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Elastic scattering by finitely many point-like obstacles

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Abstract

This paper is concerned with the time-harmonic elastic scattering by a finite number N of point-like obstacles in $\mathbb{R}^n$ $(n=2,3)$. We analyze the N-point interactions model in elasticity and derive the associated Green's tensor (integral kernel) in terms of the point positions and the scattering coefficients attached to them, following the approach in quantum mechanics for modeling N-particle interactions. In particular, explicit expressions are given for the scattered near and far fields corresponding to elastic plane waves or point-source incidences. As a result, we rigorously justify the Foldy method for modeling the multiple scattering by finitely many point-like obstacles for the Lamé model. The arguments are based on the Fourier analysis and the Weinstein-Aronszajn inversion formula of the resolvent for the finite rank perturbations of closed operators in Hilbert spaces.

1 Introduction

We consider the time-harmonic elastic scattering by N point-like scatterers located at $y^{(j)}, j=1,\cdots,N$ in $\mathbb{R}^n(n=2,3)$. We set $Y:=\{y^{(j)}:j=1,2,\cdots,N\}$. Physically, such point-like obstacles are related to highly concentrated inhomogeneous elastic medium with sufficiently small diameters compared to the wave-length of incidence. Define the Navier operator

$$H_\omega\, u \quad := \quad (-\Delta^*-\omega^2)\, u, \quad \Delta^*:=\mu\Delta+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} \tag{1}$$

where λ, μ are the Lamé constants of the background homogeneous medium, and $\omega>0$ denotes the angular frequency. Denote by $U^{tol}=U^I+U^S$ the sum of the incident field U^I and the scattered field U^S. The N-point interactions mathematical model we wish to analyze is the following: find the total elastic displacement U^{tol} such that

$$H_\omega(U^{tol}) \quad = \quad \sum_{j=1}^N a_j\,\delta(x-y^{(j)})\,\mathbf{I} \quad \text{in } \mathbb{R}^n\backslash Y, \tag{2}$$

$$\lim_{r\to\infty} r^{(n-1)/2}(\frac{\partial U_p}{\partial r}-ik_pU_p) \quad = \quad 0, \quad \lim_{r\to\infty} r^{(n-1)/2}(\frac{\partial U_s}{\partial r}-ik_sU_s)=0,\ r=|x|, \tag{3}$$

where the last two limits are uniform in all directions $\hat{x}:=x/r\in\mathbb{S}:=\{|\hat{x}|=1\}$. Here $k_p:=\omega/\sqrt{\lambda+2\mu}$, $k_s:=\omega/\sqrt{\mu}$ are the compressional and shear wavenumbers, and

$$U_p:=-k_p^{-2}\mathrm{grad}\,\mathrm{div}\,U^S, \quad U_s=-k_s^{-2}\mathrm{curlcurl}\,U^S$$

denote the longitudinal and transversal parts of the scattered field, respectively. The conditions in (3) are referred to as the Kupradze radiation condition in elasticity [10].

The equation (2) formally describes the elastic scattering by N obstacles with densities concentrated on the point $y^{(j)}$. This concentration is modeled by the Dirac impulses $\delta(\cdot-y^{(j)})$. In (2), the notation $\mathbf{I}$ stands for the $n \times n$ identity matrix, and $a_j \in \mathbb{C}$ is the coupling constant (scattering strength) attached to the j-th scatterer, which can be viewed as the limit of the density coefficients for approximating the idealized δ-functions in (2).

Let us describe the Foldy method, see Refs. [4, 11] for more details in the acoustic case, to solve the problem (2)-(3). Let $\Gamma_\omega(x,z)$ be the fundamental tensor of the Lamé model. Using (2) and (3), we obtain formally the following representation

$$U^{tol}(x) = U^I(x) + \sum_{j=1}^{N} a_j \, \Gamma_\omega(x, y_j) \, U^{tol}(y_j), \; x \neq y_j, \; j = 1, 2, \ldots, N. \tag{4}$$

There is no easy way to calculate the values of $U^{tol}(y_j), \; j = 1, 2, \ldots, N$, and we cannot evaluate (4). There are several approximations to handle this point. We can cite the Born, Foldy and also the intermediate levels of approximations, see Refs. [11, 3] for more details about these approximations. Here, we only discuss the Foldy method. Following this method, proposed in [4] to model the multiple interactions occuring in the acoustic scattering, see also [11] for more details, the total field $U^{tol}(x)$ has the form

$$U^{tol}(x) = U^I(x) + \sum_{j=1}^{N} a_j \, \Gamma_\omega(x, y_j) \, U_j(y_j), \tag{5}$$

where the approximating terms $U_j(y_j)$'s can be calculated from the Foldy linear algebraic system given by

$$U_j(y_j) = U^I(y_j) + \sum_{\substack{m=1 \\ m \neq j}}^{N} a_m \, \Gamma_\omega(y_j, y_m) \, U_m(y_m), \; \forall \, j = 1, \ldots, N. \tag{6}$$

This last system is invertible except for some particular distributions of the points y_j's, see Ref. [3] for a discussion about this issue. Hence the systems (5)-(6) provides us with a close form of the solution to the scattering by N-point scatterers. As it can be seen, the system (6) is obtained from (5) by taking the limits of x to the points y_m's and removing the singular part.

Our objective is to rigorously justify and give sense to this method in the framework of elastic propagation. To do it, we follow the approaches, presented in Ref. [1], known in quantum mechanics for describing the interaction of N-particles. As pointed out in Ref. [1] for quantum mechanical systems, the Dirac potentials on the right hand side of (2) cannot be regarded as an operator or quadratic form perturbation of the Laplacian operator in $\mathbb{R}^n$. This is also our main difficulty to deal with the scattering problem in elasticity. One way to solve this problem is to employ the self-adjoint extensions of symmetric operators and the Krein's inversion formula of the resolvents; see e.g. Ref. [1, Part I] and Ref. [5] for the basic mathematical framework in quantum mechanics. An alternative approach is the *renormalization techniques*, see Ref. [1, Part 2], based on introducing appropriate coupling constants which vanish in a suitable way in the process of approximation such that the resolvent of the model makes sense. Precisely, replacing the scattering coefficients a_j by parameter dependent coefficients $a_j(\epsilon), \epsilon \in \mathbb{R}_+$, decaying in a suitable way when $\epsilon \to 0$, and the Fourier transform of the delta distribution by its truncated part, up to $\frac{1}{\epsilon}$, one obtains a parameter family of self-adjoint operators, with ϵ as a parameter, in the Fourier variable. These operators are finite-rank perturbations of the multiplication operator (which is the Fourier transform of the Laplacian). Based on the Weinstein-Aronszajn inversion formula, one shows that the resolvent of this family of operators converges, as $\epsilon \to 0$, to the resolvent of a closed and self-adjoint operator. This last operator is taken to be the Fourier transform of the operator modeling the finitely many pointlike obstacles scattering problem.

The purpose of this paper is to develop the counterpart in elasticity for the model (2), following this renormalization procedure. As a result, we show that the Foldy system (5)-(6) is indeed a natural model to describe the scattering by N-point scatterers provided that we take the coefficients a_j of the form $(c_j - \kappa)^{-1}$ with c_j being real valued and

$$\kappa := \begin{cases} -\frac{1}{4\pi}\left[\frac{\lambda+3\mu}{\mu(\lambda+2\mu)}C + \frac{\lambda+\mu}{\mu(\lambda+2\mu)} - \frac{1}{2}\left(\frac{\ln\mu}{\mu} + \frac{\ln(\lambda+2\mu)}{\lambda+2\mu}\right)\right] & \text{in } \mathbb{R}^2, \\ i\omega\frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)} & \text{in } \mathbb{R}^3. \end{cases} \tag{7}$$

The constant C in (7) denotes Euler's constant. Let us finally mention that the system (5)-(6) is used in Refs. [3, 6] as a model for the detection of point-like obstacles from the longitudinal or the transversal parts of the far field pattern.

The rest of the paper is organized as following. In Section 2 we present a detailed investigation of the N point interactions in elasticity in $\mathbb{R}^2$. Section 2.1 gives the Green's tensors for the Navier and Lamé equations, in the absence of the obstacles, and the limit of their difference as the argument tends to origin. Such a limit will be used in Section 2.2 for deriving the Green's tensor (integral kernel) of the model in the presence of the obstacles. An immediate consequence of this tensor is the explicit far field pattern for plane wave incidence in terms of point positions and the associated scattering coefficients; see Section 2.2. Finally, in Section 3 we extend the main Theorem 2.6 in two-dimensions to the case of three-dimensions.

2 Elastic scattering by point-like obstacles in $\mathbb{R}^2$

Throughout the paper the notation $(\cdot)^\top$ means the transpose of a vector or a matrix, and $e_j, j = 1, 2, \cdots, N$ denote the Cartesian unit vectors in $\mathbb{R}^n$. We first review some basic properties of the fundamental solutions to the Navier and Lamé equations in $\mathbb{R}^2$.

