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Preprint ISSN 0946 – 8633

Few-cycle optical solitons in dispersive media beyond the envelope approximation

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submitted: July 20, 2012

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No. 1723
Berlin 2012

2010 *Physics and Astronomy Classification Scheme.* 42.81.Dp, 42.65.Tg, 05.45.Yv, 42.65.Re.

Key words and phrases. Optical solitons, Ultrashort pulses, Dispersion.

Acknowledgments. Sh.A. gratefully acknowledges support by the DFG Research Center MATHEON under project D 14. N.A. acknowledges the support of the Australian Research Council (Discovery Project DP110102068) and the Volkswagen Stiftung. He is also a recipient of the Alexander von Humboldt Award.

Edited by
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Abstract

We study the propagation of few-cycle optical solitons in nonlinear media with an anomalous, but otherwise arbitrary, dispersion and a cubic nonlinearity. Our theory extends beyond the slowly varying envelope approximation. The optical field is derived directly from the Maxwell equations under the assumption that generation of the third harmonic is a non-resonant process or at least cannot destroy the pulse prior to inevitable linear damping. The solitary wave solutions are obtained numerically up to nearly single-cycle duration using a modification of the spectral renormalisation method originally developed for the envelope solitons.

1 Introduction

Optical solitons are robust localized pulses arising from the interplay of nonlinearity and dispersion. They are usually described by the pulse envelope governed by the nonlinear Schrödingier equation (NLSE) [15, 2]. The latter is known to be integrable [34] with its fundamental soliton solution being the central item of studies in nonlinear optics. As the NLSE is derived using the slowly varying envelope approximation (SVEA), such a fundamental soliton must contain sufficiently large number of optical oscillations. Recent progress in generation of few-cycle [9, 4] and even sub-cycle [16, 25, 10] optical pulses boosted interest to ultrashort optical solitons. Several exact solutions have been found both for the generalized NLSE beyond the SVEA [29, 14, 11, 35, 26, 5] as well as for the non-envelope propagation equations derived specifically to describe ultrashort pulses [21, 18, 27, 20, 30, 8, 23]. These solutions and the present work presuppose that the pulse carrier frequency is well separated from any resonant frequency of the medium. Another important case is that of the resonant ultrashort solitons [17, 24, 22].

Non-envelope solitons are less universal then the fundamental soliton of the NLSE. The exact solutions known so far require either specific relations between the equation parameters in the case of the generalised NLSE or simplified dispersion profiles. For example, Drude's dispersion in a wide spectral window is required for the so called short pulse equation [19, 28]. Typically non-envelope solutions belong to continuous families and they are characterised by the pulse duration t_0, among other parameters. As the pulse duration increases, the ultrashort soliton moving along z-axis with the velocity V approaches the standard $\cosh^{-1}(\tau/t_0)$ shape for $\tau = t - z/V$ and the fundamental soliton of the NLSE is recovered. This simple limit motivates a backward search for the few-cycle solitons. Let's first start with a fundamental soliton of the NLSE and transform it into an exact solution of the properly simplified set of Maxwell equations. By doing so we avoid the use of the envelope and the NLSE. Thereafter one can trace the solution shape while solving Maxwell equations and decreasing the soliton duration. This turns out to be an effective strategy at least for systems, where nonlinear generation of the third harmonics can be ignored even for spectrally wide short pulses.

We have found that envelope solitons can be naturally transformed into ultrashort solitary solutions. These solutions can be traced down to nearly single-cycle duration but only as long as one can ignore contributions of both the higher harmonic generation and the soliton-specific Cherenkov radiation [3]. If this is the case, one can even observe formation of a non-smooth cusp profile which prohibits existence of sub-cycle solitons. This fact agrees with the properties of the exact ultrashort solitary solutions obtained previously both for the generalized NLSE [5] and for the non-envelope propagation model with the idealized dispersion [30, 8]. From the practical side, the ultrashort solitons presented in this paper provide natural initial conditions for direct modeling of solitons using Maxwell equations, where imperfect ultrashort fundamental solitons are used instead [12, 13, 31]. Last but not least, the non-envelope ultrashort solitons are obtained using a simple and effective numerical procedure. The latter is a natural generalization of the spectral renormalization method, originally developed for the NLSE [1].

