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Periodic solutions of isotone hybrid systems

Dedicated to the memory of A.V. Pokrovskii

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Abstract

Suggested by conversations in 1991 (Mark Krasnosel'skiĭ and Aleksei Pokrovskiĭ with TIS), this paper generalizes earlier work [7] of theirs by defining a setting of hybrid systems with isotone switching rules for a partially ordered set of modes and then obtaining a periodicity result in that context. An application is given to a partial differential equation modeling calcium release and diffusion in cardiac cells.

1 Introduction

This work was inspired by discussions in 1991 with Mark Krasnosel'skiĭ and Aleksei Pokrovskiĭ, considering periodic solutions of systems involving discontinuous modal switching through the *standard relay* operator of ideal hysteresis. In comparison with other early work on periodic solutions of such systems, Krasnosel'skiĭ and Pokrovskiĭ drew attention to their paper [7]: where the other work (cf., e.g., [2, 3, 4, 10, 13, 12, 14] had focused on the thermostat setting (using continuity between switching times to seek a two-phase periodic solution, i.e., with a pair of complementary modal switches in each period), the paper [7] used the known order preserving property of the standard relay to avoid dependence on continuity. Since the thermostat setting is intended to provide stabilizing negative feedback, it involved antitone use of the relay, whereas with isotone use of the operator, one may hope to apply the Birkhoff Fixpoint Theorem:

BFT [1]: *“Every isotone selfmap of a complete lattice has a fixed point,”*

The principal result in [7] concerned scalar systems of the form

$$\mathbf{P}y = \hat{f}(t, y, w) \quad w(\cdot) = \mathbf{W}[y(\cdot)] \quad (1.1)$$

where \mathbf{P} is an ordinary differential operator $p(d/dt)$ and \mathbf{W} is a standard relay operator; one assumes, of course, that \hat{f} is periodic in any explicit dependence on t . [Both the differential equation and the relay operator are normally expressed as initial value problems so, although we are seeking periodic solutions of (1.1), we would need initial data $[\eta, \omega]$ for y, w .] More generally, a hybrid system considers a state which evolves by some mode (e.g., given by a differential equation — in (1.1) determined by the function \hat{f}) on time intervals where the modal specification $w(t)$ remains constant. Note that it has long been known (cf., e.g., [8]) that the standard relay operator is isotone and, while generalized relays embodying switching rules for more general sets of modes than $\{0, 1\}$ have been discussed in the literature of hybrid systems (cf., e.g., [5]), there seems not to have been consideration¹ of these from the viewpoint of isotonicity in connection with periodicity.

¹It seems plausible that arguments along our present line could give existence of periodic solutions for the use of a Preisach operator replacing \mathbf{W} in (1.1), but this does not quite fit the present formulation and we have not pursued the possibility.

The argument of [7] involves defining a suitable period map \mathbf{F} . Due to involvement of the relay, the map \mathbf{F} is not continuous. Nevertheless, as \mathbf{F} can be shown to be an isotone selfmap of a complete lattice, the Birkhoff Fixpoint Theorem can be applied to provide a periodic solution of the system. This use of the BFT remains the core of our present argument, although we use a slightly stronger statement:

Theorem 1.1. *If \mathcal{X} is a complete lattice and $\mathbf{F} : \mathcal{X} \rightarrow \mathcal{X}$ is isotone, then $\bar{x} = \inf\{x : \mathbf{F}x \prec x\}$ is a fixed point of \mathbf{F} , hence the minimum fixed point.*

Those discussions, seeking some interesting application in a more general setting, never resulted in joint publication on this topic. In this paper, in comparison with [7],

- 1 we replace the scalar state variable by a vector or function,
- 2 we take into consideration more recent development of hybrid systems and formulate the problem with a *generalized relay operator* $\mathbf{W}[\cdot, \cdot]$, presented in terms of *switching rules* among a set M of modes,
- 3 we consider more general dynamics, e.g., defined through evolutionary partial differential equations,
- 4 we assume a sensor map \mathbf{Y} (not necessarily continuous) and distinguish between the state $x(t)$ determined by the dynamics and the sensor values $y(t)$ used to determine the modal switching,
- 5 we include application to a model problem of some biological interest, involving calcium waves in cardiac cells.

2 Isotone generalized relay operators

To some extent we are following [10, 11, 5] here in the description of generalized relay operators through switching rules.² We will impose assumptions on the switching rules to ensure that the modal index function $w(\cdot)$ will be piecewise constant with a finite number of pieces (non-Zeno), although allowing degenerate interswitching intervals.

We must define $\mathbf{W} : M \times \mathcal{Y} \rightarrow \mathcal{M}$ where \mathcal{Y} is a space of left-continuous Y -valued functions and \mathcal{M} is a space of non-Zeno M -valued functions with M a finite set of *modes*, the state space of the generalized relay. Later we will also introduce spaces X, Z and spaces \mathcal{X}, \mathcal{Z} of admissible X, Z -valued functions³ which will be needed for the dynamics, then restricting attention to subsets $X_* \subset X$ and $Z_* \subset Z$. We will make the assumptions:

²Note that our present concerns are rather different from those: for example, it was important in [10, 5] to preserve closure by admitting ambiguities leading to multi-valued operators whereas that is inconsistent with our focus here.

³We will work with these as functions on $[0, T]$, but also view them as defined on $[0, \infty)$ — either by considering a differential equation there or repeating $[0, T]$ periodically.

