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Spatio-temporal pulse propagation in nonlinear dispersive optical media

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Abstract

We discuss state-of-art approaches to modeling of propagation of ultrashort optical pulses in one and three spatial dimensions. We operate with the analytic signal formulation for the electric field rather than using the slowly varying envelope approximation, because the latter becomes questionable for few-cycle pulses. Suitable propagation models are naturally derived in terms of unidirectional approximation.

1 Introduction

The shortest events ever created by humans are ultrashort laser pulses with the current record being less then the Bohr-orbit time in hydrogen [1, 2]. These extreme pulses have dramatically triggered both fundamental and applied science. Their numerical treatment also created new challenges, since the phenomena in question involve many different time scales, such that a straightforward solution of the underlying field and material equations becomes impractical. On the other hand, one cannot eliminate the fastest time-scale of a single optical oscillation using the slowly varying envelope approximation (SVEA), because SVEA is no longer valid for ultrashort pulses. Therefore new models allowing for efficient numerical treatment have to be developed [3, 4, 5, 6]. We discuss several such models starting from a relatively simple case of a single-mode waveguide in which the field structure in the radial direction is fixed and only one propagation coordinate is involved [7, 8, 9, 10, 11]. Thereafter we turn to the full three-dimensional modeling of propagation of ultrashort pulses in gases [12, 13].

2 Scalar case

We start with an exemplary field-level numerical solution for an ultrashort pulse propagating in a single-mode fiber [8], Fig. 1. As one can see, the envelope structure is destroyed in the course of propagation. Another observation is that the pulse carrier frequency is gradually shifted and therefore not well defined. Such extreme propagation regimes require more careful treatment than the traditional envelope description. A natural approach is to take advantage of the nearly unidirectional character of pulse propagation.

In principle, an optical pulse in a single-mode waveguide can be described by a single field component $E(\vec{r},t)$ which, to a good approximation, is governed by a scalar wave equation

$$\vec{\nabla}^2 E - \frac{1}{c^2}\partial_t^2\left(\hat{\epsilon}E\right) = \mu_0 \partial_t^2 P_{\mathrm{NL}}, \tag{1}$$

where the dispersion operator $\hat{\epsilon}$ is defined in the frequency domain $\left(\hat{\epsilon}E\right)_\omega = \epsilon(\omega)E_\omega$ and P_{NL} denotes the nonlinear part of the induced polarization. For a given $\omega > 0$, the standard forward/backward solutions $E \sim e^{i(\pm\beta z - \omega t)}$ of the linearized Eq. (1) yield the propagation constant $\beta(\omega) = \omega n(\omega)/c$, where the refractive index $n(\omega) = \sqrt{\epsilon(\omega)}$.

We now relate $\beta(\omega)$ to the operator $\hat{\beta}$ such that

$$\left(\hat{\beta}E\right)_\omega = |\omega|\frac{n(\omega)}{c}E_\omega \qquad \text{and} \qquad \frac{1}{c^2}(i\partial_t)^2\hat{\epsilon} = \hat{\beta}^2,$$

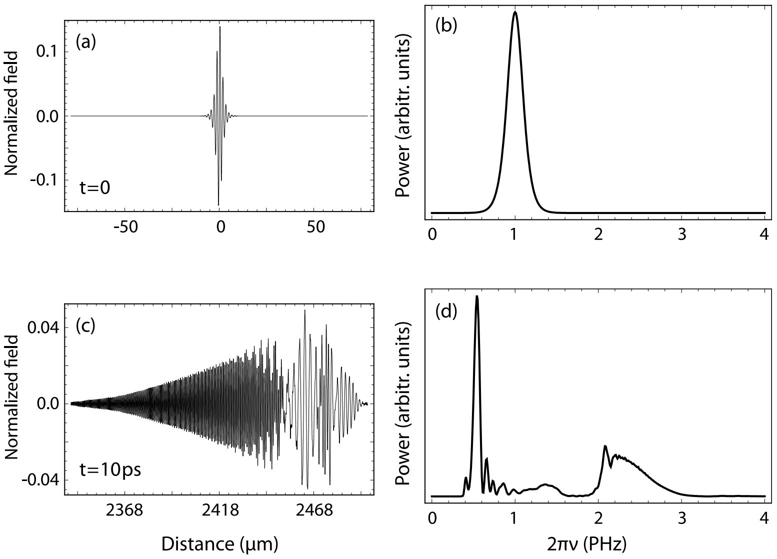

Figure 1: Electric field (a) and spectrum (b) of the initial pulse. (c) and (d), the same after 10 ps of propagation in a bulk fluoride glass. The envelope structure of the initial pulse is gradually destroyed in the course of propagation. For further details see [8].

