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Parabolic equations with dynamical boundary conditions and source terms on interfaces

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ABSTRACT. We consider parabolic equations with mixed boundary conditions and domain inhomogeneities supported on a lower dimensional hypersurface, enforcing a jump in the conormal derivative. Only minimal regularity assumptions on the domain and the coefficients are imposed. It is shown that the corresponding linear operator enjoys maximal parabolic regularity in a suitable L^p-setting. The linear results suffice to treat also the corresponding nondegenerate quasilinear problems.

1. INTRODUCTION

In this article we are interested in the linear parabolic initial-boundary value problem

$$\varepsilon\partial_t u - \nabla\cdot\mu\nabla u = f_\Omega \qquad \text{in } J\times\Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } J\times(\partial\Omega\setminus\Gamma), \tag{1.2}$$

$$\varepsilon\partial_t u + \nu\cdot\mu\nabla u + bu = f_\Gamma \qquad \text{on } J\times\Gamma, \tag{1.3}$$

$$\varepsilon\partial_t u + [\nu_\Sigma\cdot\mu\nabla u] = f_\Sigma \qquad \text{on } J\times\Sigma, \tag{1.4}$$

$$u(0) = u_0 \qquad \text{in } \Omega\cup\Gamma, \tag{1.5}$$

and in its quasilinear variants. Here $J = (0,T)$ is a bounded time interval, $\Omega\subset\mathbb{R}^d$ is a bounded domain, $\Gamma\subseteq\partial\Omega$ is a part of the boundary with outer normal ν, and $\Sigma\subset\Omega$ is e.g. a finite union of hypersurfaces, equipped with a normal field ν_Σ. By $[\nu_\Sigma\cdot\mu\nabla u]$ we denote the jump of $\nu_\Sigma\cdot\mu\nabla u$ over Σ. The case that Γ or Σ is an empty set is not excluded. We treat a nonsmooth geometry; e.g., it suffices that Γ and Σ satisfy certain Lipschitz conditions. Nothing is supposed on the Dirichlet part $\partial\Omega\setminus\Gamma$ of the boundary, and the boundary parts Γ and $\partial\Omega\setminus\Gamma$ are allowed to meet.

Also on the coefficients we impose only low regularity conditions. The (possibly nonsymmetric) coefficient matrix μ is bounded and uniformly elliptic, ε is positive, bounded and bounded away from zero, and b only has to live in an L^p-space. The (possibly nonautonomous) inhomogeneities f_Ω, f_Γ, f_Σ and the initial value u_0 are assumed to be given.

Parabolic problems with dynamical boundary conditions are considered by many authors, see e.g. [AmE], [Esc], [Hin] [AQRB], [BBR] and [BC], but there always severe assumptions on the data, as smoothness, are imposed (compare also [FGGR] and [VV], where the boundary condition on $J\times\Gamma$ is understood as Wentzell's boundary condition). It is the aim of this work to show that any smoothness assumption on the domain and the coefficient function μ can be avoided. In particular, the domain Ω does not need to be a Lipschitz domain. Moreover, we allow the inhomogeneities not only to live in the volume of the domain, but to incorporate a part which is supported on the set Σ of lower dimension $d-1$. This largely extends the applicability of the theory to real-world problems. The reader may think, e.g., of a heat source which is concentrated on an interface. Alternatively, one meets such constellations in electricity: surface charge

densities induce a jump in the normal component of the dielectric displacement, see e.g. [Tam, Chapter 1].

Our approach to (1.1)–(1.5) is essentially based on the theory of sesquilinear forms and the suitable incorporation of the boundary conditions into an L^p-space.

Let us consider the approach in more detail. The boundary part $\overline{\Gamma}$ is Lipschitz regular, and the interface $\Sigma \subset \Omega$ is a $(d-1)$-set in the sense of Jonsson–Wallin [JW] (cf. Assumptions 2.2 and 2.4). For the equations we first treat the case $\varepsilon \equiv 1$, and consider the sesquilinear form

$$\mathfrak{t}[u,v] = \int_\Omega \mu \nabla u \cdot \overline{\nabla v}\, dx,$$

which is defined on the space $W^{1,2}_\Gamma$ of $W^{1,2}(\Omega)$-functions vanishing on $\partial\Omega \setminus \Gamma$. Note that this reflects the Dirichlet conditions. For $u \in W^{1,2}_\Gamma$ we define the trace $\operatorname{tr} u$ on $\Gamma \cup \Sigma$ in a suitable sense (based on [JW]), and show that the map $\mathfrak{J}u = (u, \operatorname{tr} u)$ is continuous and has dense range from $W^{1,2}_\Gamma$ into $\mathbb{L}^2 := L^2(\Omega) \oplus L^2(\Gamma \cup \Sigma; d\mathcal{H}_{d-1})$ (see Lemma 2.10). Here $\mathcal{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. These properties of the trace are a consequence of the regularity of Γ and Σ. As the form $\mathfrak{t}$ satisfies an ellipticity condition with respect to $\mathfrak{J}$, the results in [AE] imply that $\mathfrak{t}$ induces an operator A_2 on $\mathbb{L}^2$, and $-A_2$ generates a holomorphic C_0-semigroup of contractions.

In Remark 2.15 we show that in a smooth situation, the operator A_2 consists of three parts, corresponding to the spatial derivatives occurring in (1.1), (1.3) and (1.4), respectively: the operator $-\nabla \cdot \mu \nabla$ on Ω, the conormal derivative $\nu \cdot \mu \nabla$ on Γ, and the jump $[\nu_\Sigma \cdot \mu \nabla u]$ of the conormal derivative on Σ. Hence the dynamic equations on Γ and Σ are modelled by the part $L^2(\Gamma \cup \Sigma; d\mathcal{H}_{d-1})$ of the base space $\mathbb{L}^2$.

The semigroup generated by $-A_2$ turns out to be submarkovian, and may thus be extended to a semigroup of contractions on $\mathbb{L}^p$ for all $p \in [1,\infty]$. Denoting the corresponding generators by $-A_p$, it turns out that for all $p \in (1,\infty)$ the operator $-\varepsilon^{-1}A_p$ generates a holomorphic C_0-semigroup of contractions on a suitably renormed $\mathbb{L}^p$-space. This has two important consequences. First, applying an abstract result that is presented e.g. in [LX, Proposition 2.2], we obtain a bounded holomorphic functional calculus for $\varepsilon^{-1}A_p$ with angle strictly smaller than $\frac{\pi}{2}$, and in particular the boundedness of the purely imaginary powers (see Theorem 2.23). Moreover, the pioneering theorem of Lamberton [Lam] gives us maximal parabolic regularity for $\varepsilon^{-1}A_p$ in Theorem 3.4, which we consider as the main result of this work. The introduction of temporal weights as in [PS] further allows to reduce the regularity of the initial data almost up to the base space $\mathbb{L}^p$. This yields the solution of (1.1)–(1.5) in an adequate manner, see Theorem 3.10 and again the considerations in Remark 2.15.

Based on these linear results we treat a nondegenerate quasilinear variant of (1.1)–(1.5), even if the right hand side explicitly and discontinuously depends on time (Theorem 4.5). Here a difficulty is that the domain of the realization of the operator $-\nabla \cdot \mu \nabla$ on $\mathbb{L}^p$ is not independent

of the coefficients μ. We therefore consider a problem which is obtained when applying the Kirchhoff transform to the original one, and which involves only one fixed operator (see Definition 4.2). Maximal parabolic regularity then allows to apply a result of Prüss [Prü] (see also [ClL]) to the transformed problem, giving local existence and uniqueness of solutions in a suitable sense. Throughout it is essential that $\mathrm{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$ for large p and θ sufficiently close to 1, which is a consequence of ultracontractivity estimates for the semigroup (see Lemma 2.20 and Proposition 3.5). The quasilinear problems may be of relevance for the applications: the heat source on the hypersurface can depend on the solution itself, and, additionally, explicitly on time.

Let us briefly compare the approach in this paper with those in [Gri2], [HaR1] and [HaR3] for static Robin boundary conditions. There the Banach space under consideration is a negatively indexed Sobolev space of type $H^{-\theta,q}$ or a Sobolev–Morrey space. In contrast to that settings, in $\mathbb{L}^p$ one may form the dual pairing of the above parabolic equation with the indicator function χ_Λ of suitable subsets $\Lambda \subset \Omega$. Then one may, additionally, apply Gauss' theorem to $\langle -\nabla \cdot \mu\nabla u, \chi_\Lambda\rangle = \int_\Lambda -\nabla\cdot\mu\nabla u\,dx + \int_{\Lambda\cap\Sigma} -\nabla\cdot\mu\nabla u\,d\mathcal{H}_{d-1}$. This allows to recover the underlying physical balance law for the parabolic equation, which is the starting point for the numerical treatment of such problems. For more details we refer to Remark 3.13.

This paper is organized as follows. In Section 2 we introduce the spaces $\mathbb{L}^p$, define an appropriate realization of $-\nabla\cdot\mu\nabla$ and show that it admits a bounded holomorphic functional calculus. In Section 3 we show that in this setting (1.1)–(1.5) enjoys maximal parabolic regularity, and in Section 4 we treat the quasilinear case.

2. Elliptic operators on $\mathbb{L}^p$

2.1. **Notation.** Throughout this paper $\mathcal{L}(X;Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are Banach spaces. If $X = Y$, then we abbreviate $\mathcal{L}(X)$. Note that if X and Y are two Banach spaces spaces such that $X \subset Y$ as vector spaces, and both X and Y are continuously embedded in a Hausdorff locally convex space, then the inclusion map from X into Y is continuous by the closed graph theorem.

In the sequel let Ω be a bounded domain in $\mathbb{R}^d$ with $d > 1$ and Γ an open part of its boundary $\partial\Omega$, which may be empty. If $p \in [1,\infty)$, then $L^p(\Omega)$ is the space of complex-valued, Lebesgue measurable, p-integrable functions on Ω, and for all $\theta \in [0,1]$ we denote by $W^{\theta,p}(\Omega)$ the usual Sobolev–Slobodetskii spaces, see [Gri] or [Maz]. Moreover, $L^\infty(\Omega)$ is the space of Lebesgue measurable, essentially bounded functions on Ω. The $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$ is denoted by $\mathcal{H}_{d-1}$. We denote by $B(x,r)$ the ball in $\mathbb{R}^d$ centred at x with radius r.

2.2. **The function spaces.** In this subsection we consider the function spaces on which (1.1)–(1.5) will be posed.

Definition 2.1. For all $q \in [1,\infty]$ we define $W^{1,q}_\Gamma$ as the closure in $W^{1,q}(\Omega)$ of the set

$$C^\infty_\Gamma(\Omega) \stackrel{\text{def}}{=} \Big\{ u|_\Omega : u \in C^\infty_c(\mathbb{R}^d),\, \operatorname{supp}(u) \cap (\partial\Omega \setminus \Gamma) = \emptyset \Big\}.$$

Throughout this paper we make the following assumption on Γ.

Assumption 2.2. For all $x \in \overline{\Gamma}$ there is an open neighbourhood $\mathcal{V}_x$ of x and a bi-Lipschitz mapping F_x from $\mathcal{V}_x$ onto the open unit cube E in $\mathbb{R}^d$, such that $F_x(x) = 0$ and $F_x(\Omega \cap \mathcal{V}_x)$ is equal to the lower open half cube $E_- = (-1,1)^{d-1} \times (-1,0)$ of E.

The reader should notice that the domain Ω does not need to be Lipschitzian. Moreover, nothing is supposed on the boundary of Γ within $\partial\Omega$.

An important technical tool is an extension operator for the $W^{1,q}_\Gamma$-spaces.

