6Y_2 - Q_5\sqrt{P_2} &= \widetilde{\kappa}_6(Q_6\sqrt{P_3} - Q_5\sqrt{P_2}) = \widetilde{C}_6,\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
\tag{7}
$$

By definition, the matrices $X_1, \ldots, X_4, P_1, \ldots, P_4$ are pairwise distinct. This implies that there is a uniform bound $\widetilde{\kappa}_0 > 1$ such that $\widetilde{\kappa}_i \geq \widetilde{\kappa}_0$ holds for every $i = 1, 2, \ldots$. The equations in (7) present a way how to construct a laminate $\nu \in \mathcal{M}^{\mathrm{rc}}(K_1 \cup K_2)$ such that both sets $\mathrm{supp}(\nu) \cap K_1$ and $\mathrm{supp}(\nu) \cap K_2$ are non-empty. Recall the “classical” construction of laminates for T4 configurations, see, for example, [Mül99a, §2.5]. In fact, the same procedure can be used here. This shows that ¬(a) holds and finishes the proof. □

6 Main results

Let $K \subseteq \mathbb{M}^{2\times 2}$ be a compact and frame-invariant set of matrices with positive determinant. We construct a set which can be used to determine the structure of the quasiconvex hull K^{qc}. Therefore, let $\Delta \subseteq \mathbb{R}$ be the union of all closed intervals

$$\big[\min\{\det(A), \det(B)\}, \max\{\det(A), \det(B)\}\big]$$

where A and B are rank-one connected matrices in K together with all closed intervals

$$\big[\min\{\sqrt{\det(A_i)} \mid i = 1, \ldots, 4\}, \max\{\sqrt{\det(A_i)} \mid i = 1, \ldots, 4\}\big]$$

where $(A_1, \ldots, A_4)$ are T4 configurations over $\mathcal{C}_K = \{X^tX \mid X \in K\}$.

Theorem 6.1. *Let $K \subseteq \mathbb{M}^{2\times 2}$ be a compact and frame-invariant set of matrices with positive determinant and $A, B \in K$. Then the following properties are equivalent:*

(i) A and B lie in the same connected component of K^{rc}.

(ii) A and B lie in the same connected component of K^{qc}.

(iii) A and B lie in the same connected component of K^{pc}.

(iv) $\det(A)$ and $\det(B)$ lie in the same connected component of Δ.

Proof. If necessary, we exchange the roles of A and B so that $0 < \det(A) \leq \det(B)$ holds. We know that (i) ⇒ (ii) ⇒ (iii) is true. Condition ¬(iv) implies that there is a real number $d \in \mathbb{R}$ such that $\det(A) < d < \det(B)$ holds and $d \notin \Delta$. Consider the disjoint sets

$$K_1 = K \cap \{A \in \mathbb{M}^{2\times 2} \mid \det(A) < d\}, \qquad K_2 = K \cap \{C \in \mathbb{M}^{2\times 2} \mid d > \det(C)\}.$$

Clearly, we have $K_1 \cup K_2 = K$. Since K is compact, so are K_1 and K_2. The number d is chosen in such a way that the sets K_1 and K_2 are incompatible for first-order laminates and, in addition, $\mathcal{C}_{K_1}$ and $\mathcal{C}_{K_2}$ are incompatible for T4 configurations. Proposition 5.2 implies that K_1 and K_2 can be separated by a polyconvex set. In particular, A and B must lie in different connected components of K^{pc}. We have ¬(iii).

Assume that ¬(i) is fulfilled. If there is a real number $d \in \mathbb{R}$ such that $\det(A) < d < \det(B)$ as well as $d \notin \Delta$ holds, then we get ¬(iv) and we are done. If not, we must have that the whole interval $[\det(A), \det(B)]$ is contained in Δ. In order to get a contradiction, it is sufficient to show that the set

$$M = \{X \in K^{\mathrm{rc}} \mid \det(A) \le \det(X) \le \det(B)\}$$

is connected. Since K is compact and frame invariant, so is K^{rc} and M. If M fails to be connected, then there exist compact, disjoint and frame-invariant sets M_1 and M_2 such that the union $M_1 \cup M_2$ is M and the intersection $M_1 \cap M_2$ is empty. On one hand, we can find matrices $X_1 \in M_1$ and $X_2 \in M_2$ with $\det(X_1) = \det(X_2)$. On the other hand, for every $X_1 \in M_1$ and $X_2 \in M_2$ the sets $\mathrm{SO}(2)X_1$ and $\mathrm{SO}(2)X_2$ must be incompatible for first-order laminates and, hence, $\det(X_1) = \det(X_2)$ is impossible in view of Lemma 4.4. The set M must be connected. This finishes the proof. □

Remark 6.2. *The set Δ is constructed in such a way that for every $X \in K^{\mathrm{qc}}$ we have that $\det(X)$ lies in Δ.*

Proof. Let $X \in K^{\mathrm{qc}}$ be given and $\nu \in \mathcal{M}^{\mathrm{qc}}(K)$ a homogeneous gradient Young measure such that $X = \bar{\nu}$ holds. Since the functions given by $A \mapsto \det(A)$ and $A \mapsto -\det(A)$ are polyconvex (and, in particular, quasiconvex), we can fix matrices $A, B \in \mathrm{supp}(\nu)$ such that $\det(A) \le \det(X) \le \det(B)$ holds. As an application of Theorem 1.1, we know that $\mathrm{supp}(\nu)^{\mathrm{qc}}$ is connected. This means that A and B lie in the same connected component of K^{qc}. By Theorem 6.1, $\det(A)$ and $\det(B)$ lie in the same connected component of Δ. Then $\det(X)$ must lie in Δ. □

Now we prove a generalization of Proposition 5.2.