where $|\omega|$ is used to ensure that $\hat{\beta}$ transforms a real field into a real field [c.f., $\epsilon(-\omega) = \epsilon^*(\omega)$ and $\beta(-\omega) = -\beta^*(\omega)$]. In the following we also assume that the time-averaged pulse field vanishes, $\langle E \rangle = 0$, which is compatible to Eq. (1). For such pulses $E_\omega \to 0$ as $\omega \to 0$, and one can safely define $\hat{\beta}^{-1} E$. Using $\hat{\beta}$, Eq. (1) can be transformed to the form

$$\left(i\partial_z + \hat{\beta}\right)\left(i\partial_z - \hat{\beta}\right) E = \vec{\nabla}_\perp^2 E - \mu_0 \partial_t^2 P_{\rm NL}, \tag{2}$$

where $\vec{\nabla}_\perp^2 = \partial_x^2 + \partial_y^2$ is the transverse Laplace operator. At this point it is profitable:
(i) to decompose the real-valued electric field

$$E(\vec{r}, t) = \sum_\omega E_\omega(\vec{r}) e^{-i\omega t}, \qquad E_{-\omega} = E_\omega^*,$$

into the complex-valued negative- and positive-frequency parts

$$E = \sum_{\omega<0} E_\omega e^{-i\omega t} + \sum_{\omega>0} E_\omega e^{-i\omega t} = E^{(-)} + E^{(+)},$$

(ii) to introduce the analytic signal $\mathcal{E}$ for the electric field such that $E = \mathrm{Re}[\mathcal{E}]$

$$\mathcal{E}(\vec{r}, t) = 2E^{(+)}(\vec{r}, t) = 2 \sum_{\omega>0} E_\omega(\vec{r}) e^{-i\omega t},$$

(iii) and to replace Eq. (2) with

$$\left(i\partial_z + \hat{\beta}\right)\left(i\partial_z - \hat{\beta}\right) \mathcal{E} = \vec{\nabla}_\perp^2 \mathcal{E} - 2\mu_0 \partial_t^2 P_{\rm NL}^{(+)}. \tag{3}$$

For a weakly nonlinear pulse consisting of forward waves propagating along the fiber we have $\left(i\partial_z - \hat{\beta}\right)\mathcal{E} \approx -2\hat{\beta}\mathcal{E}$ and therefore, neglecting the backward waves, one can simplify Eq. (3) to the form

$$\left(i\partial_z + \hat{\beta}\right)\mathcal{E} + \frac{\hat{\beta}^{-1}}{2} \vec{\nabla}_\perp^2 \mathcal{E} = \mu_0 \hat{\beta}^{-1} \partial_t^2 P_{\rm NL}^{(+)}.$$

For instance, neglecting the third harmonics generation in a cubic medium with $P_{\rm NL} = \epsilon_0\chi^{(3)}E^3$, one can approximate the positive frequency part of E^3 by $\frac{3}{8}\left(|\mathcal{E}|^2\mathcal{E}\right)^{(+)}$. The propagation equation

$$\left(i\partial_z + \hat{\beta}\right)\mathcal{E} + \frac{\hat{\beta}^{-1}}{2}\vec{\nabla}_\perp^2\mathcal{E} = \frac{3\chi^{(3)}}{8c^2}\hat{\beta}^{-1}\partial_t^2\left(|\mathcal{E}|^2\mathcal{E}\right)^{(+)} \tag{4}$$

looks then similar to the generalized nonlinear Schrödingier equation (NSE) and can be solved with the same numerical technique. However, Eq. (4) is completely independent on SVEA. If the pulse in question can be characterized by a narrow spectrum with the carrier frequency ω_0 and the corresponding wave vector $\beta_0 = \beta(\omega_0)$, then $\partial_t^2 \approx -\omega_0^2$, $\beta^{-1} \approx 1/\beta_0$, and

