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On elliptic and parabolic regularity for mixed boundary value problems

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ABSTRACT. We study second order equations and systems on non-Lipschitz domains including mixed boundary conditions. The key result is interpolation for suitable function spaces.

1. INTRODUCTION

In this paper we first establish interpolation properties for function spaces that are related to mixed boundary value problems. Afterwards, from this and a fundamental result of Sneiberg [52] we deduce elliptic and parabolic regularity results for divergence operators.

In recent years it became manifest that the appearance of mixed boundary conditions is not the exception when modelling real world problems, but more the rule. For instance, in semiconductor theory, models with only pure Dirichlet or pure Neumann conditions are meaningless, see [50].

One geometric concept, which proved of value for the analysis of mixed boundary value problems, is that introduced by Gröger in [23] (compare also [47] and references therein). It demands, roughly speaking, that the domain under consideration is a Lipschitz domain and that the 'Dirichlet part' $D \subset \partial\Omega$ of the boundary is locally separated from the rest by a Lipschitzian hypersurface within $\partial\Omega$. (For a simplifying characterisation in the most relevant cases of two and three space dimensions see [26].) Within this concept, several properties for differential operators, well-known from smooth constellations, were re-established. This concerns elliptic regularity (in particular Hölder continuity) [23], [20], [22], [26], maximal parabolic regularity [21], [27], [28] and interpolation [19].

Gröger's concept covers many realistic constellations occurring in real-world problems; and the procedure of proof for the above listed results was always this: localise the problem and then pass via bi-Lipschitz diffeomorphisms and (if necessary) by reflection to a corresponding – homogeneous – Dirichlet problem of the same quality. Take the information, known for the Dirichlet problem, and go back.

In this paper we, for the first time, dispense the Lipschitz property of the domain as concerns interpolation of the corresponding function spaces, and, additionally, pose a condition on the Dirichlet part of the boundary which is extremely general; in particular it is far beyond Gröger's condition. The possibility to do this rests on a new extension result, which allows to extend functions, defined on the domain Ω and possessing a certain boundary behaviour at the Dirichlet part $D \subset \partial\Omega$, to the whole of $\mathbb{R}^d$, (cf. Lemma 2.6 below, compare also [10]). A relevant example is contained in Figure 1.

Roughly spoken, our setting is as follows: we demand that the Dirichlet part has only to be a $(d-1)$-set in the sense of Jonsson/Wallin while its complement has to admit local Lipschitzian charts. We think that this concept should cover nearly everything what is needed for the treatment of real-world problems – as long as the domain does not include cracks or things like that.

The outline of the paper is as follows: in the next section we introduce some preliminaries. In Section 3 we reproduce interpolation within the sets of spaces $\{W^{1,p}_D(\Omega)\}_{p\in]1,\infty[}$, and, as a consequence, in the set $\{W^{-1,p}_D(\Omega)\}_{p\in]1,\infty[}$. Rather unexpectedly, this follows directly from the pioneering results of Jonsson/Wallin, combined with a classical interpolation principle for complemented subspaces. Knowing only this, in Section 4 we succeed in reproducing Gröger's elliptic regularity result from [23], namely that an arbitrary elliptic divergence operator $-\nabla \cdot \mu \nabla + 1$ provides a topological isomorphism between a space $W^{1,p}_D(\Omega)$ and $W^{-1,p}_D(\Omega)$ for some $p > 2$. Note that the main result from [23] was used in some tens of papers in order to treat successfully (mostly two dimensional) problems, stemming from real world applications. Having this regularity result at hand, one succeeds in proving that divergence operators of this type generate analytic semigroups on spaces $W_D^{-1,p}(\Omega)$, as long as p is chosen close to 2. Clearly, this can serve as the adequate instrument for the treatment of corresponding parabolic problems, compare e.g. [29, Ch. 2], [2], [42].

In Section 5 we extend the discussion to a class of elliptic systems comprising the equations for linear elasticity and for Cosserat models. Relying on the interpolation results it is shown that the corresponding differential operators provide topological isomorphisms between $\mathbb{W}_D^{1,p}(\Omega)$ and $\mathbb{W}_D^{-1,p}(\Omega)$ for suitable $p > 2$. Moreover, under an additional symmetry assumption on the coefficient tensor, uniform estimates are derived for classes of coefficient tensors satisfying certain uniform bounds. Since in the case of systems the coercivity of the operator not necessarily entails the positivity of the coefficient tensor, the pointwise arguments from [23] have to be modified and transferred to arguments dealing with the whole operator. In this way also the results from [30] are extended to more general geometric situations.
Finally, in Ch. 6, we point out a broad class of possible applications for our regularity results.

2. Notations, Preliminaries

If X and Y are two Banach spaces, then we use the symbol $\mathcal{L}(X;Y)$ for the space of linear, continuous operators from X to Y. In case of $X = Y$ we abbreviate $\mathcal{L}(X)$.

Definition 2.1. We denote the open unit cube $]-\frac{1}{2}, \frac{1}{2}[^d \subset \mathbb{R}^d$, centred at 0, by E_d, its upper half $E_d \times]0, \frac{1}{2}[$ by E_d^+ and its midplate $E_d \cap \{\mathrm{x} : x_d = 0\}$ by P.

Our central geometric supposition on the domain Ω and the boundary part Γ we include in the following

Assumption 2.2. $\Omega \subset \mathbb{R}^d$ is always a bounded domain and Γ is a (relatively) open subset of its boundary $\partial\Omega$. For every $\mathrm{x} \in \overline{\Gamma}$ there is an open neighbourhood U_x and a bi-Lipschitz mapping Φ_x from U_x onto the unit cube E_d, such that $\Phi_\mathrm{x}(\Omega \cap U_\mathrm{x}) = E_d^+$, $\Phi_\mathrm{x}(\partial\Omega \cap U_\mathrm{x}) = P$ and $\Phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^d$.

Remark 2.3. It is well-known that the bi-Lipschitzian charts around the points from $\overline{\Gamma}$ induce the boundary measure on $\partial\Omega \cap \left(\cup_{\mathrm{x}\in\overline{\Gamma}} U_\mathrm{x}\right)$, compare [25] and [11, Ch. 3.3.4 C]).

Definition 2.4. Let Υ be a bounded domain or $\Upsilon = \mathbb{R}^d$ and let D be a closed subset of $\overline{\Upsilon}$. Then we define

$$C_D^\infty(\Upsilon) := \{\psi|_\Upsilon : \psi \in C^\infty(\mathbb{R}^d), \operatorname{supp}\psi \cap D = \emptyset\}. \tag{2.1}$$

Moreover, for $p \in [1, \infty]$, we denote the closure of $C_D^\infty(\Upsilon)$ in $W^{1,p}(\Upsilon)$ by $W_D^{1,p}(\Upsilon)$.

Remark 2.5. If Ω is a bounded Lipschitz domain, then the definition of $W_D^{1,p}(\Omega)$ is in coincidence with the fact that in the space $W^{1,p}(\Omega)$ the restrictions of $C^\infty(\mathbb{R}^d)$-functions are dense.

Lemma 2.6. *Let Ω and Γ satisfy Assumption 2.2 and set $D := \partial\Omega \setminus \Gamma$. Then there is a continuous extension operator $\mathfrak{E} : W_D^{1,1}(\Omega) \to W_D^{1,1}(\mathbb{R}^d)$ whose restriction simultaneously provides a continuous extension operator from $W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$ for all $p \in]1, \infty]$.*

Proof. Fix $p \in [1, \infty]$. Let, for every $\mathrm{x} \in \overline{\Gamma}$ the set U_x be an open neighbourhood which satisfies the condition from Assumption 2.2. Let $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_\ell}$ be a finite subcovering of $\overline{\Gamma}$ and let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a function that is identically one in a neighbourhood of $\overline{\Gamma}$ and has its support in $U := \bigcup_{j=1}^\ell U_{\mathrm{x}_j}$.

Assume $\psi \in C_D^\infty(\Omega)$; then we can write $\psi = \eta\psi + (1-\eta)\psi$. By the definition of $C_D^\infty(\Omega)$ and η it is clear that the support of $(1-\eta)\psi$ is contained in Ω, thus this function may be extended by 0 to the whole space $\mathbb{R}^d$ while its $W^{1,p}$-norm is preserved.