2 Basic equations

For the sake of simplicity, we consider a single-mode waveguide and characterize the propagating pulse by a single field component. The latter depends on the propagation distance and time, $E = E(z,t)$, as prescribed by the scalar nonlinear wave equation

$$\partial_z^2 E - \frac{1}{c^2}\partial_t^2 \left(\hat{\epsilon} E + \chi^{(3)} E^3\right) = 0 \tag{1}$$

where c is the speed of light and parameter $\chi^{(3)}$ is the nonlinear susceptibility of the third order. A nonlocal pseudo-differential operator $\hat{\epsilon}$ is defined by a suitable medium response function $h(t)$

$$\hat{\epsilon} E(z,t) = E(z,t) + \int_0^\infty E(z, t-t')h(t')dt',$$

such that for a single harmonic $E \sim e^{-i\omega t}$ we obtain

$$(\hat{\epsilon} E)_\omega = \epsilon(\omega) E_\omega, \quad \epsilon(\omega) = 1 + \int_0^\infty h(t') e^{i\omega t'} dt', \tag{2}$$

where $E_\omega = E_\omega(z)$ denotes the Fourier component of $E(z,t)$. The dielectric constant $\epsilon(\omega)$ accounts for the combined medium and waveguide dispersion.

Note, that any nontrivial $\epsilon(\omega)$ from Eq. (2) is a complex-valued function. The corresponding dispersion relation $\beta^2(\omega)c^2 = \omega^2\epsilon(\omega)$ for the monochromatic linear wave $E \sim \exp i(\beta z - \omega t)$ provides a complex-valued propagation constant $\beta(\omega)$ for each frequency. In other words, damping is in the very nature of Eq. (1) and all optical solitons are *a priory* approximate solutions in the transparency window. Consequently, looking for a soliton, one has a right to ignore, e.g., higher harmonics and Cherenkov radiation, if their effect is below the linear damping. This is particularly valid for ultrashort solitons which are wide in the frequency domain and experience higher rates of linear and nonlinear damping. In practice, such a soliton is a long-living robust solution that persists despite other pulses being destroyed by nonlinearity and dispersion but it still can be destroyed by damping.

Optical pulses which propagate with a constant velocity without changing their shape, were examined in Ref. [33] based on Eq. (1) with a real-valued positive, but otherwise arbitrary, dielectric constant. These solutions may only be expected if $\mathrm{Re}[\epsilon(\omega)]$ has either a maximum at some ω for a focusing nonlinearity or a minimum for a defocusing one. Unfortunately this is not the case for the vast majority of material dispersions. Besides, all extrema of $\mathrm{Re}[\epsilon(\omega)]$ typically correlate to a high damping. The lack of the ultra-short solitons is in a sharp contrast to the variety of the envelope solitons. The latter exist for any carrier frequency $\omega = \omega_0$ in the transparency window provided that $\chi^{(3)}$ and $\beta_2 = \beta''(\omega_0)$ are of different sign. To be specific, we assume focusing nonlinearity and anomalous dispersion:

$$\chi^{(3)} > 0 \quad \text{and} \quad \beta''(\omega) < 0. \tag{3}$$

The latter condition applies to some interval of frequencies. Normal dispersion is allowed in the transparency window outside this region.

In what follows, we show how to obtain non-envelope solitary wave solutions for Eq. (1). To this end, one has to ignore contributions of higher harmonics. The most natural way to do so is to use the analytic signal representation for the electric field [32]. This approach is also a useful tool for the treatment of the pulse propagation problem such as the one presented in [7].

3 Reduction

Keeping in mind that we are interested in describing solitons for arbitrary dispersion, at some stage, we have to deal with the numerical solution. It is then convenient to replace a continuous ω with a sufficiently dense discrete set of frequencies by introducing a large period T in the time domain. Thus, the electric field is represented by a discrete sum

$$E(z,t) = \sum_{\omega} E_{\omega}(z) e^{-i\omega t}, \qquad \omega \in \frac{2\pi}{T}\mathbb{Z}, \tag{4}$$

where

$$E_{\omega}(z) = \int_{-T/2}^{+T/2} E(z,t) e^{i\omega t} \frac{dt}{T}, \qquad E_{-\omega} = E_{\omega}^{*}, \tag{5}$$

such that E and E_{ω} have the same physical dimension. Performing numerical calculations, we actually keep only part of harmonics, namely those belonging to the transparency window

$$\omega_L < \omega < \omega_R. \tag{6}$$

Of course, all our results can be easily rewritten to apply to the continuous spectrum.