2.1 Fundamental solutions

We begin with the Green's tensor for the operator H_ω, given by

$$\Gamma_\omega(x,y) := \frac{i}{4\mu}H_0^{(1)}(k_s|x-y|)\,\mathbf{I} + \frac{i}{4\omega^2}\mathrm{grad}_x\mathrm{grad}_x^\top[H_0^{(1)}(k_s|x-y|) - H_0^{(1)}(k_p|x-y|)] \tag{8}$$

for $x, y \in \mathbb{R}^2, x \neq y$, where $H_0^{(1)}(t)$ denotes the Hankel function of the first kind and of order zero. For $u = (u_1, u_2)^\top$ and $\omega = 0$, we have the Lamé operator

$$H_0\,u := -\Delta^* u = -\begin{pmatrix} (\lambda+2\mu)\partial_1^2 u_1 + \mu\partial_2^2 u_1 + (\lambda+\mu)\partial_1\partial_2 u_2 \\ (\mu\partial_1^2 u_2 + (\lambda+2\mu)\partial_2^2 u_2 + (\lambda+\mu)\partial_1\partial_2 u_1 \end{pmatrix}, \quad \partial_j := \partial x_j,\ j = 1, 2.$$

Define the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \frac{1}{2\pi}\lim_{R\to\infty}\int_{|x|\le R} f(x)e^{-ix\cdot\xi}dx, \quad \xi = (\xi_1, \xi_2)^\top.$$

Its inverse transform is given by

$$(\mathcal{F}^{-1}g)(x) := \frac{1}{2\pi}\lim_{R\to\infty}\int_{|\xi|\le R} g(\xi)e^{ix\cdot\xi}d\xi.$$

With simple calculations, we obtain

$$(\mathcal{F}H_0)u = \begin{pmatrix}(\lambda+2\mu)\xi_1^2+\mu\xi_2^2 & (\lambda+\mu)\xi_1\xi_2\\ (\lambda+\mu)\xi_1\xi_2 & \mu\xi_1^2+(\lambda+2\mu)\xi_2^2\end{pmatrix}\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix} =: M_0(\xi)\hat{u}$$

where $\hat{u} := \mathcal{F}u$. Moreover, we have the non-vanishing determinant of M_0:

$$\det(M_0) = (\lambda+2\mu)\,\mu\,|\xi|^4 \neq 0, \quad \text{if} \quad |\xi| \neq 0,$$

implying that M_0 is invertible, with its inverse M_0^{-1} given by

$$\begin{aligned} M_0^{-1}(\xi) &= \frac{1}{(\lambda+2\mu)\,\mu\,|\xi|^4}\begin{pmatrix}\mu\xi_1^2+(\lambda+2\mu)\xi_2^2 & -(\lambda+\mu)\xi_1\xi_2\\ -(\lambda+\mu)\xi_1\xi_2 & (\lambda+2\mu)\xi_1^2+\mu\xi_2^2\end{pmatrix}\\ &= \frac{1}{\mu|\xi|^2}\mathbf{I} - \frac{\lambda+\mu}{\mu(\lambda+2\mu)|\xi|^4}\Xi(\xi) \end{aligned} \tag{9}$$

for $|\xi| \neq 0$, where

$$\Xi(\xi) := \xi^\top\xi = \begin{pmatrix}\xi_1^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2^2\end{pmatrix} \quad \text{for} \quad \xi = (\xi_1,\xi_2)\in\mathbb{R}^2.$$

Setting $M_\omega = M_0 - \omega^2\,\mathbf{I}$, we then analogously have

$$M_\omega^{-1}(\xi) = \frac{1}{\mu|\xi|^2-\omega^2}\,\mathbf{I} - \frac{\lambda+\mu}{(\mu|\xi|^2-\omega^2)\,[(\lambda+2\mu)|\xi|^2-\omega^2]}\Xi(\xi). \tag{10}$$

Let $\Gamma_0(x,y) = \Gamma_0(|x-y|)$ be the Green's tensor to the operator H_0, i.e., the Kelvin matrix of fundamental solutions to the Lamé system, given by (see Ref. [7, Chapter 2.2])

$$\Gamma_0(x,0) = \frac{1}{4\pi}\left[-\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\ln|x|\,\mathbf{I} + \frac{\mu+\lambda}{\mu(2\mu+\lambda)\,|x|^2}\Xi(x)\right]. \tag{11}$$

Then, there holds

$$\frac{1}{2\pi}(\mathcal{F}^{-1}M_0^{-1})(x) = \Gamma_0(x,0), \quad \frac{1}{2\pi}(\mathcal{F}^{-1}M_\omega^{-1})(x) = \Gamma_\omega(x,0), \quad |x|\neq 0,\ x\in\mathbb{R}^2.$$

The following lemma gives the entries of the matrix $\Gamma_\omega - \Gamma_0$ taking the value at the origin.

Lemma 2.1. *There holds the limit*

$$\lim_{|x|\to 0}[\Gamma_\omega(x,0) - \Gamma_0(x,0)] = \eta\,\boldsymbol{I},$$

where

$$\eta := -\frac{1}{4\pi}\left[\frac{\lambda+3\mu}{\mu(\lambda+2\mu)}(\ln\frac{\omega}{2} + C - \frac{i\pi}{2}) + \frac{\lambda+\mu}{\mu(\lambda+2\mu)} - \frac{1}{2}(\frac{\ln\mu}{\mu} + \frac{\ln(\lambda+2\mu)}{\lambda+2\mu})\right], \tag{12}$$

with $C = 0.57721\cdots$ being Euler's constant.

Proof. Recall Ref. [9] that Γ_ω can be decomposed into

$$\Gamma_\omega(x,0) = \frac{1}{\pi}\ln(|x|)\ \tilde{\Gamma}_1(x) + \tilde{\Gamma}_2(x), \tag{13}$$

with the matrices $\tilde{\Gamma}_j$ taking the form

$$\tilde{\Gamma}_1(x) := \tilde{\Psi}_1(|x|)\ \mathbf{I} + \tilde{\Psi}_2(|x|)\frac{1}{|x|^2}\Xi(x),\ \tilde{\Gamma}_2(x) := \chi_1(|x|)\ \mathbf{I} + \chi_2(|x|)\frac{1}{|x|^2}\Xi(x), \tag{14}$$

where $\chi_j(\tau)$ $(j = 1, 2)$ are C^∞ functions on $\mathbb{R}^+$ and

$$\begin{aligned} \tilde{\Psi}_1(\tau) &= -\frac{1}{2\mu}J_0(k_s\tau) + \frac{1}{2\omega^2\tau}\left[k_sJ_1(k_s\tau) - k_pJ_1(k_p\tau)\right], \\ \tilde{\Psi}_2(\tau) &= \frac{1}{2\omega^2}\left[k_s^2J_0(k_s\tau) - \frac{2k_s}{\tau}J_1(k_s\tau) - k_p^2J_0(k_p\tau) + \frac{2k_p}{\tau}J_1(k_p\tau)\right]. \end{aligned}$$

Here J_n denotes the Bessel function of order n. Moreover, making use of the asymptotic behavior

$$J_0(t) = 1 - \frac{1}{4}t^2 + \frac{1}{64}t^4 + \mathcal{O}(t^6), \quad J_1(t) = \frac{1}{2}t - \frac{1}{16}t^3 + \mathcal{O}(t^5), \quad t \to 0^+,$$

we get (see also Ref. [9])

$$\begin{aligned} &\tilde{\Psi}_1(\tau) = -\eta_1 + \eta_2\ \tau^2 + \mathcal{O}(\tau^4), \quad \tilde{\Psi}_2(\tau) = \eta_3\ \tau^2 + \mathcal{O}(\tau^4), \\ &\chi_1(\tau) = \eta + \mathcal{O}(\tau^2), \quad \chi_2(\tau) = \eta_4\frac{1}{\pi} + \mathcal{O}(\tau^2) \end{aligned} \tag{15}$$

as $\tau \to 0$, where

$$\begin{aligned} \eta &:= -\tfrac{1}{4\pi\omega^2}\left[k_s^2\ln\tfrac{k_s}{2} + k_p^2\ln\tfrac{k_p}{2} + \tfrac{k_s^2-k_p^2}{2} + (C - \tfrac{i\pi}{2})(k_s^2+k_p^2)\right], \\ \eta_1 &:= \tfrac{1}{4\omega^2}(k_s^2+k_p^2),\ \eta_2 := \tfrac{1}{32\omega^2}(3k_s^4+k_p^4),\ \eta_3 := \tfrac{1}{16\omega^2}(k_p^4-k_s^4),\ \eta_4 := \tfrac{k_s^2-k_p^2}{4\omega^2}, \end{aligned}$$

with Euler's constant $C = 0.57721\cdots$. Note that the coefficients η, η_1, η_4 can be respectively rewritten as (12) and

$$\eta_1 = \frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}, \quad \eta_4 = \frac{\lambda+\mu}{4\mu(\lambda+2\mu)}$$

in terms of the Lamé constants λ and μ. Insertion of (14) and (15) into (13) yields the asymptotic behavior

$$\Gamma_\omega(x,0) = \frac{1}{\pi}\ln|x|\left[-\eta_1 + o(1)\right]\mathbf{I} + \eta\,\mathbf{I} + \frac{\eta_4}{\pi\,|x|^2}\Xi(x) + o(1) \quad \text{as } |x| \to 0,$$

which together with (11) proves Lemma 2.1. □

2.2 Solvability of elastic scattering by N point-like obstacles

Consider a new operator

$$H\,u = H_0u - \sum_{j=1}^N a_j\,\delta(x - y^{(j)})\,\mathbf{I}, \quad y^{(j)} = (y_1^{(j)}, y_2^{(j)})^\top \in \mathbb{R}^2.$$

The objective of this section is to give a mathematically rigorous meaning of this operator and describe the scattered field corresponding to incident plane waves or point-sources. As mentioned in the introduction, our arguments are in the lines of the approach known in quantum mechanics for describing the point interactions of N particles; see Ref. [1].