Proposition 2.3. *There is an extension operator $\mathfrak{E}\colon L^1(\Omega) \to L^1(\mathbb{R}^d)$ such that the restriction $\mathfrak{E}|_{W^{1,q}_\Gamma}$ maps $W^{1,q}_\Gamma$ continuously into $W^{1,q}(\mathbb{R}^d)$ for all $q \in [1,\infty]$, the restriction $\mathfrak{E}|_{L^q(\Omega)}$ maps $L^q(\Omega)$ continuously into $L^q(\mathbb{R}^d)$ for all $q \in [1,\infty]$ and $\operatorname{supp}\mathfrak{E}u \subset B(0,2R)$ for all $u \in L^1(\Omega)$, where $R = \sup\{|x| : x \in \Omega\}$.*

Proof. The proof is given in [ER, Lemma 3.4] for the case $q = 2$, but carries over to all $q \in [1,\infty]$. Moreover, the second assertion is also easily checked. The last statement follows by multiplication with a suitable $C^\infty_c(\mathbb{R}^d)$-function. □

It turns out that a classical condition from geometric measure theory is tailor made in order to define a geometric assumption on a $(d-1)$-dimensional shape Σ in Ω.

Assumption 2.4. Let $\Sigma \subset \Omega$ be a $(d-1)$-set in the sense of Jonsson–Wallin [JW, Subsection VII.1.1]. Precisely: the set Σ is Borel measurable and there exist $c_1, c_2 > 0$ such that

$$c_1 r^{d-1} \le \mathcal{H}_{d-1}\big(B(x,r) \cap \Sigma\big) \le c_2 r^{d-1} \tag{2.1}$$

for all $x \in \Sigma$ and $r \in (0,1)$.

Remark 2.5. We emphasize that Σ does not have to be closed. Nevertheless has Σ *finite* $(d-1)$-dimensional Hausdorff measure, according to (2.1). The prototype of Σ is the finite union $\bigcup_j \Sigma_j$ of Lipschitzian hypersurfaces. In that case the restriction of the Hausdorff measure $\mathcal{H}_{d-1}$ to Γ or to Σ_j can be constructed explicitly in terms of the local bi-Lipschitz charts (compare [EG, Section 3.3.4 C]). In particular, if Σ is a finite union of Lipschitz graphs, then (2.1) is easily verified using this representation of $\mathcal{H}_{d-1}$. Moreover, Assumption 2.4 implies for general Σ that Σ is of (d-dimensional) Lebesgue measure 0.

Throughout this paper we always presume Assumptions 2.2 and 2.4.

Definition 2.6. We denote by ρ the restriction of the Hausdorff measure $\mathcal{H}_{d-1}$ to $\Gamma \cup \Sigma$.

If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $F \subset \mathbb{R}^d$ is a set, then define the function $\operatorname{tr}_F u$ as in [JW, Page 15] by

$$(\operatorname{tr}_F u)(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy,$$

for all $x \in F$ for which the limit exists. The domain $\operatorname{Dom}(\operatorname{tr}_F u)$ of $\operatorname{tr}_F u$ is the set of all $x \in F$ for which this limit exists.

Lemma 2.7. *Let $q, r \in [1, \infty)$ and $\theta \in [0,1]$. Let $\mathfrak{E}$ be the extension operator as in Proposition 2.3.*

(i) *If $\frac{1}{q} - \frac{1-\theta}{d} \leq \frac{1}{r}$, then $\mathfrak{E}$ maps $W^{1,q}_\Gamma$ continuously into $W^{\theta,r}(\mathbb{R}^d)$.*

(ii) *If $\frac{1}{q} - \frac{1-\theta}{d} < \frac{1}{r}$, then $\mathfrak{E}$ maps $W^{1,q}_\Gamma$ compactly into $W^{\theta,r}(\mathbb{R}^d)$.*

(iii) *If $\theta \in (\frac{1}{q}, 1]$, then the trace map $u \mapsto \operatorname{tr}_{\Gamma\cup\Sigma} u$ is continuous from $W^{\theta,q}(\mathbb{R}^d)$ into $L^q(\Gamma \cup \Sigma; d\rho)$.*

Proof. '(i)' and '(ii)'. This follows from Proposition 2.3, the support property of $\mathfrak{E}$ and the usual Sobolev embedding.

'(iii)'. Since Γ and Σ are disjoint, the natural map from the space $L^q(\Gamma\cup\Sigma; d\rho)$ into $L^q(\Sigma; d\mathcal{H}_{d-1}) \times L^q(\Gamma; d\mathcal{H}_{d-1})$ is a linear, topological isomorphism. Therefore, it suffices to show that the trace maps $u \mapsto \operatorname{tr}_\Gamma u$ and $u \mapsto \operatorname{tr}_\Sigma u$ are continuous from $W^{\theta,q}(\mathbb{R}^d)$ into $L^q(\Gamma; d\mathcal{H}_{d-1})$ and $L^q(\Sigma; d\mathcal{H}_{d-1})$.

It follows from [JW, Chapter VIII, Proposition 1] that property (2.1) inherits to the closure $\overline{\Sigma}$ of Σ. Then the trace operator $u \mapsto \operatorname{tr}_{\overline{\Sigma}} u$ is bounded from $W^{\theta,q}(\mathbb{R}^d)$ into $L^q(\overline{\Sigma}; d\mathcal{H}_{d-1})$ by [JW, Chapter V, Theorem 1]. But the set difference $\overline{\Sigma} \setminus \Sigma$ is of $\mathcal{H}_{d-1}$ measure 0 (see again [JW, Chapter VIII, Proposition 1]). Consequently the spaces $L^q(\overline{\Sigma}; d\mathcal{H}_{d-1})$ and $L^q(\Sigma; d\mathcal{H}_{d-1})$ are identical.

Next we consider the set Γ. Using the notation as in Assumption 2.2, for every $x \in \overline{\Gamma}$ the map F_x provides a bi-Lipschitz parametrization of $\partial\Omega \cap \mathcal{V}_x$, where the parameters run through the upper plate $P := (-1,1)^{d-1} \times \{0\}$ of the half cube E_-. Moreover, the Hausdorff measure $\mathcal{H}_{d-1}$ on $\partial\Omega \cap \mathcal{V}_x$ is the surface measure, and the latter is obtained from the Lebesgue measure on $(-1,1)^{d-1} \times \{0\}$ via the bi-Lipschitzian parametrization, see [EG, Section 3.3.4 C]. Define $\mathcal{W}_x = F_x\big((-\frac{1}{2}, \frac{1}{2})^{d-1} \times \{0\}\big)$. Then $\mathcal{W}_x \subset \partial\Omega$. There exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \overline{\Gamma}$ such that $\mathcal{W}_{x_1}, \ldots, \mathcal{W}_{x_n}$ is a finite cover of $\overline{\Gamma}$. Obviously, $\overline{\mathcal{W}_{x_1}}, \ldots, \overline{\mathcal{W}_{x_n}}$ is also a finite cover of $\overline{\Gamma}$. Moreover, it is not hard to see that $\bigcup_{j=1}^n \overline{\mathcal{W}_{x_j}}$ is a $(d-1)$-set in the sense of Jonsson–Wallin (compare [HaR2, Lemma 3.2]). Hence by [JW, Chapter V, Theorem 1] there exists a continuous trace operator from $W^{\theta,q}(\mathbb{R}^d)$ into $L^q(\cup_{j=1}^n \overline{\mathcal{W}_{x_j}}; d\mathcal{H}_{d-1})$. Combining this operator with the restriction operator to Γ, one obtains the desired trace operator into $L^q(\Gamma; d\mathcal{H}_{d-1})$. □

For all $u \in L^1_{\mathrm{loc}}(\Omega)$ define the function $\operatorname{tr} u$ as in [JW, Section VIII.1.1] by

$$\operatorname{Dom}(\operatorname{tr} u) = \Big\{ x \in \Gamma \cup \Sigma : \lim_{r \to 0} \frac{1}{|B(x,r) \cap \Omega|} \int_{B(x,r)\cap\Omega} u(y)\, dy \quad \text{exists} \Big\}$$

and

$$(\operatorname{tr} u)(x) = \lim_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \int_{B(x,r)\cap\Omega} u(y)\,dy$$

for all $x \in \operatorname{Dom}(\operatorname{tr} u)$.

The above defined trace enjoys the following mapping properties.

Proposition 2.8. *Let $q, r \in (1,\infty)$ and suppose that $\frac{d-q}{q} < \frac{d-1}{r}$. Then $\operatorname{tr} u \in L^r(\Gamma\cup\Sigma; d\rho)$ for all $u \in W^{1,q}_\Gamma$, and the map $u \mapsto \operatorname{tr} u$ is compact from $W^{1,q}_\Gamma$ into $L^r(\Gamma\cup\Sigma; d\rho)$.*

Proof. Let $\mathfrak{E}$ be the extension operator as in Proposition 2.3. Then it follows from Lemma 2.7 that $u \mapsto \operatorname{tr}_{\Gamma\cup\Sigma} \mathfrak{E}u$ maps $W^{1,q}_\Gamma$ compactly into $L^r(\Gamma\cup\Sigma; d\rho)$. But if $u \in W^{1,q}_\Gamma$, then we claim that

$$(\operatorname{tr} u)(x) = (\operatorname{tr}_{\Gamma\cup\Sigma} \mathfrak{E}u)(x) \tag{2.2}$$

for $\mathcal{H}_{d-1}$-a.e. $x \in \Gamma\cup\Sigma$. Obviously, this identity holds for $\mathcal{H}_{d-1}$-a.e. $x \in \Sigma$ since $\Sigma \subset \Omega$. For $\mathcal{H}_{d-1}$-a.e. $x \in \Gamma$ we can argue as in the proof of [JW, Chapter VIII, Proposition 2], where the case $\Gamma = \partial\Omega$ is considered. Indeed, the arguments given there are purely local. Since $\mathfrak{E}u \in W^{1,q}(\mathbb{R}^d)$ it follows that for $\mathcal{H}_{d-1}$-a.e. $x \in \Gamma$ there exists a Borel set $E \subset \mathbb{R}^d$ such that $\mathcal{H}_{d-1}(E \cap B(x,r)) = o(r^{d-1})$ and $(\mathfrak{E}u)(x) = \lim\limits_{y\to x,\, y\notin E} (\mathfrak{E}u)(y)$. Using these properties of E, the same arguments as in the last part of the proof given in [JW] establish (2.2). □

The space on which (1.1)–(1.5) will be realized is given as follows.

Definition 2.9. For all $p \in [1,\infty]$, denote by $\mathbb{L}^p$ the Lebesgue space $L^p(\Omega\cup\Gamma; dx + d\rho)$. We denote the space of all real valued functions in $\mathbb{L}^p$ by $\mathbb{L}^p_{\mathbb{R}}$.

Observe that there is a natural topological isomorphism between $\mathbb{L}^p$ and the direct sum $L^p(\Omega)\oplus L^p(\Gamma\cup\Sigma; d\rho)$ and we will identify $\mathbb{L}^p$ with $L^p(\Omega)\oplus L^p(\Gamma\cup\Sigma; d\rho)$ through this natural map.

By Proposition 2.8 we can define the map $\mathfrak{J}\colon W^{1,2}_\Gamma \to \mathbb{L}^2$ by

$$\mathfrak{J}u = (u, \operatorname{tr} u) \in L^2(\Omega)\oplus L^2(\Gamma\cup\Sigma; d\rho) \cong \mathbb{L}^2.$$

Note that one can choose $p > 2$ in Statement (ii) of the next lemma.

Lemma 2.10.

