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Symmetry Breaking in Dynamical Systems

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Abstract

Symmetry breaking bifurcations and dynamical systems have obtained a lot of attention over the last years. This has several reasons: real world applications give rise to systems with symmetry, steady state solutions and periodic orbits may have interesting patterns, symmetry changes the notion of structural stability and introduces degeneracies into the systems as well as geometric simplifications. Therefore symmetric systems are attractive to those who study specific applications as well as to those who are interested in a the abstract theory of dynamical systems. Dynamical systems fall into two classes, those with continuous time and those with discrete time. In this paper we study only the continuous case, although the discrete case is as interesting as the continuous one. Many global results were obtained for the discrete case. Our emphasis are heteroclinic cycles and some mechanisms to create them. We do not pursue the question of stability. Of course many studies have been made to give conditions which imply the existence and stability of such cycles. In contrast to systems without symmetry heteroclinic cycles can be structurally stable in the symmetric case. Sometimes the various solutions on the cycle get mapped onto each other by group elements. Then this cycle will reduce to a homoclinic orbit if we project the equation onto the orbit space. Therefore techniques to study homoclinic bifurcations become available. In recent years some efforts have been made to understand the behaviour of dynamical systems near points where the symmetry of the system was perturbed by outside influences. This can lead to very complicated dynamical behaviour, as was pointed out by several authors. We will discuss some of the technical difficulties which arise in these problems. Then we will review some recent results on a geometric approach to this problem near steady state bifurcation points.

1 Introduction

In this paper we would like to investigate the effects of symmetry breaking in dynamical systems. One theme which seems to b closely related to it is the occurrence of structurally stable heteroclinic cycles in equivariant systems. There are several well known examples, see for example GUCKENHEIMER & HOLMES [23] and the work of KRUPA and MELBOURNE [29, 30, 31, 40] which is directly related to heteroclinic cycles, and the papers by LAUTERBACH & ROBERTS [36] and also [35, 39]. Heteroclinic cycles may be generated in various ways, we distinguish

1. the "invariant plane case",

2. problems with higher codimension,

3. forced symmetry breaking.

Of course there is some relation with the topics of the very nice survey on heteroclinic cycles by KRUPA [29]. But our perspectives are somewhat different, our main emphasis are methods in equivariant systems, heteroclinic cycles are to be considered as a spin off. Let us briefly discuss the items mentioned before:

1. With the invariant plane case we mean a scenario for the occurrence of heteroclinic cycles in fixed point spaces of subgroups which was first described by Melbourne, Chossat & Golubitsky [41]. We shall see later the example by GUCKENHEIMER & HOLMES [23], which fits very nicely into this framework, where the group $\mathbf{T} \oplus \mathbf{Z}_2$ acts irreducibly on $\mathbf{R}^3$. However in many cases such a situation occurs when the group action is reducible, compare ARMBRUSTER & CHOSSAT [2, 8], CHOSSAT & GUYARD [10, 24].

2. In systems without symmetry it is well known that complicated dynamical behaviour can occur if the system under consideration has higher codimension. Of course the same is true for equivariant systems, however, since symmetric systems automatically have some degeneracies, it becomes increasingly difficult to study problems with higher codimension. We present a example due to LAUTERBACH & SANDERS [38], where invariant theory has been used to study a problem with topological codimension 3. Again, a heteroclinic cycle occurs. This cycle is constructed for the equation on the orbit space. Therefore the issue of lifting it back to the full space becomes important. With respect to this problem finite and infinite groups show a different behaviour. In the finite case it is clear that a heteroclinic cycle lifts to a heteroclinic cycle, which might involve more equilibria and more heteroclinic connections than the one on the orbit space, but in principle we find the same object. However, in the case of a continuous group this changes. Even an equilibrium does not necessarily lift to an equilibrium but to a so called relative equilibrium. The heteroclinic connections just connects two group orbits. There is another important difference between the discrete and the continuous case: the behaviour with respect to perturbations which do not respect the (full) symmetry. This leads to our last topic.

3. We speak of forced symmetry breaking when we perturb the system with terms having less symmetry than the original problem. We shall see that this is a natural problem from the application point of view. It leads to interesting dynamical effects and again heteroclinic cycles come up. Our techniques evolved from the work of LAUTERBACH &

ROBERTS [36, 37]. We shall look at some questions concerning group theoretic conditions for the existence of heteroclinic cycles and moreover how to prove the existence of them for PDE's when the symmetry has been slightly perturbed.

The example 3.3 was found in a discussion with Karin Gatermann, Frédéric Guyard and Matthias Rumberger.

2 Symmetric dynamical systems – why?

It is well known that many physical systems can be modeled in terms of dynamical systems, just consider the classical problems of mechanics. In the course of the last decades the applicability of dynamical system theory has widely expanded. Systems in chemistry, biology, economy and other sciences were translated into mathematical language and can be written as dynamical systems. In the course of this translation process many simplifying and abstracting assumptions are being made. In many cases these abstractions and simplifications lead to additional structures in the equations, which were not present in the original problem. One of those structures could be the occurrence of symmetries. However, symmetries do not only come into the game by the process of mathematical idealization, but can also be a very natural ingredient of the problem under consideration. Experiments can take place in a symmetric surrounding, the nature often finds beautifully symmetric forms or patterns. From this we see that symmetries can be a natural context for the study of real world phenomena. By now it is well known that the steady state or periodic solutions of a symmetric system can reflect less symmetry than we find originally in the system. This has been observed a long time ago, we usually refer to this as *spontaneous symmetry breaking*, see for example SATTINGER [45]. In contrast to this we can also imagine situations where a system, on the first glance, has a certain symmetry, but a closer look reveals that in fact some of these symmetries are present only approximately. Therefore, the full problem has less or no symmetry whatsoever. A typical example would be a problem in geophysics where the earth, in the first approximation, has the symmetry of a ball, if we look more closely we observe the flattening of the poles, reducing the symmetry of the ball to the symmetry of a circular disc, in the group theory language, which we will adopt, it has symmetry $\mathbf{O}(2)$. Taking the rotation into account reduces the symmetry to the group $\mathbf{SO}(2)$ and finally looking from very a close perspective, the typical human approach, we see no symmetry at all. Nevertheless we expect that a decent theory takes into account that we are close to a symmetric problem. We call this type of problem *forced symmetry breaking.* In fact in the example of our planet we described a *hierarchy* of forced symmetry breakings. As we shall point out the problem of forced symmetry breaking leads to very severe conceptual and computational difficulties. A global understanding of these problems is not in reach. However it has been noted and we shall see that this can lead to extremely rich and difficult dynamical behaviour.