In what follows, we assume that the four-wave resonance conditions

$$\begin{gathered} \beta(\omega_1) + \beta(\omega_2) + \beta(\omega_3) = \beta(\omega_4), \\ \omega_1 + \omega_2 + \omega_3 = \omega_4, \qquad \omega_i > 0, \end{gathered} \tag{7}$$

are not satisfied simultaneously for any four positive frequencies from the transparency window (6) whether they are discrete or not. Consequently, generation of the third harmonics is a non-resonant process.

To utilise the above feature, we introduce a complex field $\mathcal{E}(z,t)$ which *by definition* is governed by the equation

$$\partial_z^2\mathcal{E} - \frac{1}{c^2}\partial_t^2\left(\hat{\epsilon}\mathcal{E} + \frac{3}{4}\chi^{(3)}|\mathcal{E}|^2\mathcal{E} + \frac{1}{4}\chi^{(3)}\mathcal{E}^3\right) = 0, \tag{8}$$

where $\hat{\epsilon}\mathcal{E}$ is naturally defined in the frequency domain

$$\hat{\epsilon}\mathcal{E}(z,t) = \sum_{\omega}\epsilon(\omega)\mathcal{E}_\omega(z)e^{-i\omega t}, \tag{9}$$

and $\mathcal{E}_\omega(z)$ denotes Fourier components of $\mathcal{E}(z,t)$. We stress that $\mathcal{E}$ is a complex-valued field and therefore $\mathcal{E}_{-\omega}$ and $(\mathcal{E}_\omega)^*$ are different in contrast to the standard relation (5). Actually the positive-frequency part of $\mathcal{E}(z,t)$ will dominate over the negative-frequency part (see below).

In what follows, $\mathcal{E}_\omega^*$ always denotes a complex conjugate of $\mathcal{E}_\omega$. Fourier components of the conjugated field $\mathcal{E}^*$ are denoted by $(\mathcal{E}^*)_\omega$. One can check that

$$\mathcal{E}_\omega^* = (\mathcal{E}_\omega)^* = (\mathcal{E}^*)_{-\omega}. \tag{10}$$

In addition, using (9), (10), and the standard relation $\epsilon(-\omega) = \epsilon^*(\omega)$ one can directly check that

$$(\hat{\epsilon}\mathcal{E})^* = \hat{\epsilon}(\mathcal{E}^*), \tag{11}$$

which extends the standard fact that $\hat{\epsilon}$ transforms an arbitrary real field into a real one.

Equation (8) may be considered as an odd way to solve Eq. (1). Indeed, adding Eq. (8) to its complex conjugate, using Eq. (11), and defining

$$E(z,t) = \frac{\mathcal{E}(z,t) + \mathcal{E}^*(z,t)}{2}, \tag{12}$$

we immediately recover Eq. (1) for the real electric field. To solve Eq. (1) one may introduce the complex field such that initially $E = \mathrm{Re}[\mathcal{E}]$, then solve Eq. (8), and then recover $E(z,t)$ from (12).

Now we take advantage of the presumably non-resonant dispersion profile by choosing a special initial value for $\mathcal{E}(z,t)$. As usual, we deal with the waves propagating along the z-axis so that the initial state of the system is that for $z=0$. We present the incoming field $E(z=0,t)$ as a sum of harmonics (4) and define $\mathcal{E}(z=0,t)$ as the positive-frequency part of this sum

$$E(0,t) \quad\rightarrow\quad \mathcal{E}(0,t) = 2\sum_{\omega>0}E_\omega(0)e^{-i\omega t}, \tag{13}$$

where the factor 2 is for $E = \mathrm{Re}[\mathcal{E}]$. The time-averaged field $\langle E\rangle = E_{\omega=0}$ is set to zero, the initial value of the derivative $\partial_z\mathcal{E}$ is related to $\partial_z E$ in a manner similar to Eq. (13).