- (A)
- 1 The spaces M, X, Y, \mathcal{Z} are partially ordered (so also $X \times M$ is partially ordered with the product order); we take the partial order of the function spaces \mathcal{X}, \mathcal{Y} as induced pointwise. With this, X and \mathcal{Z} are to be conditionally complete lattices (i.e., each nonempty bounded subset has a sup and inf).
 - 2 M, X_*, \mathcal{Z}_* are complete lattices (i.e., every nonempty subset has a sup and inf).
 - 3 X, Y are Hausdorff spaces; \mathcal{X} consists of continuous functions and \mathcal{Y} consists of functions having at most jump discontinuities: both $y(t-)$ and $y(t+)$ exist; we assume $y(t) = y(t-)$ and that $y(T+)$ is also given.

[Note that we do not require that Y, X, \mathcal{Z} , or the function spaces $\mathcal{M}, \mathcal{X}, \mathcal{Y}$ should have the inf, sup property considered in (A-2).] For verifying the completeness in (A-2) it may be useful to note that

Lemma 2.1. *If \mathcal{I} is any conditionally complete lattice, then each order interval $(\mathcal{I}_* = \{s \in \mathcal{I} : s^- \prec s \prec s^+\}$ with $s^- \prec s^+$ in \mathcal{I}) will be a complete lattice. If $\mathbf{F} : \mathcal{I} \rightarrow \mathcal{I}$ is isotone and there exist $s^- \prec s^+$ in \mathcal{I} with $\mathbf{F}(s^-) \succ s^-$ and $\mathbf{F}(s^+) \prec s^+$, then \mathbf{F} is a selfmap of this order interval \mathcal{I}_* .*

PROOF: The first part is immediate. For $s \in \mathcal{I}_*$ so $s \succ s^-$, the isotonicity of \mathbf{F} gives $\mathbf{F}(s) \succ \mathbf{F}(s^-) \succ s^-$; similarly, $\mathbf{F}(s) \prec s^+$. Thus $\mathbf{F}(s) \in \mathcal{I}_*$. ■

The key to the definition⁴ of a *generalized relay operator* $\mathbf{W} : y(\cdot) \mapsto w(\cdot)$ will then be the specification of a collection \mathcal{S} of subsets of the observation space Y :

$$\mathcal{S} = \{C(\omega, \hat{\omega}) : \omega, \hat{\omega} \in M\} \quad (2.1)$$

so $w = \mathbf{W}[\omega_*, y]$ is to be determined by the switching rules:

- an initial state ω_* is given.
- $y(t) \in C(\omega, \omega)$ on every open interval on which $w(\cdot) \equiv \omega$ — equivalently, w remains constant = ω so long as $y(\cdot)$ remains in $C(\omega, \omega)$.
- switching $\omega \rightsquigarrow \hat{\omega}$ occurs at t_* if $y(\cdot)$ enters $C(\omega, \hat{\omega})$, giving $w(t_*) = \hat{\omega}$.

Statically, these rules simply characterize an admissibility relation between $y(\cdot)$ and $w(\cdot)$. We actually expect application of these rules (assuming $w(\cdot)$ is to be piecewise constant⁵) to enable the dynamic construction of $w(\cdot)$ from $y(\cdot)$ and it is this input/output map \mathbf{W} , when properly defined, which we call a *generalized relay operator*.

We recognize that $y(\cdot)$ will also be undergoing dynamic construction so we will need still further assumptions on the coupled dynamics to ensure that this will work properly to define the pair

⁴Because of the nature of our arguments, we view \mathbf{W} as associated with a fixed interval $[0, T]$, but, being concerned with periodicity, think of the arguments as potentially defined on \mathbb{R} so it would be meaningful to consider $y(0-)$ and $\omega_* = w(0-)$. There is nothing special about the initial and terminal times, so we admit the possibility that $t_* = 0, T$ could be switching times.

⁵We permit some of the ‘pieces’ here to be degenerate intervals (i.e., length 0): the point is a finite number of switches.

$[x(\cdot), w(\cdot)]$. At this point, however, we make the following assumptions on the switching rules and inputs:

- 1 For each ω the collection $\mathcal{S}_\omega = \{C(\omega, \hat{\omega}) : \hat{\omega} \in M\}$ is a partition of Y with $C(\omega, \omega)$ open and nonempty. For each $\hat{\omega}$ we have $C(\omega, \hat{\omega}) \subset C(\hat{\omega}, \hat{\omega})$.
- 2 Suppose $\eta, \eta' \in Y$, $\omega, \omega', \hat{\omega}, \hat{\omega}' \in M$ with $\eta \in C(\omega, \hat{\omega}), \eta' \in C(\omega', \hat{\omega}')$
(B) Then $[\eta \succ \eta', \omega \succ \omega']$ implies $\hat{\omega} \succ \hat{\omega}'$.
- 3 For some $\hat{M} \subset M \times M$, there is a bound on the length of a sequence $[\omega_1, \omega_2, \dots]$ such that: $\omega_{k+1} \neq \omega_k$, $C(\omega_k, \omega_{k+1}) \neq \emptyset$, and $[\omega_k, \omega_{k+1}] \notin \hat{M}$.
- 4 There is some $\varepsilon > 0$ such that: if $y(\cdot) \in \mathcal{Y}$ with $y(t_*) \in C(\omega, \hat{\omega})$ for some $[\omega, \hat{\omega}] \in \hat{M}$, then y stays in $C(\hat{\omega}, \hat{\omega})$ for $t_* < t < t_* + \varepsilon$.