To start, we set

$$\widetilde{H} := \mathcal{F}H\mathcal{F}^{-1} = \mathcal{F}H_0\mathcal{F}^{-1} - \sum_{j=1}^{N} a_j\, \mathcal{F}[\delta(x-y^{(j)})\,\mathbf{I}]\mathcal{F}^{-1}.$$

For $f=(f_1,f_2)^\top \in L^2(\mathbb{R}^2)^2$, we have $(\mathcal{F}H_0\mathcal{F}^{-1})\,\hat{f} = (\mathcal{F}H_0)f = M_0\hat{f}$, and formally

$$\begin{aligned}
(\mathcal{F}\delta(x-y^{(j)})\,\mathcal{F}^{-1}\,\hat{f})(\xi) &= (\mathcal{F}\delta(x-y^{(j)})f)(\xi)\\
&= (2\pi)^{-1} f(y^{(j)}) e^{-iy^{(j)}\cdot\xi}\\
&= (2\pi)^{-1} e^{-iy^{(j)}\cdot\xi}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{f}(\xi)e^{iy^{(j)}\cdot\xi}d\xi\right)\\
&= \left\langle \hat{f},\varphi^1_{y^{(j)}}\right\rangle \varphi^1_{y^{(j)}}(\xi) + \left\langle \hat{f},\varphi^2_{y^{(j)}}\right\rangle \varphi^2_{y^{(j)}}(\xi),
\end{aligned}$$

where $\varphi^i_{y^{(j)}}(\xi) := \phi_{y^{(j)}}(\xi)\,(e_i)^\top$ for $i=1,2$, with $\phi_{y^{(j)}}(\xi) := (2\pi)^{-1}e^{-iy^{(j)}\cdot\xi}$. Here we used the inner product

$$\langle f,g\rangle := \int_{\mathbb{R}^2} f(\xi)\cdot\overline{g(\xi)}\,d\xi, \quad \text{for } f,g\in L^2(\mathbb{R}^2)^2.$$

Therefore, formally we have

$$\widetilde{H}f = (\mathcal{F}H_0\mathcal{F}^{-1})\,f = M_0(\xi)\,f - \sum_{j=1}^{N}\left\{\left\langle a_j\,f,\varphi^1_{y^{(j)}}\right\rangle \varphi^1_{y^{(j)}}(\xi) + \left\langle a_j\,f,\varphi^2_{y^{(j)}}\right\rangle \varphi^2_{y^{(j)}}(\xi)\right\}.$$

Our aim is to prove the existence of the resolvent of $\widetilde{H}$ and to deduce an explicit expression of its Green's tensor. To make the computations rigorous, we introduce the cut-off function

$$\chi_\epsilon(\xi) = \begin{cases} 1, & \text{if } \ \epsilon\le|\xi|\le 1/\epsilon,\\ 0, & \text{if } \ |\xi|<\epsilon \ \text{ or } \ |\xi|>1/\epsilon,\end{cases} \qquad \text{for some } 0<\epsilon<1,$$

and define the operator

$$\widetilde{H}^\epsilon f := M_0(\xi)\,f - \sum_{j=1}^{N}\sum_{i=1}^{2}\left\langle a_j(\epsilon)\,f,\varphi^{\epsilon,i}_{y^{(j)}}\right\rangle \varphi^{\epsilon,i}_{y^{(j)}}(\xi), \quad \varphi^{\epsilon,i}_{y^{(j)}}(\xi) := \chi_\epsilon(\xi)\,\varphi^i_{y^{(j)}}(\xi). \tag{16}$$

We will choose the coupling constants $a_j(\epsilon)$ in a suitable way such that the resolvent of $\widetilde{H}^\epsilon$ has a reasonable limit as ϵ tends to zero. Let us first recall the Weinstein-Aronszajn determinant formula from Ref. [1, Lemma B.5], which is our main tool for analyzing the resolvent of $\widetilde{H}^\epsilon$.

Lemma 2.2. *Let $\mathcal{H}$ be a (complex) separable Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$. Let A be a closed operator in $\mathcal{H}$ and $\Phi_j,\ \Psi_j\in\mathcal{H}$, $j=1,...,m$. Then*

$$\Big(A+\sum_{j=1}^{m}\langle\cdot,\Phi_j\rangle\Psi_j - z\Big)^{-1} = \big(A-z\big)^{-1} - \sum_{j=1}^{m}\big[\Pi(z)\big]^{-1}_{j,j'}\big\langle\cdot,\ [\big(A-z\big)^{-1}]^*\Phi_{j'}\big\rangle\big(A-z\big)^{-1}\Psi_j \tag{17}$$

for z in the resolvent set of A such that $\det\big[\Pi(z)\big] \neq 0$, with the entries of $\Pi(z)$ given by

$$\big[\Pi(z)\big]_{j,j'} := \delta_{j,j'} + \big\langle \big(A-z\big)^{-1}\Psi_{j'}, \Phi_j\big\rangle. \tag{18}$$

Note that in Lemma 2.2, the notation $\big[\Pi(z)\big]^{-1}_{j,j'}$ denotes the (j,j')-th entry of the matrix $\big[\Pi(z)\big]^{-1}$, and $[\,]^*$ stands for the adjoint operator of $[\,]$. To apply Lemma 2.2, we take $\mathcal{H} := L^2(\mathbb{R}^2)^2$, $A := M_0$, $m := 2N$ and $\Phi_j := \Phi_j^\epsilon$, $\Psi_j = -\tilde{a}_j\Phi_j^\epsilon$ for $j = 1, \cdots, 2N$, with $\tilde{a}_j$ and Φ_j^ϵ defined as follows:

$$\tilde{a}_j(\epsilon) = a_l(\epsilon) \quad \text{if} \quad j \in \{2l-1, 2l\}, \quad \Phi_j^\epsilon := \begin{cases} \varphi^{\epsilon,1}_{y^{(l)}} & \text{if } j = 2l-1, \\ \varphi^{\epsilon,2}_{y^{(l)}} & \text{if } j = 2l, \end{cases}$$

for some $l \in \{1, 2, \cdots, N\}$. The multiplication operator A is closed with a dense domain

$$\mathcal{D}(A) := \big\{\hat{u} \in L^2(\mathbb{R}^2)^2, \quad M_0\hat{u} \in L^2(\mathbb{R}^2)^2\big\}$$

in $L^2(\mathbb{R}^2)^2$ hence $\widetilde{H}^\epsilon$, $\epsilon > 0$, is also closed with the same domain. Moreover, we set $z := \omega^2$ for $\omega \in \mathbb{C}$ such that $\operatorname{Im}\omega > 0$. For such complex-valued number ω, one can observe that $\det(M_0 - \omega^2\mathbf{I}) \neq 0$ so that $(M_0 - \omega^2\mathbf{I})^{-1}$ always exists. Further, it holds that

$$[(M_0 - \omega^2\mathbf{I})^{-1}]^* = [(M_0 - \omega^2\mathbf{I})^{-1}]^T = (M_0 - \overline{\omega}^2\mathbf{I})^{-1},$$

where $[\,]^T$ denotes the conjugate transpose of a matrix, and $\overline{\omega}$ denotes the conjugate of ω. Simple calculations show that

$$\big(A-z\big)^{-1}\Psi_j = -\tilde{a}_j(M_0 - \omega^2\mathbf{I})^{-1}\Phi_j^\epsilon$$

and

$$\delta_{j,j'} + \big\langle\big(A-z\big)^{-1}\Psi_{j'}, \Phi_j\big\rangle = \tilde{a}_j\,[\tilde{a}_j^{-1}\,\delta_{j,j'} - \big\langle(M_0 - \omega^2\mathbf{I})^{-1}\Phi^\epsilon_{j'}, \Phi^\epsilon_{j'}\big\rangle].$$

Therefore, by Lemma 2.2 we arrive at an explicit expression of the inverse of $\widetilde{H}^\epsilon - \omega^2$, given by

$$(\widetilde{H}^\epsilon - \omega^2)^{-1}f = (M_0 - \omega^2\mathbf{I})^{-1}f + \sum_{j,j'=1}^{2N}[\Pi_\epsilon(\omega)]^{-1}_{j,j'}\left\langle f, \chi_\epsilon\, F^{(j')}_{-\overline{\omega}}\right\rangle \chi_\epsilon\, F^{(j)}_\omega, \quad \operatorname{Im}\omega > 0, \tag{19}$$

with

$$\Pi_\epsilon(\omega) := \big[\tilde{a}_j^{-1}\delta_{j,j'} - \big\langle(M_0 - \omega^2\mathbf{I})^{-1}\Phi^\epsilon_{j'},\ \Phi^\epsilon_j\big\rangle\big]^N_{j,j'=1}, \quad \chi_\epsilon F^{(j)}_\omega := (M_0 - \omega^2\mathbf{I})^{-1}\Phi^\epsilon_j, \tag{20}$$

provided that $\operatorname{Im}\omega > 0$ and $\det[\Pi_\epsilon(\omega)] \neq 0$.

In order to obtain $(\widetilde{H} - \omega^2)^{-1}$, we need to remove the cut-off function in (19) by evaluating the limits of $\Pi_\epsilon(\omega)$ and $\left\langle f, \chi_\epsilon F^{(j')}_{-\overline{\omega}}\right\rangle \chi_\epsilon F^{(j')}_\omega$ as $\epsilon \to 0$. This will be done in the subsequent Lemmas 2.3 and 2.5.