(i) *The map $\mathfrak{J}$ is continuous and has dense range.*
(ii) *If $p \in [1,\infty)$ and $(d-2)p < 2(d-1)$, then $\mathfrak{J}W^{1,2}_\Gamma \subset \mathbb{L}^p$.*
(iii) *The map $\mathfrak{J}$ is compact.*

Proof. '(i)'. The continuity follows from Proposition 2.8. Let $f = (f_\Omega, f_\partial) \in L^2(\Omega)\oplus L^2(\Gamma\cup\Sigma; d\rho)$ and suppose that $(\mathfrak{J}u, f)_{L^2(\Omega)\oplus L^2(\Gamma\cup\Sigma; d\rho)} = 0$ for all $u \in W^{1,2}_\Gamma$. We show that $f = 0$. For all $u \in C^\infty_c(\Omega\setminus\overline{\Sigma})$ one has $0 = (\mathfrak{J}u, f) = \int_\Omega u\,\overline{f_\Omega}\,dx$. Since $C^\infty_c(\Omega\setminus\overline{\Sigma})$ is dense in $L^2(\Omega\setminus\overline{\Sigma}) = L^2(\Omega)$ one deduces that $f_\Omega = 0$. Therefore $0 = \int_{\Gamma\cup\Sigma} \operatorname{tr} u\,\overline{f_\partial}\,d\rho$ for all $u \in W^{1,2}_\Gamma$

and in particular for all $u \in C^\infty_\Gamma(\Omega)$. But $\{u|_{\Gamma\cup\Sigma} : u \in C^\infty_\Gamma(\Omega)\}$ is dense in $L^2(\Gamma \cup \Sigma; d\rho)$. So $f_\partial = 0$.

'(ii)'. If $\mathfrak{E}$ is the extension operator as in Proposition 2.3 then it follows from Lemma 2.7(i) that $\mathfrak{E}$ maps $W^{1,2}_\Gamma$ continuously into $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$ with $(d-2)p \leq 2d$. So $W^{1,2}_\Gamma \subset L^p(\Omega)$. Now the statement follows from Proposition 2.8.

'(iii)'. It follows immediately from Lemma 2.7(ii) that the restriction $\mathfrak{E}|_\Omega$ maps $W^{1,2}_\Gamma$ compactly into $L^2(\Omega)$. So the embedding of $W^{1,2}_\Gamma$ into $L^2(\Omega)$ is compact. Also the map tr is compact from $W^{1,2}_\Gamma$ into $L^2(\Gamma \cup \Sigma; d\rho)$ by Proposition 2.8. Therefore the map $\mathfrak{J}$ is compact. □

We end this subsection with a truncation lemma.

Lemma 2.11. *Let $u \in W^{1,2}_\Gamma$ be real-valued. Then $u \wedge \mathbb{1}_\Omega \in W^{1,2}_\Gamma$ and $\mathfrak{J}(u \wedge \mathbb{1}_\Omega) = (\mathfrak{J}u) \wedge \mathbb{1}_{\Omega\cup\Gamma}$.*

Proof. The first statement is shown in the proof of [ER, Theorem 3.1]. The second statement is obvious for real-valued $u \in C^\infty_\Gamma(\Omega)$. Since the maps $u \mapsto \mathfrak{J}(u \wedge \mathbb{1}_\Omega)$ and $u \mapsto (\mathfrak{J}u) \wedge \mathbb{1}_{\Omega\cup\Gamma}$ are continuous on the real version of $W^{1,2}_\Gamma$, the identity carries over to the general case by density. □

2.3. **The operator on $\mathbb{L}^p$.** In this subsection we introduce a differential operator on $\mathbb{L}^p$ that corresponds to the spatial derivatives in (1.1), (1.3) and (1.4).

Throughout the remaining of this paper we adopt the next assumption.

Assumption 2.12. Let $\mu = \{\mu_{k,l}\}_{k,l} \colon \Omega \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ be a measurable map from Ω into the set of real $d \times d$ matrices. We assume that there are $\mu_\bullet, \mu^\bullet > 0$ such that

$$\|\mu(x)\|_{\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)} \leq \mu^\bullet \quad \text{and} \quad \sum_{k,l=1}^d \mu_{k,l}(x)\, \xi_k\, \xi_l \geq \mu_\bullet \sum_{k=1}^d \xi_k^2$$

for all $x \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

We emphasize that μ does not have to be symmetric.

Definition 2.13. Define the sesquilinear form $\mathfrak{t} \colon W^{1,2}_\Gamma \times W^{1,2}_\Gamma \to \mathbb{C}$ by

$$\mathfrak{t}[u, v] = \int_\Omega \mu \nabla u \cdot \overline{\nabla v}\, dx.$$

The form $\mathfrak{t}$ is continuous and

$$\mathrm{Re}\, \mathfrak{t}[u, u] + \|\mathfrak{J}u\|^2_{\mathbb{L}^2} \geq (\mu_\bullet \wedge 1)\|u\|^2_{W^{1,2}_\Gamma} \tag{2.3}$$

for all $u \in W^{1,2}_\Gamma$. Therefore by Lemma 2.10 and [AE, Theorem 2.1] there exists a unique operator A_2 in $\mathbb{L}^2$ such that for all $\varphi, \psi \in \mathbb{L}^2$ one has $\varphi \in \mathrm{Dom}(A_2)$ and $A_2\varphi = \psi$ if and

only if there exists a $u \in W^{1,2}_\Gamma$ such that $\mathfrak{J}u = \varphi$ and

$$\mathfrak{t}[u,v] = (\psi, \mathfrak{J}v)_{\mathbb{L}^2} \tag{2.4}$$

for all $v \in W^{1,2}_\Gamma$.

Proposition 2.14. *The operator A_2 is m-sectorial with vertex 0 and semi-angle $\arctan \frac{\mu^\bullet}{\mu_\bullet}$. Moreover, A_2 has compact resolvent.*

Proof. It follows from [AE, Theorem 2.1] that A_2 is m-sectorial. Let $\varphi \in \mathrm{Dom}(A_2)$ and $u \in W^{1,2}_\Gamma$ with $\mathfrak{J}u = \varphi$. Then $\mathrm{Re}(A_2\varphi,\varphi)_{\mathbb{L}^2} = \mathrm{Re}\,\mathfrak{t}[u,u] \geq 0$. Hence the vertex is 0. Further, one has $\mathrm{Re}\,\mathfrak{t}[u,u] \geq \mu_\bullet \int_\Omega |\nabla \mathrm{Re}\,u|^2 + |\nabla \mathrm{Im}\,u|^2\,dx$ and

$$|\mathrm{Im}\,\mathfrak{t}[u,u]| \leq 2\mu^\bullet \int_\Omega |\nabla \mathrm{Re}\,u||\nabla \mathrm{Im}\,u|\,dx \leq \mu^\bullet \int_\Omega |\nabla \mathrm{Re}\,u|^2 + |\nabla \mathrm{Im}\,u|^2\,dx.$$

Thus $|\arg(A_2\varphi,\varphi)_{\mathbb{L}^2}| \leq \arctan \frac{\mu^\bullet}{\mu_\bullet}$ if $\varphi \neq 0$.

Since the map $\mathfrak{J}$ is compact by Lemma 2.10(iii), the generator has compact resolvent by [AE, Lemma 2.7]. □

Remark 2.15. Let us give a heuristics for the operator A_2. For this purpose, let Ω be a domain with Lipschitz boundary, and let Σ be part of a surface Λ which is piecewise C^1 and decomposes Ω into two subdomains Ω_1 and Ω_2. Assume that the outer normal ν_1 of Ω_1 across Λ equals ν_Σ on $\partial\Omega_1 \cap \Sigma$. Let $\varphi \in \mathrm{Dom}(A_2)$. For simplicity we identify $u \in W^{1,2}_\Gamma$ with $\mathfrak{J}u \in \mathbb{L}^2$. We have by definition

$$\int_{\Omega\cup\Sigma\cup\Gamma} (A_2\varphi)\,\overline{\psi}\,(dx + d\rho) := \int_\Omega \mu\nabla\varphi\cdot\overline{\nabla\psi}\,dx = \int_{\Omega_1} \mu\nabla\varphi\cdot\overline{\nabla\psi}\,dx + \int_{\Omega_2} \mu\nabla\varphi\cdot\overline{\nabla\psi}\,dx$$

for all $\psi \in C^\infty_\Gamma(\Omega)$. Since ψ vanishes on $\partial\Omega\setminus\Gamma$ one can apply Gauss' theorem in the following manner:

$$\int_{\Omega_1} \mu\nabla\varphi\cdot\overline{\nabla\psi}\,dx = \int_{\Omega_1} (-\nabla\cdot\mu\nabla\varphi)\,\overline{\psi}\,dx + \int_{\partial\Omega_1\cap\Gamma} (\nu\cdot\mu\nabla\varphi)\,\overline{\psi}\,d\mathcal{H}_{d-1} + \int_{\Lambda\cap\Omega} (\nu_1\cdot\mu\nabla\varphi)\,\overline{\psi}\,d\mathcal{H}_{d-1}. \tag{2.5}$$

An equation, analogous to (2.5), can also be written for Ω_2. Then the unit normal ν_2 of Ω_2 across Λ equals $-\nu_1$ and one deduces

$$\begin{aligned}\int_{\Omega\cup\Sigma\cup\Gamma} (A_2\varphi)\,\overline{\psi}\,(dx + d\rho) = \int_\Omega (-\nabla\cdot\mu\nabla\varphi)\,\overline{\psi}\,dx + \int_\Gamma (\nu\cdot\mu\nabla\varphi)\,\overline{\psi}\,d\mathcal{H}_{d-1} \\ + \int_{\Lambda\cap\Omega} [\nu_\Sigma\cdot\mu\nabla\varphi]\,\overline{\psi}\,d\mathcal{H}_{d-1},\end{aligned} \tag{2.6}$$

where $[\nu_\Sigma\cdot\mu\nabla\varphi] = \nu_\Sigma\cdot\big(\mu\nabla\varphi|_{\partial\Omega_1\cap\Sigma} - \mu\nabla\varphi|_{\partial\Omega_2\cap\Sigma}\big)$ is the jump in the conormal derivative. Note that $[\nu_\Sigma\cdot\mu\nabla\varphi]$ vanishes on $\Lambda\setminus\overline{\Sigma}$ since the measure ρ is supported in $\overline{\Gamma}\cup\overline{\Sigma}$. Thus, varying ψ suitably and comparing both sides of (2.6), one recognizes that A_2 has in fact three 'components', namely

1 the divergence of the vector field $\mu\nabla\varphi$, which takes $L^2(\Omega)$-functions as values,
2 the conormal derivative on Γ, which takes $L^2(\Gamma; d\mathcal{H}_{d-1})$-functions as values,
3 the jump in the conormal derivative on Σ, which takes $L^2(\Sigma; d\mathcal{H}_{d-1})$-functions as values.

Remark 2.16. Heuristically there is another relation involving $A_2\varphi$, where again $\varphi \in \mathrm{Dom}(A_2)$. Let $f = (f_\Omega, f_\partial) \in \mathbb{L}^2$. Then $f_\Omega \in L^2(\Omega) \subset L^1(\Omega)$ and $f_\partial \in L^2(\Gamma \cup \Sigma, d\rho) \subset L^1(\Gamma \cup \Sigma, d\rho)$. Define the complex measure η on the Borel σ-algebra $\mathcal{B}(\Omega)$ of Ω by

$$\eta(\Lambda) = \int_\Lambda f_\Omega \, dx + \int_{\Lambda\cap\Sigma} f_\partial \, d\rho = \int_\Lambda f \, (dx + d\rho)...$$

Note that η does not depend on the representatives of f_Ω and f_∂. We emphasize that Λ is a subset of Ω, thus $\Lambda \cap \Gamma = \emptyset$. Decomposing f as a linear combination of four positive functions, the monotone convergence theorem implies that

$$\int_\Omega v \, d\eta = \int_\Omega v \, f_\Omega \, dx + \int_\Sigma v \, f_\partial \, d\rho$$

for every bounded Borel-measurable $v\colon \Omega \to \mathbb{C}$. If $v \in W^{1,2}_\Gamma \cap C_c(\Omega)$ then $(\mathrm{tr}\, v)_\Sigma = v|_\Sigma$ pointwise. Therefore

$$\int_\Omega v \, d\eta = \int_{\Omega\cup\Sigma} (\mathfrak{J}v) \, f \, (dx + d\rho) = \int_{\Omega\cup\Gamma\cup\Sigma} (\mathfrak{J}v) \, f \, (dx + d\rho) = (f, \overline{\mathfrak{J}v})_{\mathbb{L}^2}$$

for all $v \in W^{1,2}_\Gamma \cap C_c(\Omega)$.