It is, of course, (B-2) which constitutes the isotonicity of the switching rules; we will easily observe that this ensures isotonicity in $\omega_*, y(\cdot)$ of the operator \mathbf{W} . The conditions (A-3) and (B-3,4) take the form given here since we wish to allow for the occurrence of discontinuous observation functions $y(\cdot)$ in the example of Section 5; it is then necessary⁶ to verify these assumptions for such examples. We will take the construction below as uniquely defining the operator \mathbf{W} and our interpretation of the switching rules.

Theorem 2.2. *The operator $\mathbf{W} : [\omega_*, y(\cdot)] \rightarrow w(\cdot) : M \times \mathcal{Y} \rightarrow \mathcal{M}$ with*

$$\begin{aligned} \mathcal{M} := \{w : [0, T] \rightarrow M \mid \exists 0 = t_0 < \dots < t_K = T, \omega_0, \dots, \omega_{2K} \in M : \\ w(\cdot) \equiv \omega_{2k+1} \text{ on } (t_k, t_{k+1}), \text{ for } k = 0, \dots, K-1, \\ w(t_k) = \omega_{2k} \text{ for } k = 0, \dots, K\}^7 \end{aligned} \quad (2.2)$$

is well-defined and isotone.

PROOF: Let $\omega_* \in M$ and $y(\cdot) \in \mathcal{Y}$ be given. We construct a (finite) set $t_0 = 0 < t_1 < \dots < t_K = T$ and corresponding $\{\omega_k\}$ such that w of the form considered in (2.2) satisfies the switching rules.

Start with $t_0 = 0$ and $k = 0$. There is a unique ω_0 such that $y(0) \in C(\omega_*, \omega_0)$. If $y(\cdot) \in C(\omega_0, \omega_0)$ on $(0, s')$ for some $s' > 0$, then set $\omega_1 := \omega_0$. Otherwise, there is a unique $\hat{\omega} \neq \omega_0$ such that $y(0+) \in C(\omega_0, \hat{\omega})$, and we set $\omega_1 := \hat{\omega}$.

Now, we proceed recursively in k . Given t_k, s_k and ω_{2k+1} , let t_* be the maximal element in $(t_k, T]$ such that $y(t) \in C(\omega_{2k+1}, \omega_{2k+1})$ on (t_k, t_*) ; set $t_{k+1} = t_*$. Because $C(\omega_{2k+1}, \omega_{2k+1})$ is open, maximality of t_* ensures, by (A-3), that one must have either

- $t_* = T$, in which case set $\omega_{2k+2} := \hat{\omega}$ with $\hat{\omega}$ satisfying $y(t_*) \in C(\omega_{2k+1}, \hat{\omega})$, set $K = k + 1$, and stop, or

⁶Here we have not required that $\mathbf{Y} : X \rightarrow Y$ be continuous. If y were continuous, one ensures avoidance of Zeno phenomena by having each $C(\omega, \omega)$ open, but, when inputs may jump, the last of the switching rules requires some interpretation.

⁷The condition in the last line of (2.2) is equivalent to request that $w(\cdot) \equiv \omega_{2k}$ on the degenerate interval $[t_k, t_k]$ for $k = 0, \dots, K$.

- $y(t_*) \in C(\omega_{2k+1}, \omega_{2k+1})$, $t_* < T$ and $y(t_*+) \in C(\omega_{2k+1}, \hat{\omega})$ with $\omega_{2k+1} \neq \hat{\omega}$ in which case set $\omega_{2k+2} := \omega_{2k+1}$ and $\omega_{2k+3} = \hat{\omega}$, or
- $y(t_*) \in C(\omega_{2k+1}, \hat{\omega})$ with $\hat{\omega} \neq \omega_{2k+1}$, $t_* < T$. In this case set $\omega_{2k+2} := \hat{\omega}$.
 - If in this case $y(\cdot) \in C(\hat{\omega}, \hat{\omega})$ holds on (t_*, s') for some $s' \in (t_*, T]$, then set $\omega_{2k+3} := \hat{\omega}$.
 - Otherwise, consider ω' with $y(t_*+) \in C(\hat{\omega}, \omega')$ and set $\omega_{2k+3} := \omega'$.

(B-1) ensures that each ω_i is uniquely determined and that when using $y(t_*+)$ to define ω_{2k+3} , we have some $s' > t_*$ such that $y(\cdot)$ is in the open set $C(\hat{\omega}, \hat{\omega})$ on (t_*, s') . (B-3,4) ensures that this recursion must terminate with K finite and $t_K = T$ with a well-defined terminal value $w(T) = \omega_{2K}$.

To see the isotonicity, we take $\omega_* \prec \omega'_*$ and $y \prec y'$ and, for t_* defined by maximality for one of (either) w_1 or w_2 , we must verify the appropriate comparisons. In each of the cases (all combinations of the construction and of the determination of t_*) one easily sees that the assumption (B-2) ensures the correctly isotonic comparison of the “new” values⁸ $\hat{\omega}$ and $\hat{\omega}'$ so we again proceed recursively in k to get $w = \mathbf{W}(\omega_*, y) \prec w' = \mathbf{W}(\omega'_*, y')$. ■

[It is also interesting to observe that — with $w = \mathbf{W}(\omega_*, y)$ and defining $y_\tau(s) = y(s + \tau)$ — we then get

$$\mathbf{W}(w(\tau), y_\tau)(t) = \mathbf{W}(\omega_*, y)(t + \tau) \quad (\tau > 0).] \quad (2.3)$$