Theorem 6.3. *Let $L_1, L_2 \subseteq \mathbb{M}^{2\times 2}$ be compact and frame-invariant sets of matrices with positive determinant. Then the following properties are equivalent:*

(a) L_1 and L_2 are incompatible for homogeneous gradient Young measures.

(b) L_1 and L_2 are incompatible for first-order laminates. In addition, $\{A^tA \mid A \in L_1\}$ and $\{C^tC \mid C \in L_2\}$ are incompatible for T4 configurations.

(c) L_1 and L_2 can be separated by a polyconvex set.

Proof. Theorem 4.1 implies that (c) ⇒ (a) holds. The implication ¬(b) ⇒ ¬(a) can be shown like in the proof of Proposition 5.2. It remains to show that (b) ⇒ (c) holds. Like before, given any $K \subseteq \mathbb{M}^{2\times 2}$ we set $\mathcal{C}_K = \{X^tX \mid X \in K\}$. Assume that L_1 and L_2 are incompatible for first-order laminates and, in addition, $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$ are incompatible for T4 configurations.

Set $K = L_1 \cup L_2$. We will use the set Δ which was defined above. For $i = 1, 2$ let Z_i be a connected component of L_i. In particular, Z_1 and Z_2 are incompatible for first-order laminates and, in addition, $\mathcal{C}_{Z_1}$ and $\mathcal{C}_{Z_2}$ are incompatible for T4 configurations. Then there is a real number $d \notin \Delta$ such that

$$\max\{\det(A) \mid A \in Z_1\} < d < \min\{\det(C) \mid C \in Z_2\}$$

where we exchange the roles of Z_1 and Z_2 if necessary. Consider the disjoint sets

$$K_1 = K \cap \{A \in \mathbb{M}^{2\times 2} \mid \det(A) < d\}, \qquad K_2 = K \cap \{C \in \mathbb{M}^{2\times 2} \mid d > \det(C)\}.$$

By definition, K_1 and K_2 are compact and frame invariant. The union $K_1 \cup K_2$ is equal to K. The sets K_1 and K_2 are incompatible for homogeneous gradient Young measures. Otherwise, in view of Theorem 1.1, there is a matrix X with $\det(X) = d$. But then Remark 6.2 implies that d lies in Δ, which gives a contradiction. Hence, we can apply Proposition 5.2 to K_1 and K_2. Then K_1 and K_2 and, in particular, Z_1 and Z_2 can be separated by a polyconvex set. Now we vary Z_1 over all connected components of L_1 and Z_2 over all connected components of L_2. Every choice of Z_1 and Z_2 gives a polyconvex set. If we take the intersection of all those polyconvex sets, we end up with a polyconvex set which separates L_1 and L_2. $\square$

References

[Bal77] John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1977.

[CDLMR03] Sergio Conti, Camillo De Lellis, Stefan Müller, and Mario Romeo. Polyconvexity equals rank-one convexity for connected isotropic sets in $\mathbb{M}^{2\times 2}$. *C. R. Acad. Sci. Paris, Sér. I*, 337(4):233–238, 2003.

[Dac89] Bernard Dacorogna. *Direct Methods in the Calculus of Variations*. Springer-Verlag, Berlin, 1989.

[DGK63] Ludwig Danzer, Branko Grünbaum, and Victor Klee. Helly's theorem and its relatives. In *Proc. Sympos. Pure Math.*, volume VII, pages 101–180. Amer. Math. Soc., 1963.

[Dol03] Georg Dolzmann. *Variational Methods for Crystalline Microstructure - Analysis and Computation*. Springer-Verlag, 2003.

[FS08] Daniel Faraco and László Székelyhidi. Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. *Acta Math.*, 200:279–305, 2008. 10.1007/s11511-008-0028-1.

[Hei11] Sebastian Heinz. Quasiconvexity equals rank-one convexity for isotropic sets of 2x2 matrices. WIAS Preprint 1637, 2011.

[Kir03] Bernd Kirchheim. Rigidity and Geometry of Microstructures. University of Leipzig, 2003. Habilitation thesis.

[KP91] David Kinderlehrer and Pablo Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rational Mech. Anal.*, 115:329–365, 1991.

[Kri99] Jan Kristensen. On the non-locality of quasiconvexity. *Ann. I. H. Poincaré – AN*, 16(1):1–13, 1999.

[Mil04] Graeme W. Milton. *The Theory of Composites*. Cambridge University Press, 2004.

[Mor52] Charles B. Morrey, Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2:25–53, 1952.

[Mül99a] S. Müller. Variational models for microstructure and phase transitions. In *Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996)*, pages 85–210. Springer, Berlin, 1999.

[Mül99b] Stefan Müller. Rank-one convexity implies quasiconvexity on diagonal matrices. *Internat. Math. Res. Notices*, 20:1087–1095, 1999.

[Ped93] Pablo Pedregal. Laminates and microstructure. *Eur. J. Appl. Math.*, 4:121–149, 1993.

[Šve92] Vladimír Šverák. Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(1-2):185–189, 1992.

[Szé05] László Székelyhidi. Rank-one convex hulls in $\mathbb{R}^{2\times 2}$. *Calculus of Variations and Partial Differential Equations*, 22:253–281, 2005. 10.1007/s00526-004-0272-y.

[Tar93] Luc Tartar. Some remarks on separately convex functions. In *Microstructure and phase transition*, pages 183–189. Springer, New York, 1993.