Lemma 2.3. *The coefficients $\tilde{a}_j(\epsilon)$ can be chosen in such a way that the limit $\Pi_{B,Y}(\omega) := \lim_{\epsilon\to 0}\Pi_\epsilon(\omega)$ exists and takes the form*

$$\Pi_{B,Y}(\omega) = \begin{pmatrix} (b_1-\eta)\mathbf{I} & -\Gamma_\omega(y^{(1)}-y^{(2)}) & \cdots & -\Gamma_\omega(y^{(1)}-y^{(N)}) \\ -\Gamma_\omega(y^{(2)}-y^{(1)}) & (b_2-\eta)\mathbf{I} & \cdots & -\Gamma_\omega(y^{(2)}-y^{(N)}) \\ \vdots & \vdots & \vdots & \vdots \\ -\Gamma_\omega(y^{(N)}-y^{(1)}) & -\Gamma_\omega(y^{(N)}-y^{(2)}) & \dots & (b_N-\eta)\mathbf{I} \end{pmatrix}, \tag{21}$$

where η is given in (12) and $B := (b_1, \cdots, b_N)$ is an arbitrary vector in $\mathbb{C}^{1\times N}$. If in addition we choose

$$b_l := c_l - \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}(\ln\frac{\omega}{2} - \frac{i\pi}{2}), \quad l = 1, 2, \cdots, N, \tag{22}$$

with $c_l \in \mathbb{R}$ arbitrary, then we have

$$(\Pi_{B,Y}(\omega))^* = \Pi_{B,Y}(-\bar{\omega}). \tag{23}$$

Remark 2.4. *In this paper, the number η is referred to as the normalizing constant and $b_l \in \mathbb{C}$ is viewed as the scattering coefficient attached to the l-th scatterer. The coefficient b_l characterizes the scattering density concentrated at $y^{(l)}$. The relation between the scattering coefficient b_l and the scattering strength a_l will be given in Remark 2.7.*

Proof. The proof will be carried out in the following three cases of $j, j' \in \{1, \cdots, 2N\}$.

Case 1: $|j' - j| = 1$, and $j, j' \in \{2l-1, 2l\}$ for some $l \in \{1, \cdots, N\}$.

We have $j' - j = 1$ if j is an odd number, and $j - j' = 1$ if j is an even number. Assume firstly that $j = 2l-1, j' = 2l$ for some $l = 1, \cdots, N$. Then, we have $\Phi_j^\epsilon = \chi_\epsilon \varphi_{y^{(j)}}(1,0)^\top$, $\Phi_{j'}^\epsilon = \chi_\epsilon \varphi_{y^{(j)}}(0,1)^\top$. Hence

$$\left\langle (M_0 - \omega^2 \mathbf{I})^{-1}\Phi_j^\epsilon, \Phi_{j'}^\epsilon \right\rangle = \left\langle (M_0 - \omega^2 \mathbf{I})^{-1}\chi_\epsilon(1,0)^\top,\ \chi_\epsilon(0,1)^\top \right\rangle$$

since $\varphi_{y^{(l)}}(\xi)\ \overline{\varphi_{y^{(l)}}(\xi)} = 1$. Consequently, it holds that

$$\left\langle (M_0 - \omega^2 \mathbf{I})^{-1}\Phi_j^\epsilon, \Phi_{j'}^\epsilon \right\rangle = \int_{\epsilon<|\xi|<1/\epsilon} (M_0 - \omega^2 \mathbf{I})^{-1}(1,0)^\top \cdot (0,1)^\top d\xi = 0$$

because the scalar function $(M_0 - \omega^2 \mathbf{I})(\xi)^{-1}(1,0)^\top \cdot (0,1)^\top$ is odd in both ξ_1 and ξ_2; see (10). By symmetry, we have also $\left\langle (M_0 - \omega^2 \mathbf{I})^{-1}\Phi_{j'}^\epsilon, \Phi_j^\epsilon \right\rangle = 0$.

Case 2: $j = j' \in \{2l-1, 2l\}$ for some $l \in \{1, 2, \cdots, N\}$.

In this case, we set

$$\tilde{a}_j^{-1}(\epsilon) := \frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon} M_0(\xi)^{-1}\Phi_j^\epsilon \cdot \overline{\Phi_j^\epsilon}\, d\xi + b_l,\ b_l \in \mathbb{C}.$$

Hence, if $j = 2l-1$ is an odd number, then by (9) we have

$$\begin{aligned}
\tilde{a}_j^{-1}(\epsilon) &= \frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon} M_0(\xi)^{-1}(1,0)^\top \cdot (1,0)^\top d\xi + b_l \\
&= \frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon} \frac{\mu\xi_1^2 + (\lambda+2\mu)\xi_2^2}{\mu(\lambda+2\mu)|\xi|^4} d\xi + b_l \\
&= -\frac{(\lambda+3\mu)}{2\pi(\lambda+2\mu)\mu}\ln\epsilon + b_l. \qquad (24)
\end{aligned}$$

Moreover, by the choice of $\tilde{a}_j(\epsilon)$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left[\tilde{a}_j^{-1}(\epsilon) - \left\langle (M_0 - \omega^2 \mathbf{I})^{-1}\Phi_j^\epsilon,\ \Phi_j^\epsilon \right\rangle\right] \\
=\ &\lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon}\left[M_0(\xi)^{-1} - (M_0(\xi) - \omega\,\mathbf{I})^{-1}\right](1,0)^\top \cdot (1,0)^\top d\xi + b_l \\
=\ &\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left[M_0(\xi)^{-1} - (M_0(\xi) - \omega\,\mathbf{I})^{-1}\right](1,0)^\top \cdot (1,0)^\top d\xi + b_l \qquad (25)
\end{aligned}$$

From the definition of the inverse Fourier transformation, we have

$$\begin{aligned}\lim_{|x|\to 0}\left[\,\Gamma_0(x,0)-\Gamma_\omega(x,0)\right] &= \frac{1}{4\pi^2}\lim_{|x|\to 0}\int_{\mathbb{R}^2}\left[M_0(\xi)^{-1}-(M_0(\xi)-\omega\,\mathbf{I})^{-1}\right]e^{i\xi\cdot x}d\xi\\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left[M_0(\xi)^{-1}-(M_0(\xi)-\omega\,\mathbf{I})^{-1}\right]d\xi,\end{aligned}$$

where the last step follows from the uniform convergence

$$\left[M_0(\xi)^{-1}-(M_0(\xi)-\omega\,\mathbf{I})^{-1}\right]_{m,n}(1-e^{i\xi\cdot x})\to 0,\quad \text{as } |x|\to 0,\ m,n=1,2,$$

in $\xi\in\mathbb{R}^2$, which can be easily proved using the expressions of $M_0(\xi)^{-1}$ and $(M_0(\xi)-\omega\,\mathbf{I})^{-1}$ given in (9) and (10). Therefore, the first term on the right hand side of (25) is just the $(1,1)$-th entry of the matrix $\Gamma_0(x,0)-\Gamma_\omega(x,0)$ taking the value at $|x|=0$. Recalling Lemma 2.1, we obtain

$$\lim_{\epsilon\to 0}\left[\tilde{a}_j^{-1}(\epsilon)-\left\langle (M_0-\omega^2\mathbf{I})^{-1}\Phi_j^\epsilon,\ \Phi_j^\epsilon\right\rangle\right]=-\eta+b_l,$$

where η is given in (12).

Analogously, if $j=2l$ for some $l=1,\cdots,N$, then $\tilde{a}_j^{-1}$ takes the same form as in (24) and

$$\begin{aligned}&\lim_{\epsilon\to 0}\left[\tilde{a}_j^{-1}(\epsilon)-\left\langle (M_0-\omega^2\mathbf{I})^{-1}\Phi_j^\epsilon,\ \Phi_j^\epsilon\right\rangle\right]\\ =\ &\lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon}\left[M_0(\xi)^{-1}-(M_0(\xi)-\omega^2\,\mathbf{I})\right]^{-1}d\xi\ (0,1)^\top\cdot(0,1)^\top+b_l\\ =\ &\left[\Gamma_0(x,0)-\Gamma_\omega(x,0)\right]_{|x|=0}\ (0,1)^\top\cdot(0,1)^\top+b_l\\ =\ &-\eta+b_l.\end{aligned}$$

To sum up Cases 1 and 2, we deduce that the 2×2 diagonal blocks of the matrix $\Pi_{B,Y}:=\lim_{\epsilon\to 0}\Pi_\epsilon(\omega)$ are given by the 2×2 matrices

$$\begin{pmatrix}-\eta+b_l & 0\\ 0 & -\eta+b_l\end{pmatrix},\quad l=1,2,\cdots,N.$$

Case 3: $j\in\{2l-1,2l\}$, $j'\in\{2l'-1,2l'\}$ for some $l,l'\in\{1,\cdots,N\}$ such that $|l-l'|\geq 1$, i.e. the element $[\Pi_{B,Y}]_{j,j'}$ lies in the off diagonal-by-2×2-blocks of $\Pi_{B,Y}$.