Symmetries can be described in the language of group theory. The most obvious way of doing it is to consider a domain (or a compact embedded manifold without boundary) $\Omega \subset \mathbf{R}^n$ and its symmetry group G_Ω defined by

$$G_\Omega = \{A \in \mathrm{O}(n) \mid A\omega \in \Omega \ \forall \omega \in \Omega\} . \tag{1}$$

If the mathematical formulation of the problem leads to a partial differential equation of the type

$$\frac{\partial u}{\partial t} + Lu = f(u, \lambda) \tag{2}$$

with a sectorial operator L and f sufficiently smooth and "reasonable" boundary conditions, then (2) defines a semidynamical system on $L^2(\Omega)$ (or $H^{1,2}(\Omega)$). The group G_Ω acts on function spaces X (for example $X = L^2(\Omega)$ or a Sobolev space $W^{k,p}(\Omega)$ over Ω) simply by

$$G_\Omega \times X : (\gamma, u) \mapsto \gamma u, \ \gamma u(\omega) = u\left(\gamma^{-1}\omega\right) . \tag{3}$$

We assume that the linear operator is *equivariant* with respect to this action, i.e.

$$L(\gamma u) = \gamma(Lu), \text{ for all } u \in X, \ \gamma \in G_\Omega \tag{4}$$

This assumption is always satisfied if L is the Laplace operator. Smaller groups allow some more general partial differential operators. In fact a reasonable modeling should lead to G_Ω equivariant operators.

We say that equation (2) is equivariant with respect to G_Ω if

$$-L(\gamma u) + f(\gamma u, \lambda) = \gamma(-Lu + f(u, \lambda)) \tag{5}$$

for all $u \in X$ and all $\gamma \in G_\Omega$ and if the boundary conditions are invariant under the action of G_Ω on u. Typical examples are the buckling of spheres, where the space X is a function space over the 2–sphere and no boundary condition are present or the spherical Bénard problem, where Ω is a spherical annulus and we have boundary conditions on the inner and the outer sphere. Let us just recall that the Bénard problem is to describe a fluid flow between two infinite plates, where the temperature on the plates is spatially constant and different, let T_l denote the temperature at the lower plate and T_u denote the temperature at the upper plate we requite $T_l > T_u$. It is known that if the difference exceeds a certain value some interesting states occur. In the spherical Bénard problem we consider a fluid confined between two concentric spherical shells with inner and outer temperatures T_i and T_o, respectively. If $T_i - T_o$ is sufficiently large, we again observe new and interesting states. The Navier equations, describing these problems, are equivariant with respect to the Euclidean group in the planar case and with respect to $\mathbf{O}(3)$ in the spherical case.

Observe that the equivariance was assumed for the operator L, the nonlinearity is automatically equivariant if it does not explicitly depend on the spatial variable ω. Therefore adding in small terms which are spatially non constant leads to forced symmetry breaking. A typical scenario for the Bénard problem is to assume a small deviation from spatially homogeneous temperatures on the boundary, which can be rewritten in terms of small perturbations in the interior with explicit space dependence. We will come back later to these issues. Before we go on, we collect some simple properties of dynamical systems with symmetry, which are easily verified.

Some simple facts

1. If $u(t)$ is a solution, then $\gamma u(t)$ is again a solution.
2. For $u \in X$ let H_u denote its isotropy subgroup, i.e.
$$H_u = \{\gamma \in G_\Omega \,\|\, \gamma u = u\}.$$
Then $H_{\gamma u} = \gamma H_u \gamma^{-1}$.
3. Along trajectories the isotropy is not decreasing. i.e. if $0 < s < t$ then $H_{u(s)} \subset H_{u(t)}$. If backward uniqueness holds, then we have equality.

The main issues to be studied are to characterize the symmetry type of bifurcating solutions, structural stability in equivariant systems and global behaviour. For the local questions singularity theory proved to be very successful, compare GOLUBITSKY, STEWART & SCHAEFFER [22]. The main ingredient in a local theory are the center manifold theory or Lyapunov–Schmidt reduction. It is important to note that these tools carry over directly to the equivariant context. We just recall these results. For this we need some group theory language. A *representation* of a group G is a homomorphism ρ into $\mathbf{Gl}(n)$ for some n. We also say that G *acts* on $\mathbf{R}^n$. If there is a continuous homomorphism into the bounded linear operators on a Banach space we speak about an *infinite dimensional* representation. Actions on the function space as described before are such infinite dimensional representations. A subspace U of $\mathbf{R}^n$ (or X) is called invariant if for all $\gamma \in G$ $\rho(\gamma)u \in U$ for all $u \in U$. A representation is called reducible if it has a nontrivial invariant subspace, otherwise it is *irreducible.* A representation is absolutely irreducible if the only equivariant linear mappings are scalar multiples of the identity. Two representations ρ_1, ρ_2 on spaces V_1, V_2 of a group G are called *equivalent* if there is an isomorphism $\tau : V_1 \to V_2$ such that for all $\gamma \in G$ we have $\tau \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \tau$ as mappings $V_1 \to V_2$. For any finite group there are up to equivalence only finitely many irreducible representations, any representation can be written as a sum of irreducible ones. A similar statement is true for infinite dimensional representations of compact Lie groups. An important tool is the *character* of a representation. It is a function $\chi : G \to \mathbf{C}^* = \{z \in \mathbf{C} \mid |z| = 1\}$, defined by

$$\chi(g) = \operatorname{tr}(\rho(g)). \tag{6}$$

For an introduction into character theory, see for example SERRE [47]. A very nice tool for doing actual computations with characters is the program GAP [18].

Some more facts

1. Any (closed) invariant subspace has a (closed) invariant complement.
2. Any absolutely irreducible representation is irreducible. Over $\mathbf{C}$ the reversed statement is also true. The group $\mathbf{SO}(2)$ acts (by rotations) irreducibly on $\mathbf{R}^2$, but not absolutely irreducibly.
3. The kernel of an equivariant linear mapping is invariant.
4. Consider $L^2(G)$ to be space of square integrable (with respect to Haar measure, see HEWITT & ROSS [25]) complex valued functions. The characters of all irreducible representations form a complete orthonormal system.

GOLUBITSKY, STEWART & SCHAEFFER [22] show that in generic one parameter families of linear equivariant mappings on $\mathbf{R}^n$ the kernel is either trivial or absolutely irreducible. From this it follows that in one parameter families of equivariant, finite dimensional bifurcation problems the kernels are generically absolutely irreducible representations of G_Ω. A similar theorem is true for one parameter families of sectorial operators with compact resolvent, see LAUTERBACH [34]. However this does not imply that it is sufficient to study only one parameter bifurcations with absolutely irreducible kernels. For an example, see the section on local bifurcations.