Now let $\varphi \in \mathrm{Dom}(A_2)$ and choose $f = A_2\varphi$. If $v \in C^\infty_c(\Omega)$ then

$$\int_\Omega \overline{v} \, d\eta = (A_2\varphi, \mathfrak{J}v)_{\mathbb{L}^2} = \int_\Omega \mu\nabla u \cdot \overline{\nabla v} \, dx,$$

where we used (2.4) in the last step and $u \in W^{1,2}_\Gamma$ is such that $\mathfrak{J}u = \varphi$. Therefore the distributional divergence of the vector field $\mu\nabla u$ is equal to the complex measure $-\eta$, as distributions on Ω (see also [CTZ2, Definition 2.18]). Next let $\Lambda \Subset \Omega$ be a measurable set with finite perimeter. Although the complex measure η is not absolutely continuous with respect to the Lebesgue measure, one can apply (under certain further assumptions on $\mu\nabla u$ and/or Λ) the generalized Gauss–Green theorems of e.g. [CTZ1], [Fug] and [Zie] to obtain

$$\int_\Lambda d\eta = \int_{\partial\Lambda} \nu \cdot \mu\nabla u \, d\mathcal{H}_{d-1},$$

where $\nu \cdot \mu\nabla u \in L^1(\partial\Lambda; d\mathcal{H}_{d-1})$ is 'the generalized normal component of the corresponding flux', see ibidem. For instance, the application of [Zie, Theorem 5.2] requires the boundedness of $\mu\nabla u$ (what is not true in general), whilst in [Fug] the admissibility of Λ depends on $\mu\nabla u$ itself. Then

$$\int_\Lambda (A_2\varphi) \, (dx + d\rho) = \eta(\Lambda) = \int_{\partial\Lambda} \nu \cdot \mu\nabla u \, d\mathcal{H}_{d-1}. \tag{2.7}$$

We emphasize that the above discussion is heuristic. The formula (2.7) can be used to obtain the original local flux balances for (1.1)–(1.5) from the abstract equation $u' + A_2u = f$, see Remark 3.13.

We continue with the analysis of the operator A_2. By Proposition 2.14 the operator A_2 is m-sectorial with vertex 0 and semi-angle $\arctan \frac{\mu^\bullet}{\mu_\bullet}$. Hence by [Kat, Theorem IX.1.24] the operator $-A_2$ generates a holomorphic semigroup, denoted by S, which is holomorphic and contractive on the sector with semi-angle $\arctan \frac{\mu^\bullet}{\mu_\bullet}$.

Proposition 2.17. *The semigroup S leaves $\mathbb{L}^2_{\mathbb{R}}$ invariant, it is submarkovian and positive.*

Proof. Clearly the set $\mathbb{L}^2_{\mathbb{R}}$ is closed and convex in $\mathbb{L}^2$. Moreover, $\varphi \mapsto \operatorname{Re}\varphi$ is the projection from $\mathbb{L}^2$ onto $\mathbb{L}^2_{\mathbb{R}}$ and $\operatorname{Re}\mathfrak{t}(u, u - \operatorname{Re} u) = 0$ for all $u \in W^{1,2}_\Gamma$. Since the form $\mathfrak{t}$ is accretive, the set $\mathbb{L}^2_{\mathbb{R}}$ is invariant under the semigroup by [AE, Proposition 2.9(ii)]

Next, let $C = \{u \in \mathbb{L}^2 : u \text{ is real valued and } u \leq \mathbb{1}\}$. Then C is closed and convex. Let $P\colon \mathbb{L}^2 \to C$ denote the orthogonal projection. Then $Pu = (\operatorname{Re} u) \wedge \mathbb{1}_{\Omega\cup\Gamma}$. Let $u \in W^{1,2}_\Gamma$. By Lemma 2.11 one has $(\operatorname{Re} u) \wedge \mathbb{1}_\Omega \in W^{1,2}_\Gamma$ and $P\mathfrak{J}u = \mathfrak{J}((\operatorname{Re} u) \wedge \mathbb{1}_\Omega)$. Moreover, an easy calculation gives

$$\operatorname{Re}\mathfrak{t}[(\operatorname{Re} u) \wedge \mathbb{1}_\Omega, u - (\operatorname{Re} u) \wedge \mathbb{1}_\Omega] = 0.$$

Observing that the form $\mathfrak{t}$ is accretive, it follows from [AE, Proposition 2.9(ii)] that C is invariant under the semigroup S. Now let $\varphi \in \mathbb{L}^2 \cap \mathbb{L}^\infty$ and $t > 0$. There exists an $\alpha \in \mathbb{R}$ such that $\|S_t\varphi\|_{\mathbb{L}^\infty} = \|\operatorname{Re}(e^{i\alpha}S_t\varphi)\|_{\mathbb{L}^\infty}$. But $\operatorname{Re}(e^{i\alpha}S_t\varphi) = S_t \operatorname{Re}(e^{i\alpha}\varphi)$. Therefore

$$\|S_t\varphi\|_{\mathbb{L}^\infty} = \|S_t \operatorname{Re}(e^{i\alpha}\varphi)\|_{\mathbb{L}^\infty} \leq \|\operatorname{Re}(e^{i\alpha}\varphi)\|_{\mathbb{L}^\infty} \leq \|\varphi\|_{\mathbb{L}^\infty}$$

and S is submarkovian.

Finally, if $\varphi \in \mathbb{L}^2_{\mathbb{R}}$ and $\varphi \leq 0$, then $n\varphi \in C$ for all $n \in \mathbb{N}$. So $S_t(n\varphi) \leq \mathbb{1}$ for all $t > 0$ and $n \in \mathbb{N}$. Therefore $S_t\varphi \leq 0$ and S is positive. □

Corollary 2.18. *For all $p \in [1, \infty]$ the semigroup S extends consistently to a contraction semigroup $S^{(p)}$ on $\mathbb{L}^p$. The semigroup $S^{(p)}$ is strongly continuous for all $p \in [1, \infty)$.*

Proof. Observe that if the coefficient matrix μ satisfies the conditions of Assumption 2.12, then its transpose satisfies these as well. Thus the dual semigroup S^* shares the same properties as S. Now the assertion follows from Proposition 2.17 and standard interpolation and duality arguments, see e.g. [Ouh2, page 56]. □

We denote the generator of $S^{(p)}$ by $-A_p$. Then $-A_p$ is dissipative by the Lumer–Phillips theorem. If no confusion is possible we write $S = S^{(p)}$.

Remark 2.19. It is possible to prove the dissipativity of $-A_p$ also by showing that the form $-\mathfrak{t}$ is p-dissipative, cf. [CM].

Lemma 2.20.

(i) *The semigroup S is ultracontractive. Moreover, for all $\beta > d - 1$ and $\omega > 0$ there exists a $c > 0$ such that*

$$\|S_t\varphi\|_{\mathbb{L}^q} \leq c\, t^{-\beta(\frac{1}{p}-\frac{1}{q})} e^{\omega t}\|\varphi\|_{\mathbb{L}^p}$$

for all $t > 0$, $\varphi \in \mathbb{L}^p$ and $p, q \in [1, \infty]$ with $p \leq q$.

(ii) *If $1 \leq p < q \leq \infty$ and $j \in \mathbb{N}$ are such that $\frac{d-1}{j}\,(\frac{1}{p} - \frac{1}{q}) < 1$, then the operator $(A_p + 1)^{-j}$ maps $\mathbb{L}^p$ continuously into $\mathbb{L}^q$.*

(iii) *The operator A_p has compact resolvent for all $p \in (1, \infty)$.*

(iv) *If the matrix of coefficients μ is symmetric, then the operator A_2 is self-adjoint and positive.*

Proof. '(i)'. Let $r \in (2, \infty)$ be such that $(d-2)r < 2(d-1)$. Then it follows from Lemma 2.10(ii) that $\mathfrak{J}W^{1,2}_\Gamma \subset \mathbb{L}^r$, and the inclusion is continuous by the closed graph theorem. Let $\varphi \in \mathbb{L}^2$ and $t > 0$. Since $S_t\varphi \in \mathrm{Dom}(A_2)$, there is a $u \in W^{1,2}_\Gamma$ such that $S_t\varphi = \mathfrak{J}u$. For given $\omega > 0$ one has

$$\begin{aligned}\|S_t\varphi\|^2_{\mathbb{L}^r} = \|\mathfrak{J}u\|^2_{\mathbb{L}^r} \leq C\,\|u\|^2_{W^{1,2}_\Gamma} &\leq C(\mu_\bullet \wedge 1)^{-1}\big(\operatorname{Re}\mathfrak{t}[u,u] + \|\mathfrak{J}u\|^2_{\mathbb{L}^2}\big)\\ &= C(\mu_\bullet \wedge 1)^{-1}\big(\operatorname{Re}(A_2S_t\varphi, S_t\varphi)_{\mathbb{L}^2} + \|S_t\varphi\|^2_{\mathbb{L}^2}\big)\\ &\leq C'\,t^{-1}e^{2\omega t}\|\varphi\|^2_{\mathbb{L}^2},\end{aligned}$$

for suitable $C, C' > 0$, using (2.3), the definition of A_2, the Cauchy–Schwarz inequality and the holomorphy and contractivity of S_t. Therefore the semigroup S is ultracontractive, and by [Ouh2, Lemma 6.1] there exists a $c > 0$ such that

$$\|e^{-\omega t}S_t\varphi\|_{\mathbb{L}^\infty} \leq c\,t^{-\frac{r}{2(r-2)}}\,\|\varphi\|_{\mathbb{L}^2}$$

for all $t > 0$ and $\varphi \in \mathbb{L}^2$. Now duality and interpolation give Statement (i).

Statement (ii) follows from (i) and the well-known formula

$$(A_p + 1)^{-j} = \frac{1}{(j-1)!}\int_0^\infty t^{j-1}e^{-t}S_t\,dt.$$

Statement (iii) is a consequence of Proposition 2.14 and interpolation. The last statement of the lemma is easy to verify. □

2.4. **Multipliers acting on Lebesgue spaces.** In order to solve (1.1)–(1.5), we divide (1.1) (at first formally) by ε. Obviously, one is then confronted with the necessity to investigate the functional analytic properties of operators of the type ςA_p, where ς is a bounded strictly positive measurable function. Concerning the generator property of an analytic semigroup in a space $L^p(\Omega)$ this was carried out in [GKR] and concerning maximal parabolic regularity on $L^p(\Omega)$ in [HiR]. In the latter case the decisive instrument was the insight from [DO] that a suitable multiplicative perturbation does not destroy upper Gaussian estimates, which in turn imply maximal parabolic regularity on $L^p(\Omega)$. Unfortunately, we cannot apply this here, since our Lebesgue space does not only live 'on the volume'. But a surprisingly simple trick allows us to overcome the problem in the present context.

The next proposition is of independent interest.

Proposition 2.21. *Let $(X, \mathcal{B}, \lambda)$ be a measure space and let $\tau\colon X \to (0,\infty)$ be a measurable function such that both τ and τ^{-1} are bounded. Let $p \in [1,\infty)$ and let T be an operator in $L^p(X, d\lambda)$.*

(i) *If T is dissipative on $L^p(X, d\lambda)$, then τT is dissipative on $L^p(X, \tau^{-1}d\lambda)$.*

(ii) *If T generates a strongly continuous contraction semigroup on $L^p(X, d\lambda)$, then τT generates a strongly continuous contraction semigroup on $L^p(X, \tau^{-1}d\lambda)$.*

(iii) *If $p = 2$, $\theta \in (0, \frac{\pi}{2}]$ and T generates a holomorphic semigroup in $L^2(X, d\lambda)$ which is contractive in the sector with semi-angle θ, then τT generates a holomorphic semigroup in $L^2(X, \tau^{-1}d\lambda)$ which is contractive in the sector with semi-angle θ.*

Now suppose that $p = 2$ and T generates a strongly continuous contraction semigroup S on $L^2(X, d\lambda)$. Denote the semigroup generated by τT on $L^2(X, \tau^{-1}d\lambda)$ by S^τ.