It remains to define the extension of the function $\eta\psi$, what we will do now. For this, let $\eta_1, \dots, \eta_\ell$ be a partition of unity on $\operatorname{supp}(\eta)$, subordinated to the covering $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_\ell}$. Then we can write $\eta\psi = \sum_{r=1}^{\ell} \eta_r\eta\psi$ and have to define an extension for every function $\eta_r\eta\psi$. For doing so, we first transform the corresponding function under the mapping Φ_{x_r} to the function $\widetilde{\eta_r\eta\psi} = (\eta_r\eta\psi) \circ \Phi_{\mathrm{x}_r}^{-1}$ on the half unit cube E_d^+. Afterwards, by even reflection, one obtains a function $\widehat{\eta_r\eta\psi} \in W^{1,p}(E_d)$ on the unit cube E_d. It is clear by construction that $\operatorname{supp}(\widehat{\eta_r\eta\psi})$ has a positive distance to ∂E_d. Transforming back, one ends up with a function $\overline{\eta_r\eta\psi} \in W^{1,p}(U_{\mathrm{x}_r})$ whose support has a positive distance to $\partial U_{\mathrm{x}_r}$. Thus, this function may also be extended by 0 to the whole of $\mathbb{R}^d$, preserving again the $W^{1,p}$ norm.

Lastly, one observes that all the mappings $W^{1,p}(U_{\mathrm{x}_r} \cap \Omega) \ni \eta_r\eta\psi \mapsto \widetilde{\eta_r\eta\psi} \in W^{1,p}(E_d^+)$, $W^{1,p}(E_d^+) \ni \widetilde{\eta_r\eta\psi} \mapsto \widehat{\eta_r\eta\psi} \in W^{1,p}(E_d)$ and $W^{1,p}(E_d) \ni \widehat{\eta_r\eta\psi} \mapsto \overline{\eta_r\eta\psi} \in W^{1,p}(U_{\mathrm{x}_r})$ are continuous. Thus, adding up, one arrives at an extension of ψ whose $W^{1,p}(\mathbb{R}^d)$-norm may be estimated by $c\|\psi\|_{W^{1,p}(\Omega)}$ with c independent from ψ. Hence, the mapping $\mathfrak{E}$, up to now defined on $C_D^\infty(\Omega)$, continuously and uniquely extends to a mapping from $W_D^{1,p}$ to $W^{1,p}(\mathbb{R}^d)$.

It remains to show that the images in fact even ly in $W_D^{1,p}(\mathbb{R}^d)$. For doing so, one first observes that, by construction of the extension operator, for any $\psi \in C_D^\infty(\Omega)$, the support of the extended function $\mathfrak{E}\psi$ has a positive distance to D – but $\mathfrak{E}\psi$ need not be smooth. Clearly, one may convolve $\mathfrak{E}\psi$ suitably in order to obtain an appropriate approximation in the $W^{1,p}(\mathbb{R}^d)$-norm – maintaining a positive distance of the support to the set D. Thus, $\mathfrak{E}$ maps $C_D^\infty(\Omega)$ continuously into $W_D^{1,p}(\mathbb{R}^d)$, what is also true for its extension to the whole of $W_D^{1,p}(\Omega)$. $\square$

Remark 2.7. It is clear by the construction of $\mathfrak{E}$ that all functions $\mathfrak{E}\psi$ have their support in $\Omega \cup \bigcup_{j=1}^{\ell} U_{\mathrm{x}_j}$. Let from now on $B \subset \mathbb{R}^d$ be a fixed ball around $0 \in \mathbb{R}^d$ such that $\frac{1}{2}B \supset \Omega \cup \left(\bigcup_{j=1}^{\ell} U_{\mathrm{x}_j}\right)$. In particular, one then has $D \subseteq \partial\Omega \subset B$.

Remark 2.8. It is not hard to see that the operator $\mathfrak{E}$ extends to a continuous operator from $L^p(\Omega)$ to $L^p(\mathbb{R}^d)$, where $p \in [1, \infty]$. Using this, one can establish the corresponding Sobolev embeddings $W_D^{1,p}(\Omega) \to L^q(\Omega)$ (compactness, included) in a straightforward manner.

Remark 2.9. Combining the mapping $\mathfrak{E}$ with the operator that restricts any function on $\mathbb{R}^d$ to B, one obtains an operator that maps $W_D^{1,p}(\Omega)$ continuously into the space $W_D^{1,p}(B)$; we maintain the notation $\mathfrak{E}$ for the resulting operator.

Definition 2.10. We denote by $\mathfrak{R} : W^{1,p}(B) \to W^{1,p}(\Omega)$ the canonic restriction operator.

Remark 2.11. It is not hard to see that the canonic restriction operator $\mathfrak{R} : W^{1,p}(B) \to W^{1,p}(\Omega)$ gives rise to a restriction operator $\mathfrak{R} : W_D^{1,p}(B) \to W_D^{1,p}(\Omega)$ – for which we also maintain the notation $\mathfrak{R}$. Note that $\mathfrak{E}$ and $\mathfrak{R}$ are consistent on the sets $\{W_D^{1,p}(B)\}_{p\in[1,\infty]}$ and $\{W_D^{1,p}(\Omega)\}_{p\in[1,\infty]}$: if $q > p$, then $\mathfrak{R} : W_D^{1,q}(B) \to W_D^{1,q}(\Omega)$ is the restriction of $\mathfrak{R} : W_D^{1,p}(B) \to W_D^{1,p}(\Omega)$ and $\mathfrak{E} : W_D^{1,q}(\Omega) \to W_D^{1,q}(B)$ is the restriction of $\mathfrak{E} : W_D^{1,p}(\Omega) \to W_D^{1,p}(B)$.

Finally, one observes that, for every $p \in [1, \infty]$, the operators $\mathfrak{R} : W_D^{1,p}(B) \to W_D^{1,p}(\Omega)$ and $\mathfrak{E} : W_D^{1,p}(\Omega) \to W_D^{1,p}(B)$ form a retraction/coretraction pair, see [54, Ch. 1.2.4].

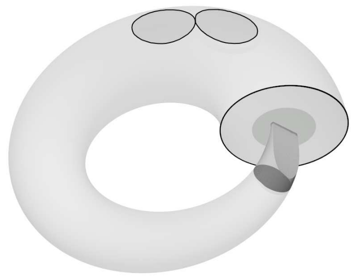

FIGURE 1. A geometric non-Lipschitzian setting which fulfills our assumptions, if the grey apex and the three shaded circles carry the Dirichlet condition

We are now going to impose the adequate condition on the Dirichlet boundary part $D = \partial\Omega \setminus \Gamma$. For this we first recall the notion of an l-set, cf. Jonsson/Wallin [33, II.1.1].

Definition 2.12. Assume $0 < l \le d$. Let $M \subset \mathbb{R}^d$ be closed and ϱ a Borel measure on M. Then ϱ is an l-measure, and M is called an l-set, if there exist two positive constants c_1, c_2 that satisfy

$$c_1 r^l \le \varrho\big(B(\mathrm{x}, r) \cap M\big) \le c_2 r^l, \ \mathrm{x} \in M, r \in \,]0, 1[, \tag{2.2}$$

where $B(\mathrm{x}, r)$ is the ball with center x and radius r in $\mathbb{R}^d$.

Let us further introduce

Assumption 2.13. $D = \partial\Omega \setminus \Gamma$ is a $(d-1)$-set.

Remark 2.14. Of course, in the case of Lipschitz domains, the boundary measure, restricted to D, plays the role of ϱ, compare [11, Ch. 3.3.4 C] or see [25, Ch. 3].

Since the ultimate instrument for almost everything in the next section is a pioneering result of Jonsson/Wallin (see [33, Ch. VII]) we quote this here for the convenience of the reader:

Proposition 2.15. *Let $F \subset \mathbb{R}^d$ be closed and, additionally, a $(d-1)$-set.*

i) *Then there is a continuous restriction operator $\mathcal{R}_F$ which maps every space $W^{1,p}(\mathbb{R}^d)$ continuously onto the Besov space $B^{1-\frac{1}{p}}_{p,p}(F)$ as long as $p \in \,]1, \infty[$.*
ii) *Conversely, there is an extension operator $\mathcal{E}_F$ which maps each space $B^{1-\frac{1}{p}}_{p,p}(F)$ continuously into $W^{1,p}(\mathbb{R}^d)$, provided $p \in \,]1, \infty[$.*
iii) *The operator $\mathcal{E}_F$ maps the space of Lipschitz continuous functions on F continuously into the space of Lipschitz continuous functions on $\mathbb{R}^d$.*
iv) *By construction, $\mathcal{E}_F$ is a right inverse for $\mathcal{R}_F$, i.e. $\mathcal{R}_F\mathcal{E}_F$ is the identity operator on $B^{1-\frac{1}{p}}_{p,p}(F)$, cf.* [33, Ch.V.1.3].