Theorem 2.1 *Let X be a Banach space, $F : X \times \mathbf{R} \to X$ be G–equivariant and sufficiently smooth. Assume $F(0, \lambda) = 0$ for all $\lambda \in \mathbf{R}$ and $D_xF(0,0)$ has a nontrivial kernel K. Let $f : K \times \mathbf{R} \to K$ denote the mapping obtained via a Lyapunov–Schmidt reduction, then f is G-equivariant.*

For the center manifold reduction we note, that if all choices are made in a reasonable way, then the equation on the center manifold is G–equivariant. Just choose a cut–off which makes the center manifold unique and apply the group elements to get another center manifold, which by uniqueness coincides with the first one. This sets the stage for a local theory, which of course if well known. For further reference and for setting up the notation we recall some of the fundamental results in the next section. First, however, we need some more notation. If G acts on the space V, then for $v \in V$ the *orbit* is given by

$$\mathcal{O}(v) = \{\gamma v \,|\, \gamma \in G\}. \tag{7}$$

The set of all orbits is denoted by V/G. To each point we associate the *orbit type* as the class of subgroups $[H]$ conjugate to the isotropy subgroup H of v, i.e.

$$[H] = \{\gamma H \gamma^{-1} \,|\, \gamma \in G\}, \tag{8}$$

where H is the isotropy subgroup of v. From our previous observation that the isotropy subgroup of γv is given by $\gamma H \gamma^{-1}$ we conclude that the orbit type is constant along orbits, which justifies this nomenclature.

3 Some Aspects of Local Bifurcations

The last section prepared to consider finite dimensional equivariant dynamical systems, which we consider to be the reduced equations obtained via the equivariant center manifold reduction. There are several methods to study and classify these problems. In [22] an equivariant singularity theory was developed, however in practical computations it is often difficult to get to satisfactory answers. In BUZANO, GEYMONAT & POSTON [6] this theory is applied to the low dimensional representations of the dihedral group D_n, which are used to study the buckling of thin rods with a cross–section of a regular n-gon. An attempt to give a classification of G–equivariant problems by their codimension (in the sense of contact equivalence, see [22]) is made in GATERMANN & LAUTERBACH [19]. Here computer algebra is used to construct G–equivariant bifurcation problems, ordered by codimension. The calculations use Poincaré–series and give lists of generating elements for the ring of invariant functions and the module of equivariant mappings. However, even here one cannot expect to treat large groups or high dimensional representations. A second approach is based on isotropy subgroups and the geometry of fixed point subspaces. Let V be a finite dimensional vector space and $\rho : G \to \mathbf{Gl}(V)$ be a representation of a group G. Let $f : V \times \mathbf{R} \to V$ be G–equivariant, i.e. $f(\rho(\gamma)v, \lambda) = \rho(\gamma) f(v, \lambda)$ and consider the differential equation

$$\dot{v} = f(v, \lambda). \tag{9}$$

A subgroup $H \subset G$ is called an isotropy subgroup, if there exists some $v \in V$, $v \neq 0$, such that H is the isotropy subgroup of v, i.e.

$$H = \{\gamma \in G \,|\, \rho(\gamma) v = v\}. \tag{10}$$

An isotropy subgroup H is called *maximal* if it is a maximal element in the partial ordered set of all isotropy subgroup. It is easy to see that subgroups with one dimensional fixed point space are always maximal. Due to the so called equivariant branching lemma these groups play an important rôle, sometimes they are called *axial* subgroups. The following result goes back to VANDERBAUWHEDE [48] and CICOGNA [12]. The present formulation is due to IHRIG & GOLUBITSKY [28].

Theorem 3.1 *Let v be an absolutely irreducible representation of G, then we have*

1. $f(0, \lambda) = 0$,
2. $D_v f(0, \lambda) = c(\lambda) \mathbf{1}_V$ *for some scalar* $c(\lambda)$.

Suppose $c(0) = 0$, $c'(0) \neq 0$ *and H is a maximal isotropy subgroup of G, then there exists a bifurcating branch of solutions having isotropy H.*

The proof of this important result is very simple. A similar theorem holds in the context of Hopf bifurcation, however it is much less trivial. Again we just recall the statement. It is due to GOLUBITSKY & STEWART [21]. It describes the bifurcation of periodic solutions with spatial–temporal symmetry. Bifurcation of periodic solutions can be studied using a Lyapunov–Schmidt reduction in a space of 2π periodic functions. The equations obtained by this reduction do not only allow G–equivariance, but due to the fact that the problem is invariant under time shifts, the group $G \times S^1$ acts on the kernel and the bifurcation equation is equivariant with respect to this action. Note that this action is not absolutely irreducible, in general it is an irreducible sum of two absolutely irreducible representations. The set of commuting matrices is isomorphic to $\mathbf{C}$. Such representations are called *simple* ([22]). In fact a genericity result similar to the one that kernels at an eigenvalues zero are generically absolutely irreducible one shows that invariant with pure imaginary eigenvalues lead to generically to simple representations, see [22]. A subgroup H of $\Gamma = G \times S^1$ is called $\mathbf{C}$-*axial* if it has a two dimensional fixed point space.

Theorem 3.2 *Assume the trivial solution of equation(17) looses its stability through a pair of conjugate complex eigenvalues which cross the imaginary axis with nontrivial speed, and suppose that the representation of Γ on the kernel is simple. Then, if $H \subset \Gamma$ is a subgroup with* $\dim\mathrm{Fix}(H) = 2$, *then a branch of periodic solutions with isotropy H bifurcates.*

These results have been used to classify bifurcating solutions in many applications. Especially the equivariant Hopf theorem leads to very interesting patterns in coupled oscillators. One assumes that there are n identical oscillators and there are various possibilities of coupling. In general one has the internal symmetry of the single oscillator Σ and a group S of permutations acting on the set of oscillators. Depending on the coupling one can have a direct product structure $\Sigma \times S$ or a wreath-product $G \wr S$. For both types of coupling DIONNE, GOLUBITSKY & STEWART [14, 15] give a characterization of the $\mathbf{C}$–axial subgroups. Although further solutions could occur, this gives a broad class of solutions leading to interesting patterns.

Example 3.3 *Now we would like to discuss an example which shows that although generically the kernels at bifurcation points are absolutely irreducible representations, it might be important to understand also bifurcations with non absolutely irreducible group actions. Look at the five dimensional absolutely irreducible representation of A_5, the group of even permutations of five letters, also known as the symmetry group of the icosahedron. The isomorphism is given by viewing the symmetry group of the icosahedron as a permutation group on its subgroups of order 12, which are isomorphic to the symmetry group of a tetrahedron. In this representation the 2-Sylow subgroup D_2 of A_5 (see* LANG *[32] for the notion of Sylow groups) of A_5 is a maximal isotropy subgroup and has a two dimensional fixed point space. The normalizer of D_2 is the group A_4 (isomorphic to the symmetry group of the tetrahedron) and it acts on* $\mathrm{Fix}(D_2)$. *This is a general fact that the normalizer of a subgroup H acts on the fixed point space of H. Since D_2 acts trivially on its fixed point space, the effective action is the action of $\mathbf{Z}_3$. This action is irreducible but not absolutely irreducible. The bifurcation within this fixed point subspace is not the generic $\mathbf{Z}_3$ bifurcation in $\mathbf{R}^2$. The later one would allow for a Hopf bifurcation which is not possible in the A_5-context. To compute the actual bifurcations in* $\mathrm{Fix}(D_2)$ *requires an understanding of the module of equivariant mappings. These computations touch the limitations of todays workstations.*

A different method of studying equivariant dynamical systems is to project these equations onto the orbit space. This can be combined with various other techniques. For continuous groups it leads to a reduction of dimension. Let us briefly describe the essential features.