(iv) *If S leaves the real valued functions invariant, then S^τ also leaves the real valued functions invariant.*

(v) *If S is positive, then S^τ is also positive.*

(vi) *Suppose S is submarkovian. Then S^τ is also submarkovian. Hence for all $q \in [2,\infty)$ the semigroups S and S^τ extend consistently to a strongly continuous semigroup $S^{(q)}$ and $S^{(\tau,q)}$ on $L^q(X, d\lambda)$ and $L^q(X, \tau^{-1}d\lambda)$, respectively. Let T_q and $T_{\tau,q}$ denote the generators. Then $T_{\tau,q} = \tau T_q$ for all $q \in [2,\infty)$.*

Proof. '(i)'. The operator T is dissipative on $L^p(X, d\lambda)$ if and only if

$$\operatorname{Re} \int_{\{f\neq 0\}} (Tf)\, |f|^{p-2}\, \overline{f}\, d\lambda \leq 0$$

for all $f \in D(T)$. This implies the dissipativity of τT on $L^p(X, \tau^{-1}d\lambda)$.

'(ii)'. Since T generates a contraction semigroup on $L^p(X, d\lambda)$, it follows that T is dissipative. Therefore τT is dissipative on $L^p(X, \tau^{-1}d\lambda)$ by Statement (i). So by the Lumer–Phillips theorem it remains to show that the operator $\tau T - 1$ is surjective on $L^p(X, \tau^{-1}d\lambda)$.

Let $\delta > 0$ be such that $\tau^{-1} - \delta \geq 0$. Then the multiplication operator $-(\tau^{-1} - \delta)$ is dissipative on $L^p(X, d\lambda)$ and has a relative bound equal to zero with respect to T. Therefore $T - (\tau^{-1} - \delta)$ generates a strongly continuous contraction semigroup on $L^p(X, d\lambda)$ by the perturbation result [Dav, Theorem 3.7]. Hence $T - \tau^{-1}$ is surjective on $L^p(X, d\lambda)$ by the Lumer–Phillips theorem. But this implies that $\tau T - 1$ is surjective on $L^p(X, \tau^{-1}d\lambda)$.

'(iii)'. For all $\alpha \in (-\theta, \theta)$ the above applies to the operator $e^{i\alpha}T$. Therefore $e^{i\alpha}\tau T$ generates a strongly continuous contraction semigroup on $L^2(X, \tau^{-1}d\lambda)$. Hence by [Kat, Theorem IX.1.23] the operator τT generates a holomorphic semigroup in $L^2(X, \tau^{-1}d\lambda)$ which is contractive on the sector with semi-angle θ.

Now suppose $p = 2$ and T generates a strongly continuous contraction semigroup S on $L^2(X, d\lambda)$. Let C be a closed convex subset of $L^2(X, d\lambda)$. Then C is also closed and convex in $L^2(X, \tau^{-1}d\lambda)$. Since T is m-dissipative it follows from [Ouh1, Theorem 2.2] that C is invariant under S if and only if $\mathrm{Re}(Tf, f-Pf)_{L^2(X,d\lambda)} \leq 0$ for all $f \in D(T)$, where P is the orthogonal projection in $L^2(X, d\lambda)$ onto C. Similarly, since τT is m-dissipative on $L^2(X, \tau^{-1}d\lambda)$, the set C is invariant under S^τ if and only if $\mathrm{Re}(\tau Tf, f-P^\tau f)_{L^2(X,\tau^{-1}d\lambda)} \leq 0$ for all $f \in D(\tau T)$, where P^τ is the orthogonal projection in $L^2(X, \tau^{-1}d\lambda)$ onto C. But $D(\tau T) = D(T)$. Hence if $P = P^\tau$, then C is invariant under S if and only if C is invariant under S^τ.

Then for the proof of Statement (iv) choose $C = \{f \in L^2(X, d\lambda) : f \text{ is real valued}\}$ and note that the projection is $Pf = \mathrm{Re}\, f = P^\tau f$. For the proof of Statement (v) choose $C = \{f \in L^2(X, d\lambda) : f \text{ is real valued and } f \geq 0\}$ and note that the projection is $Pf = (\mathrm{Re}\, f)^+ = P^\tau f$. For the submarkovian part in the proof of Statement (vi) choose $C = \{f \in L^2(X, d\lambda) : |f| \leq 1 \text{ a.e.}\}$ and note that the projection is $Pf = (|f| \wedge \mathbb{1}) \operatorname{sgn} f = P^\tau f$.

It remains to prove the second part of Statement (vi). Let $q \in [2, \infty)$. Let $u \in D(T_{\tau,q}) \cap D(T_{\tau,2})$. Write $v = T_{\tau,2}u$. Then $u \in L^2(X, d\lambda) \cap L^q(X, d\lambda)$ and $T_{\tau,q}u = T_{\tau,2}u = v$. So $v \in L^q(X, d\lambda)$ and $\tau^{-1}v \in L^q(X, d\lambda)$ since τ^{-1} is bounded. Moreover, $T_{\tau,2} = \tau T_2$, so $u \in D(T_2)$ and $T_2u = \tau^{-1}v$. Therefore

$$t^{-1}(S^{(q)}_t u - u) = t^{-1}(S^{(2)}_t u - u) = t^{-1}\int_0^t S^{(2)}_s T_2 u\, ds = t^{-1}\int_0^t S^{(q)}_s T_2 u\, ds$$

for $t > 0$. As $t \downarrow 0$, the latter term converges to T_2u in $L^q(X, d\lambda)$ by the strong continuity of $S^{(q)}$. Hence $u \in D(T_q)$ and $T_qu = T_2u = \tau^{-1}v$. Then $\tau T_q u = v = T_{\tau,q}u$. We proved that

$$D(T_{\tau,q}) \cap D(T_{\tau,2}) \subset D(\tau T_q) \cap D(\tau T_2)$$

and $T_{\tau,q}u = \tau T_q u$ for all $u \in D(T_{\tau,q}) \cap D(T_{\tau,2})$. Similarly the converse inclusion is valid, so

$$D(T_{\tau,q}) \cap D(T_{\tau,2}) = D(\tau T_q) \cap D(\tau T_2) = D(T_q) \cap D(T_2).$$

We claim that $D(T_q) \cap D(T_2)$ is dense in $D(T_q) = D(\tau T_q)$. Consider the set

$$\mathcal{D} = \{t^{-1}\int_0^t S^{(q)}_s u\, ds : u \in L^q(X, d\lambda) \cap L^2(X, d\lambda), t \in (0, \infty)\}.$$

Then $\mathcal{D} \subset D(T_q)$. Since $S^{(q)}$ and $S^{(2)}$ are consistent, also $\mathcal{D} \subset D(T_2)$. So $\mathcal{D} \subset D(T_q) \cap D(T_2)$. Moreover, $\lim_{t\downarrow 0} t^{-1}\int_0^t S^{(q)}_s u\, ds = u$ in $L^q(X, d\lambda)$ for all $u \in L^q(X, d\lambda) \cap L^2(X, d\lambda)$ and $L^q(X, d\lambda) \cap L^2(X, d\lambda)$ is dense in $L^q(X, d\lambda)$. Therefore $\mathcal{D}$ is dense in $L^q(X, d\lambda)$. Clearly $\mathcal{D}$ is invariant under $S^{(q)}$. Hence $\mathcal{D}$ is a core for T_q by [EN, Proposition II.1.7]. This implies that $D(T_q) \cap D(T_2)$ is dense in $D(T_q)$. The same arguments show that $D(T_{\tau,q}) \cap D(T_{\tau,2})$ is dense in $D(T_{\tau,q})$. Hence $T_{\tau,q} = \tau T_q$. □

Let $\varsigma\colon \Omega \cup \Gamma \to (0, \infty)$ be a measurable function such that $\varsigma, \varsigma^{-1} \in \mathbb{L}^\infty$. We write

$$\mathbb{L}^p_\varsigma := L^p(\Omega \cup \Gamma; \varsigma^{-1}(dx + d\rho)).$$

Proposition 2.21 allows to transfer the conclusion of Corollary 2.18 to the operators ςA_p.

Theorem 2.22. *For all $p \in [1,\infty)$ the operator $-\varsigma A_p$ generates a strongly continuous positive semigroup $S^{(\varsigma,p)}$ of contractions on the space $\mathbb{L}^p_\varsigma$. The semigroups are consistent. Moreover, $S^{(\varsigma,2)}$ is holomorphic and contractive in the sector with semi-angle $\arctan \frac{\mu^\bullet}{\mu_\bullet}$.*

Proof. For $p \geq 2$ all follows from Propositions 2.14, 2.17 and 2.21. The dual of the operator ςA_2 on $\mathbb{L}^2_\varsigma$ is given by $\varsigma A^\#$, where $A^\#$ is the operator obtained with coefficient matrix equal to the transpose of the matrix μ. Hence by Proposition 2.21 the dual semigroup $(S^{(\varsigma,2)})^*$ is submarkovian and extends consistently to a strongly continuous semigroup on $\mathbb{L}^p_\varsigma$ for all $p \in [2,\infty)$. Then by duality the semigroup $S^{(\varsigma,2)}$ extends consistently to a strongly continuous semigroup on $\mathbb{L}^p_\varsigma$ for all $p \in [1,2]$. □

2.5. **Consequences for the operators ςA_p on $\mathbb{L}^p$.** We have the following abstract properties for ςA_p.

Theorem 2.23. *Let $p \in (1,\infty)$. Then one has the following.*

(i) *The operator ςA_p admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. In particular, it admits bounded imaginary powers.*

(ii) *For all $\theta \in (0,1)$ one has*

$$(\varsigma A_p + 1)^{-\theta} = \frac{\sin \pi\theta}{\pi} \int_0^\infty t^{-\theta} (\varsigma A_p + 1 + t)^{-1}\, dt.$$

(iii) *If $\theta \in (0,1]$, then $\mathrm{Dom}\big((\varsigma A_p)^\theta\big) = [\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p)]_\theta = \mathrm{Dom}(A_p^\theta)$, where $[\cdot,\cdot]_\theta$ denotes complex interpolation.*

Proof. '(i)'. For all $p \in [1,\infty)$ denote by $S^{(\varsigma,p)}$ the contraction semigroup on $\mathbb{L}^p_\varsigma$ generated by $-\varsigma A_p$. Then the semigroups $S^{(\varsigma,p)}$ with $p \in [1,\infty)$ are consistent. Moreover, $S^{(\varsigma,2)}$ is holomorphic and bounded on a sector. Let $p \in (1,\infty)$. Then it follows from [Ouh2, Proposition 3.12] and duality that $S^{(\varsigma,p)}$ is holomorphic and bounded in a sector (which depends on p). Also $S^{(\varsigma,p)}$ is a positive contraction semigroup. Hence the operator $S_t^{(\varsigma,p)}$ is contractively regular for all $t > 0$. So by [LX, Proposition 2.2] the operator ςA_p admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. This is then also the case on $\mathbb{L}^p$, since $\mathbb{L}^p = \mathbb{L}^p_\varsigma$ as vector spaces, with equivalent norms.

'(ii)'. For the integral representation see [Lun, (4.41)].

'(iii)'. Since ςA_p admits bounded imaginary powers, it follows from [Lun, Theorem 4.17] that

$$\mathrm{Dom}\big((\varsigma A_p)^\theta\big) = [\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p)]_\theta.$$

Since $\mathrm{Dom}(\varsigma A_p) = \mathrm{Dom}(A_p)$, one has $\mathrm{Dom}\big((\varsigma A_p)^\theta\big) = [\mathbb{L}^p, \mathrm{Dom}(A_p)]_\theta$, and the result follows. □

3. LINEAR PARABOLIC EQUATIONS

In this section we will draw conclusions for linear parabolic equations, which, in particular, allow to give (1.1)–(1.5) a precise meaning and afterwards to solve it.