Proof. Only iii) is not explicitely contained in [33], therefore we give a proof here, referring to [33] for more background and details.
The extension operator used in the theorem is of Whitney type, and we need some facts about the Whitney decomposition of $\mathbb{R}^d \setminus F$ and a related partition of unity $\{\phi_i\}$. The decomposition is a collection of closed, dyadic cubes Q_i, with sidelength 2^N for some integer N, and with mutually disjoint interiours, such that $\cup Q_i = \mathbb{R}^d \setminus F$, and

$$\mathrm{diam} Q_i \le d(Q_i, F) \le 4 \mathrm{diamQ_i}, \tag{2.3}$$

where $d(Q_i, F)$ is the distance between Q_i and F. Denote the diameter of Q_i by l_i, its sidelength by s_i, and its center by x_i, and let $Q_i^\star$ denote the cube obtained by expanding Q_i around its center with a factor ρ, $1 < \rho < 5/4$. It follows from (2.3) that

$$1/4l_i \le l_k \le 4l_i \tag{2.4}$$

if Q_i and Q_k touch. This means that $Q_i^\star$ intersects a cube Q_k only if Q_i and Q_k touch, and that each point in $\mathbb{R}^d \setminus F$ is contained in at most N_0 cubes $Q_i^\star$, where N_0 is a number depending only on the dimension d. Next, nonnegative $C^\infty-$ functions ϕ_i are constructed in such a way that $\phi_i(\mathrm{x}) = 0$ if $\mathrm{x} \notin Q_i^\star$, $\sum_i \phi_i(\mathrm{x}) = 1, \mathrm{x} \in \mathbb{R}^d \setminus F$, and so that $|D^j\phi_i| \le c l_i^{-|j|}$ for any j, where c depends on j.

Let I denote those i such that $s_i \le 1$, let ϱ be a $(d-1)$-measure on F, and put $c_i = \varrho(B(\mathrm{x}_i, 6l_i))^{-1}$. Note that it follows from (2.2) and (2.3), that $\varrho(B(\mathrm{x}_i, 6l_i)) > 0$. The extension operator used in Proposition 2.15 is given by

$$\mathcal{E}_F f(\mathrm{x}) = \sum_{i\in I} \phi_i(\mathrm{x}) c_i \int_{|\mathrm{t}-\mathrm{x}_i|\le 6_{l_i}} f(\mathrm{t}) d\varrho(\mathrm{t}),\ \mathrm{x} \in \mathbb{R}^d \setminus F, \tag{2.5}$$

and $\mathcal{E}_F f(\mathrm{x}) = f(\mathrm{x}), \mathrm{x} \in F$.

Let x and y be in cubes with sides $\le 1/4$, then $\sum \phi_i(\mathrm{x}) = \sum \phi_k(\mathrm{y}) = 1$. Using this, one obtains, for any constant b,

$$\mathcal{E}_F f(\mathrm{x}) - b = \sum_i \phi_i(\mathrm{x}) c_i \int_{|\mathrm{t}-\mathrm{x}_i|\le 6_{l_i}} (f(\mathrm{t}) - b) d\varrho(\mathrm{t}), \tag{2.6}$$

and taking $b = \mathcal{E}_F f(\mathrm{y})$

$$\mathcal{E}_F f(\mathrm{x}) - \mathcal{E}_F f(\mathrm{y}) = \sum_i \sum_k \phi_i(\mathrm{x})\phi_k(\mathrm{y}) c_i c_k \int\int_{|\mathrm{t}-\mathrm{x}_i|\le 6_{l_i}, |\mathrm{s}-\mathrm{x}_k|\le 6_{l_k}} (f(\mathrm{t}) - f(\mathrm{s})) d\varrho(\mathrm{t}) d\varrho(\mathrm{s}). \tag{2.7}$$

We also have

$$D^j(\mathcal{E}_F f)(\mathrm{x}) = \sum_i D^j\phi_i(\mathrm{x}) c_i \int_{|\mathrm{t}-\mathrm{x}_i|\le 6_{l_i}} f(\mathrm{t}) d\varrho(\mathrm{t}), \tag{2.8}$$

and, for $|j| > 0$, since then $\sum D^j\phi_i(\mathrm{x}) = 0$, so we can subtract $\mathcal{E}_F f(\mathrm{y})$ from the integrand,

$$D^j(\mathcal{E}_F f)(\mathrm{x}) = \sum_i \sum_k D^j\phi_i(\mathrm{x})\phi_k(\mathrm{y}) c_i c_k \int\int_{|\mathrm{t}-\mathrm{x}_i|\le 6_{l_i}, |\mathrm{s}-\mathrm{x}_k|\le 6_{l_k}} (f(\mathrm{t}) - f(\mathrm{s})) d\varrho(\mathrm{t}) d\varrho(\mathrm{s}). \tag{2.9}$$

Assume now that f is Lipschitz continuous with Lipschitz norm 1. Let $\mathrm{x} \in Q_\nu$, $\mathrm{y} \in Q_\eta$, where, say, $s_\nu \ge s_\eta$, and assume first $s_\nu \le 1/4$. If $|\mathrm{x}-\mathrm{y}| < l_\nu/2$, then by the mean value theorem $f(\mathrm{x}) - f(\mathrm{y}) = \nabla f(\xi)\cdot(\mathrm{x}-\mathrm{y})$ for some ξ with $|\mathrm{x}-\xi| < l_\nu/2$. Take κ so that $\xi \in Q_\kappa$. Now we use, if $s_\kappa \le 1/4$ (otherwise, see below), (2.9) with x and y equal to ξ, and recall that if $\phi_i(\xi) \ne 0$, then Q_i and Q_κ touch. For nonzero terms we then have, for t and s in the domain of integration, $|\mathrm{t}-\mathrm{s}| \le |\mathrm{t}-\mathrm{x}_i| + |\mathrm{x}_i - \mathrm{x}_\kappa| + |\mathrm{x}_\kappa - \mathrm{x}_k| + |\mathrm{x}_k - s| \le 7l_i + 2l_\kappa + 7l_k$, and also, by (2.4), that, l_i and l_k are comparable to l_κ. Recalling that $0 \le \phi_i \le 1$, $|D^j\phi_i| \le c l_i^{-1}$, $|j| = 1$ and using $|f(\mathrm{s}) - f(\mathrm{t})| \le |\mathrm{t}-\mathrm{s}|$, one immediately obtains $|D^j(\mathcal{E}f)(\xi)| \le c$, $|j| = 1$, so

$$|\mathcal{E}f(\mathrm{x}) - \mathcal{E}f(\mathrm{y})| \le c|\mathrm{x}-\mathrm{y}|. \tag{2.10}$$

If $|\mathrm{x}-\mathrm{y}| \ge l_\nu/2$, we use (2.7) together with the observation that now $|\mathrm{t}-\mathrm{s}| \le |\mathrm{t}-\mathrm{x}_i| + |\mathrm{x}_i - \mathrm{x}| + |\mathrm{x}-\mathrm{y}| + |\mathrm{y}-\mathrm{y}_k| + |\mathrm{y}_k - \mathrm{s}| \le 7l_i + l_\nu + l_\eta + 7l_k + |\mathrm{x}-\mathrm{y}| \le 58l_\nu + |\mathrm{x}-\mathrm{y}| \le c|\mathrm{x}-\mathrm{y}|$ if $\phi(\mathrm{x})$ and $\phi(\mathrm{y})$ are nonzero, and obtain again (2.10). If instead $\mathrm{y} \in F$ we get the same result using (2.6) with $b = f(\mathrm{y})$ and $|\mathrm{t}-\mathrm{y}| \le |\mathrm{t}-\mathrm{x}_i| + |\mathrm{x}_i - \mathrm{x}| + |\mathrm{x}-\mathrm{y}| \le 7l_i + l_\nu + |\mathrm{x}-\mathrm{y}| \le c|\mathrm{x}-\mathrm{y}|$, since, by (2.3), $l_\nu \le |\mathrm{x}-\mathrm{y}|$.