Initially $\mathcal{E}(z=0,t)$ contains only positive frequencies and thus is an analytic signal for $E(z=0,t)$. Strictly speaking, this is not true for $z>0$, because negative frequency components of

$\mathcal{E}(z,t)$ still may be generated by the $|\mathcal{E}|^2\mathcal{E}$ term. However, the process is non-resonant. For example, let us assume that a harmonic $e^{-i\omega t}$ results from the nonlinear interaction of three positive-frequency harmonics $e^{-i\omega_i t}$ through the $|\mathcal{E}|^2\mathcal{E}$ term, such that

$$\beta(\omega) = \beta(\omega_1) - \beta(\omega_2) + \beta(\omega_3),$$
$$\omega = \omega_1 - \omega_2 + \omega_3, \qquad \omega_i > 0.$$

If $\omega < 0$, the latter conditions are equivalent to Eq. (7) and the resonance excitation does not occur. Therefore, to a large extent, $\mathcal{E}(z,t)$ contains only positive frequencies also for $z > 0$. More precisely, if the pulse field scales as $O(\varepsilon)$, the induced negative-frequency part scales as $O(\varepsilon^3)$, its backward effect on the positive-frequency part might be resonant but scales as $O(\varepsilon^5)$ which is beyond the accuracy of the initial Eq. (1).

Up to this very moment the transformation from Eq. (8) to Eq. (1) was exact. Now we take advantage of the above considerations and neglect generation of the third harmonics by ignoring the $\mathcal{E}^3$ term in Eq. (8). The resulting propagation equation

$$\partial_z^2\mathcal{E} - \frac{1}{c^2}\partial_t^2\left(\hat{\epsilon}\mathcal{E} + \frac{3}{4}\chi^{(3)}|\mathcal{E}|^2\mathcal{E}\right) = 0, \tag{14}$$

is the one that will be used below. All meaningful solutions of Eq. (14) are subject to a simple criterion: the negative-frequency part of $\mathcal{E}(z,t)$, being zero at $z = 0$ by construction, must remain small for $z > 0$. Numerical examples demonstrate that this is a fairly good approximation at least for the propagation distances typical for ultrashort pulses.

The simplest solution of Eq. (14) is given by a monochromatic nonlinear wave

$$\mathcal{E} = \mathcal{E}_0 e^{i[\beta(\omega)z - \omega t]},$$

which must satisfy the nonlinear dispersion relation

$$\beta^2(\omega) = \frac{\omega^2}{c^2}\left[\epsilon(\omega) + \frac{3\chi^{(3)}}{4}|\mathcal{E}_0|^2\right].$$

Furthermore, one can introduce a standard nonlinear correction to the refraction index $n(\omega) = \sqrt{\epsilon(\omega)}$ such that

$$n_{\text{eff}}(\omega) = n(\omega) + \frac{3\chi^{(3)}}{8n(\omega)}|\mathcal{E}_0|^2. \tag{15}$$

Equation (14) has the same nonlinearity as the NLSE and possesses a rich set of solitary solutions which are naturally connected to the fundamental solitons of the NLSE. On the other hand, Eq. (14) is much more general than the NLSE as it is of the second order, operates with the electric field, accounts for arbitrary dispersion, and is not the result of the SVEA. We now turn to the description of solitary wave solutions. To this end, it is convenient to write Eq. (14) in the frequency representation

$$\partial_z^2\mathcal{E}_\omega + \beta^2(\omega)\mathcal{E}_\omega + \frac{3\chi^{(3)}\omega^2}{4c^2}\left(|\mathcal{E}|^2\mathcal{E}\right)_\omega = 0, \tag{16}$$

where $\left(|\mathcal{E}|^2\mathcal{E}\right)_\omega$ denotes a spectral component of $|\mathcal{E}|^2\mathcal{E}$.

4 Solitons

Up until now the dielectric constant was an arbitrary complex-valued function for which $\mathrm{Re}[\epsilon(\omega)] \gg \mathrm{Im}[\epsilon(\omega)]$ in the transparency window (6). From now on we neglect the imaginary part completely and simply write $\epsilon(\omega)$ instead of $\mathrm{Re}[\epsilon(\omega)]$. All derived values, e.g., the propagation constant and the refraction index, are then real.

To describe solitary wave solutions we introduce a shape function $\mathfrak{e}(\tau)$ for the electric field

$$\mathfrak{e}(\tau) = \sum_{\omega} \mathfrak{e}_{\omega} e^{-i\omega\tau},$$

where τ refers to the retarded time $\tau = t - z/V$ and parameter V is the velocity of the soliton. We will consider two different cases.