Without loss of generality we assume $j=2l-1, j'=2l'-1$. Then,

$$\begin{aligned}\Phi_j^\epsilon=\chi_\epsilon\varphi^1_{y^{(l)}}=\chi_\epsilon(1,0)^\top\phi_{y^{(l)}},\quad \Phi_{j+1}^\epsilon=\chi_\epsilon\varphi^2_{y^{(l)}}=\chi_\epsilon(0,1)^\top\phi_{y^{(l)}},\\ \Phi_{j'}^\epsilon=\chi_\epsilon\varphi^1_{y^{(l')}}=\chi_\epsilon(1,0)^\top\phi_{y^{(l')}},\quad \Phi_{j'+1}^\epsilon=\chi_\epsilon\varphi^2_{y^{(l')}}=\chi_\epsilon(0,1)^\top\phi_{y^{(l')}},\end{aligned}$$

Define the 2×2 matrix $\Upsilon_l:=(\Phi_j^\epsilon,\Phi_{j+1}^\epsilon)=\chi_\epsilon\phi_{y^{(l)}}\,\mathbf{I}$. A short computation shows

$$\begin{aligned}&\left\langle (M_0-\omega^2\mathbf{I})^{-1}(\xi)\Upsilon_l(\xi),\ \Upsilon_{l'}(\xi)\right\rangle\\ =\ &\int_{\epsilon<|\xi|<1/\epsilon}(M_0-\omega^2\mathbf{I})^{-1}(\xi)\,\phi_{y^{(l)}}(\xi)\,\overline{\phi_{y^{(l')}}(\xi)}\,d\xi\\ =\ &\frac{1}{4\pi^2}\int_{\epsilon<|\xi|<1/\epsilon}(M_0-\omega^2\mathbf{I})^{-1}(\xi)\exp[i(y^{(l')}-y^{(l)})\cdot\xi]\,d\xi\\ \to\ &\left[\Gamma_\omega(y^{(l')}-y^{(l)})\right]\quad\text{as } \epsilon\to 0,\end{aligned}$$

where the last step follows from the inverse Fourier transformation.

Finally, combining Cases 1-3 gives the matrix (21).

In addition, if we choose the vector B in the form (22) with $c_l \in \mathbb{R}$, $l = 1, ..., N$, then from the explicit forms of $\Pi_{B,Y}(\omega)$ in (21) and η in (2.1), we obtain $(\Pi_{B,Y}(\omega))^* = \Pi_{B,Y}(-\overline{\omega})$. $\square$

We next prove the convergence of the operator $K^{\epsilon}_{j,j'}:\ L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ defined by

$$K^{\epsilon}_{j,j'}(f) := \left\langle f,\ \chi_\epsilon F^{(j')}_{-\overline{\omega}} \right\rangle \chi_\epsilon F^{(j)}_{\omega}, \quad f \in L^2(\mathbb{R}^2)^2.$$

To be consistent with the definitions of Φ^{ϵ}_j and $\chi_\epsilon F^{(j)}_\omega$, we introduce the functions

$$\Phi_j(\xi) := \begin{cases} (2\pi)^{-1} e^{-i\xi\cdot y^{(l)}} (1,0)^\top & \text{if} \quad j = 2l-1, \\ (2\pi)^{-1} e^{-i\xi\cdot y^{(l)}} (0,1)^\top & \text{if} \quad j = 2l, \end{cases}, \ F^{(j)}_\omega := (M_0 - \omega^2 \mathbf{I})^{-1} \Phi_j(\xi), \tag{26}$$

With these notations, we define the matrix

$$\Theta_{l,\omega}(\xi) := (F^{(2l-1)}_\omega, F^{(2l)}_\omega) = (M_0 - \omega^2 \mathbf{I})^{-1} \exp(-i\xi \cdot y^{(l)})(2\pi)^{-1} \in \mathbb{C}^{2\times 2},\ \operatorname{Im}\omega > 0 \tag{27}$$

for $l = 1, ..., N$.

Lemma 2.5. *Suppose that* $\operatorname{Im} w > 0$. *Then the operator* $K^{\epsilon}_{j,j'}$ *converges to* $K_{j,j'}$ *in the operator norm, where the operator* $K_{j,j'} : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ *is defined by*

$$K_{j,j'}(f) := \left\langle f,\ F^{(j')}_{-\overline{\omega}} \right\rangle F^{(j)}_\omega.$$

Proof. It is easy to see

$$(\mathcal{F}^{-1}\Theta_{l,\omega})(x) \ = \ \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (M_0 - \omega^2 \mathbf{I})^{-1}(\xi) \exp(i\xi \cdot (x - y^{(l)}))\, d\xi = \Gamma_\omega(x - y^{(l)}). \tag{28}$$

By the definition of Γ_ω and the asymptotic behavior of Hankel functions for a large complex argument, it follows that both $F^{(j)}_\omega$ and $F^{(j)}_{-\overline{\omega}}$ belong to $L^2(\mathbb{R}^2)^2$ for every j provided $\operatorname{Im}(\omega) > 0$. Then obviously $\chi_\epsilon F^{(j')}_{-\overline{\omega}} - F^{(j')}_{-\overline{\omega}}$ and $\chi_\epsilon F^{(j)}_{-\overline{\omega}} - F^{(j)}_{-\overline{\omega}}$ tend to zero in $L^2(\mathbb{R}^2)^2$ when ϵ tends to 0. We write

$$\begin{aligned} & \left\langle f,\ \chi_\epsilon F^{(j')}_{-\overline{\omega}} \right\rangle \chi_\epsilon F^{(j)}_\omega - \left\langle f, F^{(j')}_{-\overline{\omega}} \right\rangle F^{(j)}_\omega \\ = \ & \left\langle f,\ \left(\chi_\epsilon F^{(j')}_{-\overline{\omega}} - F^{(j')}_{-\overline{\omega}}\right) \right\rangle \left(\chi_\epsilon F^{(j)}_\omega - F^{(j)}_\omega\right) - \left\langle f,\ F^{(j')}_{-\overline{\omega}} \right\rangle \left(F^{(j)}_\omega - \chi_\epsilon F^{(j)}_\omega\right) \\ & + \left\langle f,\ \left(\chi_\epsilon F^{(j')}_{-\overline{\omega}} - F^{(j')}_{-\overline{\omega}}\right) \right\rangle F^{(j)}_\omega, \end{aligned}$$

which combined with the Cauchy-Schwartz inequality implies the convergence

$$\sup_{f\in L^2(\mathbb{R}^2)^2} \frac{\| \left\langle f,\ \chi_\epsilon F^{(j')}_{-\overline{\omega}} \right\rangle \chi_\epsilon F^{(j)}_\omega - \left\langle f, F^{(j')}_{-\overline{\omega}} \right\rangle F^{(j)}_\omega \|_{L^2(\mathbb{R}^2)^2}}{\|f\|_{L^2(\mathbb{R}^2)^2}} \to 0, \quad \epsilon \to 0.$$

This proves the convergence $||K_{j,j'} - K^{\epsilon}_{j,j'}||_{L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2} \to 0$ as $\epsilon \to 0$. $\square$

Combining Lemmas 2.3 and 2.5, we obtain the convergence in the operator norm of $(\widetilde{H}^\epsilon - \omega^2)^{-1}$ to

$$L(\omega)\hat{f} := (M_0 - \omega^2 \mathbf{I})^{-1}\hat{f} + \sum_{j,j'=1}^{2N} \Big[\Pi_{B,Y}(\omega)\Big]^{-1}_{j,j'} \left\langle \hat{f}\,, F^{(j')}_{-\overline{\omega}} \right\rangle F^{(j)}_{\omega}, \quad \forall\, \hat{f} \in L^2(\mathbb{R}^2)^2, \tag{29}$$

for all $\operatorname{Im} w > 0$ such that $\det[\Pi_{B,Y}(\omega)] \neq 0$. Recall again that $[\Pi_{B,Y}(\omega)]^{-1}_{j,j'}$ stands for the (j,j')-th entry of the matrix $[\Pi_{B,Y}(\omega)]^{-1}$. The main theorem of this paper is stated as the following.

Theorem 2.6. *Suppose that the operator $\widetilde{H}^\epsilon$ is given by (16), with*

$$\tilde{a}_j(\epsilon) = \left[-\frac{(\lambda+3\mu)}{2\pi(\lambda+2\mu)\mu}\ln\epsilon + b_j\right]^{-1}, \quad b_j \in \mathbb{C},\ j \in \{1,2,\cdots,N\}.$$

Write $B = (b_1,\cdots,b_N)$ satisfying the condition (22), and let $\Pi_{B,Y}, F^{(j)}_\omega$ be defined by (21), (26) respectively. Then

(i) *The operator $\widetilde{H}^\epsilon$ converges in norm resolvent sense to a closed and self-adjoint operator $\hat{\Delta}_{B,Y}$ as $\epsilon \to 0$, where the resolvent of $\hat{\Delta}_{B,Y}$ is given by (29). That is, for $\operatorname{Im}\omega > 0$ such that $\det[\Pi_{B,Y}(\omega)] \neq 0$,*

$$(\hat{\Delta}_{B,Y} - \omega^2)^{-1} = (M_0 - \omega^2 \boldsymbol{I})^{-1} + \sum_{j,j'=1}^{2N} \Big[\Pi_{B,Y}(\omega)\Big]^{-1}_{j,j'} \left\langle \cdot\,, F^{(j')}_{-\overline{\omega}} \right\rangle F^{(j)}_{\omega}.$$

(ii) *For $\omega > 0$ such that $\det[\Pi_{B,Y}(\omega)] \neq 0$, the resolvent of $\Delta_{B,Y}$ reads as*

$$(\Delta_{B,Y} - \omega^2)^{-1} = \Gamma_\omega + \sum_{l,l'=1}^{N} \Gamma_\omega(\cdot - y^{(l)})\,\big[\Pi^{-1}_{B,Y}(\omega)\big]_{l,l'}\,\langle \cdot\,, \overline{\Gamma_\omega(\cdot - y^{(l')})}\rangle,$$

with the Green's tensor

$$(\Delta_{B,Y} - \omega^2)^{-1}(x,y) = \Gamma_\omega(x,y) + \sum_{l,l'=1}^{N} \Gamma_\omega(x, y^{(l)})\ \big[\Pi^{-1}_{B,Y}(\omega)\big]_{l,l'}\,\Gamma_\omega(y^{(l')}, y),$$

for $x \neq y$ and $x, y \neq y^{(l)}$. Here $\big[\Pi^{-1}_{B,Y}\big]_{l,l'}$ denotes the 2-by-2 blocks of the matrix $\big[\Pi_{B,Y}\big]^{-1}$.