In the following, $J = (0, T)$ denotes a bounded interval and X a Banach space. Throughout we fix the numbers

$$1 < s < \infty \qquad \text{and} \qquad \frac{1}{s} < \alpha \leq 1.$$

We introduce the weighted space

$$L^s_\alpha(J; X) = \{u\colon J \to X \ : \ [t \mapsto t^{1-\alpha} u(t)] \in L^s(J; X)\},$$

and the corresponding weighted Sobolev space

$$W^{1,s}_\alpha(J; X) = \{u \in L^s_\alpha(J; X) : u' \in L^s_\alpha(J; X)\},$$

where here and below the time derivative is understood in the sense of X-valued distributions (see [Ama, Subsection III.1.1]). These are Banach spaces when equipped with their canonical norm, respectively. Note that $\alpha = 1$ corresponds to the unweighted case, i.e., $L^s_1 = L^s$. By [PS, Lemma 2.1] one has $W^{1,s}_\alpha(J; X) \subset W^{1,1}(J; X)$, which implies that each element of $W^{1,s}_\alpha(J; X)$ has a well-defined trace at $t = 0$.

Definition 3.1. Let A be a closed linear operator on X with dense domain $\mathrm{Dom}(A)$. We say that A has *maximal parabolic $L^s_\alpha(J; X)$-regularity*, if for all $f \in L^s_\alpha(J; X)$ there is a unique solution $u \in W^{1,s}_\alpha(J; X) \cap L^s_\alpha(J; \mathrm{Dom}(A))$ of

$$u' + Au = f, \qquad u(0) = 0.$$

We write $MR^s_\alpha(J; X)$ for the class of all operators on X with this property.

We proceed with some comments concerning maximal parabolic regularity.

1 It is shown in [PS, Theorem 2.4] that $A \in MR^s_1(J; X)$ if and only if $A \in MR^s_\alpha(J; X)$ for all $\alpha \in (\frac{1}{s}, 1]$, i.e., maximal parabolic L^s_α-regularity is independent of the weight. (In fact, in [PS] only the case $J = (0, \infty)$ is treated, but the arguments given there also apply to bounded J.) In this sense it is natural to consider the temporal weights in the context of parabolic problems.

2 If $A \in MR^{s_0}_1(J_0; X)$ for an interval $J_0 = (0, T_0)$, where $T_0 \in (0, \infty)$ and $s_0 \in (1, \infty)$, then $A \in MR^s_\alpha(J; X)$ for all $T \in (0, \infty)$, $s \in (1, \infty)$ and $\alpha \in (\frac{1}{s}, 1]$. This is shown in [Dor, Corollary 5.4 and Theorem 7.1]. In this spirit, we then simply say that A satisfies maximal parabolic regularity on X.

3 The notion 'maximal parabolic regularity' does not depend on the concrete norm of the Banach space. In other words: an operator A, satisfying maximal parabolic regularity on X, remains to satisfy maximal parabolic regularity if X is equipped with an equivalent norm.

4 If A satisfies maximal parabolic regularity on X, then $-A$ generates an analytic C_0-semigroup (cf. [Dor, Corollary 4.4]). If X is a Hilbert space, then the converse is also true, cf. [DeS].

For the case of nontrivial initial values, the following has been proved in [PS, Theorem 3.2]. We denote by $(\cdot,\cdot)_{\theta,s}$ the real interpolation functor, cf. [Tri, Sections 1.3 and 1.6].

Proposition 3.2. *Suppose that A satisfies maximal parabolic regularity on X. Then for all $f \in L^s_\alpha(J;X)$ and $u_0 \in (X,\mathrm{Dom}(A))_{\alpha-\frac{1}{s},s}$ the Cauchy problem*

$$u' + Au = f, \qquad u(0) = u_0,$$

has a unique solution $u \in W^{1,s}_\alpha(J;X) \cap L^s_\alpha(J;\mathrm{Dom}(A))$, and the estimate

$$\|u'\|_{L^s_\alpha(J;X)} + \|u\|_{L^s_\alpha(J;\mathrm{Dom}(A))} \le c\big(\|u_0\|_{(X,\mathrm{Dom}(A))_{\alpha-\frac{1}{s},s}} + \|f\|_{L^s_\alpha(J;X)}\big) \tag{3.1}$$

is valid for some constant c, independent of f and u_0.

By working in temporally weighted spaces one can thus reduce the regularity of the initial values u_0 almost up to the base space X.

We have the following embeddings for the weighted maximal regularity class. The space of γ-Hölder continuous functions is denoted by C^γ.

Proposition 3.3. *If A satisfies maximal parabolic regularity on X, then*

$$W^{1,s}_\alpha(J;X)\cap L^s_\alpha(J;\mathrm{Dom}(A)) \subset BUC(\overline{J};(X,\mathrm{Dom}(A))_{\alpha-\frac{1}{s},s})\cap C(J;(X,\mathrm{Dom}(A))_{1-\frac{1}{s},s}).$$

Moreover, for every $\theta \in [0,\alpha-\frac{1}{s})$ there is a $\gamma \in (0,1)$ such that

$$W^{1,s}_\alpha(J;X) \cap L^s_\alpha(J;\mathrm{Dom}(A)) \subset C^\gamma(\overline{J};[X,\mathrm{Dom}(A)]_\theta).$$

Proof. The first inclusion is shown in [PS, Proposition 3.1]. The second one can be proved along the lines of [DMRT, Lemma 1]. □

We apply a classical result of Lamberton [Lam] to the operators ςA_p.

Theorem 3.4. *Let $\varsigma\colon \Omega\cup\Gamma \to (0,\infty)$ be a measurable function such that $\varsigma, \varsigma^{-1} \in \mathbb{L}^\infty$. Then for all $p \in (1,\infty)$ the operator ςA_p satisfies maximal parabolic regularity on $\mathbb{L}^p$.*

Proof. Theorem 2.22 states that the semigroup generated by $-\varsigma A_2$ on $\mathbb{L}^2_\varsigma$ is bounded and analytic, and that it extents consistently to a contractive semigroup on $\mathbb{L}^q_\varsigma$ for all $q \in [1,\infty]$. Now the result is a consequence of [Lam, Corollary. 1.1]. □

In order to include lower order terms into the boundary and interface conditions we need some preparation.

Proposition 3.5. *Let $p \in (1,\infty)$ and $\theta \in (0,1)$ be such that $d-1 < \theta\, p$. Then one has $\mathrm{Dom}\big((\varsigma A_p)^\theta\big) \subset \mathbb{L}^\infty$.*

Proof. Since $\mathrm{Dom}\big((\varsigma A_p)^\theta\big) = \mathrm{Dom}\big((A_p+1)^\theta\big)$ by Theorem 2.23(iii) and [Lun, Lemma 4.1.11], it suffices to show that $(A_p+1)^{-\theta}$ maps $\mathbb{L}^p$ into $\mathbb{L}^\infty$. In [Paz, Section 2.6] it is shown that

$$(A_p+1)^{-\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1}\, e^{-t}\, S_t\, dt.$$

Now the assertion follows from the estimate of Lemma 2.20(i). □

Corollary 3.6. *Suppose $p \in (\frac{d-1}{\alpha-\frac{1}{s}}, \infty)$. Then $(\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p))_{\alpha-\frac{1}{s},s}$ continuously embeds into $\mathbb{L}^\infty$.*

Proof. Fix $\theta \in (\frac{d-1}{p}, \alpha - \frac{1}{s})$. Then Proposition 3.5 yields $\mathrm{Dom}\big((\varsigma A_p)^\theta\big) \subset \mathbb{L}^\infty$. But

$$(\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p))_{\alpha-\frac{1}{s},s} \subset (\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p))_{\theta,1} \subset [\mathbb{L}^p, \mathrm{Dom}(\varsigma A_p)]_\theta$$

by [Lun, Propositions 1.1.3, 1.3.2 and Corollary 2.1.8], and the latter interpolation space equals $\mathrm{Dom}\big((A_p)^\theta\big)$ by Theorem 2.23(iii). □

Definition 3.7. Fix $b \in L^p(\Gamma \cup \Sigma; d\rho)$. Define the operator $B\colon \mathbb{L}^\infty \to \mathbb{L}^p$ by

$$B(f_\Omega, f_\partial) = (0, b\, f_\partial)$$

for all $f = (f_\Omega, f_\partial) \in L^p(\Omega) \oplus L^p(\Gamma \cup \Sigma; d\rho) \cong \mathbb{L}^p$.

Note that b is allowed to be complex valued.

Theorem 3.8. *Let $p \in (d-1, \infty)$. Then the operator $\varsigma(A_p + B)$ satisfies maximal parabolic regularity on $\mathbb{L}^p$.*

Proof. One deduces from Corollary 3.6 that the operator ςB acts continuously on an interpolation space between $\mathrm{Dom}(\varsigma A_p)$ and $\mathbb{L}^p$. Then the result follows from the perturbation theorem [Dor, Theorem 6.2]. □

Remark 3.9. In a somewhat more general concept B may also depend explicitly on time, see [ACFP].

Now we are in the position to solve the parabolic equation (1.1)–(1.5) from the introduction of the paper, see also Remark 2.15.

Theorem 3.10. *Let $T \in (0, \infty)$ and set $J = (0, T)$. Let $p \in (d-1, \infty)$, $s \in (1, \infty)$ and $\alpha \in (\frac{1}{s}, 1]$. Let Ω be a bounded domain in $\mathbb{R}^d$ with $d > 1$, let Γ be an open part of its boundary $\partial\Omega$ and $\Sigma \subset \Omega$. Adopt the Assumptions 2.2, 2.4 and 2.12. Let $\varepsilon \in \mathbb{L}^\infty$ be a positive function with a positive essential lower bound and let b as in Definition 3.7. Then the initial value problem (1.1)–(1.5) admits a solution in the following sense: for all $f \in L^s_\alpha(J; \mathbb{L}^p)$ and $u_0 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ there is a unique function $u \in W^{1,s}_\alpha(J; \mathbb{L}^p) \cap L^s_\alpha(J; \mathrm{Dom}(A_p))$ satisfying*

$$\varepsilon u' + A_p u + Bu = f, \qquad u(0) = u_0. \tag{3.2}$$

Proof. One reformulates (3.2) as

$$u' + \varepsilon^{-1} A_p u + \varepsilon^{-1} B u = \varepsilon^{-1} f, \qquad u(0) = u_0.$$

Obviously, $\varepsilon^{-1} f$ satisfies the same assumptions as f. Moreover, one has $\mathrm{Dom}(A_p) = \mathrm{Dom}(\varepsilon^{-1} A_p) = \mathrm{Dom}(\varepsilon^{-1}(A_p + B))$, with equivalent norms. This implies that

$$(\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha - \frac{1}{s}, s} = (\mathbb{L}^p, \mathrm{Dom}(\varepsilon^{-1}(A_p + B)))_{\alpha - \frac{1}{s}, s}.$$

The assertion then follows from Proposition 3.2 and Theorem 3.8. □

Remark 3.11. In the situation of the theorem, the solution depends continuously on the data, due to (3.1). Proposition 3.3 gives further regularity properties of a solution. Moreover, again by (3.1), it is straightforward to see that the solution depends continuously on the function ε, with respect to the $\mathbb{L}^\infty$-norm.

Remark 3.12. Since the coefficient function μ is real valued, the resolvent of ςA_p commutes with complex conjugation on the spaces $\mathbb{L}^p$. The latter is also true for the semigroup operators $e^{-t\varsigma A_p}$. Thus, the restriction of ςA_p to real spaces $\mathbb{L}^p_{\mathbb{R}}$ also satisfies maximal parabolic regularity. If B is induced by a real valued function, then the same is true for the operator $\varsigma(A_p + B)$.

Remark 3.13. At the end of this section, let us give more detailed, partly heuristic arguments what the real advantage is of the treatment of our parabolic equations in the spaces $\mathbb{L}^p$.