It will be convenient to consider Cartesian products: if \mathbf{W}_1 and \mathbf{W}_2 are given by switching sets \mathcal{S}_1 in Y_1 and \mathcal{S}_2 in Y_2 , we may let $\mathbf{W} = \mathbf{W}_1 \otimes \mathbf{W}_2$ be the generalized relay operator induced by taking $M = M_1 \times M_2$ and $Y = Y_1 \times Y_2$ with⁹ $C([\omega_1, \omega_2], [\omega'_1, \omega'_2]) = C(\omega_1, \omega'_1) \times C(\omega_2, \omega'_2) \subset Y$, etc. Obviously one could similarly consider products $\mathbf{W} = \mathbf{W}_1 \otimes \cdots \otimes \mathbf{W}_N$. It is again easy to verify that

Lemma 2.3. *Suppose each of the factors $\mathbf{W}_1, \dots, \mathbf{W}_N$ is an isotone generalized relay operator. Then the product $\mathbf{W} = \mathbf{W}_1 \otimes \cdots \otimes \mathbf{W}_N$ is also an isotone generalized relay operator.*

3 Coupled dynamics

We are considering problems here in which, for piecewise constant $w \in \mathcal{M}$, we have an X -valued state x which, on each (nondegenerate) interval of constancy of $w(\cdot)$ (interswitching interval), is to satisfy an abstract ODE of the form

$$\dot{x} = \mathbf{L}x + \mathbf{B}z \quad \text{with } z(t) = f(t, x, w). \quad (3.1)$$

⁸Note that we take $\hat{\omega} = \omega$ if t_* would not be a switching time for w_1 , etc.

⁹Observe that whenever we might have simultaneous switchings in \mathbf{W}_1 and in \mathbf{W}_2 , the pair here appears in the product $\mathbf{W}_1 \otimes \mathbf{W}_2$ as a single switch to the resultant pair.

Here $\mathbf{B} : Z \rightarrow X$ and $\mathbf{L} : X \supset \mathcal{D}(\mathbf{L}) \rightarrow X$ are suitable linear operators and $f : [0, T] \times X \times M \rightarrow Z$; we use $z(t) = f(t, x(t), w(t))$ pointwise to define $\Phi : [x, w] \mapsto z$ for functions.

We actually wish to construct $x(\cdot)$ and $w(\cdot)$ simultaneously. We begin by assuming

- 1 Given $\xi \in X_*$ and a function $z(\cdot) \in \mathcal{Z}_*$, there is a (unique) solution $x(\cdot) \in \mathcal{X}$ (with each $x(t) \in X_*$) of the ODE

$$(C) \quad \dot{x} = \mathbf{L}x + \mathbf{B}z \quad \text{with } x(0) = \xi. \quad (3.2)$$

- 2 We assume 1. gives¹⁰ $\mathbf{Y}(x) \in \mathcal{Y}$.

Lemma 3.1. *Under the hypotheses above, there is a well-defined map*

$$\mathbf{G} : [\xi, \omega, z] \mapsto [x(\cdot), w(\cdot)] : X_* \times M \times \mathcal{Z}_* \rightarrow \mathcal{X} \times \mathcal{M}$$

defined dynamically so that $x(\cdot)$ satisfies (3.2) (with the given z) and with $w(\cdot) = \mathbf{W}(\omega_*, \mathbf{Y}(x(\cdot))) \in \mathcal{M}$ given by the switching rules.

PROOF: Given ξ, z we solve (3.2), noting (C-1), to get $x \in \mathcal{X}$. By (C-2) and Theorem 2.2 we then get $w = \mathbf{W}(\omega, y)$, defined dynamically. \blacksquare

We denote the resulting $[x, w]$ by $\mathbf{G}_{[\xi_*, \omega_*]}(z)$, to emphasize consideration of the map: $z \mapsto [x, w]$ for fixed initial data.

Using $z(t) = f(t, x(t), w(t))$ pointwise in t to define $\Phi : [x, w] \mapsto z$, the coupled dynamics for a solution with data $[\xi_*, \omega_*]$ take the implicit form

$$[x, w] = \mathbf{G}_{[\xi_*, \omega_*]}(z) \quad \text{with } z = \Phi(x, w). \quad (3.3)$$

[This is just the distinction between (3.1) and (3.2).] Rather than working with the state taken implicitly as in (3.3) above, it is convenient to introduce the composition

$$\Psi_{[\xi_*, \omega_*]} = \Phi \circ \mathbf{G}_{[\xi_*, \omega_*]}. \quad (3.4)$$

Any fixed point of this map is a solution of (3.3) with the given data. We now impose isotonicity assumptions:

- 1 The nonlinearity $f(t, \cdot, \cdot)$ is isotone¹¹ from $X_* \times M$ for $t \in [0, T]$.
- 2 For $[x, w] = \mathbf{G}_{[\xi, \omega]}(z)$ with $z \in \mathcal{Z}_*$, one has $\Phi(x, w) \in \mathcal{Z}_*$, i.e., \mathcal{Z}_* is invariant under Ψ .
- (D) 3 The ordinary differential equation defining \mathbf{G} is isotone, i.e., if $\xi \succ \xi'$ in X_* and $z(t) \succ z'(t)$ in Z for each $t \in [a, b] \subset [0, T]$, then the solutions x, x' of $\dot{x} = \mathbf{L}x + \mathbf{B}z$ with $x(0) = \xi$ and of $\dot{x}' = \mathbf{L}x' + \mathbf{B}z'$ with $x'(0) = \xi'$ give $x(t) \succ x'(t)$ on $[0, T]$.