Proof. (i) Let us first show that $L(\omega)$ is invertible for $\operatorname{Im}\omega > 0$ and $\det\Big[\Pi_{B,Y}(\omega)\Big] \neq 0$. We recall that $\widetilde{H}^\epsilon$ are densely defined and closed operators. From Lemma 2.2, we know that $\widetilde{H}^\epsilon - \omega^2$, and hence $(\widetilde{H}^\epsilon - \omega^2)^{-1}$, are invertible for $\operatorname{Im}\omega > 0$ and $\det\Big[\Pi_\epsilon(\omega)\Big] \neq 0$. In particular, $(\widetilde{H}^\epsilon - \omega^2)^{-1}$ are surjective for $\operatorname{Im}\omega > 0$ and $\det\Big[\Pi_\epsilon(\omega)\Big] \neq 0$. Hence its limiting operator $L(\omega)$ is also surjective for $\operatorname{Im}\omega > 0$ and $\det\Big[\Pi_{B,Y}(\omega)\Big] \neq 0$. Remark that, due to Lemma 2.3, if $\det\Big[\Pi_{B,Y}(\omega)\Big] \neq 0$ then $\det\Big[\Pi_\epsilon(\omega)\Big] \neq 0$ for ϵ small enough. From the explicit form (29) we can show that $L(\omega)$ is also injective. Indeed, let $\hat{f} \in L^2(\mathbb{R}^2)^2$ such that $L(\omega)\hat{f} = 0$. Then $(M_0 - \omega^2\mathbf{I})L(\omega)\hat{f} = 0$, which we can write as

$$\hat{f}(\xi) = -\sum_{j,j'=1}^{2N} \Big[\Pi_{B,Y}(\omega)\Big]^{-1}_{j,j'} \int_{\mathbb{R}^2} \Big[\overline{(M_0 - \overline{\omega}^2)^{-1}(\zeta)\,\Phi_{j'}(\zeta)\hat{f}(\zeta)}\Big]\, d\zeta\, \Phi_j(\xi) \tag{30}$$

using (29). However, the left hand side of (30) cannot be in $L^2(\mathbb{R})^2$ unless $\hat{f}=0$. Hence the operator L is injective.

Since $L(\omega)$ is invertible, in the open set of $\omega\in\mathbb{C}$ such that $\operatorname{Im}\omega>0$ and $\det\Big[\Pi_{B,Y}(\omega)\Big]\neq 0$, and it is the limit, in the operator norm sense, of the resolvent of closed operators, i.e. $\widetilde{H}^\epsilon$, then from Theorem VIII.1.3 of Ref. [8] we deduce that it is the resolvent of a closed operator. We denote this operator by $\hat{\Delta}_{B,Y}$, i.e.

$$L(\omega)=(\hat{\Delta}_{B,Y}-\omega^2)^{-1},\quad \operatorname{Im}\omega>0 \text{ and } \det\Big[\Pi_{B,Y}(\omega)\Big]\neq 0. \tag{31}$$

Let us show that $\hat{\Delta}_{B,Y}$ is densely defined and self-adjoint. Since $L(\omega)$ is invertible, then its range is given by the domain of its inverse. Then **D**$(\hat{\Delta}_{B,Y})=$ **R**$(L(\omega))$. Hence $g\in$ **D**$(\hat{\Delta}_{B,Y})$ can be written as $g=L(\omega)h$ with $h\in L^2(\mathbb{R}^2)^2$. Let $f\in$ **D**$(\hat{\Delta}_{B,Y})^\perp$, then $<f,g>=0,\ \forall g\in$ **D**$(\hat{\Delta}_{B,Y})$ and then $<f,L(\omega)h>=0,\ \forall h\in L^2(\mathbb{R}^2)^2$. From (29) and (23), we see that $L^*(\omega)=L(-\bar{\omega})$ (remark that $\operatorname{Im}(-\bar{\omega})=\operatorname{Im}\omega>0$), which implies that $<L(-\bar{\omega})f,h>=0;\ \forall h\in L^2(\mathbb{R}^2)^2$. We deduce that $L(-\bar{\omega})f=0$ and then $f=0$, i.e., $\overline{\mathbf{D}(\hat{\Delta}_{B,Y})}=L^2(\mathbb{R}^2)^2$.

Regarding the self-adjointness, we write

$$(\hat{\Delta}_{B,Y})^*-\hat{\Delta}_{B,Y}=(L^{-1}(\omega))^*+\bar{\omega}^2-L^{-1}(\omega)-\omega^2=L^{-1}(-\bar{\omega})-L^{-1}(\omega)+\bar{\omega}^2-\omega^2 \tag{32}$$

based on (31) and (29). From the resolvent identity

$$(\hat{\Delta}_{B,Y}(\alpha)-\alpha^2)^{-1}-(\hat{\Delta}_{B,Y}(\beta)-\beta^2)^{-1}=(\alpha^2-\beta^2)\,(\hat{\Delta}_{B,Y}(\alpha)-\alpha^2)^{-1}\,(\hat{\Delta}_{B,Y}(\beta)-\beta^2)^{-1}$$

for $\alpha,\beta\in\mathbb{C}$, we deduce that

$$(\hat{\Delta}_{B,Y}(\beta)-\beta^2)-(\hat{\Delta}_{B,Y}(\alpha)-\alpha^2)=\alpha^2-\beta^2,$$

which implies $\hat{\Delta}_{B,Y}(\beta)=\hat{\Delta}_{B,Y}(\alpha)$. Therefore, the operator $\hat{\Delta}_{B,Y}$ given in (31) is independent of ω, and it follows from (31) that

$$L^{-1}(\omega)-L^{-1}(-\bar{\omega})=(\hat{\Delta}_{B,Y}-\omega^2)-(\hat{\Delta}_{B,Y}-\bar{\omega}^2)=\bar{\omega}^2-\omega^2. \tag{33}$$

Combining (32) and (33) gives the relation $\hat{\Delta}^*_{B,Y}-\hat{\Delta}_{B,Y}=0$, i.e. $\hat{\Delta}_{B,Y}$ is self-adjoint.

This proves the first assertion.

(ii) To prove the second assertion, we define the matrix $\big[\Pi^{-1}_{B,Y}\big]_{l,l'}$ as the 2×2 blocks of $\Pi_{B,Y}(\omega)$, i.e., for $\operatorname{Im}\omega>0$ and $\det\Big[\Pi_{B,Y}(\omega)\Big]\neq 0$,

$$\big[\Pi^{-1}_{B,Y}\big]_{l,l'}:=\begin{pmatrix}[\Pi_{B,Y}(\omega)]^{-1}_{2l-1,2l'-1} & [\Pi_{B,Y}(\omega)]^{-1}_{2l-1,2l'}\\ [\Pi_{B,Y}(\omega)]^{-1}_{2l,2l'-1} & [\Pi_{B,Y}(\omega)]^{-1}_{2l,2l'}\end{pmatrix}\in\mathbb{C}^{2\times 2},\quad l,l'=1,2,\cdots,N.$$

In view of the definition of $\Theta_{l,\omega}$ given in (27), via simple calculations we have

$$\sum_{j=2l-1}^{2l}\sum_{j'=2l'-1}^{2l'}\Big[\Pi_{B,Y}(\omega)\Big]^{-1}_{j,j'}\Big\langle\hat{f}\,,F^{(j')}_{-\bar{\omega}}\Big\rangle F^{(j)}_{\omega}=\Theta_{l,\omega}(\xi)\,\big[\Pi^{-1}_{B,Y}\big]_{l,l'}\int_{\mathbb{R}^2}\big[\Theta_{l',-\bar{\omega}}(\xi)\big]^T\hat{f}(\xi)\,d\xi. \tag{34}$$

Recall again that, in (34) the notation $[\,]^T$ denotes the conjugate transpose of a complex valued matrix $[\,]$. Moreover, employing (27) and the inverse Fourier transform enables us to rewrite the integral on the right hand side of (34) as

$$\begin{aligned}\int_{\mathbb{R}^2} \left[\Theta_{l',-\overline{\omega}}(\xi)\right]^T \hat{f}(\xi)\, d\xi &= (2\pi)^{-1} \int_{\mathbb{R}^2} \left[(M_0 - \overline{\omega}^2\mathbf{I})^{-1}\right]^T(\xi)\, \hat{f}(\xi) \exp(i\xi \cdot y^{(l')})\, d\xi \\ &= \mathcal{F}^{-1} \left[\left[(M_0 - \overline{\omega}^2\mathbf{I})^{-1}\right]^T \hat{f}\right] (y^{(l')}) \\ &= (2\pi)^{-1} \left(\mathcal{F}^{-1}\left[(M_0 - \overline{\omega}^2\mathbf{I})^{-1}\right]^T * f\right) (y^{(l')}) \\ &= \int_{\mathbb{R}^2} [\Gamma_{\overline{\omega}}(y^{(l')} - y)]^T\, f(y)\, dy. \qquad (35)\end{aligned}$$

Here the operator $*$ stands for the convolution. Taking the inverse Fourier transform in (29) and making use of (28), (34) and (35), we obtain

$$\begin{aligned}[(\Delta_{B,Y} - \omega^2)^{-1} f](x) &= \int_{\mathbb{R}^2} \Gamma_\omega(x-y)\, f(y)\, dy \\ &+ \sum_{l',l=1}^{N} \left\{\Gamma_\omega(x - y^{(l)})\, \left[\Pi_{B,Y}^{-1}\right]_{l,l'} \int_{\mathbb{R}^2} [\Gamma_{\overline{\omega}}(y^{(l')} - y)]^T\, f(y)\, dy\right\}\end{aligned}$$

for $\mathrm{Im}\,\omega > 0$ such that $\det[\Pi_{B,Y}(\omega)] \neq 0$.