4.1 Stationary solitons

The simplest case is that the soliton moves with a constant velocity V and is stationary in the co-moving frame. This corresponds to the classical concept of a soliton. The corresponding ansatz reads

$$\mathfrak{e}(\tau) \quad \to \quad \mathcal{E}(z,t) = \mathfrak{e}\left(t - \frac{z}{V}\right) \tag{17}$$

such that

$$\mathcal{E}_{\omega}(z) = \mathfrak{e}_{\omega} e^{i\omega z/V}, \quad \left(|\mathcal{E}|^2\mathcal{E}\right)_{\omega} = \left(|\mathfrak{e}|^2\mathfrak{e}\right)_{\omega} e^{i\omega z/V}. \tag{18}$$

We insert the ansatz (17) into Eq. (16) to derive that $\mathfrak{e}(\tau)$ is determined by the following relation

$$\left[\frac{c^2}{V^2} - \epsilon(\omega)\right] \mathfrak{e}_{\omega} = \frac{3\chi^{(3)}}{4} \left(|\mathfrak{e}|^2\mathfrak{e}\right)_{\omega}. \tag{19}$$

An equation similar to Eq. (19) has been studied in Ref. [33]. Similar analysis applies also to our case. If for some resonant frequency ω_r the corresponding phase velocity equals to that of the soliton, $c/\sqrt{\epsilon(\omega_r)} = V$, the pulse will radiate energy and, thus, cannot be a soliton. To obtain a solitary wave solution, let us assume that $\epsilon(\omega)$ has a maximum at $\omega = \omega_*$ and that $c^2/V^2 > \epsilon(\omega_*)$. The condition $\epsilon'(\omega_*) = 0$ indicates that corresponding phase and group velocities are the same,

$$V_* = V_{\text{ph}}(\omega_*) = V_{\text{gr}}(\omega_*),$$

the soliton velocity V is somewhat smaller than V_* so that the term in the square brackets in Eq. (19) is positive-defined.

Now we introduce a new function $\psi(\tau)$ such that

$$\mathfrak{e}(\tau) = \psi(\tau) e^{-i\omega_*\tau}, \qquad \psi(\tau) = \sum_{\Omega} \psi_{\Omega} e^{-i\Omega\tau}$$

and therefore

$$\mathfrak{e}_{\omega_*+\Omega} = \psi_{\Omega}, \qquad \left(|\mathfrak{e}|^2\mathfrak{e}\right)_{\omega_*+\Omega} = \left(|\psi|^2\psi\right)_{\Omega}.$$

Equation (19) yields

$$\left[\frac{c^2}{V^2} - \epsilon(\omega_* + \Omega)\right]\psi_\Omega = \frac{3\chi^{(3)}}{4}\left(|\psi|^2\psi\right)_\Omega. \tag{20}$$

for $\omega = \omega_* + \Omega$.

Equation (20) is an exact consequence of Eq. (19). The SVEA and the fundamental soliton of the NLSE come into play when $\Omega \ll \omega_*$ and c^2/V^2 is just slightly higher than $\epsilon(\omega_*)$. Expanding $\epsilon(\omega_* + \Omega)$ and transforming Eq. (20) to the time domain we obtain a simple equation

$$\left[\frac{c^2}{V^2} - \epsilon(\omega_*)\right]\psi + \frac{\epsilon''(\omega_*)}{2}\partial_\tau^2\psi = \frac{3\chi^{(3)}}{4}|\psi|^2\psi. \tag{21}$$

Equation (21) yields the standard $\cosh^{-1}(t/t_0)$ shape solitary wave solution for the focusing nonlinearity and $\epsilon''(\omega_*) < 0$. One can check the relationship

$$\epsilon'(\omega_*) = 0 \quad \Rightarrow \quad \frac{\epsilon''(\omega_*)}{\epsilon(\omega_*)} = 2\frac{\beta''(\omega_*)}{\beta(\omega_*)}$$

indicating that the stationary moving solitons are subject to the condition (3). The soliton duration t_0 is determined by the relation

$$t_0^2 = \frac{-\beta''(\omega_*)/\beta(\omega_*)}{V_*^2/V^2 - 1},$$

where ω_* and V_* are determined solely by dispersion, $V < V_*$ parametrizes the family of solitary solutions.