$$\left(i\partial_z + \hat{\beta}\right)\mathcal{E} + \frac{1}{2\beta_0}\vec{\nabla}_\perp^2\mathcal{E} + \frac{3\omega_0\chi^{(3)}}{8cn(\omega_0)}|\mathcal{E}|^2\mathcal{E} = 0,$$

where the latter equation can be related to the 1D NSE for the traditional pulse envelope $\psi(z,\tau)$ by a standard transformation to the pulse-comoving frame and projection onto the dominant waveguide mode [14]

$$\mathcal{E}(\vec{r},t) = \mathfrak{R}(x,y)\psi(z,\tau)e^{i(\beta_0 z - \omega_0 t)} + \text{h.o.t.},$$

where $\tau = t - \beta_1 z$ is the retarded time, $\beta_1 = \beta'(\omega_0)$ is the inverse group velocity, and $\mathfrak{R}(x,y)$ is the approximately fixed radial field structure. The same projection technique can be applied directly to Eq. (4), the result has the same basic structure

$$i\partial_z\mathcal{E}(z,t) + \hat{\beta}\mathcal{E}(z,t) = \frac{3\chi^{(3)}}{8c^2}\hat{\beta}^{-1}\partial_t^2\left(|\mathcal{E}|^2\mathcal{E}\right)^{(+)},$$

but attributes to the so-called effective dispersion $\beta_{\rm eff}(\omega)$ which depends on the fiber in question.

3 Vectorial case

In homogeneous media without (linear) waveguiding one has to account, in principle, for a fully vectorial description of the electric field $\vec{E}$, e.g., in the frequency domain by

$$[\vec{\nabla}^2 + \beta^2(\omega)]\vec{E}_\omega = \vec{S}_\omega. \tag{5}$$

The source term $\vec{S}_\omega$ is given by

$$\vec{S}_\omega = -\mu_0\omega^2\vec{P}_{\rm NL,\omega} + i\mu_0\omega\vec{J}_\omega + \frac{1}{\epsilon_0}\vec{\nabla}\left(\rho - \vec{\nabla}\cdot\vec{P}_\omega\right), \tag{6}$$

and takes account of the nonlinear part $\vec{P}_{\rm NL,\omega}$ of the total polarization density $\vec{P}_\omega$, the existence of free carriers with density ρ and current density $\vec{J}_\omega$, respectively. The last term on the r.h.s. of Eq. (6) models vectorial effects which become important for strongly divergent beams occuring under extreme focusing conditions. Scalar approximations can be restored for many experimental situations of interest. Nonlinear self-focusing effects may increase the optical intensity to trigger photoionization, which requires to include free carrier terms. Analogous to the previous

Section, we wish to introduce the notion of directional propagation. As for the one-dimensional case, for this purpose we refer to the linear regime where $\vec{J}_\omega = \vec{P}_{\mathrm{NL},\omega} = \rho = \vec{\nabla} \cdot \vec{P}_\omega \equiv 0$. In this case, the source term $\vec{S}_\omega$ vanishes. In Fourier space parametrized by the wavevector $\vec{k}^\top = (k_x, k_y, k_z)$ and angular frequency ω, the resulting Helmholtz equation is then solved by the ansatz $\vec{E}_{\omega,k_x,k_y,k_z} = \vec{\mathcal{E}}^{+}_{\omega,k_x,k_y} + \vec{\mathcal{E}}^{-}_{\omega,k_x,k_y}$ if the generalized functions $\mathcal{E}^\pm$ are chosen as

$$\vec{\mathcal{E}}^{\pm}_{\omega,k_z,k_x,k_y} = \vec{A}^{\pm}_{\omega,k_x,k_y,k_z} \delta \left(k_z \mp \sqrt{\beta^2(\omega) - k_\perp^2} \right) \tag{7}$$

with amplitudes $\vec{A}^\pm$. The chosen decomposition into generalized functions having support on the upper (lower) half-sphere defined by $||\vec{k}|| = |\beta|$ and $k_z > 0$ ($k_z < 0$) then naturally defines forward and backward running fields $\vec{A}^{\pm}_{\omega,k_x,k_y}$. A factorization of the nonlinear equation (5) proceeds analogous to the fiber optical case, and yields, according to [12], the coupled directional equations for the electric field,

$$(i\partial_z \pm |k_z|)\mathcal{E}^{\pm}_{\omega} = \vec{S}_\omega \tag{8}$$

where $|k_z| = \sqrt{\beta^2 - k_\perp^2}$. According to [13], we may rewrite the source terms as