If $s_\nu > 1/4$, or $s_\kappa > 1/4$, we can no longer use (2.9), (2.7), and (2.6). In the case $|\mathrm{x}-\mathrm{y}| < l_\nu/2$, (2.8) together with $|f| \le 1$ gives the desired estimate $|D^j(\mathcal{E}f)(\xi)| \le cl_\kappa^{-1} \le c$, $|j| = 1$. Using (2.5) we see that $|\mathcal{E}f| \le c$ everywhere, which in particular implies (2.10) in the remaining cases. $\square$

Remark 2.16. The proof of iii) does not require much about the measure ϱ. The only thing needed is that the measure of any ball with center in F is positive, which in particular holds for any l-measure ϱ, $0 < l \le n$.

Remark 2.17. Since the detailed structure of the Besov spaces $B_{p,p}^{1-\frac{1}{p}}(F)$ is not of interest here, we refer to [33, Ch. V.1] for a definition.

In the sequel we consider in our case $F = D$ the restriction/extension operators $\mathcal{R}_F/\mathcal{E}_F$ not only on all of $\mathbb{R}^d$, but also on the ball B. Since $D \subset B$ and the restriction operator $\mathcal{R}_D$ takes into account only the local behaviour of functions near D, $\mathcal{E}_D$ remains a right inverse of $\mathcal{R}_D$ in this understanding. In this spirit, we also maintain the notations $\mathcal{E}_D, \mathcal{R}_D$.

Definition 2.18. We use the symbol $\mathcal{W}_D^{1,p}(B)$ for the space

$$\{\psi \in W^{1,p}(B) : \mathcal{R}_D\psi = 0 \text{ a.e. on } D \text{ with respect to } \varrho\},$$

cf. Definition 2.12.

It is a natural question whether $\mathcal{W}_D^{1,p}(B) = W_D^{1,p}(B)$ holds. An affirmative answer for our situation will be given in Theorem 3.2.

3. Interpolation

In this section we establish the interpolation results that are well-known for $\mathbb{R}^d$ or smooth domains, for the spaces $W_D^{1,p}(\Omega)$.

What is of use for us from the Jonsson/Wallin result is the following: the right inverse property of $\mathcal{E}_D$ for $\mathcal{R}_D$ implies that $\mathcal{E}_D\mathcal{R}_D : W^{1,p}(B) \to W^{1,p}(B)$ is a (continuous) projection. Furthermore, it is straightforward to verify that $\mathcal{E}_D\mathcal{R}_D\varphi = 0$, iff $\mathcal{R}_D\varphi = 0$. This implies that $\varphi \in \mathcal{W}_D^{1,p}(B)$, if and only if $\varphi \in W^{1,p}(B)$ and $(1 - \mathcal{E}_D\mathcal{R}_D)\varphi = \varphi$. Consequently, the operator $\mathcal{P} := 1 - \mathcal{E}_D\mathcal{R}_D$ is a (continuous) projection from $W^{1,p}(B)$ onto $\mathcal{W}_D^{1,p}(B)$.

The existence of the projector $\mathcal{P}$ allows to deduce the desired interpolation properties for the spaces $\mathcal{W}_D^{1,p}(\Omega)$ by purely functorial properties.

Theorem 3.1. *Let $p \in]1,\infty[$ and let $D \subset B$ be a $(d-1)$-set. Then the spaces $\mathcal{W}_D^{1,p}(B)$ interpolate according to the same rules as the spaces $W^{1,p}(B)$ do.*

Proof. Let $\mathcal{P}$ be the projection from above. Since, for any $p \in]1,\infty[$, $\mathcal{P}$ maps $W^{1,p}(B)$ onto $\mathcal{W}_D^{1,p}(B)$, interpolation carries over from the spaces $W^{1,p}(B)$ to the spaces $\mathcal{W}_D^{1,p}(B)$ by a classical interpolation principle for complemented subspaces, see [54, Ch. 1.17.1]. $\square$

In order to obtain this also for the spaces $W_D^{1,p}(\Omega)$, we will prove the following:

Theorem 3.2. *Let $B \subset \mathbb{R}^d$ be a ball and $D \subset B$ be a $(d-1)$-set. Then the spaces $W^{1,p}_D(B)$ and $\mathcal{W}^{1,p}_D(B)$ in fact coincide for $p \in]1,\infty[$.*

Proof. The inclusion $W^{1,p}_D(B) \subseteq \mathcal{W}^{1,p}_D(B)$ is implied by the Jonsson/Wallin result: all functions from $C^\infty_D(\Omega)$ vanish in a neighbourhood of D and, hence, have trace 0 on D. Since the trace is a continuous operator into $L^1(D;\varrho)$, this remains true for all elements from $W^{1,p}_D(\Omega)$.

Conversely, assume $\psi \in \mathcal{W}^{1,p}_D(B)$, and let $\hat{\psi}$ be a $W^{1,p}$-extension of ψ to all of $\mathbb{R}^d$ with compact support K. By the definition of the projector $\mathcal{P} = 1 - \mathcal{E}_D\mathcal{R}_D$ one has $\mathcal{P}\hat{\psi} = \hat{\psi}$. Since $\hat{\psi} \in \mathcal{W}^{1,p}_D(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$, there is a sequence $\{\psi_k\}_k$ from $C^\infty(\mathbb{R}^d)$ that converges towards $\hat{\psi}$ in the $W^{1,p}(\mathbb{R}^d)$ topology. Modulo multiplication with a suitable cutoff function we can arrange that all the functions ψ_k have their supports in a common compact set. Clearly, then $\mathcal{P}\psi_k \to \mathcal{P}\hat{\psi} = \hat{\psi}$, and the elements $\mathcal{P}\psi_k$ fulfill, by the definition of $\mathcal{P}$, the condition $\mathcal{P}\psi_k = 0$ a.e. on D with respect to ϱ.

We fix k and denote $\mathcal{P}\psi_k$ by f for brevity. Our intention is to show:

$$\text{There exists } g \in C^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp}(g) \cap D = \emptyset \text{ and } \|f - g\|_{W^{1,p}(\mathbb{R}^d)} \le \frac{1}{k}. \tag{3.1}$$

By the construction of the projector $\mathcal{P} = 1 - \mathcal{E}_D\mathcal{R}_D$ and the Jonsson/Wallin results in Proposition 2.15 the function f is Lipschitzian and vanishes almost everywhere on D. We will now show that, in fact, it vanishes identically on D. Let $x \in D$ be an arbitrary point. Then, for every $r > 0$, one has $\varrho(D \cap B(x,r)) > 0$ because D is a $(d-1)$-set. Thus, in this ball there is a point $y \in D$ for which $f(y) = 0$ holds. Hence, x is an accumulation point of the set on which f vanishes, and the claim follows from the continuity of f.

Let now $\{\zeta_n\}_n$ be the sequence of cut-off functions defined on $\mathbb{R}_+$ by

$$\zeta_n(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1/n, \\ nt - 1, & \text{if } 1/n \le t \le 2/n, \\ 1, & \text{if } 2/n < t. \end{cases}$$

Note that for $t \neq 0$ the values $\zeta_n(t)$ tend to 1 as $n \to \infty$. Moreover, one has $0 \le t\zeta_n'(t) \le 2$ and $t\zeta_n'(t)$ tends to 0 for all t. We denote by $\operatorname{dist}_D : \mathbb{R}^d \to \mathbb{R}_+$ the function which measures the distance to the set D. Note that dist_D is Lipschitzian with Lipschitz constant 1, and hence, from $W^{1,\infty}(\mathbb{R}^d)$ with $|\nabla \operatorname{dist}_D| \le 1$, see [11, Ch. 4.2.3]. Define $w_n := \zeta_n \circ \operatorname{dist}_D$. Note that $w_n \to 1$ almost everywhere in $\mathbb{R}^d$ when $n \to \infty$. Moreover, since ζ_n is piecewise smooth, one calculates, according to the chain rule (see [17, Ch. 7.4]),

$$\nabla w_n(x) = \begin{cases} 0, & \text{if } \operatorname{dist}_D(\mathrm{x}) \in \{\frac{1}{n}, \frac{2}{n}\}, \\ \zeta_n'(\operatorname{dist}_D(\mathrm{x}))\nabla \operatorname{dist}_D(\mathrm{x}), & \text{else.} \end{cases}$$