¹⁰We expect $x \in \mathcal{X}$ to be continuous in t for the X -topology but, since we have not assumed that $\mathbf{Y} : X \rightarrow Y$ is continuous, having $\mathbf{Y}(x) \in \mathcal{Y}$ is a more delicate assumption.

¹¹Note that we have no need to assume any continuity of f .

4 The sensor map $\mathbf{Y} : X_* \rightarrow Y$ is isotone.

Lemma 3.2. *Under the hypotheses above the implicit system (3.3) has a unique minimum solution for each initial $[\xi_*, \omega_*] \in X_* \times M$.*

PROOF: By (D-2), each $\Psi_{[\xi_*, \omega_*]}$ is a selfmap of the complete lattice \mathcal{Z}_* . Since the partial orders of $\mathcal{X}, \mathcal{M}, \mathcal{Z}$ were defined pointwise, (D-1) gives isotonicity of Φ . Similarly, the assumed isotonicities of (D-1,3) ensure isotone dependence of $x(t)$ on ξ and on z whence $y(t) = \mathbf{Y}(x(t))$ will depend isotonically on these by (D-4). As \mathbf{W} is isotone, we also have isotonicity of the w -component. Thus, the map \mathbf{G} of Lemma 3.1 is isotone in all variables, and therefore also $\Psi_{[\cdot, \cdot]}$, whence $\Psi_{[\xi_*, \omega_*]}$ is isotone in z and so has a minimum fixed point, necessarily unique, by the Birkhoff Fixpoint Theorem 1.1. For any fixed point of $\Psi_{[\xi_*, \omega_*]}$, (3.2) and (3.1) are equivalent, so we have a solution of (3.3) and the minimum fixed point given by Theorem 1.1 is necessarily unique. ■

Lemma 3.3. *Under our hypotheses, the map*

$$\mathbf{Z} : [\xi, \omega] \mapsto z = [\text{minimum fixed point of } \Psi_{[\xi, \omega]}] : X_* \times M \rightarrow \mathcal{Z}_*, \quad (3.5)$$

is well-defined and isotone.

PROOF: That \mathbf{Z} is well-defined follows from the proof of Lemma 3.2 and we must show isotonicity, using Theorem 1.1: given $[\xi, \omega] \succ [\xi', \omega']$ in $X_* \times M$ and setting

$$z = \mathbf{Z}[\xi, \omega] = \Psi_{[\xi, \omega]}(z) \quad z' = \mathbf{Z}[\xi', \omega'] = \Psi_{[\xi', \omega']}(z'),$$

we show that $z \succ z'$. Since $z = \mathbf{Z}(\xi, \omega)$ is the minimum fixed point of $\Psi_{[\xi, \omega]}$, and $\Psi_{[\cdot, \cdot]}$ is isotone by the proof of Lemma 3.2 we have $\Psi_{[\xi', \omega']}(z) \prec \Psi_{[\xi, \omega]}(z) = z$ so $z \in \{\hat{z} \in \mathcal{Z}_* : \Psi_{[\xi', \omega']}(\hat{z}) \prec \hat{z}\}$. Hence $z \succ \inf\{\hat{z} \in \mathcal{Z}_* : \Psi_{[\xi', \omega']}(\hat{z}) \prec \hat{z}\} = \mathbf{Z}(\xi', \omega') = z'$. ■

The solution of the initial value problem (3.3) with data $[\xi_*, \omega_*]$ is given by

$$[x, w] = \mathbf{G}_{[\xi_*, \omega_*]}(\mathbf{Z}(\xi_*, \omega_*)) \quad (3.6)$$

and to consider periodicity, we consider the Poincaré *period map* for the coupled dynamics:

$$\mathbf{F} : [\xi_*, \omega_*] \mapsto [x(T), w(T)] \quad \text{with } x, w \text{ given by (3.6)}. \quad (3.7)$$

With this machinery we are now ready for the periodicity theorem.

Theorem 3.4. *Assume the coupled dynamics are as described with all the hypotheses above satisfied. Then the map \mathbf{F} is isotone and has a fixed point. If, also, any explicit t -dependence of f is T -periodic, i.e., if*

$$f(T + t, \xi, \omega) = f(t, \xi, \omega) \quad \text{for each } t \in \mathbb{R}, \quad (3.8)$$

then the system has at least one periodic solution with period T .

PROOF: Starting with $[\xi_*, \omega_*] \in X_* \times M$, we begin by observing that $z = \mathbf{Z}(\xi_*, \omega_*)$ is well defined in \mathcal{Z}_* by Lemma 3.3 so $[x, w] = \mathbf{G}_{[\xi_*, \omega_*]}(z)$ is well defined by Lemma 3.1. Since $x(\cdot)$ is continuous by (A-1), there is no difficulty in evaluating it at T ; we note that $x(T) \in X_*$ by (C-1). Similarly, the construction in the proof of Theorem 2.2 ensures that we have a well defined value for $w(T)$. Thus, \mathbf{F} is well defined by (3.7) as a selfmap of $X_* \times M$.

Since \mathbf{Z} is isotone in $[\xi, \omega]$ and \mathbf{G} is isotone in its variables by Lemmas 3.3, 3.1, we have isotonicity of \mathbf{F} and so existence of a fixed point by the Birkhoff Fixpoint Theorem.

Given (3.8), it is quite standard to note that a solution of (3.3), now viewed as periodically extended to all of \mathbb{R} , is a periodic solution when the data $[\xi_*, \omega_*]$ is a fixed point of the Poincaré period map \mathbf{F} . \blacksquare

4 The example of [7]

As our first example of the use of Theorem 3.4, we return to (1.1) and [7].