When considering the solution u of a parabolic equation $u' + Au = f$ on a Banach space X one can form the dual pairing with any $\psi \in X^*$ to obtain

$$\frac{\partial}{\partial t} \langle u, \psi \rangle + \langle Au, \psi \rangle = \langle f, \psi \rangle. \tag{3.3}$$

E.g., if $X = W^{-1,2}(\Omega)$, then one can choose for ψ as any element of $W^{1,2}_0(\Omega)$, but *not* an indicator function of a subset of Ω. In our setting, the situation is different: if $X = \mathbb{L}^p$, then the dual pairing with the indicator function χ_Λ of a measurable set $\Lambda \subset \Omega$ is admissible. Now suppose that $p = 2$. Then (3.3) with $A = A_2$ gives

$$\frac{\partial}{\partial t} \int_\Lambda u \, (dx + d\rho) + \int_\Lambda (A_2 u) \, (dx + d\rho) = \int_\Lambda f \, (dx + d\rho). \tag{3.4}$$

Note that

$$\int_\Lambda (A_2 u) \, (dx + d\rho) = \int_{\partial\Lambda} \nu \cdot \mu \nabla u \, d\mathcal{H}_{d-1}$$

by the heuristic argument in (2.7). Substituting (2.7) in (3.4) gives the desired balance law, as is classical when $\nabla \cdot \mu \nabla u$ is an $L^2(\Omega)$-function; compare [Som, Chapter 21] and [ChaL]. This shows that the concept of $\mathbb{L}^p$-spaces is in principal correspondence with the wish to recover the physical local balance law – modulo technical difficulties which should be overcome.

As already mentioned in the introduction, this is the basis for local flux balances, which are crucial for the foundation of Finite Volume methods for the numerical solution of such problems, compare [BRF], [FL] and [Gär].

4. QUASILINEAR PARABOLIC EQUATIONS

In this section we treat a nondegenerate, quasilinear variant of (1.1)–(1.5), including nonlinear terms in the dynamic equations on Γ and Σ, i.e.,

$$\varepsilon\partial_t \mathfrak{b}(u) - \nabla\cdot\mu\mathfrak{a}(u)\nabla u = F_\Omega(t,u) \qquad \text{in } J\times\Omega, \tag{4.1}$$

$$u = 0 \qquad \text{on } J\times(\partial\Omega\setminus\Gamma), \tag{4.2}$$

$$\varepsilon\partial_t \mathfrak{b}(u) + \nu\cdot\mu\mathfrak{a}(u)\nabla u = F_\Gamma(t,u) \qquad \text{on } J\times\Gamma, \tag{4.3}$$

$$\varepsilon\partial_t \mathfrak{b}(u) + [\nu_\Sigma\cdot\mu\mathfrak{a}(u)\nabla u] = F_\Sigma(t,u) \qquad \text{on } J\times\Sigma, \tag{4.4}$$

$$u(T_0) = u_0 \qquad \text{in } \Omega\cup\Gamma, \tag{4.5}$$

where $J = (T_0, T_1)$ is a bounded interval. Interesting examples for the nonlinearities on the left-hand side are e.g. when $\mathfrak{b}$ and $\mathfrak{a}$ are an exponential, or the Fermi–Dirac distribution function $\mathcal{F}_{1/2}$, which is given by

$$\mathcal{F}_{1/2}(t) := \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\sqrt{s}}{1+e^{s-t}}\,ds.$$

Further, in phase separation problems a rigorous formulation as a minimal problem for the free energy reveals that $\mathfrak{a} = \mathfrak{b}'$ is appropriate. This topic has been thoroughly investigated in [Qua], [QRV], [GL1], and [GL2], see also [GS] and [Gri1].

We consider from now on the real part $\mathbb{L}^p_{\mathbb{R}}$ of the spaces $\mathbb{L}^p$ and the operators A_p. For simplicity we write $\mathbb{L}^p$ for $\mathbb{L}^p_{\mathbb{R}}$. As in the linear case we give the quasilinear equation a suitable functional analytic formulation, and within this framework the problem will then be solved (see Definition 4.2 and Theorem 4.5 below). Again throughout this section we fix the numbers

$$1 < s < \infty \qquad \text{and} \qquad \frac{1}{s} < \alpha \le 1.$$

We impose the following conditions on the coefficients on the left-hand side of (4.1)–(4.5).

Assumption 4.1. The coefficient matrix μ is real-valued, $\mathfrak{b} \in W^{2,\infty}_{\mathsf{loc}}(\mathbb{R})$ is such that $\mathfrak{b}'$ is positive, and $\mathfrak{a} \in W^{1,\infty}_{\mathsf{loc}}(\mathbb{R})$ is positive and satisfies $\int_0^\infty \mathfrak{a}(\zeta)\,d\zeta = \infty = \int_{-\infty}^0 \mathfrak{a}(\zeta)\,d\zeta$.

Note that we do not require monotonicity for $\mathfrak{a}$. In particular, terms of the form $\mathfrak{a}(u) = \eta + |u|^m$ with $\eta > 0$ and $m \ge 1$ can be treated, that arise e.g. as a regularization of the porous media equation.

It is in general not to expect that the domain of the realization of $-\nabla\cdot\mu\mathfrak{a}(v)\nabla$ on $\mathbb{L}^p$ as in Section 2.3 is independent of $v \in L^\infty(\Omega)$. Consider, e.g., the case of a smooth geometry with $\mu\mathfrak{a}(v)$ equal to a constant on the one hand and a nonsmooth $\mu\mathfrak{a}(v)$ on the other hand. This observation motivates our definition of a solution of (4.1)–(4.5), which we describe in the following. We put

$$\mathfrak{K}(\xi) := \begin{cases} \int_0^\xi \mathfrak{a}(\zeta)\,d\zeta, & \text{if } \xi \ge 0,\\ -\int_\xi^0 \mathfrak{a}(\zeta)\,d\zeta, & \text{if } \xi < 0. \end{cases}$$

Then the assumptions on $\mathfrak{a}$ imply that

$$\mathfrak{K}\colon \mathbb{R} \to \mathbb{R} \text{ is bijective}, \quad \mathfrak{K}, \mathfrak{K}^{-1} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}), \quad \mathfrak{K}' = \mathfrak{a}, \quad \text{and} \quad \mathfrak{K}(0) = 0 = \mathfrak{K}^{-1}(0).$$

In the sequel we identify the functions $\mathfrak{b}, \mathfrak{K}, \mathfrak{K}^{-1}$ with the Nemytzkii operators they induce. The reformulation of (4.1)–(4.5) is based on the so-called Kirchhoff transform $w = \mathfrak{K}(u)$. This (formally) gives $\mathfrak{a}(u)\nabla u = \nabla w$ and $\partial_t(\mathfrak{b}(u)) = \frac{\mathfrak{b}'}{\mathfrak{a}}(\mathfrak{K}^{-1}(w))\partial_t w$. Since $\mathfrak{K}(0) = 0$, the problem (4.1)–(4.5) thus transforms into

$$\begin{aligned}
\partial_t w - \eta(w)\nabla \cdot \mu\nabla w &= \eta(w) F_\Omega(t, \mathfrak{K}^{-1}(w)) && \text{in } J \times \Omega,\\
w &= 0 && \text{on } J \times (\partial\Omega \setminus \Gamma),\\
\partial_t w + \eta(w)\nu \cdot \mu\nabla w &= \eta(w) F_\Gamma(t, \mathfrak{K}^{-1}(w)) && \text{on } J \times \Gamma,\\
\partial_t w + \eta(w)[\nu_\Sigma \cdot \mu\nabla w] &= \eta(w) F_\Sigma(t, \mathfrak{K}^{-1}(w)) && \text{on } J \times \Sigma,\\
w(T_0) &= \mathfrak{K}(u_0) && \text{in } \Omega \cup \Gamma,
\end{aligned}$$

where we have set

$$\eta(w) := \varepsilon^{-1} \frac{\mathfrak{a}}{\mathfrak{b}'} \mathfrak{K}^{-1}(w).$$

For all $t \in J$, let us further define the operator

$$R(t,w) := \begin{cases} \eta(w|_\Omega) F_\Omega(t, \mathfrak{K}^{-1}(w|_\Omega)) & \text{on } \Omega \setminus \Sigma,\\ \eta(w|_\Gamma) F_\Gamma(t, \mathfrak{K}^{-1}(w|_\Gamma)) & \text{on } \Gamma,\\ \eta(w|_\Sigma) F_\Sigma(t, \mathfrak{K}^{-1}(w|_\Sigma)) & \text{on } \Sigma, \end{cases} \tag{4.6}$$

acting on real-valued functions defined on $\Omega \cup \Gamma$.

Definition 4.2. Let $p \in (\frac{d-1}{\alpha - \frac{1}{s}}, \infty)$, and let A_p be the realization of $-\nabla \cdot \mu\nabla$ on $\mathbb{L}^p$ as in Section 2.3. We say that $u \in C([T_0, T_1]; \mathbb{L}^\infty)$ is a solution of (4.1)–(4.5) on J if

$$\mathfrak{K}(u) \in W^{1,s}_\alpha(J; \mathbb{L}^p) \cap L^s_\alpha(J; \mathrm{Dom}(A_p)),$$

and if $w = \mathfrak{K}(u)$ satisfies

$$\partial_t w + \eta(w) A_p w = R(\cdot, w) \quad \text{on } J, \qquad w(T_0) = \mathfrak{K}(u_0). \tag{4.7}$$

If $\mathfrak{K}(u)$ is as above, then $u \in C([T_0, T_1]; \mathbb{L}^\infty)$ is already a consequence of Proposition 3.3, Corollary 3.6 and the regularity of $\mathfrak{K}$. Proposition 3.3 shows that in fact $u \in C^\gamma([T_0, T_1]; \mathbb{L}^\infty)$ for some $\gamma > 0$. For specific choices of $\mathfrak{K}$ additional regularity may carry over from $\mathfrak{K}(u)$ to u. In any case one has $u(t, \cdot) \to u_0$ as $t \to T_0$ in the $\mathbb{L}^\infty$-norm.

Observe further that in the definition it is necessary that $\mathfrak{K}(u_0) \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha - \frac{1}{s}, s}$. It would be interesting to find another description for this condition for a class of nonlinearities $\mathfrak{a}$. If $\mathfrak{a}$ is constant, then a solution in the above sense can be defined for all $u_0 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha - \frac{1}{s}, s}$.

If $\mathfrak{a} = \mathfrak{b}'$, then (4.7) is in fact a semilinear problem. This is in particular the case for the phase separation problems from above.

To solve (4.7) we intend to use the following abstract existence and uniqueness result, which is proved in [Prü] for the temporally unweighted case $\alpha = 1$. The proof in [Prü] literally carries over to the weighted case $\alpha < 1$.

Proposition 4.3. *Let $1 < s < \infty$ and let A be a closed operator on a Banach space X with dense domain $D = \mathrm{Dom}(A)$, satisfying maximal parabolic regularity. Suppose $w_0 \in (X,D)_{\alpha-\frac{1}{s},s}$ and that $\mathcal{A}\colon (X,D)_{\alpha-\frac{1}{s},s} \to \mathcal{L}(D,X)$ is such that $A = \mathcal{A}(w_0)$ and for all $M > 0$ there is a constant $C_M > 0$ with*

$$\|\mathcal{A}(w_1) - \mathcal{A}(w_2)\|_{\mathcal{L}(D,X)} \le C_M \, \|w_1 - w_2\|_{(X,D)_{\alpha-\frac{1}{s},s}}$$

for all $w_1, w_2 \in (X,D)_{\alpha-\frac{1}{s},s}$ with $\|w_1\|_{(X,D)_{\alpha-\frac{1}{s},s}} \le M$ and $\|w_2\|_{(X,D)_{\alpha-\frac{1}{s},s}} \le M$. Suppose further that $\mathcal{R}\colon J \times (X,D)_{\alpha-\frac{1}{s},s} \to X$ is such that $\mathcal{R}(\cdot, w_0)$ is measurable for all $w_0 \in (X,D)_{\alpha-\frac{1}{s},s}$, that $\mathcal{R}(\cdot,0) \in L^s_\alpha(J;X)$, and that for all $M > 0$ there is an $r_M \in L^s_\alpha(J)$ with

$$\|\mathcal{R}(t,w_1) - \mathcal{R}(t,w_2)\|_X \le r_M(t) \, \|w_1 - w_2\|_{(X,D)_{\alpha-\frac{1}{s},s}} \qquad \text{for a.e. } t \in J,$$

for all $w_1, w_2 \in (X,D)_{\alpha-\frac{1}{s},s}$ with $\|w_1\|_{(X,D)_{\alpha-\frac{1}{s},s}} \le M$ and $\|w_2\|_{(X,D)_{\alpha-\frac{1}{s},s}} \le M$. Then for all $w_0 \in (X,D)_{\alpha-\frac{1}{s},s}$ there are $T^* \in (T_0, T_1]$ and a unique maximal solution w of

$$w' + Aw = \mathcal{R}(\cdot, w) \quad \text{on } (T_0, T^*), \qquad w(T_0) = w_0,$$

such that $w \in W^{1,s}_\alpha(T_0, T; X) \cap L^s_\alpha(T_0, T; D)$ for all $T \in (T_0, T^*)$.