Definition 3.4 *Let G act on a space V (or on a manifold, topological space etc.). The orbit space is given by $V/\sim$, where $\sim$ is the equivalence relation, $v_1 \sim v_2$ iff v_1, v_2 are on the same group orbit. This definition is equivalent to the previous definition of V/G.*

If V is a Hausdorff space and if G is compact, then V/G is again Hausdorff. In general V/G is not a manifold, but stratified, where each stratum is a smooth manifold. Since the orbit type is constant along orbits, we associate to each element in V/G an orbit type. The strata for the smooth stratification are given by orbits of the same orbit type. That this stratification is finite follows from the finiteness of orbit types which is part of the following theorem.

Theorem 3.5 *Let G be a compact Lie group, ρ a representation on a vector space V. Then we have:*

1. *There are only finitely many orbit types.*
2. *There is a uniquely determined minimal obit type, called the* principal orbit type.
3. *The set $P_G(V)$ of points of minimal orbit type is open and dense in V.*
4. *If G is connected, then $P_G(V)$ is connected.*

Proof: See BREDON [5]. □

SCHWARZ [46] shows that there is a C^∞–structure on the orbit space, that each smooth, stratum preserving vector field on V/G lifts to a G–equivariant vector field on V. This will be useful later. At the moment we are interested in projecting vector fields onto the orbit space. This was used in LAUTERBACH & SANDERS [38] to study a certain degenerate case of O(3)–equivariant bifurcations. As we shall see later the orbit space is also an important tool for the study of forced symmetry breakings. In order to get differential equations on the orbit space, we recall some facts from invariant theory. As we have seen, to each action of a group G on a space V we have a natural action of G on function spaces over V. This defines an action on polynomials on V. A polynomial is called *invariant* if it is fixed under this action. Obviously the invariant polynomials form a ring.

Definition 3.6 *Let $\mathcal{R}_V$ denote the ring of all G–invariant polynomials. $\mathcal{M}_V$ is the module of G–equivariant polynomial mappings from V into itself.*

It is a fundamental result of HILBERT that $\mathcal{R}_V$ is a finitely generated algebra, as well as $\mathcal{M}_V$ is finitely generated over this ring. From here it follows that equivariant equations can be rewritten as

$$\dot{v} = f(v, \lambda) = \sum_{j=1}^{t_0} f_j(\pi_1(v), \pi_2(v), \ldots, \pi_s(v), \lambda)e_j(v). \tag{11}$$

The finiteness result together with this rewriting of the equation and the following lemma constitute the basis for the reduction to the orbit space.

Lemma 3.7 *$\mathcal{R}_V$ separates orbits, i.e. for two different orbits τ_1, τ_2 there exists an invariant polynomial p such p is 0 on τ_1 and 1 on τ_2.*

Proof: It is easy to construct a continuous function which is 0 on τ_1, and 1 on τ_2. From Weierstraß approximation theorem we know that for each $\varepsilon > 0$ there is a polynomial q, such that $q(v) > 1 - \varepsilon$ for all $v \in \tau_2$ and $q(v) < \varepsilon$ for $v \in \tau_1$. Averaging this polynomial over the group gives p (up to a multiplicative constant). □

Together with Hilbert's finiteness theorem this tells us, that there are finitely many invariant polynomials $\pi_1, \ldots, \pi_s$ which generate the algebra $\mathcal{R}_V$ and which separate orbits. Therefore the map

$$\Pi : V \to \mathbf{R}^s : v \mapsto (\pi_1(s), \ldots, \pi_s(v)) \tag{12}$$

gives rise to a continuous and injective mapping from $V/G \to \mathbf{R}^s$. Therefore the range of Π is a homeomorphic image of V/G and we can use the map Π to derive a differential equation on the orbit space. This can be done as follows: choose coordinates on $\mathbf{R}^s$ using $\pi_1, \ldots, \pi_s$ and compute for $i = 1, \ldots, s$

$$\frac{\partial \pi_i}{\partial t} =< \nabla\pi_i, \dot{v} >=< \nabla\pi_i, f(v,\lambda) = \sum_{j=1}^{t_0} f_j(\pi_1, \ldots, \pi_s, \lambda) < \nabla\pi_i, e_j > . \tag{13}$$

Now, we have reduced the computation of the equation on the orbit space to a computation of the scalar products $< \nabla\pi_i, e_j >$. Sometimes it is possible to compute these scalar products without knowing explicitly the functions π_i, e_j. In any case it is possible to derive these equations automatically from the invariants and equivariants. The explicit expressions for the invariants and equivariants are often very cumbersome, while the reduced equation has a reasonable form. However, it might be difficult to give a precise interpretation of the results on the orbit space to the full equation.

A *relative equilibrium* is a solution $v(t)$ which is part of a group orbit. Steady states on the orbit space are relative equilibria for the original equation,but need not be equilibria. In a similar fashion periodic orbits on the orbit space are *relative periodic orbits* for the full equation.

There are a few points to be observed. It is in general not true, that $s < \dim(V)$. This seems to indicate a gain in dimension rather than a loss. However it can be shown that the maximal number of algebraically independent generators is $r = \dim(V) - \dim G \leq \dim(V)$. In the reduction process described before one gets r differential equations and $s - r$ algebraic equations. Therefore the reduction to the orbit space leads to an algebro–differential equation, a feature which has not yet been exploited.

The reduction to the orbit space gives some extra tools, which we want to describe by the way of an example, compare [38]. If we look at the local bifurcation for the natural action of the group D_3 on $\mathbf{R}^2$ (which is the same theory as the local bifurcation theory for the 5–dimensional irreducible representation of $\mathbf{O}(3)$, see [22]) then one finds two algebraic independent generators of the algebra on invariant functions, i.e. $r = s = 2$, the orbit space is a subset of $\mathbf{R}^2$. Since one of the invariants can always be chosen as $\pi_1(v) = \|v\|^2$, the range of Π is in the right half plane. It is easy to check, that the invariants can be chosen of degree 2 and 3. Up to a scaling of the invariants one has the following lemma

Lemma 3.8 *The range of Π is equal to $\Delta(\pi_1, \pi_2) \geq 0$ with $\Delta(\pi_1, \pi_2) = \pi_1^3 - 27\pi_2^2$.*

Proof: See [38], or compute the invariants and check. □

Now this function Δ satisfies a differential equation, which can be easily seen:

$$\dot{\Delta} = 6 f_1(\pi_1, \ldots, \pi_s, \lambda)\Delta. \tag{14}$$

It is a general fact, that the algebraic relations describing the boundary of the range of Π give rise to differential equations. In our D_3 example this equation can be used to derive *global* information in a bifurcation problem with topological codimension 3, see [38].