By construction, the operator $(\Delta_{B,Y} - \omega^2)^{-1}$ is well defined from $L^2(\mathbb{R}^2)^2$ to $L^2(\mathbb{R}^2)^2$ and it is invertible when $\mathrm{Im}\,\omega > 0$ and $\det[\Pi_{B,Y}(\omega)] \neq 0$. It is also a bounded operator between the Agmon's spaces $L^2_\sigma(\mathbb{R}^2)^2$ and $L^2_{-\sigma}(\mathbb{R}^2)^2$ for $\sigma > 1$, where the weighted space $L^2_\sigma(\mathbb{R}^2)^2$ is defined by

$$L^2_\sigma(\mathbb{R}^2)^2 := \{f : ||(1 + |x|^2)^{\sigma/2} f||_{L^2(\mathbb{R}^2)^2} < \infty\}.$$

In addition, from the explicit form of $(\Delta_{B,Y} - \omega^2)^{-1}$, the limiting (absorption) operator $\lim_{\mathrm{Im}\,\omega\to 0}(\Delta_{B,Y} - \omega^2)^{-1}$ is also well defined and bounded in the above mentioned Agmon's spaces. More precisely, for $f \in L^2_\sigma(\mathbb{R}^2)^2$, $\sigma > 1$, we have

$$\lim_{\mathrm{Im}\,\omega\to 0}[(\Delta_{B,Y} - \omega^2)^{-1} f](x) = (\Gamma_{\omega_r} * f)(x) + \sum_{l,l'=1}^{N} \Gamma_{\omega_r}(x, y^{(l)})\, \left[\Pi_{B,Y}^{-1}(\omega_r)\right]_{l,l'}\, (\Gamma_{\omega_r} * f)(y^{(l')}) \qquad (36)$$

where $\omega_r := \mathrm{Re}\,\omega$ denotes the real part of ω, whenever $\det \Pi_{B,Y}(\omega_r) \neq 0$. Note that in deriving (36), we have used the symmetry

$$[\Gamma_{\omega_r}]^T = \frac{1}{2\pi} \mathcal{F}^{-1} \left[\left[(M_0 - \omega_r^2\mathbf{I})^{-1}\right]^T\right] = \frac{1}{2\pi} \mathcal{F}^{-1} \left[(M_0 - \omega_r^2\mathbf{I})^{-1}\right] = \Gamma_{\omega_r}$$

for the real number ω_r. The formula (36) reveals the resolvent of the operator modeling the scattering by the point scatterers $y^{(j)}, j = 1, 2, ..., N$. Obviously, the kernel (Green's tensor) of the operator (36) is given by

$$\mathbb{G}_\omega(x, y) := \Gamma_\omega(x, y) + \sum_{l,l'=1}^{N} \Gamma_\omega(x, y^{(l)})\, \left[\Pi_{B,Y}^{-1}\right]_{l,l'}(\omega)\, \Gamma_\omega(y^{(l')}, y) \qquad (37)$$

for $\omega \in \mathbb{R}_+$ such that $\det \Pi_{B,Y}(\omega) \neq 0$. $\square$

In classical scattering theory, (37) describes the total field by the collection of point like scatterers Y corresponding to the incident point source $\Gamma_\omega(x,y)$ located at y. As an application of the Green's tensor (37), we next derive the scattered near and far fields for elastic plane waves.

For a fixed vector $d \in \mathbb{S}$, the far-field pattern $\Gamma^\infty_\omega(\hat{y};x,d)$ of the function $y \to \Gamma_\omega(x,y)d$ is given by (after some normalization)

$$\Gamma^\infty_\omega(\hat{y};x,d) = \exp(-ik_p\hat{y}\cdot x)(\hat{y}\cdot d)\,\hat{y} + \exp(-ik_s\hat{y}\cdot x)(\hat{y}^\perp\cdot d)\,\hat{y}^\perp \tag{38}$$

as $y \to \infty$, where $\hat{y} := y/|y| = (\cos\theta, \sin\theta)^\top$ for some $\theta \in (0, 2\pi]$, and $\hat{y}^\perp := (-\sin\theta, \cos\theta)^\top$. We refer to the first resp. second term of (38) as the pressure resp. shear part of $\Gamma\omega^\infty(\hat{y};x,d)$. We define the elastic plane pressure wave, $U^I_p(x,-\hat{y})$, as the far field of the point source $\Gamma_\omega(x,y)(-\hat{y})$ when the source y is far away from x (i.e. y tends to infinity), that is,

$$U^I_p(x,d) = d\exp(ik_p x\cdot d), \quad d := -\hat{y}. \tag{39}$$

Then, multiplying (37) by $-\hat{y}$ and letting $y \to \infty$ we obtain the total field

$$U^{tol}(x,d) = U^I_p(x,d) + \sum_{l,l'=1}^{N} \Gamma_\omega(x,y^{(l)})\ \left[\Pi^{-1}_{B,Y}(\omega)\right]_{l,l'} U^I_p(y^{(l')},d) \tag{40}$$

for scattering of the elastic plane wave (39) by point-like scatterers $y^{(l)}, l = 1,2,\cdots,N$. Here $d := -\frac{y}{|y|}$ is referred to as the direction of incidence.

Analogously, defining the plane shear wave $U^I_s(x,d)$, with the incident direction $d = -\hat{y}$, as $\Gamma^\infty(\hat{y};x,d^\perp)$, i.e.

$$U^I_s(x,d) = d^\perp\exp(ik_s x\cdot d), \quad d := -\hat{y}.$$

we end up with the same formula as in (40) with U^I_p replaced by U^I_s. For the general elastic plane wave of the form

$$U^I(x,d,\alpha,\beta) := \alpha\, d\exp(ik_p x\cdot d) + \beta\, d^\perp\exp(ik_s x\cdot d), \quad \alpha,\beta\in\mathbb{C}, \quad d\in\mathbb{S}, \tag{41}$$

by superposition principle we have $U^{tol}(x,d,\alpha,\beta) = U^I(x,d,\alpha,\beta) + U^S(x,d,\alpha,\beta)$, where the scattered field is given by

$$U^S(x,d,\alpha,\beta) = \sum_{l,l'=1}^{N} \Gamma_\omega(x,y^{(l)})\ \left[\Pi^{-1}_{B,Y}(\omega)\right]_{l,l'} U^I(y^{(l')},d,\alpha,\beta). \tag{42}$$

In view of (38), we get the longitudinal and transversal parts of the far-field pattern of (42)

$$U^\infty_p(\hat{x}) = \hat{x}\left\{\sum_{l,l'=1}^{N} \exp(-ik_p y^{(l)}\cdot\hat{x})\ \left(\left[\Pi^{-1}_{B,Y}(\omega)\right]_{l,l'} U^I(y^{(l')},d,\alpha,\beta)\right)\cdot\hat{x}\right\}, \tag{43}$$

$$U^\infty_s(\hat{x}) = \hat{x}^\perp\left\{\sum_{l,l'=1}^{N} \exp(-ik_s y^{(l)}\cdot\hat{x})\ \left(\left[\Pi^{-1}_{B,Y}(\omega)\right]_{l,l'} U^I(y^{(l')},d,\alpha,\beta)\right)\cdot\hat{x}^\perp\right\}. \tag{44}$$

Obviously, there holds the reciprocity relations

$$U^I(x,d,\alpha,\beta) = \Gamma^\infty(-d;x,\alpha d+\beta d^\perp), \quad U^S(x,d,\alpha,\beta) = (\mathbb{G}_\omega - \Gamma_\omega)^\infty(-d;x,\alpha d+\beta d^\perp)$$

where $(\mathbb{G}_\omega - \Gamma_\omega)^\infty(\hat{y};x,d)$ denotes the far field pattern corresponding to the scattered field $(\mathbb{G}_\omega(x,y) - \Gamma_\omega(x,y))\,d$ due to the point source incidence $\Gamma_\omega(x,y)\,d$ (cf. (37)).

Remark 2.7. *Comparing (42) with the Foldy model (5)-(6), we see that the scattering coefficient b_j is related to the scattering strength a_j via*

$$a_j = (b_j - \eta)^{-1} = (c_j - \kappa)^{-1}, \quad j = 1, 2, \cdots, N,$$

where c_j and κ are given in (22) and (7) respectively in the case of 2D.