Since $|\nabla \operatorname{dist}_D| \le 1$ a.e., $\operatorname{dist}_D \nabla w_n$ is uniformly (in n) bounded a.e. and converges almost everywhere to 0 as $n \to \infty$. Let $f_n = f w_n$. We claim that $f_n - f = f(1 - w_n) \to 0$ in $W^{1,p}(\mathbb{R}^d)$. By the dominated convergence theorem, $f(1 - w_n) \to 0$ in $L^p(\mathbb{R}^d)$ since $w_n \to 1$. Now, for the gradient we have

$$\nabla(f_n - f) = (1 - w_n)\nabla f + f\nabla w_n \quad \text{a.e. on } \mathbb{R}^d.$$

Again by the dominated convergence theorem, the first term converges to 0 in $L^p(\mathbb{R}^d)$. It remains to prove that $\|f\nabla w_n\|_{L^p} \to 0$. We have

$$\|f\nabla w_n\|_{L^p}^p = \int_{\mathbb{R}^d} \Big|\frac{f}{\mathrm{dist}_D}\Big|^p \Big|\mathrm{dist}_D \nabla w_n\Big|^p \mathrm{dx}. \tag{3.2}$$

Due to the fact that f vanishes identically on D and the Lipschitz property of f, the function $\frac{f}{\mathrm{dist}_D}$ is bounded. Hence, again dominated convergence yields $f\nabla w_n \to 0$ in $L^p(\mathbb{R}^d)$. The support of each function f_n has a positive distance to the set D. Thus, it suffices to convolve a function f_n (according to a sufficiently high index n) with a smooth mollifying function with small support to obtain g, which proves (3.1). Thus, according to (3.1), for any $\varepsilon > 0$ there is a function $\xi_\varepsilon \in C^\infty(\mathbb{R}^d)$, such that $\mathrm{supp}(\xi_\varepsilon) \cap D = \emptyset$ and

$$\|\xi_\varepsilon|_B - \psi\|_{W^{1,p}(B)} \le \|\xi_\varepsilon - \hat{\psi}\|_{W^{1,p}(\mathbb{R}^d)} \le \varepsilon. \qquad \square$$

Remark 3.3. i) The basic idea of this proof is analogous to that in [32, Prop. 3.12].

ii) Seemingly, the coincidence of the spaces $W^{1,p}_D(B)$ and $\mathcal{W}^{1,p}_D(B)$ is only of limited, more technical interest. This, however, is not the case: on the one hand it is often considerably simpler to prove that a certain function belongs to the space $\mathcal{W}^{1,p}_D$, compare [33, Ch. VIII.1] or [43, Ch. 6.6]. On the other hand, it is of course often more comfortable, if one has to prove a certain property for all elements from $\mathcal{W}^{1,p}_D(B)$ and may confine oneself, by density, to the functions from $C^\infty_D(B)$.

iii) Theorem 3.2 heavily rests on the property of D to be a $(d-1)$-set: suppose e.g. $p > d$ and assume that $\mathrm{x} \in D$ is an isolated point. Then, for every $\psi \in C^\infty_D(\Omega)$ one has $\psi(\mathrm{x}) = 0$, what clearly extends to all $\psi \in W^{1,p}_D(\Omega)$, since the Dirac measure δ_x is a continuous linear form on $W^{1,p}(\Omega)$. On the other hand, the condition $\mathcal{R}_D\psi = 0$ a.e. on D does not impose a condition on ψ in the point x because $\{\mathrm{x}\}$ is of measure 0 with respect to ϱ.

Corollary 3.4. *Concerning real and complex interpolation, Theorem 3.1 remains true, if there* $\mathcal{W}^{1,p}_D(B)$ *is replaced by* $W^{1,p}_D(B)$. *In particular, one has for* $p_0, p_1 \in \,]1,\infty[$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

$$\big[W^{1,p_0}_D(B), W^{1,p_1}_D(B)\big]_\theta = W^{1,p}_D(B) = \big(W^{1,p_0}_D(B), W^{1,p_1}_D(B)\big)_{\theta,p},$$

compare [54, Ch. 2.4.2].

Proof. The assertion concerning complex interpolation is immediate from Theorem 3.1 and Theorem 3.2, which also imply the right equality. Considering real interpolation, one gets

$$\big(W^{1,p_0}_D(B), W^{1,p_1}_D(B)\big)_{\theta,q} = \big(\mathcal{W}^{1,p_0}_D(B), \mathcal{W}^{1,p_1}_D(B)\big)_{\theta,q}. \tag{3.3}$$

According to Theorem 3.1, the right hand side is some Besov space (see [54, Ch. 2.4.2]) including again the trace-zero condition on D. It is clear that $C^\infty_D(\Omega)$ is contained in this space. What remains to show is that $C^\infty_D(B)$ is also dense in this space.

Let us suppose, without loss of generality, $p_1 > p_0$. By definition, $C^\infty_D(B)$ is dense in $W^{1,p_1}_D(B)$ with respect to its natural topology. Moreover, $W^{1,p_1}_D(B)$ is dense in the interpolation space (3.3) (see [54, Ch. 1.6.2]), and the topology of this interpolation space is weaker than that of $W^{1,p_1}_D(B)$. Hence, $C^\infty_D(B)$ is indeed dense in the corresponding interpolation space, or, in other words: the interpolation space is the closure of $C^\infty_D(B)$ with respect to the corresponding Besov topology. $\square$

Theorem 3.5. *Let Assumptions 2.2 and 2.13 be satisfied. Then complex and real interpolation between the spaces of the set* $\{W^{1,p}_D(\Omega)\}_{p\in]1,\infty[}$ *acts as if one formally replaces* Ω *by* B *and* D *by* ∂B*. In particular, one has for* $p_0, p_1 \in]1,\infty[$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

$$\big[W^{1,p_0}_D(\Omega), W^{1,p_1}_D(\Omega)\big]_\theta = W^{1,p}_D(\Omega) = \big(W^{1,p_0}_D(\Omega), W^{1,p_1}_D(\Omega)\big)_{\theta,p},$$

Proof. Let B be the ball from Remark 2.7. Firstly, Corollary 3.4 shows how the spaces from the set $\{W^{1,p}_D(B)\}_{p\in]1,\infty[}$ interpolate. Secondly, the extension/restriction operators $\mathfrak{E}/\mathfrak{R}$ together with the retraction/coretraction theorem, see [54, Ch. 1.2.4], allow to carry over interpolation between spaces from $\{W^{1,p}_D(B)\}_{p\in]1,\infty[}$ to the spaces from $\{W^{1,p}_D(\Omega)\}_{p\in]1,\infty[}$. ☐

Corollary 3.6. *Let* $\hat{W}^{-1,q}_D(\Omega)$ *denote the dual of* $W^{1,q'}_D(\Omega)$*,* $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $W^{-1,q}_D(\Omega)$ *denote the space of continuous antilinear forms on* $W^{1,q'}_D(\Omega)$*,* $\frac{1}{q} + \frac{1}{q'} = 1$*. For* $p_0, p_1 \in]1,\infty[$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *one has*

$$\big[\hat{W}^{-1,p_0}_D(\Omega), \hat{W}^{-1,p_1}_D(\Omega)\big]_\theta = \hat{W}^{-1,p}_D(\Omega), \tag{3.4}$$

and

$$\big[W^{-1,p_0}_D(\Omega), W^{-1,p_1}_D(\Omega)\big]_\theta = W^{-1,p}_D(\Omega). \tag{3.5}$$

Proof. Concerning (3.4), one employs the duality formula for complex interpolation in case of reflexive Banach spaces (see [54, Ch. 1.11.3]), which reads as $[X', Y']_\theta = [X, Y]'_\theta$. In order to conclude (3.5), one associates to any linear form T an antilinear form T_a defining $\langle T_a, \psi\rangle := \langle T, \overline{\psi}\rangle$. It is clear that the mappings $T \mapsto T_a$ and $T_a \mapsto T$ form a retraction/coretraction pair, thus (3.5) may be derived from (3.4) by the retraction/coretraction theorem for interpolation. ☐

4. Elliptic and parabolic regularity

In this section we prove that alone the interpolation property of the spaces $W^{1,p}_D(\Omega)$ – in conjuction with a deep result of Sneiberg [52] – already leads to suitable regularity results within this scale of spaces.