It would be standard to rewrite (1.1) as a first order system $\dot{x} = Ax + f$ where A is the $d \times d$ companion matrix of p for p of degree d so the state space is $X = \mathbb{R}^d$. We set $D_t : y(\cdot) \mapsto [y(t), \dots, y^{[d-1]}(t)]$ so, letting $y(\cdot)$ be the solution (denoted by $S(z, \xi)$) of

$$p(d/dt)y = z \quad D_0 y = \xi, \quad (4.1)$$

the state $x(t)$ is $D_t y$.

We use the somewhat unusual partial ordering of \mathbb{R}^d defined by setting

$$\xi \succ 0 \quad \iff \quad S(0, \xi)(t) \geq 0 \text{ for each } t \geq 0.$$

Assuming that p has only (distinct) real roots r_1, \dots, r_d , we set $b_k = [1, r_k, \dots]$ and note that each $S(0, b_k) = e^{r_k t}$ will be positive; thus, $\sum_k \eta_k b_k \succ 0$ if each $\eta_k \geq 0$, although the converse does not hold. As $\{b_k\}$ is a basis, we can now define a norm $\|\sum_k \eta_k b_k\| = \max_k \{|\eta_k|\}$ and note that $\pm \xi \prec (\|\xi\| b_*)$ with $b_* = \sum_k b_k$. We now take $X = \mathbb{R}^d$ with this partial order, $\mathbf{Y} : \xi \mapsto y = \xi_1$, $Z = \mathbb{R}$ and then $\mathcal{X} = \mathcal{C}([0, T] \rightarrow X)$, $\mathcal{Z} = L^\infty([0, T] \rightarrow Z)$. Note that (4.1) becomes

$$\dot{x} = \mathbf{L}x + \mathbf{B}z \quad x(0) = \xi \quad (4.2)$$

with $\mathbf{L} = \text{diag}\{r_1, \dots\}$ using the basis $\{b_k\}$, i.e., $\mathbf{L} \left(\sum_{k=1}^d \eta_k b_k \right) = \sum_{k=1}^d r_k \eta_k b_k$. Thus, with $-\alpha = \max\{r_k\}$, we have

$$\|x(t)\| \leq e^{-\alpha t} \|\xi\| + \int_0^t e^{-\alpha(t-s)} \|\mathbf{B}z(s)\| ds. \quad (4.3)$$

Lemma 4.1. *If $z \geq 0$ and $\xi \succ 0$, then $y = S(z, \xi) \geq 0$ and $x(\tau) = D_\tau y \succ 0$ for each $\tau \geq 0$*

PROOF: It is sufficient to take $\xi = 0$ and we then proceed by induction on d . For $d = 1$, $y' - ry = z \geq 0$ gives $y(t) = \int_0^t e^{r(t-s)} z(s) ds \geq 0$. Next write $p(r) = (r - r_1)\hat{p}(r)$ so

$y = S(z, 0)$ satisfies $y' - r_1 y = \hat{y}$ with $\hat{p}(d/dt)\hat{y} = z$; the inductive hypothesis gives $\hat{y} \geq 0$ as \hat{p} has degree $(d-1)$ whence $y \geq 0$. For $\tau > 0$ we can apply this to $z_\tau = \{z \text{ on } [0, \tau]; 0 \text{ else}\}$ so $y_\tau = S(z_\tau, 0) \geq 0$ and $y_\tau(\cdot + \tau) = S(0, D_\tau y_\tau) \geq 0$ whence $D_\tau y_\tau \succ 0$ by our definition of the partial order; since $D_\tau y_\tau = D_\tau y$, we have $x(\tau) \succ 0$. ■

We are considering the standard relay so we have $M = \{0, 1\}$ with, e.g., switching at 0, 1 so, in terms of switching rules, we have set

$$\begin{aligned} C(0, 0) &= (-\infty, 1), & C(0, 1) &= [1, \infty), \\ C(1, 1) &= (0, \infty), & C(1, 0) &= (-\infty, 0]. \end{aligned} \quad (4.4)$$

Note that (B-2) holds as well as (B-1).

We next introduce the nonlinearity \hat{f} . If we assume that this is isotone and Lipschitzian, then the ODE with suitable initial data is well-posed. [Further, the solution $y(\cdot)$ will be Lipschitzian; there is then a lower bound $\varepsilon > 0$ (uniform in the presence of a bound on the data) on each interswitching interval as y moves between 0, 1 so we have (B-4) with $\hat{M} = \{[0, 1], [1, 0]\}$; there can be no Zeno phenomenon.] We also assume that \hat{f} satisfies the growth condition

$$\|\mathbf{B}\hat{f}(t, \mathbf{Y}(\xi), \omega)\| \leq \kappa + \lambda\|\xi\| \quad (4.5)$$

with $0 \leq \lambda < \alpha = -\max\{r_k\}$ (requiring each $r_k < 0$).