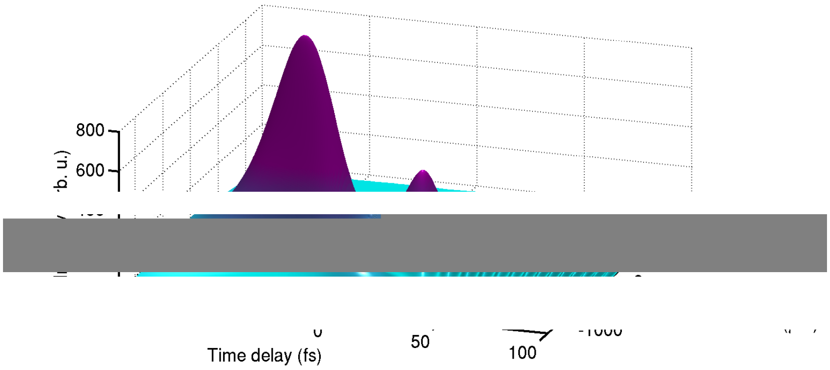

Figure 2: Numerical simulation of optical wavebreaking in a femtosecond filament in argon.

$$S_\omega = -\mu_0 \omega^2 \left[\mathbf{1} - \frac{\vec{k} \otimes \vec{k}}{k^2} \right] \vec{\mathcal{F}}_{\mathrm{NL},\omega}, \tag{9}$$

where in the frequency domain, $\vec{\mathcal{F}}_{\mathrm{NL},\omega} = \vec{P}_{\mathrm{NL},\omega} + i\vec{J}_\omega/\omega$. The operator $\mathbf{1} - \vec{k} \otimes \vec{k}/k^2$ projects out the longitudinal parts of a vector field, and its presence in the propagation equation involves a very costly numerics. Similar to the waveguiding case, one therefore usually employs the unidirectional limit by letting

$$\frac{\vec{k} \otimes \vec{k}}{k^2} \vec{\mathcal{F}}_{\mathrm{NL},\omega} \approx 0, \qquad \vec{\mathcal{E}}^{-}_{\omega} \approx 0. \tag{10}$$

The first relation states that we can neglect longitudinal field components. Furthermore, it means that orthogonal polarization states are not coupled, allowing to go back to a scalar description, $\vec{\mathcal{E}} \rightarrow \mathcal{E}$. The second relation states that backward propagating waves are weak and can be neglected. The resulting unidirectional propagation equation is comparable to the propagation Eq. (4) in the waveguiding case. Using the approximation $|k_z| \approx \beta(1 - k_\perp^2/2\beta^2)$ valid for

$k_\perp \ll \beta$, we obtain the paraxial approximation of Eq. (8). This approximation also restores the idenfication of directional fields with the analytic signal. Neglecting the backward propagating wave then yields the forward Maxwell equation (FME) [15]. The latter is successfully used in the context of femtosecond filamentation. However, as it fails to describe the propagation of the dc-component of the optical field, optical rectification processes in filaments, like the emission of Teraherz radation [16], are more suitably described by Eq. (8) using the approximations of Eq. (10).

The formation of femtosecond filaments can be observed when sufficiently intense pulsed femtosecond laser radiation is loosely focused into a dielectric medium. These filaments are narrow, longitudinally extended structures of dilute plasma and light. In many cases of interest, filamentation is suitably described by the FME. Numerically, this is solved using a pseudospectral split-step scheme. A characteristic radially symmetric field configuration arising in a simulation of femtosecond filamentation [17] is shown in Fig. 2. In fact, this figure depicts the higher dimensional analogue of optical wavebreaking occurring during fiber propagation [14]. It is observed due to a modulation instability in the interplay of temporal dispersion and nonlinear self-focusing and also known as hyperbolic shock-wave formation [18].

4 Conclusion

We discuss state-of-art approaches to modeling of propagation of ultrashort optical pulses in one (waveguide case) and three (bulk propagation in gases) spatial dimensions. As the very notion of the pulse envelope becomes questionable for a few-cycle pulse, we avoid the use of the slowly varying envelope approximation and operate with the analytic signal for the electric field. Suitable propagation models are naturally derived in terms of unidirectional approximation.

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