As one decreases the soliton velocity V, the soliton becomes shorter in time and wider in the frequency domain. For ultrashort pulses the Taylor expansion of $\epsilon(\omega_* + \Omega)$ becomes invalid [6] and one has to deal with the full Eq. (19). The solitary solutions have to be found numerically (see Section 4.3). The exact $\cosh^{-1}$ solution of Eq. (21) yields a reasonable first approximation.

The results of this section are easy to interpret. A fundamental soliton moves with the group velocity whereas it's carrier wave moves with the phase velocity. Both velocities should be equal for the solution (17). This can only happen when $\epsilon(\omega)$ has an extremum. Multi-parametric families of solutions of Eq. (19) branch out from such maxima [33]. As already mentioned above, extrema of the dielectric constant typically correlate to a high linear damping and solutions of the form (17) are unlikely to be observed. We therefore must consider the ultrashort solitons which are non-stationary in the co-moving frame.

4.2 Non-stationary solitons

In this section we apply a less restrictive two-parametric ansatz for the electric field

$$\mathcal{E}(z,t) = \mathfrak{e}\left(t - \frac{z}{V}\right)e^{iKz}, \tag{22}$$

where the soliton shape function $\mathfrak{e}(\tau)$ is determined by K, V and Eq. (14). In what follows, it is convenient to fix V and to look how $\mathfrak{e}(\tau)$ shrinks with the increase of K. Equation (22) indicates that

$$\mathcal{E}_\omega(z) = \mathfrak{e}_\omega e^{i(K+\omega/V)z}, \quad \left(|\mathcal{E}|^2\mathcal{E}\right)_\omega = \left(|\mathfrak{e}|^2\mathfrak{e}\right)_\omega e^{i(K+\omega/V)z},$$

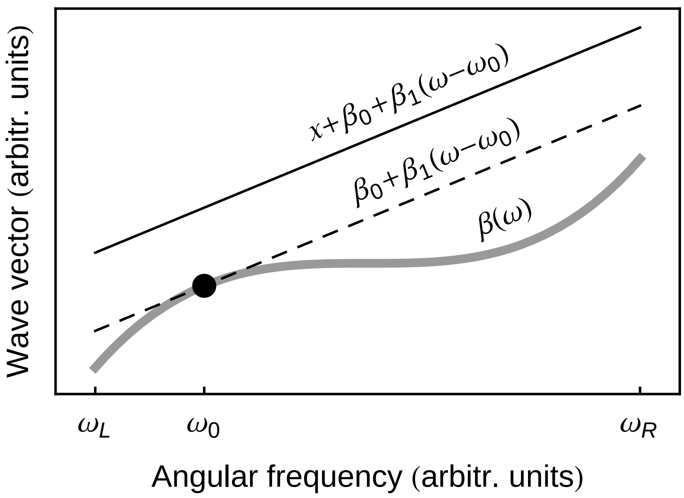

Figure 1: A schematic representation of the dispersion relation $\beta = \beta(\omega)$ in the transparency window $\omega_L < \omega < \omega_R$. Solitons with the carrier frequency ω_0 are parametrized by $\varkappa$. They can be found numerically for sufficiently small $\varkappa$ if ω_0 belongs to the anomalous dispersion domain and if the tangent line at ω_0 does not intersect with $\beta(\omega)$. The fundamental soliton of the NLSE is recovered for $\varkappa \to 0$.

c.f., Eq. (18). The corresponding soliton oscillates in the co-moving frame. Inserting $\mathcal{E}_\omega(z)$ into Eq. (14) we obtain

$$\left[\left(K + \frac{\omega}{V}\right)^2 - \beta^2(\omega)\right] \mathfrak{e}_\omega = \frac{3\chi^{(3)}\omega^2}{4c^2} \left(|\mathfrak{e}|^2\mathfrak{e}\right)_\omega . \tag{23}$$