To define X_* , Z_* as order intervals as in Lemma 2.1, we first take ξ^+ of the form $\xi^+ = cb_*$ and consider the solution x^+ of (4.2) with $z = \hat{f}(t, \mathbf{Y}(x^+), 1)$ and $x^+(0) = \xi^+$. Using (4.5) with (4.3) gives $\|x^+(t)\| \leq [\vartheta + (1 - \vartheta)e^{-\alpha t}] \leq c^+$ for $\vartheta \leq 1$ with $\vartheta = (\lambda + \kappa/c^+)/\alpha$. Thus, $x(t) \prec \|x(t)\| b_* \leq c^+ b_* = \xi^+$ (and, similarly, $x^+(t) \succ 0$) if we take c^+ large enough. Taking $\xi^- = -c^- b_*$ and so x_- for $z = \hat{f}(t, \mathbf{Y}(x^-), 0)$, we have $\xi^- \prec x^-(t) \prec 0$ for large enough c^- . We let X_* be the order interval in X between ξ^- and ξ^+ (noting $\xi^- \leq 0 \leq \xi^+$) and let Z_* be the order interval in Z between $z^- = \hat{f}(t, \mathbf{Y}(x^-), 0)$ and $z^+ = \hat{f}(t, \mathbf{Y}(x^+), 1)$ (noting $x^- \leq 0 \leq x^+$ so $z^- \leq 0 \leq z^+$ by the isotonicity of \hat{f}). This gives (C-1) and the isotonicity then shows the invariance of Z_* under $\Psi_{[\xi_*, \omega_*]}$ for $\xi_* \in X_*$, as required in (D-2). The remaining hypotheses are easy to verify.

Since the ODE here is autonomous, the Periodicity Theorem 3.4 then applies to show existence of a periodic solution.

5 Ca^{2+} waves in cardiac cells

As another example of the use of Theorem 3.4, we consider a model (somewhat modified from [6]) of the spread of calcium (Ca^{2+}) in cardiac cells.

Let $x(t, \cdot)$ be the concentration of Ca^{2+} at time t in a cardiac cell represented by $\Omega \subset \mathbb{R}^3$. Here x satisfies the diffusion equation

$$\begin{aligned} x_t &= \Delta x - \lambda x + S & \text{on } \Omega \\ x &= x^* & \text{at } \partial\Omega. \end{aligned} \quad \text{for } t \geq 0 \quad (5.1)$$

with $x = \xi \geq 0$ on Ω at $t = 0$. We assume the boundary data is T -periodic and satisfies $0 \leq x^* \leq \beta$.

For this model the source S is given by a set \mathcal{N} of *Calcium Release Units*, indexed by their positions $\nu \in \mathcal{N} \subset \Omega$; with $N = \#\mathcal{N}$ possibly large, but finite. These provide distinct point sources of calcium when active. The source S in (5.1) is then

$$S = \sum_{\mathcal{N}} z_{\nu} \delta_{\nu} = \mathbf{B}z \quad (5.2)$$

where δ_{ν} denotes the delta function (unit point source) at the point $\nu \in \Omega$ and $z = [z_{\nu} : \nu \in \mathcal{N}] \in [0, 1]^N$ gives the release rates at each CRU. [Note that \mathbf{B} maps N -tuples to measures on Ω .] Independently for each CRU $_{\nu}$, a transition to activity is triggered by the local concentration $x(\cdot, \nu)$ rising to a threshold level η_{ν}^{+} .

We begin by introducing sensors \mathbf{Y}_{ν} , for each $\nu \in \mathcal{N} \subset \Omega$, giving

$$y_{\nu}(x(t, \cdot)) = x(t, \nu), \quad \mathbf{Y} = \bigotimes_{\nu \in \mathcal{N}} \mathbf{Y}_{\nu} \quad (5.3)$$

so $y(x(t, \cdot)) = [x(t, \nu_1), \dots, x(t, \nu_N)]$. Fixing thresholds $\eta_{\nu}^{-} < \eta_{\nu}^{+}$ and considering a relay operator with these thresholds, the switching rules are now given, as earlier, by switching sets, separately for each ν :

$$C_{\nu}(0, 1) = [\eta_{\nu}^{+}, \infty], \quad C_{\nu}(1, 0) = [0, \eta_{\nu}^{-}]$$

with $C_{\nu}(0, 0)$, $C_{\nu}(1, 1)$ complementary in $Y_{\nu} = [0, \infty]$ to those so (B-1). We easily verify (B-1,2) here — indeed, this is essentially the standard relay for each ν and, collectively, this defines an isotone generalized relay operator \mathbf{W} as in Lemma 2.3, driven by the input $y = [y_{\nu}]$. [Note that we have taken $Y_{\nu} = [0, \infty]$ to include $+\infty$ since the pointwise¹² concentration at a release point is infinite whenever there is calcium release: allowing $y_{\nu} = \infty$ when $z_{\nu} \neq 0$.] In the model of [6], activation of each CRU ($w_{\nu} : 0 \curvearrowright 1$) is (stochastically) triggered when the ambient calcium level reaches a threshold η_{ν}^{+} and the CRU then remains active for a period of some fixed length. Since we have been unable to arrange this (and retain isotonicity) within our present hybrid formulation, we are retaining this trigger, albeit deterministically, but use a different mechanism to turn off the release. Somewhat arbitrarily, our version of this model will take the supply $s_{\nu}(t)$ to the CRU to be an exogenously varied part of the state, as part of our periodic forcing, and it is then this which will (indirectly) control the shutoff: we assume s_{ν} takes values 0 (no supply) or 1 (adequate supply) and the CRU $_{\nu}$ will release calcium (at a fixed rate, say 1) only when it is both active and supplied. Thus, we are taking

$$z_{\nu} = f_{\nu}(t, w_{\nu}) = \begin{cases} 1 & \text{when } w_{\nu} = 1 \text{ and } s_{\nu}(t) = 1 \\ 0 & \text{else: } w_{\nu} = 0 \text{ or } s_{\nu}(t) = 0. \end{cases} \quad (5.4)$$

Note that the dependence of f_{ν} on the exogenous variable s_{ν} is interpreted here as dependence on t so f is independent of x and isotone in w . We are assuming $s_{\nu}(\cdot)$ is piecewise constant and T -periodic in t .