3 Elastic scattering by point-like obstacles in $\mathbb{R}^3$

Let us now turn to studying the elastic scattering problem in 3D. We only need to make necessary changes related to the Green's tensor in $\mathbb{R}^3$. The Kupradze matrix $\tilde{\Gamma}_\omega$ of the fundamental solution to the Navier equation is given by (see Ref. [10, Chapter 2])

$$\widetilde{\Gamma}_\omega(x, 0) = \frac{1}{\mu}\Phi_{k_s}(x)\mathbf{I} + \frac{1}{\omega^2}\mathrm{grad}_x \, \mathrm{grad}_x^\top[\Phi_{k_s}(x) - \Phi_{k_p}(x)],$$

where $\Phi_k(x) = 1/(4\pi)\exp(ik|x|)$ denotes the free-space fundamental solution of the Helmholtz equation $\Delta + k^2 u = 0$ in $\mathbb{R}^3$. Using Taylor series expansion for exponential functions we can rewrite the matrix $\widetilde{\Gamma}_\omega(x, 0)$ as the series (see also Ref. [2])

$$\begin{aligned} \widetilde{\Gamma}_\omega(x, 0) \;\; &= \;\; \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(n+1)(\lambda+2\mu)+\mu}{\mu(\lambda+2\mu)}\frac{(i\omega)^n}{(n+2)n!}|x|^{n-1}\,\mathbf{I} \\ &\quad -\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\frac{(i\omega)^n(n-1)}{(n+2)n!}|x|^{n-3}\,\Xi(x), \end{aligned} \tag{45}$$

from which it follows that

$$\widetilde{\Gamma}_\omega(x, 0) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x|}\,\mathbf{I} + i\omega\frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}\,\mathbf{I} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x|^3}\,\Xi(x) + o(1)\omega^2 \tag{46}$$

as $|x| \to 0$. Taking $\omega \to 0$ in (45), we obtain

$$\lim_{w\to 0}\widetilde{\Gamma}_\omega(x, 0) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x|}\,\mathbf{I} + \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{1}{|x|^3}\,\Xi(x) =: \widetilde{\Gamma}_0(x, 0). \tag{47}$$

This is just the Kelvin matrix of the fundamental solution of the Lamé system in $\mathbb{R}^3$. Note that the above convergence (47) was proved in Ref. [10, Chapter 2] via the estimate

$$|\widetilde{\Gamma}_\omega(x, 0) - \widetilde{\Gamma}_0(x, 0)| \le C(\lambda, \mu)\,|\omega|$$

for some unknown constant $C(\lambda, \mu) > 0$. Combining (46) and (47) gives the limit of the entries

$$\lim_{|x|\to 0}[\widetilde{\Gamma}_\omega(x, 0) - \widetilde{\Gamma}_0(x, 0)] = \tilde{\eta}\,\mathbf{I},$$

with

$$\tilde{\eta} := i\omega\frac{2\lambda+5\mu}{12\pi\mu(\lambda+2\mu)}. \tag{48}$$

In order to generalize Theorem 2.6 to 3D , we employ the cut-off function

$$\chi_\epsilon(\xi) = \begin{cases} 1, & \text{if} \quad |\xi| \le 1/\epsilon, \\ 0, & \text{if} \quad |\xi| > 1/\epsilon, \end{cases} \qquad \text{for } \epsilon > 0, \quad \xi = (\xi_1, \xi_2, \xi_3)^\top.$$

For $\operatorname{Im}\omega > 0$, define the operator

$$\widetilde{H}^\epsilon f := \widetilde{M_0}(\xi)\, f - \sum_{j=1}^{N}\sum_{i=1}^{3} \left\langle \tilde{a}_j(\epsilon)\, f, \varphi^{\epsilon,i}_{y^{(j)}} \right\rangle \varphi^{\epsilon,i}_{y^{(j)}}(\xi), \quad \varphi^{\epsilon,i}_{y^{(j)}}(\xi) := \chi_\epsilon(\xi)\, \varphi^{i}_{y^{(j)}}(\xi), \tag{49}$$

where $\widetilde{M_0}(\xi)$ is the Fourier transform of the matrix $\widetilde{\Gamma}_0(x, 0)$, and

$$\varphi^{i}_{y^{(j)}}(\xi) := (2\pi)^{-3/2} \exp(-iy^{(j)} \cdot \xi)\, (e_i)^\top, \quad i = 1, 2, 3, \; j = 1, 2, \cdots N.$$

For $j = 1, 2, \cdots, N$, $m = 1, 2, 3$, choose

$$\begin{aligned} \tilde{a}_j^{-1}(\epsilon) \;&:=\; \tilde{a}_{j,m}^{-1}(\epsilon) = \frac{1}{8\pi^2}\left(\int_{|\xi|<1/\epsilon} \widetilde{M_0}(\xi)^{-1} e_m^\top \, d\xi\right) e_m^\top + b_j \\ &=\; \frac{1}{8\pi^2}\int_{|\xi|<1/\epsilon} \left(\frac{1}{\mu|\xi|^2} - \frac{(\lambda+\mu)\xi_m^2}{\mu(\lambda+2\mu)|\xi|^4}\right) d\xi + b_j, \end{aligned}$$

for some $b_j \in \mathbb{R}$, that is,

$$\tilde{a}_j(\epsilon) = 1/\left[(6\pi^2(\lambda+2\mu)\,\epsilon)^{-1} + b_j\right]. \tag{50}$$

Define

$$\tilde{F}^{(j)}_\omega := (2\pi)^{-3/2} \exp(-i\xi \cdot y^{(l)})(\widetilde{M_0} - \omega^2 \mathbf{I})^{-1}(e_{3-m})^\top, \quad \text{if } j = 3l - m, \; m = 0, 1, 2,$$

for some $l = 1, 2, \cdots, N$, and define the $3N \times 3N$ matrix $\widetilde{\Pi}_{B,Y}$ by

$$\widetilde{\Pi}_{B,Y}(\omega) = \begin{pmatrix} (b_1 - \tilde{\eta})\mathbf{I} & -\widetilde{\Gamma}_\omega(y^{(1)} - y^{(2)}) & \cdots & -\widetilde{\Gamma}_\omega(y^{(1)} - y^{(N)}) \\ -\widetilde{\Gamma}_\omega(y^{(2)} - y^{(1)}) & (b_2 - \tilde{\eta})\mathbf{I} & \cdots & -\widetilde{\Gamma}_\omega(y^{(2)} - y^{(N)}) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\widetilde{\Gamma}_\omega(y^{(N)} - y^{(1)}) & -\widetilde{\Gamma}_\omega(y^{(N)} - y^{(2)}) & \ldots & (b_N - \tilde{\eta})\mathbf{I} \end{pmatrix},$$

with $B = (b_1, \cdots, b_N) \in \mathbb{R}^{1\times N}$. Then, arguing analogously to Theorem 2.6 we obtain

Theorem 3.1. *Suppose that the operator $\widetilde{H}^\epsilon$ is given by (49), with*

$$\tilde{a}_j(\epsilon) = 1/\left[(6\pi^2(\lambda+2\mu)\,\epsilon)^{-1} + b_j\right], \quad b_j \in \mathbb{R}, \; j \in \{1, 2, \cdots, N\}.$$

Then

(i) *The operator $\widetilde{H}^\epsilon$ converges in a norm resolvent sense to a closed and selfadjoint operator $\hat{\Delta}'_{B,Y}$ as $\epsilon \to 0$, where the resolvent of $\hat{\Delta}'_{B,Y}$ is given by*

$$(\hat{\Delta}'_{B,Y} - \omega^2)^{-1} = (\widetilde{M_0} - \omega^2 \boldsymbol{I})^{-1} + \sum_{j,j'=1}^{3N} \left[\widetilde{\Pi}_{B,Y}(\omega)\right]^{-1}_{j,j'} \left\langle \cdot\,, \tilde{F}^{(j')}_{-\overline{\omega}} \right\rangle \tilde{F}^{(j)}_\omega.$$

(ii) *For $\omega > 0$ such that $\det[\widetilde{\Pi}_{B,Y}(\omega)] \neq 0$, the resolvent of $\Delta'_{B,Y}$ takes the form*

$$(\Delta'_{B,Y} - \omega^2)^{-1} = \widetilde{\Gamma}_\omega + \sum_{l,l'=1}^{N} \widetilde{\Gamma}_\omega(\cdot - y^{(l)}) \left[\widetilde{\Pi}^{-1}_{B,Y}(\omega)\right]_{l,l'} \left\langle \cdot , \overline{\Gamma_\omega(\cdot - y^{(l')})} \right\rangle,$$

with the Green's tensor

$$(\Delta'_{B,Y} - \omega^2)^{-1}(x,y) = \widetilde{\Gamma}_\omega(x,y) + \sum_{l,l'=1}^{N} \widetilde{\Gamma}_\omega(x, y^{(l)}) \left[\widetilde{\Pi}^{-1}_{B,Y}(\omega)\right]_{l,l'} \widetilde{\Gamma}_\omega(y^{(l')}, y), \tag{51}$$

for $x \neq y$ and $x, y \neq y^{(l)}$. Here $\left[\widetilde{\Pi}^{-1}_{B,Y}\right]_{l,l'}$ denotes the 3-by-3 blocks of the matrix $\left[\widetilde{\Pi}_{B,Y}\right]^{-1}$.

Similar to the 2D case, (51) is no thing but the Foldy model (5)-(6) taking the scattering strengths a_j of the form $(b_j - \tilde{\eta})^{-1}$ where $b_j \in \mathbb{R}$, $j = 1, ..., N$. One can also get analogous formulas to (42), (43) and (44) for the scattered near and far fields associated with incident plane waves in $\mathbb{R}^3$.

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