We apply this result to (4.7). Let $p \in (\frac{d-1}{\alpha-\frac{1}{s}}, \infty)$, choose $X = \mathbb{L}^p$ and $A = A_p$. Then

$$(\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s} \subset \mathbb{L}^\infty \tag{4.8}$$

by Corollary 3.6. Suppose $\mathfrak{b}$ and $\mathfrak{a}$ satisfy Assumption 4.1. For all $w_0 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ we put $\mathcal{A}(w_0) := \eta(w_0) A_p$. If $\|w_0\|_{(\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} \le M$ for a given number M, then it follows from (4.8) that the image of $\Omega \cup \Gamma$ under w_0 is almost everywhere contained in a compact interval that only depends on M. In particular, this gives $\eta(w_0), \eta(w_0)^{-1} \in \mathbb{L}^\infty$, and the operator $\mathcal{A}(w_0)$ satisfies maximal parabolic regularity by Theorem 3.4. Recall that $\mathrm{Dom}(\mathcal{A}(w_0)) = \mathrm{Dom}(A_p)$.

The function η is locally Lipschitz continuous on $\mathbb{R}$. Therefore

$$\begin{aligned} \|\mathcal{A}(w_1) - \mathcal{A}(w_2)\|_{\mathcal{L}(\mathrm{Dom}(A_p), \mathbb{L}^p)} &\le \|\eta(w_1) - \eta(w_2)\|_{\mathbb{L}^\infty} \\ &\le C_M \|w_1 - w_2\|_{\mathbb{L}^\infty} \le \|w_1 - w_2\|_{(\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} \end{aligned}$$

for all $w_1, w_2 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ with $\|w_j\|_{(\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} \le M$ for all $j \in \{1,2\}$. This verifies the conditions of the proposition for $\mathcal{A}$.

We next present sufficient conditions for the functions F_Ω, F_Γ and F_Σ such that the operator R, defined in (4.6), satisfies the conditions for R in Proposition 4.3.

Assumption 4.4. For all $\xi \in \mathbb{R}$ the mappings $F_\Omega(\cdot,\xi)\colon J \to \mathbb{R}$, $F_\Gamma(\cdot,\xi)\colon J \to \mathbb{R}$ and $F_\Sigma(\cdot,\xi)\colon J \to \mathbb{R}$ are measurable. For all $M > 0$ there is an $r_M \in L^s_\alpha(J)$ such that

$$|F_\Omega(t,\xi_1) - F_\Omega(t,\xi_2)| \le r_M(t)\,|\xi_1 - \xi_2|$$

for a.e. $t \in J$ and $\xi_1, \xi_2 \in \mathbb{R}$ with $|\xi_1|, |\xi_2| \le M$; and analogous conditions for F_Γ and F_Σ.

Under the above assumption, (4.8) implies that $R(\cdot,w_0)\colon J \to \mathbb{L}^p$ is measurable for all $w_0 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ and that $R(\cdot,0) \in L^s_\alpha(J)$. We verify the Lipschitz property for F_Ω. If $M > 0$, $w_1, w_2 \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ with $\|w_1\|_{(\mathbb{L}^p,\mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} \le M$ and $\|w_2\|_{(\mathbb{L}^p,\mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} \le M$, then for a.e. $t \in J$ we have

(4.9)

$$\begin{aligned}
&\|\eta(w_1|_\Omega)F_\Omega(t,\mathfrak{K}^{-1}(w_1|_\Omega)) - \eta(w_2|_\Omega)F_\Omega(t,\mathfrak{K}^{-1}(w_2|_\Omega))\|_{L^p(\Omega)}\\
&\qquad \le \|\eta(w_1|_\Omega) - \eta(w_2|_\Omega)\|_{L^\infty(\Omega)}\|\eta(w_1|_\Omega)F_\Omega(t,\mathfrak{K}^{-1}(w_1|_\Omega))\|_{L^p(\Omega)}\\
&\qquad\quad + \|\eta(w_2|_\Omega)\|_{L^\infty(\Omega)}\|F_\Omega(t,\mathfrak{K}^{-1}(w_1|_\Omega)) - F_\Omega(t,\mathfrak{K}^{-1}(w_1|_\Omega))\|_{L^p(\Omega)}\\
&\qquad \le C_M\big(\|w_1|_\Omega - w_2|_\Omega\|_{L^\infty(\Omega)} + \widetilde{r}_M(t)\|\mathfrak{K}^{-1}(w_1|_\Omega) - \mathfrak{K}^{-1}(w_2|_\Omega)\|_{L^p(\Omega)}\big)\\
&\qquad \le C_M(1+\widetilde{r}_M(t))\|w_1 - w_2\|_{(\mathbb{L}^p,\mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}},
\end{aligned}$$

for a suitable $\widetilde{r}_M \in L^s_\alpha(J)$. The same arguments apply to the components F_Γ and F_Σ, and thus R is as desired to apply the proposition.

We have proven the main result of this section.

Theorem 4.5. *Let $p \in (\frac{d-1}{\alpha-\frac{1}{s}}, \infty)$, and suppose that Ω, Γ, Σ, and ε are as in Theorem 3.10, that μ, $\mathfrak{b}$ and $\mathfrak{a}$ are as in Assumption 4.1, and that f, g and h are as in Assumption 4.4. Then for all $u_0 \in \mathbb{L}^p$ with $\mathfrak{K}(u_0) \in (\mathbb{L}^p, \mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}$ there are $T^* = T^*(u_0) \in (T_0, T_1]$ and a unique maximal solution $u \in C([T_0,T^*);\mathbb{L}^\infty)$ of (4.1)–(4.5) in the sense of Definition 4.2. This means that for all $T_0 < T < T^*$ we have*

$$\mathfrak{K}(u) \in W^{1,s}_\alpha(T_0,T;\mathbb{L}^p) \cap L^s_\alpha(T_0,T;\mathrm{Dom}(A_p)),$$

and $\mathfrak{K}(u)$ is the unique solution of

$$\partial_t w + \eta(w)A_p w = R(\cdot,w) \quad \text{on } (T_0,T), \qquad w(T_0) = \mathfrak{K}(u_0). \tag{4.10}$$

Instead of F_Ω, F_Γ and F_Σ one can easily find also non-local maps such that the corresponding operator R satisfies the condition of Proposition 4.3. One can take for example (linear or non-linear) integral operators with suitable kernel properties. Moreover, in our example, F_Ω maps $L^\infty(\Omega)$ into itself, while F_Γ maps $L^\infty(\Gamma)$ itself, and correspondingly also for F_Σ, i.e., the mapping R has no crossing terms. This is also not necessary in general.

The nonlinearity in the elliptic operator may also be a nonlocal operator. This case arises e.g. in models for the diffusion of bacteria; see [CC], [ChiL] and references therein.

We end this section with some comments on the case when (4.1)–(4.5) is semilinear, i.e., when $\mathfrak{b} = \mathfrak{K} = \mathrm{id}$, such that u itself solves the realization (4.10) of the problem.

The following is a useful criterion for the global existence of solutions.

Proposition 4.6. *Adopt the assumptions of Theorem* 4.5*. Suppose in addition that* $\mathfrak{b} = \mathfrak{K} = id$*, and let* $u \in C([T_0, T^*); \mathbb{L}^\infty)$ *be the maximal solution of* (4.10)*. If*

$$\limsup_{t \to T^*} \|u(t,\cdot)\|_{L^\infty(\Omega)} < \infty,$$

then u *is a global solution of* (4.10)*, i.e.,* $T^* = T_1$ *and* $u \in W^{1,s}_\alpha(J; \mathbb{L}^p) \cap L^s_\alpha(J; \mathrm{Dom}(A_p))$.

Proof. By Proposition 3.2, for all $T < T^*$ the solution u satisfies

(4.11)
$$\|u'\|_{L^s_\alpha(T_0,T;\mathbb{L}^p)} + \|u\|_{L^s_\alpha(T_0,T;\mathrm{Dom}(A_p))} \le c\big(\|u_0\|_{(X,\mathrm{Dom}(A_p))_{\alpha-\frac{1}{s},s}} + \|R(\cdot,u)\|_{L^s_\alpha(T_0,T;\mathbb{L}^p)}\big),$$

where c is uniform in T. Observe that $\|u(t,\cdot)\|_{\mathbb{L}^\infty} \le \|u(t,\cdot)\|_{L^\infty(\Omega)}$ for almost every t by the definition of the trace (see Section 2.2). Hence $M = \|u\|_{L^\infty(T_0,T^*;\mathbb{L}^\infty)} < \infty$. Estimates as in (4.9) yield

$$\|R(\cdot,u)\|_{L^s_\alpha(T_0,T^*;\mathbb{L}^p)} \le \|R(\cdot,0)\|_{L^s_\alpha(T_0,T^*;\mathbb{L}^p)} + C_M\big(1 + \|\widetilde{r}_M\|_{L^s_\alpha(T_0,T^*)}\big).$$

Therefore the terms on the left-hand side of (4.11) are bounded uniformly in T. By [Prü, Corollary 3.2], this implies $T^* = T_1$. □

We finally comment on the asymptotics of solutions.

Remark 4.7. Under the assumptions of Theorem 4.5, in the autonomous semilinear case the solutions form a local semiflow in the phase space $\mathrm{Dom}(A_p^\theta)$, where θ is sufficiently close to 1. Since the resolvent of A_p is compact by Lemma 2.20(iii), the solution semiflow is compact, and bounded orbits are relatively compact. This property is very useful in studying the long-time behaviour of solutions.

5. Concluding remarks

Remark 5.1. The realization of (1.1)–(1.5) in Section 3 still enjoys maximal regularity if one adds a term bu in the dynamic equation on $J \times \Sigma$ and imposes suitable conditions on b.

Remark 5.2. Everything can be done also for systems which couple in the reactions.

Remark 5.3. The fundamental result of Prüss (Proposition 4.3) allows to treat the quasilinear problem (4.1)–(4.5) also in the case where the nonlinearities $\mathfrak{b}$ and $\mathfrak{a}$ depend explicitly on time. We did not carry out this here for the sake of technical simplicity.

Remark 5.4. If one requires Ω to be a Lipschitz domain and, additionally, imposes a certain compatibility condition between Γ and its complement in the boundary (see [Grö], [HMRS]), then $(-\nabla \cdot \mu\nabla + 1)^{-1}$ maps $\breve{W}_\Gamma^{-1,q}$, i.e., the anti-dual space of $W_\Gamma^{1,q}$, into a Hölder space, if $q > d$. If s in Theorem 3.10/Theorem 4.5 is chosen sufficiently large, then the corresponding solutions are even Hölder continuous in space and time, compare [DMRT].

Remark 5.5. What cannot be treated within this framework is the case where Σ moves in Ω in time. If one wants to include this, the concept of [HaR1] should be adequate, see also [HaR3].

Remark 5.6. What also cannot be treated within this framework is the case where the function ε is not away from 0, in particular, if it is 0 on a subset of positive boundary measure. This would e.g. affect the case of inhomogeneous Neumann boundary conditions. It is known that the resulting problem is of very different functional analytic quality and requires different methods, see [Nit].

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