Let us first recall the definition of a scale of Banach spaces (see [40, Ch.1], compare also [54, Ch. 1.19.4]).

Definition 4.1. Consider a closed interval $I \subset [0,\infty[$ and a family of complex Banach spaces $\{X_\tau\}_{\tau\in I}$. One calls this family a (complex) scale (of Banach spaces) if

i) X_β embeds continuously and densely in X_α, whenever $\beta > \alpha$.
ii) For every triple $\alpha, \beta, \gamma \in I$ satisfying $\alpha < \beta < \gamma$ there is a positive constant $c(\alpha,\beta,\gamma)$ such that for all $\psi \in X_\gamma$ the following interpolation inequality holds

$$\|\psi\|_{X_\beta} \le c(\alpha,\beta,\gamma)\|\psi\|_{X_\alpha}^{\frac{\gamma-\beta}{\gamma-\alpha}} \|\psi\|_{X_\gamma}^{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{4.1}$$

We associate to the families $\{W^{1,p}_D(\Omega)\}_{p\in]1,\infty[}$ and $\{W^{-1,p}_D(\Omega)\}_{p\in]1,\infty[}$ Banach scales in the following manner

Definition 4.2. For $\tau \in]0,1[$ we define $X_\tau := W^{1,(1-\tau)^{-1}}_D(\Omega)$ and $Y_\tau := W^{-1,(1-\tau)^{-1}}_D(\Omega)$.

Lemma 4.3. *Let Assumptions 2.2 and 2.13 be satisfied. Then, for all* $\tau_1, \tau_2 \in]0,1[$ *with* $\tau_1 < \tau_2$*, the families* $\{X_\tau\}_{\tau\in[\tau_1,\tau_2]}$ *and* $\{Y_\tau\}_{\tau\in[\tau_1,\tau_2]}$ *form Banach scales.*

Proof. We show more, namely: for every $\alpha, \beta, \gamma \in]0, 1[$ with $\alpha < \beta < \gamma$ one has

$$X_\beta = [X_\alpha, X_\gamma]_{\frac{\beta-\alpha}{\gamma-\alpha}} \text{ and } Y_\beta = [Y_\alpha, Y_\gamma]_{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{4.2}$$

Putting $\theta := \frac{\beta-\alpha}{\gamma-\alpha}$, one has $1-\beta = (1-\alpha)(1-\theta) + (1-\gamma)\theta$. Thus the equalities in (4.2) follow from Theorem 3.5 and Corollary 3.6. The inequality (4.1) is then the interpolation inequality for complex interpolation. ☐

Throughout the rest of the paper we assume that the following is satisfied:

Assumption 4.4. μ is a bounded, measurable, elliptic coefficient function on Ω. This means that $\mathrm{Re}(\langle \mu(\mathrm{x})\xi, \xi\rangle) \geq c|\xi|^2$ for some positive constant c, all $\xi \in \mathbb{C}^d$ and almost all $\mathrm{x} \in \Omega$.

Definition 4.5. We define the operator $-\nabla \cdot \mu\nabla : W^{1,2}_D(\Omega) \to W^{-1,2}_D(\Omega)$ by

$$\langle -\nabla \cdot \mu\nabla v, w\rangle_{W^{-1,2}_D(\Omega)} := \int_\Omega \mu\nabla v \cdot \nabla\overline{w} \,\mathrm{dx}, \quad v, w \in W^{1,2}_D(\Omega). \tag{4.3}$$

In the sequel, we maintain the notation $-\nabla \cdot \mu\nabla$ for the restriction of this operator to the spaces $W^{1,p}_D(\Omega)$ in case of $p > 2$. If $p < 2$ and μ^* denotes the adjoint coefficient function, we define the operator $-\nabla \cdot \mu\nabla : W^{1,p}_D(\Omega) \to W^{-1,p}_D(\Omega)$ as the adjoint of $-\nabla \cdot \mu^*\nabla : W^{1,p'}_D(\Omega) \to W^{-1,p'}_D(\Omega)$.

Remark 4.6. When restricting the range space of the operator $-\nabla \cdot \mu\nabla$ to $L^2(\Omega)$, one obtains an operator for which the elements ψ of its domain satisfy the conditions $\psi|_{\partial\Omega\setminus\Gamma} = 0$ in the sense of traces and $\nu \cdot \mu\nabla\psi = 0$ on Γ in a generalized sense – ν being the outward unit normal of the domain, compare [5, Ch. 1.2] or [13, Ch II.2]).
In this spirit, the operator $-\nabla \cdot \mu\nabla$ shall be understood as one with mixed boundary conditions – as announced in the title.

Theorem 4.7. *Let Assumptions 2.2, 2.13 and 4.4 be satisfied. Then there is an open interval $I \ni 2$ such that the operator*

$$-\nabla \cdot \mu\nabla + 1 : W^{1,p}_D(\Omega) \to W^{-1,p}_D(\Omega) \tag{4.4}$$

is a topological isomorphism for all $p \in I$.

Proof. We know from Lemma 4.3 that the families $\{X_\tau\}_{\tau\in[\alpha,\beta]}$ and $\{Y_\tau\}_{\tau\in[\alpha,\beta]}$ with $\alpha, \beta \in]0, 1[$ form complex interpolation scales. The mapping in (4.4) is continuous for all p, due to the boundedness of the coefficient function μ, what is to be interpreted as the continuity of

$$-\nabla \cdot \mu\nabla + 1 : X_\tau \to Y_\tau \tag{4.5}$$

for all $\tau \in]0, 1[$. Lastly, the quadratic form $W^{1,2}_D(\Omega) \ni \psi \mapsto \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\psi} + |\psi|^2 \,\mathrm{dx}$ is coercive. Hence the Lax-Milgram lemma gives the continuity of the inverse of (4.4) in the case of $p = 2$. In the scale terminology, this is nothing else but the continuity of $(-\nabla \cdot \mu\nabla + 1)^{-1} : Y_\tau \to X_\tau$ in case of $\tau = \frac{1}{2}$. A deep theorem of Sneiberg ([52], see also [4, Lemma 4.16] or [53]) says that the set of parameters τ for which (4.5) is a topological isomorphism, is open. Since $\frac{1}{2}$ is contained in this set, it cannot be empty. ☐

Remark 4.8.

i) Again interpolation shows that the values p, for which (4.4) is a topological isomorphism, form an interval. Due to the Sneiberg result, this interval is an open one.

ii) If μ takes real, symmetric matrices as values, then the maximal restriction of $-\nabla \cdot \mu\nabla$ to $L^2(\Omega)$ is selfadjoint. Hence, the interval I must be of the form $I =]q, \frac{q}{q-1}[$ in this case, because with q also the number $\frac{q}{q-1}$ is contained in I.

iii) It is well-known that the interval I depends on the domain Ω (see [32], [6]) and on μ (see [45] or [44]), and on D (see [47]). The most important point is that the length of I may be arbitrarily small, see [9, Ch. 4] for a striking example. Even in smooth situations it cannot be expected that 4 belongs to I, as the pioneering counterexample in [51] shows.

iv) If $\mathcal{M}$ is a set of coefficient functions μ with a common L^∞ bound and a common ellipticity constant, then one can find a common open interval $I_\mathcal{M}$ around 2, such that (4.4) is a topological isomorphism for all $\mu \in \mathcal{M}$ and all $p \in I_\mathcal{M}$. Finally, one has

$$\sup_{I_\mathcal{M}} \sup_{\mu \in \mathcal{M}} \left\| (-\nabla \cdot \mu \nabla + 1)^{-1} \right\|_{\mathcal{L}(W_D^{-1,p};W_D^{1,p})} < \infty.$$

The proof of this is completely analogous to [24, Thm. 1].

v) It is interesting to observe that in the case of two space dimensions Theorem 4.7 immediately implies the Hölder continuity of the solution as long as the right hand side belongs to a space $W_D^{-1,p}(\Omega)$ with $p > 2$. The question arises whether this remains true in higher dimensions, if p exceeds the corresponding space dimension – despite the fact that the gradient of the solution does only admit integrability a bit more than 2 in general. We will prove – by entirely different methods – in a forthcoming paper [49] that this is indeed the case, at least for space dimensions up to 4.