Similar to Eq. (19), if for some $\omega = \omega_r$ it happens that $K + \omega_r/V = \beta(\omega_r)$, the pulse in question radiates energy. We therefore assume that the line $K + \omega/V$ and the curve $\beta(\omega)$, both plotted against frequency ω, do not intersect with each other as illustrated in Fig. 1. Let $\varkappa$ be the smallest value of the difference

$$\varkappa = \min_\omega \left[K + \frac{\omega}{V} - \beta(\omega)\right], \tag{24}$$

and let the frequency ω_0 yield this smallest difference; ω_0 is referred to as the carrier frequency. We use the standard notation

$$\beta_m = \beta^{(m)}(\omega_0)$$

for the derivatives of $\beta(\omega)$ at $\omega = \omega_0$. The minimization problem (24) requires that $\beta_2 < 0$ and yields $V = 1/\beta_1$. That is, we define ω_0 in such a way that the soliton velocity equals the group velocity at the carrier frequency. In addition, one identifies that

$$K + \frac{\omega}{V} = \varkappa + \bar{\beta}(\omega),$$

where

$$\bar{\beta}(\omega) = \beta_0 + \beta_1(\omega - \omega_0)$$

represents a tangent line to $\beta(\omega)$ at $\omega = \omega_0$. Equation (23) is transformed to the form

$$[\varkappa + \bar{\beta}(\omega) - \beta(\omega)]\mathfrak{e}_\omega = \frac{3\chi^{(3)}\omega^2 \left(|\mathfrak{e}|^2\mathfrak{e}\right)_\omega}{4c^2[\varkappa + \bar{\beta}(\omega) + \beta(\omega)]}, \tag{25}$$

and is considered for a fixed ω_0 and different positive values of $\varkappa$.

Equation (25) has the same basic features but is much less restrictive than Eq. (19). As expected, each suitable ($\beta_2 < 0$) carrier frequency ω_0 now generates a family of solitary wave solutions parametrized by $\varkappa > 0$. The only restriction is that $\beta(\omega)$ and $\bar{\beta}(\omega)$ intersect only at $\omega = \omega_0$, i.e., the resonance radiation does not take place. In practice, it is sufficient that the possible resonant frequency ω_r considerably differs from ω_0 and therefore Cherenkov's radiation is smaller than the linear damping. This, in turn, requires a large region with the anomalous dispersion where $\beta(\omega)$ is a convex function (see Fig. 1).

The SVEA and the fundamental soliton of the NLSE are recovered from Eq. (25) for $\varkappa \to 0$. The corresponding solutions are

$$\sqrt{n_2}\mathfrak{e}(\tau) = \sqrt{\frac{c/\omega_0}{t_0^2/|\beta_2|}}\frac{e^{-i\omega_0\tau}}{\cosh(\tau/t_0)}$$

and

$$\sqrt{n_2}\mathcal{E}(z,t) = \sqrt{\frac{c/\omega_0}{t_0^2/|\beta_2|}}\frac{\exp i[(\varkappa+\beta_0)z - \omega_0 t]}{\cosh[(t-\beta_1 z)/t_0]},$$

where in accordance with Eq. (15)

$$n_2 = \frac{3\chi^{(3)}}{8n(\omega_0)} \qquad \text{and} \qquad \varkappa = \frac{|\beta_2|}{2t_0^2}.$$

As $\varkappa$ increases, the soliton becomes shorter in time. At some $\varkappa$ one has to switch to the numerical solution of Eq. (25). The soliton can be traced up to the largest $\varkappa$ at which it vanishes usually because of the resonant radiation. For idealized dispersion profiles one may expect cusp formation as predicted by known exact solutions [30, 5, 8]. The numerical approach is described in the next subsection.

4.3 Numerical solutions

Equation (25) is now written in a form which suggests an iterative solution

$$\mathfrak{e}(\tau) = N[\mathfrak{e}(\tau)], \qquad \Big(N[\mathfrak{e}]\Big)_\omega = f(\omega)\chi^{(3)}\left(|\mathfrak{e}|^2\mathfrak{e}\right)_\omega,$$
$$f(\omega) = \frac{3\omega^2/(4c^2)}{\left[\varkappa + \bar{\beta}(\omega)\right]^2 - \beta^2(\omega)},$$

where we recall that $\chi + \bar{\beta}(\omega) > \beta(\omega)$, as illustrated in Fig. 1, so that $f(\omega)$ is positively defined in the transparency window.