Example 3.9 *Let us look at a D_3–equivariant problem on $\mathbf{R}^2$ (or at the five dimensional absolutely irreducible representation of $\mathbf{O}(3)$) and let us write the equation in form of equation (11). For both cases D_3 or $\mathbf{O}(3)$ this equation has the same form, this is why these two theories are the same. We choose $f_1(\pi_1, \pi_2, \lambda) = \lambda + B_1\pi_2$ and $f_2(\pi_1, \pi_2, \lambda) = A_2\pi_1$. Using the notion of contact equivalence this problem has C^∞–codimension 5 and topological codimension 3. For the computation of these codimensions one can follow the line of [6] or one uses a direct computation to compute the relevant modules using the Gröbner package in some computer algebra system. We do not attempt to describe the behaviour for an unfolding, we just describe an interesting region in parameter space. From the computation of the codimension we find an unfolding of the form*

$$\begin{aligned} f_1 &= \lambda + a_1\pi_1 + \varepsilon_1\pi_2 \\ f_2 &= c + a_2\pi_1 + \varepsilon_2\pi_2, \end{aligned}$$

where ε_1 is near B_1, a_2 is near A_2 and c, ε_2 and a_1 are close to 0. Choosing the parameters such that $a_1\varepsilon_2 - a_2\varepsilon_1 > 0$ and $12a_1^2 + a_2 c < 0$ is satisfied, one finds

Theorem 3.10 *At $\lambda = 0$ the trivial solution $v = 0$ looses stability and a transcritical bifurcation takes place. In the orbit space, we find a secondary bifurcation to steady states and tertiary Hopf branch.*

The proof of this result follows classical lines and is omitted here. Concerning the global behaviour of the branch of periodic solutions we use the global Hopf bifurcation theorem by Alexander & Yorke *[1]. This theorem tells us that one of the following is true*

1. *the amplitude goes to infinity, or*

2. *the the branch is unbounded in parameter space, or*

3. *the period goes to infinity, or finally*

4. *the the closure of the connected component of periodic solutions emanating at our Hopf point contains another Hopf point.*

In our example the third alternative is true, in fact we can show

Theorem 3.11 *Along the connected component of periodic solutions in the orbit space bifurcating at the tertiary Hopf point the minimal period goes to infinity, in fact the closure contains a heteroclinic cycle with two equilibria. The two equilibria have isotropy $\mathbf{Z}_2$, there is one connection in the space with isotropy type $\mathbf{Z}_2$, one connection in the space with trivial isotropy.*

Proof: *We prove the part where we have to use the differential equation for Δ. This is the part where we show that the amplitude of the periodic solutions in the connected component containing the Hopf point in its closure does not approach infinity. Suppose it did. From our assumption concerning f_1 we conclude that there is a $\pi_1^0 > 0$ such that f_1 is of one sign in the domain $\pi_1 > \pi_1^0$. Due to the equation for Δ we conclude that Δ is a Lyapunov function in the domain where $\pi_1 > \pi_1^0$.Hence there exists a number $c_0 > 0$ such that the domains $\Delta > c > c_0$ are positively or negatively invariant (depending on the sign of f_1). From the fact that the curves $\Delta = 0$ and $\Delta = c > 0$ are asymptotically equal, we find that the amplitudes of periodic solutions have to be uniformly bounded.* □

Remark 3.12 1. *Here the result for the $\mathrm{O}(3)$ case looks slightly different, the isotropy $\mathbf{Z}_2$ is replaced by $\mathrm{O}(2)$ and the trivial isotropy by D_2 (which is the principal isotropy in this example)*

2. *Since D_3 is a finite group it is easy to see that equilibria in the orbit space correspond to equilibria in the state space and periodic orbits correspond to periodic orbits. In the case of the continuous group $\mathrm{O}(3)$ this also true but less trivial to see. This property is very specific to the case of the 5-dimensional irreducible representation of $\mathrm{O}(3)$.*

3. *Another difference between the two cases occurs if we allow perturbations which destroy the equivariance property, such that the perturbed equation is only equivariant with respect to a subgroup. In the D_3–case the periodic solutions will lead to periodic solutions in the perturbed equation near the original periodics. In the continuous case a very complicated dynamical behaviour is expected near the manifold of equilibria. This has not yet been completely studied, however it is clear that this question leads to interesting topological and dynamical problems.*

4 Forced symmetry breaking

As before we consider a domain or an embedded compact manifold without boundary $\Omega \subset \mathbf{R}^n$ with a partial differential equation on Ω of the form

$$\frac{\partial u}{\partial t} + Lu = f(u, \lambda), \tag{15}$$

where L is a sectorial operator and f is sufficiently smooth. In the case of a domain Ω we also require boundary values, let us say of the form

$$Bu = \phi,$$

where B is a boundary operator of the form

$$Bu = au + b\frac{\partial u}{\partial n}, \tag{16}$$

with functions a, b, ϕ on the boundary $\partial\Omega$ and n denotes the outer normal unit vector. As said before, if a, b, ϕ are constant (homogeneous boundary conditions) and if the coefficients of L do not depend explicitly on $\omega \in \Omega$ then we have a G_Ω equivariant equation. Forced symmetry breaking may occur through several mechanism which may differ in the physical mechanism, but which can lead to similar mathematical problems. We classify according to the mathematical effects.

Let us first mention some physical situations as perturbations of boundary conditions, perturbations by adding some terms depending explicitly on the state variable or introducing drift. The first example might be physically the most important one, when outside influences perturb the boundary conditions. We might think of non homogeneous temperature distribution in the spherical Bénard problem or to speak about more recent problems the phase locking of high frequency pulses in DFB-lasers to periodic outside signals, compare WÜNSCHE, BANDELOW, FEISTE ET AL. [4, 16, 42].

Associated to Ω we have $G = G_\Omega$ the symmetry group of Ω. We assume that the equation (15) and the boundary operator B, both are equivariant under the group action of G. Let H be a (closed) subgroup of G. The forced symmetry breaking perturbations which we want to study in our mathematical framework are the following

1. Add a function of the form $\varepsilon h(x, u)$ in the equation. More specifically we add terms of the form $h_1(x)g(u)$, where g is a (non-)linear function of u and h_1 is invariant under the action of H. We refer to these perturbations as *class I* perturbations.

2. Add a term of the form $h_1(x) < e(x), g(u)\nabla u >$, where h_1 is H invariant and e is an H–equivariant mapping $\Omega \to \mathbf{R}^n$.These perturbations will be called class II perturbations.

Note that these two classes of perturbations of the symmetry lead to H–equivariant equations. If we perturb in a similar fashion the boundary operator B or the prescribed function ϕ, we can reduce these perturbations to perturbations of the equation on Ω. This does not remain true if we perturb the type of boundary condition, like Dirichlet to mixed or Neumann to mixed, by adding small terms. Then some functional analytic problems arise which have not yet been solved in general, compare ASHWIN & MEI [3].