Corollary 4.9. i) *Under the same suppositions as in Theorem 4.7*

$$-\nabla \cdot \mu \nabla + \lambda : W_D^{1,p}(\Omega) \to W_D^{-1,p}(\Omega) \tag{4.6}$$

is a topological isomorphism for all $p \in I$, if $-\lambda \in \mathbb{C}$ is not an eigenvalue of $-\nabla \cdot \mu \nabla$.

ii) *If, in particular the boundary measure of $D \cap \left(\bigcup_{\mathrm{x} \in \overline{\Gamma}} U_\mathrm{x} \right)$ is nonzero, then 0 is not an eigenvalue of $-\nabla \cdot \mu \nabla$.*

Proof. i) According to Remark 2.8, the embedding $W_D^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W_D^{-1,p}(\Omega)$ is compact. Thus (4.6) can only fail to be an isomorphism, if $-\lambda$ is an eigenvalue for $-\nabla \cdot \mu \nabla$, according to the Riesz-Schauder theory, cf. [36, Ch. III.6.8]. Observe that an eigenvalue for $-\nabla \cdot \mu \nabla$, when considered on $W_D^{-1,p}(\Omega)$ for $p > 2$ is automatically an eigenvalue when $-\nabla \cdot \mu \nabla$ is considered on $W_D^{-1,2}(\Omega)$. Since all eigenvalues are real, the assertion for $p \in I \cap]1, 2[$ follows by duality.

ii) Assume that this is false, and let $w \in \mathrm{dom}_{W_D^{-1,p}(\Omega)}(\nabla \cdot \mu \nabla) \subset W_D^{1,2}(\Omega)$ be the corresponding eigenfunction. Then, testing the equation $-\nabla \cdot \mu \nabla w = 0$ by w, one gets

$$0 = \langle -\nabla \cdot \mu \nabla w, w \rangle \geq c \int_\Omega \|\nabla w\|^2 \,\mathrm{dx},$$

thanks to the ellipticity of μ. Hence, w has to be constant on Ω. But, if the boundary measure of $D \cap \left(\bigcup_{\mathrm{x} \in \overline{\Gamma}} U_\mathrm{x} \right)$ is nonzero, the nonzero constant functions cannot belong to $W_D^{1,2}(\Omega)$ and, hence, not to $\mathrm{dom}_{W_D^{-1,p}(\Omega)}(-\nabla \cdot \mu \nabla) \subset W_D^{1,2}(\Omega)$. □

In the sequel we are going to show how to exploit the elliptic regularity result for proving resolvent estimates for the operators $-\nabla \cdot \mu \nabla$, which assure the generator property for an analytic semigroup on suitable spaces $W_D^{-1,p}(\Omega)$. It is well known that this property allows to solve parabolic equations like

$$u' - \nabla \cdot \mu \nabla u = f; \quad u(0) = u_0,$$

where the right hand side f depends Hölder continuously on time (or even suitably on the solution u itself), see [42] or [29]. Since we proceed very similar to [24] we do not point out all details but refer to that paper.

Theorem 4.10. *Let Assumptions 2.2, 2.13 and 4.4 be satisfied. Suppose, additionally, that $\overline{\Omega} \subset \mathbb{R}^d$ is a d-set. Then the following assertions hold true.*

i) *There is an open interval $J \ni 2$ such that the operator $\nabla \cdot \mu \nabla$ generates an analytic semigroup on $W^{-1,p}_D(\Omega)$, as long as $p \in J$.*
ii) *If $\mathcal{M}$ is a set of coefficient functions μ with common L^∞ bound and common ellipticity constant, one can find – in the spirit of i) – a common interval $J_\mathcal{M}$ for all these $\mu \in \mathcal{M}$.*
iii) *There is an interval $J \ni 2$ such that for all $p \in J$ one has resolvent estimates like*

$$\|\big(-\nabla \cdot \mu \nabla + 1 + \lambda\big)^{-1}\|_{\mathcal{L}(W^{-1,p}_D(\Omega))} \leq \frac{c}{1 + |\lambda|}, \tag{4.7}$$

which are uniform in $\mu \in \mathcal{M}$, $p \in J_\mathcal{M}$ and $\lambda \in H_+ := \{\vartheta \in \mathbb{C} : \mathrm{Re}(\vartheta) \geq 0\}$, i.e. the same constant c may be taken for all these parameters.

Proof. Assertion iii) implies points i) and ii), so we concentrate on this. Concerning the p's above 2 one proceeds exactly as in [24]: Let $I_0 := [2, p_0]$ be a closed interval, such that (4.4) is a topological isomorphism for all $p \in I_0$. Due to Remark 2.8, the embedding $W^{1,p}_D(\Omega) \hookrightarrow L^p(\Omega)$ is compact for every $p \in \,]1, \infty[$. Thus the resolvent of $-\nabla \cdot \mu \nabla$ is compact on every space $W^{-1,p}_D(\Omega)$, as long as $p \in I_0$. Moreover, no $\lambda \in -H_+$ is an eigenvalue of $-\nabla \cdot \mu \nabla + 1$, when this operator is considered on $W^{-1,p}_D(\Omega)$ with $p \in [2, \infty[$. Thus, the Riesz-Schauder theory tells us that

$$-\nabla \cdot \mu \nabla + 1 + \lambda : W^{1,p}_D(\Omega) \to W^{-1,p}_D(\Omega) \tag{4.8}$$

is a topological isomorphism for all $p \in I_0$ and $\lambda \in H_+$.

Defining $\widetilde{\Omega} := \Omega \times \,]0, 1[$, one obtains

$$\partial\widetilde{\Omega} = \big(\overline{\Omega} \times \{0, 1\}\big) \cup \big(\partial\Omega \times \,]0, 1[\big) = \big(\Omega \times \{0, 1\}\big) \cup \big(\partial\Omega \times [0, 1]\big). \tag{4.9}$$

We define $\widetilde{\Gamma} := \Gamma \times \,]0, 1[$, thus obtaining

$$\widetilde{D} := \partial\widetilde{\Omega} \setminus \widetilde{\Gamma} = \big(\overline{\Omega} \times \{0, 1\}\big) \cup \big(D \times \,]0, 1[\big) = \big(\overline{\Omega} \times \{0, 1\}\big) \cup \big(D \times [0, 1]\big). \tag{4.10}$$

Since $\overline{\Omega}$ is a d-set and D is a $(d-1)$-set by supposition, it is clear that $\widetilde{D}$ is a d-set. Let us next show that $\widetilde{\Gamma}$ satisfies (mutatis mutandis) the condition in Assumption 2.2. For points from $\overline{\Gamma} \times \,]0, 1[$, suitable bi-Lipschitzian charts are constructed from the bi-Lipschitzian charts for $\overline{\Gamma}$ in a straightforward manner as follows: if $\mathrm{x} \in \overline{\Gamma}$ and Φ_x is the corresponding bi-Lipschitz mapping (cf. Definition 2.2) onto the unit cube, then $\widetilde{\Phi}_\mathrm{x}$ is defined by $\widetilde{\Phi}_\mathrm{x}(\mathrm{y}, t) := (\Phi_\mathrm{x}(\mathrm{y}), t)$.