Iterations start with the fundamental soliton solution described in the previous section. The most simple iterative scheme, $\mathfrak{e}_{n+1} = N[\mathfrak{e}_n]$, diverges. To obtain a correct solution one can apply the so called spectral renormalization method developed for the NLSE [1]. In the course of iterations, the transition from $\mathfrak{e}_n$ to $\mathfrak{e}_{n+1}$ is performed in two steps:

$$\mathfrak{e}_{n+1/2} = N[\mathfrak{e}_n], \qquad \mathfrak{e}_{n+1} = \lambda_n \mathfrak{e}_{n+1/2}$$

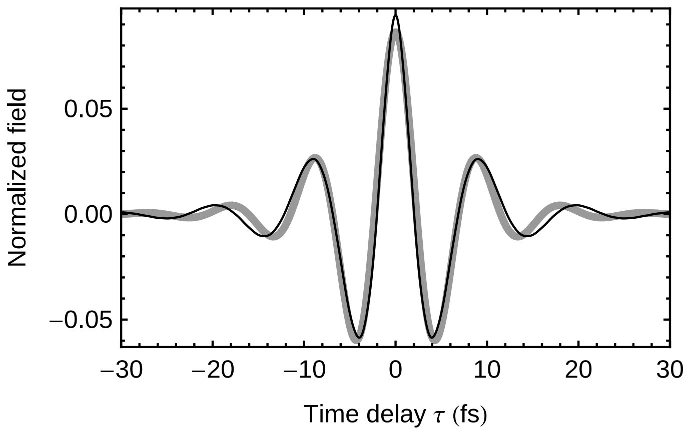

Figure 2: A nearly single-cycle solitary solution for the bulk sapphire. Electric field is normalized by $\sqrt{\chi^{(3)}}$ and is shown versus the retarded time. The carrier wave corresponds to $3\,\mu$m. Thin line: the exact solitary solution. Thick line: the best fit by a fundamental soliton.

where rescaling on the second step is performed in such a way that $\langle \mathfrak{e}_{n+1}|\mathfrak{e}_{n+1}\rangle = \langle \mathfrak{e}_n|\mathfrak{e}_n\rangle$, for a suitably defined scalar product, e.g., the time-averaged value of $|\mathfrak{e}|^2$. After rescaling, all $\mathfrak{e}_n(\tau)$ belong to a ünit sphere", which considerably improves convergence. If the iterations converge to $\mathfrak{e}_\infty$ and λ_∞, we have

$$\mathfrak{e}_\infty = \lambda_\infty N[\mathfrak{e}_\infty] \quad \Rightarrow \quad \sqrt{\lambda_\infty}\mathfrak{e}_\infty = N[\sqrt{\lambda_\infty}\mathfrak{e}_\infty],$$

just because $N[\mathfrak{e}]$ represents a cubic nonlinearity. It follows that

$$\mathfrak{e}(\tau) = \sqrt{\lambda_\infty}\mathfrak{e}_\infty(\tau)$$

solves equation $\mathfrak{e}(\tau) = N[\mathfrak{e}(\tau)]$. An illustrative single-cycle solitary solution for sapphire is shown in Fig. 2. For comparison, the best possible fit by a fundamental soliton is also shown. Surprisingly, the fundamental soliton of the NLSE remains to be a reasonable approximation to the exact solution for practically all pulse durations.

5 Conclusions

In conclusion, we demonstrated that the analytic signal representation yields a natural way to ignore generation of the third harmonic for a suitable (non-resonant) dispersion profile. By doing so the nonlinear wave equation for optical pulses in fibers is transformed to a simplified propagation equation without introducing the carrier frequency. Neither the slowly varying envelope approximation nor the unidirectional approximation are used in our analysis. The resulting Eq. (14) possesses useful similarities to the nonlinear Schrödingier equation but actually is a bidirectional nonlinear wave equation. The latter applies directly to the electric field and describes pulses with arbitrary durations, as long as one can neglect the third harmonic generation.

Using the analytic signal representation, we investigated solitary wave solutions of the nonlinear wave equation (14). These solutions directly yield the electric field for the non-envelope ultrashort solitons in the case of anomalous, but otherwise arbitrary dispersion. Moreover, each envelope soliton generates a continuous family of ultrashort solitons, the family can be effectively traced up to a few-cycle duration. These solutions are obtained numerically using a modification of the spectral renormalisation method.

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