A specific situation arises when we look at the effects of forced symmetry breaking near a bifurcation point. We begin with the discussion of a steady state bifurcation point. In principle we have different ways to proceed. We could compute the effect of the perturbation on the center manifold and then discuss the finite dimensional problem. There are two main difficulties involved with this approach. First of all symmetry in general leads to multiple eigenvalues. Perturbing the symmetry may split (some of) the eigenvalues and we have several bifurcation points. If we discuss the behaviour near the full set of bifurcation points we run into extremely messy calculations. Near the bifurcation point we will see the branches as they come out of the perturbed points, further away, when the effects of the forced symmetry breaking become smaller (compared with the hyperbolic structure of manifolds of equilibria) we see a slightly distorted picture of the original bifurcation problem. There is very complex recombination of branches and lots of secondary bifurcations going on. Even in the simple example of a spherical problem with the $\ell = 2$–representation on the kernel a perturbation to axisymmetric symmetry leads to almost unsurmountable computational difficulties. This may reflect the following fact. If we consider the G-equivariant problem within the class of H–equivariant problem we could use a singularity theory approach in the sense of [22] to classify these problems. However if G is not a finite group and if the dimension of $\dim(G/H)$ (as a homogeneous manifold) exceeds 0, then any G–equivariant bifurcation problem has codimension infinity, compare GOLUBITSKY & SCHAEFFER [20]. Therefore we study a more specific question, than describing the perturbed flow in a complete neighborhood of the bifurcation point. In order to describe the principal ideas of our approach, let us consider a G–equivariant ODE

$$\dot{x} = f(x, \lambda) \tag{17}$$

and suppose a H–equivariant vector field $h(x)$ is given, Consider

$$\dot{x} = f(x, \lambda) + \varepsilon h(x). \tag{18}$$

Furthermore suppose x_0 is a steady state solution of (17). Then the orbit

$$\mathcal{O}(x_0) = Gx_0$$

is contained in the set of equilibria of (17). Let K denote the isotropy of x_0. Then $\mathcal{O}(x_0)$ is diffeomorphic to G/K.We impose the following hypotheses

H1) $\mathcal{O}(x_0)$ is isolated in the set of equilibria.

H2) $\mathcal{O}(x_0)$ is a normally hyperbolic manifold.

HIRSCH, PUGH & SUB [26] give a detailed theory of normally hyperbolic manifolds. Here, we just need, that normally hyperbolic invariant manifolds are persistent, i.e. if M is such a manifold, then for any vector field sufficiently close to (17) there exists a unique invariant, normally hyperbolic manifold $\tilde{M}$ near M which is diffeomorphic to M, i.e. there exists a diffeomorphism

$$\Psi : M \to \tilde{M}. \tag{19}$$

For a manifold of equilibria to be normally hyperbolic it is necessary and sufficient that at each point $x \in M$ the linearization of the vector field has precisely $\dim(M)$ eigenvalues on the imaginary axis and all the others off the imaginary axis. Applying this concept to our present situation, LAUTERBACH & ROBERTS [36] have shown, that for each sufficiently small H–equivariant perturbation h (18) of (17) and for each normally hyperbolic manifold of equilibria there exists a unique invariant manifold $\tilde{M}$ for (18) near $\mathcal{O}(x_0)$ which is H–equivariantly diffeomorphic to $\mathcal{O}(x_0)$, i.e. the diffeomorphism Ψ is H–equivariant. We follow the exposition in [36]) and start with the observation that there is an action of H on $\tilde{M}$, and since $\tilde{M}$ is H–equivariantly diffeomorphic to G/K, and this homogeneous space is a H–space, i.e. there is a natural action of H on G/K we find that the a–priori unknown manifold $\tilde{M}$ is diffeomorphic to G/K, with the natural action of H on G/K. This action is given by multiplication

$$h[g]_K = [hg]_K. \tag{20}$$

In general this manifold $\tilde{M}$ does not consist of equilibria, but it carries a nontrivial flow. Our aim is to describe some properties of this flow. Now it is possible to classify H–equivariant flows on G/K, a program which was initiated and carried through in [36] for some examples with $G = \mathbf{SO}(3)$ and H, K closed subgroups of G. In that paper possible flows for $H = \mathbf{T}$ and $K = \mathbf{SO}(2)$ (or vice versa) were classified and in the case of ODE's it was possible to construct flows with heteroclinic cycles. The main observation is a description of the precise location of the fixed point space for the action of

K on G/H. The fixed point space of a subgroup H_1 of this action is given by the set of points where

$$h[g]_K = [g]_K \ \forall h \in H_1. \tag{21}$$

This is satisfied if and only if $g^{-1}hg \in K$ for all $h \in H_1$, or $g^{-1}H_1g \subset K$. Therefore we see

$$\mathrm{Fix} H_1 = \{g \in G \mid g^{-1}H_1g \subset K\}/K. \tag{22}$$

We denote this set $\{g \in G \mid g^{-1}H_1g \subset K\}$ by $N(H_1, K)$. This set was introduced by IHRIG & GOLUBITSKY [28]. Some properties of $N(H_1, K)$ were derived in [28]. In the spherical case the computations of $N(H_1, K)$ for all pairs of subgroups was started in [33], and continued in [11, 36]. Now all fixed point spaces for actions of groups H on G/K for $G = \mathbf{O}(3)$ and closed subgroups H, K are available [35, 43]. From (22) the fixed point spaces for subgroups can be characterized in a purely algebraic fashion. These fixed point spaces are flow invariant which gives severe restrictions on the flow. Pictures of the geometry of some of these spaces can be found in [36, 35, 39]. The main idea in [35] is to give group theoretic conditions for heteroclinic cycles in problems with forced symmetry breaking. This is translated into a graph theoretical problem using the stratification of the double quotient $H\backslash G/K$ into orbit types for the action of H on G/K. In this context we find a notion which is similar to Krupa's notion of a robust heteroclinic cycle [29].

Definition 4.1 *A point $[g]_K$ which is isolated in its stratum is called a* group theoretic equilibrium. *A* group theoretic connection of two equilibria *is a one dimensional fixed point space, containing both equilibria ξ_1, ξ_2 and an arc with endpoints ξ_1, ξ_2 containing no other group theoretic equilibria. A collection of group theoretic equilibria $\xi_1, \ldots, \xi_m$ and of one dimensional fixed point spaces $V_1, \ldots, V_k$ is called a* group theoretic cycle *if we can find a directed closed path consisting of group theoretic equilibria and of arcs on group theoretic connections.*

An application of the theoretical results to problems with spherical symmetry yields

Theorem 4.2 *Given an ODE of the form (17) which is equivariant with respect to $\mathbf{O}(3)$. Suppose a normally hyperbolic orbit of equilibria with isotropy type K is given. A necessary condition for the occurrence of group theoretical cycles is that either $H = \mathbf{T}$ or $K = \mathbf{T}$.*

In [43, 35] all graphs associated to forced symmetry breaking in problems with spherical symmetry are computed. From this one gets a complete list of group theoretical cycles. There is duality between the dynamics associated to the pair (H, K) and the one corresponding to the pair (K, H). Here again lifting theorem (SCHWARZ [46]) is used.