¹²Each y_{ν} is the sum of a smooth function and the convolution of the heat kernel (at $x = 0$) with z_{ν} . Since z_{ν} is non-negative, taking values in $\{0, 1\}$, we have y_{ν} continuous except for possible jumps up to $+\infty$. Thus, we have (A-3).

We recall some facts about the diffusion equation (5.1), first noting that it is well-posed to, e.g., $x \in \mathcal{X} = L^2([0, T] \times \Omega)$ for $z \in \mathcal{Z} = [L^\infty(0, T)]^N$ and $\xi_* \in X = L^2(\Omega)$. Indeed, letting ξ^* be the solution of the steady state equation

$$-\Delta \xi^* + \lambda \xi^* = \sum_{\mathcal{N}} \delta_\nu \quad \xi^*|_{\partial\Omega} = \beta,$$

we set $X_* = \{\xi \in L^2(\Omega) : 0 \leq \xi \leq \xi^*(\cdot)\}$ and note that the Maximum Principal ensures that we will have $x(t, \cdot) \in X_*$ for any $x(0, \cdot) \in X_*$ and any $z \in \mathcal{Z}_* = \{z \in \mathcal{Z} : z(t) \in \{0, 1\}^N\}$. Note that both X_* and \mathcal{Z}_* are complete lattices. By standard comparison results for diffusion equations, the solution of the resulting (5.1), (5.2) depends isototonically on z , as well as on the initial data $x(0) = \xi_*$ and \mathcal{Z}_* is invariant under $\mathbf{G}_{[\xi, \omega]}$ for $[\xi, \omega] \in X_* \times M$. Finally, we note that local regularity (see, e.g., [9](Thm.16.3)) shows that x is smooth between the singularities of S — including smoothness at ν for $z_\nu(t) \neq 1$ — so each $y_\nu = x(\cdot, \nu)$ is well-defined and¹³ continuous to the extended \mathbb{R}^+ , i.e., to $[0, \infty]$, justifying the use of (5.3) when constructing $\mathbf{W}_\nu(\omega^*, y_\nu)$ dynamically. We thus see that we have well-posed coupled dynamics for (5.1) coupled with $\mathbf{W}(\omega_*, \mathbf{Y}(x))$; (C) and (D) hold. Indeed, all the hypotheses are now easily verifiable and, as an immediate application of Theorem 3.4, we then obtain:

Theorem 5.1. *The cardiac cell model described here must support at least one T -periodic calcium distribution.* ■

REMARK: Of course this is only of interest if we would have a *nontrivial* solution, such that $w(\cdot)$ is not constant in t , and we note that the possibility of a solution being trivial can be eliminated by imposing appropriate conditions on the data. To see this, we begin with the observation:

Lemma 5.2. *Assume $\lambda > 0$. Then for T -periodic z the equation (5.1) with $S = \mathbf{B}z$ given by (5.2) has a unique T -periodic solution on \mathbb{R} , which we denote by $x(\cdot, \cdot; z)$. The resulting map: $z \mapsto x(\cdot, \cdot; z)$ is isotone; we also have isotone dependence on the periodic boundary data $x^*(\cdot)$.*

PROOF: Considering the equation with arbitrary initial data, the dissipativity ensures convergence to a periodic solution, since we have assumed x^*, z are periodic. Using the isotonicity of the initial value problem, as earlier, then gives the isotonicity in z asserted here. A similar use of the Maximum Principal gives the isotone dependence on the data x^* . ■

We now let $z^0 = f(t, 0) \equiv 0$, $z^1 = f(t, 1) = [s_\nu(t) : \nu \in \mathcal{N}]$ and then set $x^0 = x(\cdot, \cdot; z^0)$, $x^1 = x(\cdot, \cdot; z^1)$ with $y^0 = \mathbf{Y}(x^0)$, etc. [Note that these do not involve w (or any switching), but do depend on the data.] We now observe that: if $[\xi_*, \omega_*]$ — and so $[x, w]$ and z — are obtained through Theorem 3.4, then $z^0 \prec z \prec z^1$ so $x^0 \prec x = x(\cdot, \cdot; z) \prec x^1$ and

¹³Except for the possibility of an impulsive jump up to $+\infty$ if s_ν goes from 0 to 1 while $w_\nu = 1$ — which, we observe, would leave $y_\nu \in C(1, 1)$, consistent with (A-1). Note that we have (B-3,4) with $\hat{M} = \{[1, 0]\}$.

$y^0 \prec y = \mathbf{Y}(x) \prec y^1$ whence $w^0 = \mathbf{W}[0, y^0] \prec w = \mathbf{W}[\omega_*, y] \prec w^1 = \mathbf{W}[1, y^1]$. If x^* is such that $x^0(t, \nu) \geq \eta_\nu^+$ for some $t \in [0, T]$ — as will necessarily be the case if x^* is large enough on some long enough subinterval of $[0, T]$ — then we must have $w_\nu^0 \neq 0$. On the other hand, if x^* is such that $x^1(t, \nu) \leq \eta_\nu^-$ for some $t \in [0, T]$ — as will necessarily be the case if x^* is small enough and $s_\nu(t) \equiv 0$ on some long enough subinterval of $[0, T]$ — then we must have $w_\nu^1 \neq 1$. Imposing both these as conditions on the data then ensures that there will not be any periodic solution with $w(\cdot)$ constant in t , so the periodic solution given by Theorem 3.4 would necessarily be nontrivial. ■

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