In order to apply this to PDE's we do not use center manifold reductions, but we compute a approximation to the group theoretic cycle in the Banach space and determine the flow on this cycle. Let us first define the notation: suppose a G–equivariant equation (2) is given and defines a semidynamical system on $H^1(\Omega)$. Assume

1. $u = 0$ is a solution for all $\lambda \in \mathbf{R}$.
2. For λ_0 the linearization at the trivial solution has a nontivial kernel V.
3. V is an absolutely irreducible representation of G.
4. K is an isotropy subgroup for this action on V and has a one-dimensional fixed point space.
5. The hypotheses of the equivariant branching lemma are satisfied.
6. The bifurcating branch of steady states with isotropy K is normally hyperbolic. Observe that this is a generic property, compare FIELD [17].

Suppose that we perturb equation (2) by an H–equivariant term as described above. We would like to compute the group orbit of bifurcating solution and then the nearby invariant manifold for the perturbed system. However the second step is very difficult. As an approximation we compute a group orbit in the kernel V of a point with isotropy K. Observe, that all these points lie in a one dimensional subspace. Up to a scaling by a real parameter s we get a unique group orbit, of the form sGv_0, where s is the real parameter and v_0 is a unit vector with isotropy K. On this orbit we can compute the group theoretical cycle. This is a purely group theoretic data and does not depend on the equation or on its perturbation. For each s, sufficiently small, we find a unique orbit M_s of steady states of (2), just use the mapping $\sigma : V \rightarrow V^{\perp}$ describing the center manifold, here $V^{\perp}$ denotes a closed complement to V in the Banach space. For the class of problems we have studied it is possible to prove the existence of a closed invariant complement. This mapping transports the group theoretic cycle onto M_s as well. Finally we use the mapping (19) Ψ constructed via normal hyperbolicity to transport all the information to $\tilde{M}_s$. It can be shown (LAUTERBACH & ROBERTS, [37]), that for additive perturbations the flow on the group theoretic cycle can be computed, by computing the scalar product between the tangent vector to the one-dimensional pieces of the cycle and the perturbations, i.e. let $v(\tau)$ be a parametrization of an arc in the group theoretic cycle and let

$$t_\tau = \frac{d}{d\tau} v(\tau)$$

be the tangent vector to the arc at τ. We have

Theorem 4.3 *There exists some $\varepsilon_0 > 0$, such that for $|s| < \varepsilon_0, s \neq 0$ the direction of the flow on the group theoretic cycle on $\tilde{M}_s$ at $v(\tau)$ is given by the scalar product*

$$\langle t_\tau, h\rangle_{L_2(\Omega)}, \tag{23}$$

where εh denotes perturbation, of either form.

Using this result one can study what kind of perturbations h lead to heteroclinic cycles (on the group theoretic cycle). In MAIER–PAAPE & LAUTERBACH [39] this result is used to investigate forced symmetry breaking near the $\ell = 2$ bifurcation for a problem with spherical symmetry.

Theorem 4.4 *Consider a PDE of form (2) which is equivariant with respect to $\mathbf{O}(3)$. Suppose for all $\lambda \in \mathbf{R}$ $u = 0$ is a solution which changes stability at λ_0. Suppose moreover that the kernel V of the linearization at $u = 0, \lambda_0$ is the $\ell = 2$–representation of $\mathbf{O}(3)$. Then there exists a branch of axisymmetric solutions. We consider perturbations with $H = \mathbf{T} \oplus \mathbf{Z}_2^-$–equivariance. Then there exists an open set of perturbations (in the space of H-equivariant perturbations of class 1 and class 2 in the $C(\overline{\Omega})$ topology) which lead to heteroclinic cycles.*

Proof: The main difficulty is to study the type of perturbations leading to heteroclinic cycles. The classification is based on a detailed study of the invariant theory for the exceptional subgroups of $\mathbf{O}(3)$. The details can be found in [39]. □

A similar theory can be developed for the perturbations of Hopf bifurcations, this is work in progress. Different techniques to investigate perturbations of Hopf branches were developed by CHOSSAT and FIELD [9], for applications in physics, see DANGELMAYR & KNOBLOCH [13], and HIRSCHBERG & KNOBLOCH [27].

So far we have looked at forced symmetry breaking of continuous groups. In the case of finite groups these techniques cannot work. SANDSTEDE & SCHEEL [44] look at the problem of forced symmetry breaking for finite groups. By projecting on the orbit space they find a codimension 2 homoclinic bifurcation. This leads to various periodic orbits, heteroclinic cycles and even geometric Lorenz attractors.

5 Heteroclinic cycles and invariant planes

A typical scenario for the creation of heteroclinic cycles in equivariant systems is the following (in the simplest possible case). Assume that G contains subgroups H_0, H_1 and H_2 with

1. $H_0 \supset H_1$ and $H_0 \supset H_2$, and
2. (a) $\dim\mathrm{Fix}(H_0) = 1$,
   (b) $\dim\mathrm{Fix}(H_1) = 2$ and
   (c) $\dim\mathrm{Fix}(H_2) = 2$.

Moreover, we assume that there are two nontrivial hyperbolic fixed points in $\mathrm{Fix}(H_0)$, say v_1, v_2 such that the unstable manifold of v_1 intersects $\mathrm{Fix}(H_1)$ in a one dimensional manifold, and so does the stable manifold of v_1 with $\mathrm{Fix}(H_2)$. For stable and unstable manifolds of the point v_2 we require the opposite inclusions, i.e. we have

$$\begin{aligned} \dim\left(W^u(v_1) \cap \mathrm{Fix}(H_1)\right) = \dim\left(W^s(v_1) \cap \mathrm{Fix}(H_2)\right) &= \\ \dim\left(W^u(v_2) \cap \mathrm{Fix}(H_2)\right) = \dim\left(W^s(v_2) \cap \mathrm{Fix}(H_1)\right) &= 1, \end{aligned}$$

see figure 1.

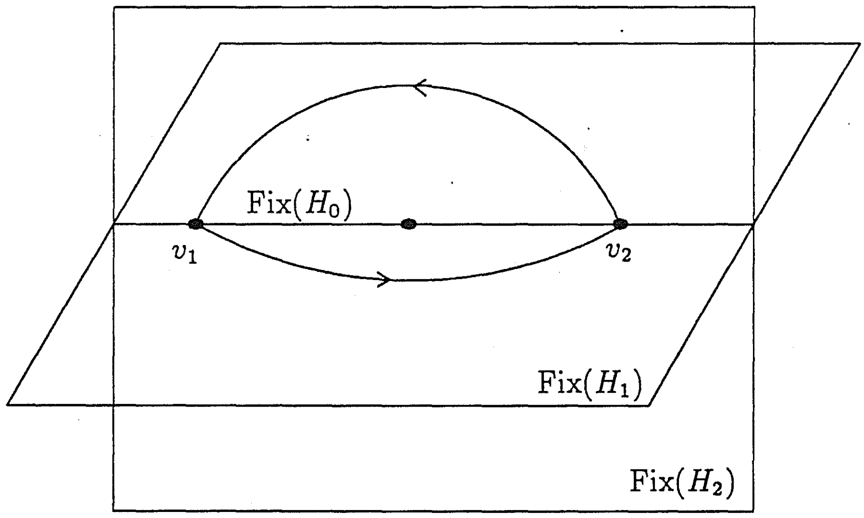

Figure 1: The geometry of the fixed point planes with a heteroclinic cycle.

This type of heteroclinic cycle is called *robust heteroclinic cycle* in KRUPA [29].

Of course, there might be several groups conjugate to H_1 or H_2 contained in H_0. In [11] the number $n(H, K)$ is introduced as the number of conjugate copies of H contained in K. CHOSSAT & GUYARD [10] make a distinction between the two cases

1. $n(H_j, H_0) = 1$ for $j = 1, 2$ and

2. $n(H_j, H_0) > 1$ for $j = 1$ or $j = 2$.

A nice and simple example for this scenario is due to GUCKENHEIMER & HOLMES [23]. They give a vector field on $\mathbf{R}^3$ which is equivariant with respect to the group $\mathbf{T} \oplus \mathbf{Z}_2^r$ of all rigid motions of a regular tetrahedron $\mathbf{T}$ together with $\mathbf{Z}_2^r$ which acts as a reflection at one of the coordinate planes. The subgroups of the form $\mathbf{Z}_2 \oplus \mathbf{Z}_2^r$ have a one dimensional fixed point subspace. They contain two subgroups of order 2. Consider the vector field

$$\begin{aligned}
\dot{x} &= \lambda x + x(ax^2 + by^2 + cz^2) \\
\dot{y} &= \lambda y + y(ay^2 + bz^2 + cx^2) \\
\dot{z} &= \lambda z + z(az^2 + bx^2 + cy^2).
\end{aligned}$$

This vector field has the right equivariance property, therefore we find the three coordinate planes as invariant subspaces. Choosing the parameter values $a < 0, , \lambda > 0$ and either $b < a < c$ or $c < a < b$ we obtain a pair of non-trivial equilibria on each coordinate line and a heteroclinic orbit connected them, compare [23, 44]. This gives a heteroclinic cycle involving 3 equilibria. This example is slightly more complicated than the scenario shown in figure 1.

An application of this technique to problems with spherical symmetry was given in CHOSSAT & GUYARD [10]. It can be shown that in irreducible representations of $\mathbf{O}(3)$ there is no possibility of a local steady state bifurcation giving rise to a heteroclinic cycle through this scenario. In fact the bifurcation equations have some variational structure ([45] to prohibit heteroclinic cycles. The interest in heteroclinic cycles in spherical problems comes partly from geophysics. Such cycles could be a model for the change of orientation of the earths magnetic field. There are some indications that the relevant bifurcations come from mode interactions involving several irreducible representations of $\mathbf{O}(3)$. A systematic investigation of the scenario described in [41] in mode interactions for problems with spherical symmetry was done by CHOSSAT & GUYARD [10, 24]. They study two types of mode interactions, the $\ell = 2, \ell = 6$ mode interaction and the interactions of type $\ell, \ell + 1$. Here ℓ stands for the $2ell + 1$-dimensional representation of $\mathbf{O}(3)$. These studies follow some earlier work of CHOSSAT & ARMBRUSTER [2, 8], where heteroclinic cycles in the $(1, 2)$ mode interaction were found. Concerning the $(\ell, \ell+1)$ mode interaction CHOSSAT & GUYARD [10] give a complete list of heteroclinic cycles which can be constructed with the invariant planes scenario. To describe the results concerning the spherical symmetric case we follow the notation in [11, 22, 28].

Theorem 5.1 *Consider the spherical Bénard problem and let $\ell > 1$. If the loss of stability of the purely heat conducting solution leads to a kernel with* *the $(\ell, \ell+1)$ mode interaction, then there exists an open region in parameter space and an open neighborhood U of the bifurcation point such that for each parameter value in the open region there exists at least one heteroclinic cycle in U connecting two $\mathbf{O}(2) \oplus \mathbf{Z}_2^c$ symmetric points. If $\ell = 8$ then besides this heteroclinic cycle there is another one connecting two $\mathbf{O} \oplus \mathbf{Z}_2^c$ symmetric points.*

Observe that the functions invariant under the group $\mathbf{O}(2)$ are axisymmetric, the ones invariant under $\mathbf{O}$ have the symmetry of a cube. $\mathbf{Z}_2^c$ stands for the group generated by $x \mapsto -x$ in R^3.

Proof: The proof consists of two parts. We begin with a group theoretic verification of the geometry of the fixed point subspaces. There is a necessary condition on the partial ordered set (po-set) of isotropy subgroups, namely the occurrence of a subgraph of the form indicated in figure 2. This gives

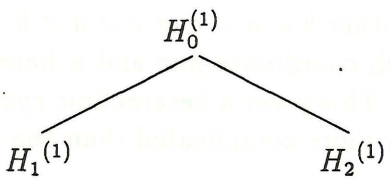

Figure 2: In order to find heteroclinic cycles a part of the po-set of isotropy subgroups has to have the indicated form The numbers indicate the dimension of the corresponding fixed point subspace.

the possibility of a heteroclinic connection in $\text{Fix}(H_1)$ and in $\text{Fix}(H_2)$ and in the fixed point subspaces of groups $H_k \subset H_0$ conjugate to H_1 or to H_2. The number of such conjugate subgroups plays a crucial rôle. In order to establish the existence of heteroclinic connections one has to look at the vector fields restricted to these subspaces. For the genericity statement one has to show that for open regions in parameter space the equations give rise to a steady state bifurcation of a pair of points in $\text{Fix}(H_0)$ which have the correct stability assignments within $\text{Fix}(H_{1,2})$ and moreover one has to show that the stable or unstable manifold cannot go off to infinity. □

Remark 5.2 *For a proof that the generic hypotheses are satisfied in a given system one has to study the specific equation. Here, the Clebsch–Gordan coefficients allow to gather sufficient information to prove the existence of the heteroclinic cycles as asserted.*

Remark 5.3 *This result cannot be directly applied to the geophysical problem, one reason is the earths rotation. Taking it into account the problem can be treated as a forced symmetry breaking to a $\mathbf{SO}(2)$-equivariant problem.*

Some work in this direction has been done by CHOSSAT [7]. *The study of the behaviour of the heteroclinic cycles under this symmetry breaking perturbations is under way and promises some interesting dynamical effects.*

Remark 5.4 *It is amusing to note that the group theoretic computations to verify the necessary condition in the mode interaction case are very similar to the group theoretic computations for the forced symmetry breaking analysis, compare* GUYARD [24], LAUTERBACH & ROBERTS [36], LAUTERBACH, MAIER & REISSNER [35] *and* REISSNER [43].

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