What is not so easy is to construct such charts around the points from $\overline{\Gamma} \times \{0\}$ and from $\overline{\Gamma} \times \{1\}$. We perform the construction for the first set, the second is treated analogously. Let, in this spirit, $\mathrm{x} \in \overline{\Gamma}$ and U_x be a corresponding neighbourhood from Assumption 2.2. We define $V_\mathrm{x} := U_\mathrm{x} \times \,]-\frac{1}{2}, \frac{1}{2}[$ and $\Psi_\mathrm{x} : V_\mathrm{x} \to E_{d+1}$ by $\Psi_\mathrm{x}(\mathrm{y}, t) = (\Phi_\mathrm{x}(\mathrm{y}), t)$. Clearly, then $V_\mathrm{x} \cap \widetilde{\Omega} = (U_\mathrm{x} \cap \Omega) \times \,]0, \frac{1}{2}[$, what implies

$$\Psi_\mathrm{x}(V_\mathrm{x} \cap \widetilde{\Omega}) = E^+_d \times \Big]0, \frac{1}{2}\Big[= E_{d-1} \times \Big]0, \frac{1}{2}\Big[\times \Big]0, \frac{1}{2}\Big[. \tag{4.11}$$

On the other hand, (4.9) gives

$$V_{\mathrm{x}} \cap \partial\widetilde{\Omega} = \big((U_{\mathrm{x}} \cap \Omega) \times \{0\}\big) \cup \Big((U_{\mathrm{x}} \cap \partial\Omega) \times \Big[0, \frac{1}{2}\Big[\Big). \tag{4.12}$$

From this we get by the definition of Ψ_{x}:

$$\begin{aligned}\Psi_{\mathrm{x}}(V_{\mathrm{x}} \cap \partial\widetilde{\Omega}) &= \big(E_d^+ \times \{0\}\big) \cup \Big(P \times \Big[0, \frac{1}{2}\Big[\Big) \\ &= \Big(E_{d-1} \times \Big]0, \frac{1}{2}\Big[\times \{0\}\Big) \cup \Big(E_{d-1} \times \{0\} \times \Big[0, \frac{1}{2}\Big[\Big) \\ &= E_{d-1} \times \Big[\Big(\Big]0, \frac{1}{2}\Big[\times \{0\}\Big) \cup \Big(\{0\} \times \Big[0, \frac{1}{2}\Big[\Big)\Big]\end{aligned} \tag{4.13}$$

and $\Psi_{\mathrm{x}}(\mathrm{x}, 0) = 0 \in \mathbb{R}^{d+1}$. The equations (4.11) and (4.13) reveal that the image sets under the bi-Lipschitz charts are not as required in Assumption (2.2) up to now. Consequently, we are going to modify them: Let us define κ_+ as the linear mapping from $\mathbb{R}^2$ onto itself which leaves the vector $(1, 1)$ invariant and maps the vector $(0, 1)$ onto the vector $(-1, 0)$. Furthermore, we define the bi-Lipschitz mapping $\kappa : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\kappa(\mathrm{z}) = \begin{cases} \mathrm{z}, & \text{if } z_2 \le z_1, \\ \kappa_+(\mathrm{z}), & \text{else.} \end{cases}$$

It is not hard to see that $\frac{1}{4}E_2 \subset \kappa(E_2)$. Writing the elements y from $\mathbb{R}^{d+1}$ as $(\underline{\mathrm{y}}, y_d, y_{d+1})$ with $y_d, y_{d+1} \in \mathbb{R}$, we define the bi-Lipschitz mapping $\Phi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by $\Phi(\mathrm{y}) := (\underline{\mathrm{y}}, \kappa(y_d, y_{d+1}))$. Moreover, we put $\widetilde{U_{\mathrm{x}}} := \Psi_{\mathrm{x}}^{-1} \circ \Phi^{-1}(\frac{1}{4}E_{d+1})$ and $\widetilde{\Phi}_{\mathrm{x}} := 4\Phi \circ \Psi_{\mathrm{x}}$. It is easily verified that then $\widetilde{U_{\mathrm{x}}}$ and $\widetilde{\Phi_{\mathrm{x}}}$ together fulfill (mutatis mutandis) Assumption 2.2.

The following considerations can be carried out in detail in exactly the same way as in [24], and we give here only a short summary of the main steps. As in [24], for every $\lambda \in H_+$, one defines a coefficient function $\widetilde{\mu}$ on $\widetilde{\Omega}$ in the following manner: Let $\mu^\bullet$ be the L^∞-bound for the coefficient function μ and $\mu_\bullet$ its ellipticity constant. Then we define the coefficient function for the auxiliary divergence operator on $\widetilde{\Omega}$ by

$$\widetilde{\mu}_{j,k}(\mathrm{x}, t) = \begin{cases} (1 - \frac{\mu_\bullet}{2\mu^\bullet}\operatorname{sign}(\operatorname{Im}(\lambda))i)\mu_{j,k}(\mathrm{x}), & \text{if } j, k \in \{1, \dots, d\}, \\ \widetilde{\mu}_{j,d+1} = \widetilde{\mu}_{d+1,j} = 0 & \text{for all } j \in \{1, \dots, d\}, \\ \widetilde{\mu}_{d+1,d+1} = \frac{\lambda}{|\lambda|}\big(\mu^\bullet - \frac{\mu_\bullet}{2}\operatorname{sign}(\operatorname{Im}(\lambda))i\big). \end{cases} \tag{4.14}$$

One easily observes that all these coefficient functions admit a common L^∞-bound and a common ellipticity constant. Thus, Remark 4.8 iv) applies to the operators $-\nabla \cdot \widetilde{\mu}\nabla + 1$. This gives an interval $I_1 := [2, p_1]$ such that the norms of the operators $(-\nabla \cdot \widetilde{\mu}\nabla + 1)^{-1} : W^{-1,p}_{\widetilde{D}}(\widetilde{\Omega}) \to W^{1,p}_{\widetilde{D}}(\widetilde{\Omega})$ are bounded, uniformly in $\lambda \in H_+$ and in $p \in I_1$.

One associates to the problem $(-\nabla \cdot \mu\nabla + 1 + \lambda)u = f$ a problem $(-\nabla \cdot \widetilde{\mu}\nabla + 1)u_\lambda = f_\lambda$ and exploits the (uniform) regularity properties of the operators $-\nabla \cdot \widetilde{\mu}\nabla + 1$ for an estimate

$$\|u\|_{W^{1,p}_D(\Omega)} \le c\|f\|_{W^{-1,p}_D(\Omega)}, \tag{4.15}$$

where c is independent from f and $\lambda \in H_+$. We already know the isomorphism property of (4.8), thus (4.15) may be expressed as

$$\sup_{\lambda \in H_+} \big\|(-\nabla \cdot \mu\nabla + 1 + \lambda)^{-1}\big\|_{\mathcal{L}(W^{-1,p}_D(\Omega); W^{1,p}_D(\Omega))} < \infty \tag{4.16}$$

for all $p \in I_0 \cap I_1$.

Finally, (4.16) allows us to deduce the estimate

$$\begin{aligned}
&\sup_{\lambda \in H_+} |\lambda| \big\| (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega))} \\
&= \sup_{\lambda \in H_+} \big\| \lambda (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega))} \\
&= \sup_{\lambda \in H_+} \big\| 1 - (-\nabla \cdot \mu \nabla + 1)(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega))} \\
&\le 1 + \| -\nabla \cdot \mu \nabla + 1 \|_{\mathcal{L}(W_D^{1,p}(\Omega); W_D^{-1,p}(\Omega))} \sup_{\lambda \in H_+} \big\| (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega); W_D^{1,p}(\Omega))} \\
&< \infty
\end{aligned}$$

for all $p \in I_0 \cap I_1$.

The case $p < 2$ can be treated as follows: first, one gets the following resolvent estimate on $W_D^{1,p}(\Omega)$ for $p > 2$:

$$\begin{aligned}
&\big\| (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{1,p}(\Omega))} \\
&= \big\| (-\nabla \cdot \mu \nabla + 1)^{-1} (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} (-\nabla \cdot \mu \nabla + 1) \big\|_{\mathcal{L}(W_D^{1,p}(\Omega))} \\
&\le \big\| (-\nabla \cdot \mu \nabla + 1)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega); W_D^{1,p}(\Omega))} \big\| (-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1} \big\|_{\mathcal{L}(W_D^{-1,p}(\Omega))} \\
&\qquad \times \| -\nabla \cdot \mu \nabla + 1 \|_{\mathcal{L}(W_D^{1,p}(\Omega); W_D^{-1,p}(\Omega))}.
\end{aligned}$$

Since the first and third factor are finite, one can use (4.7). Then, considering the adjoint of $(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}$, which is nothing else but $(-\nabla \cdot \mu^* \nabla + 1 + \overline{\lambda})^{-1}$ on $W_D^{-1,p'}(\Omega)$, one obtains the assertion for $p < 2$. □

Remark 4.11. The proof of the main result in [24] follows an old idea of Agmon in [1].

5. Elliptic regularity for systems

In this section we apply the interpolation property of the $W^{1,p}$-spaces in order to derive p-estimates for linear elliptic operators acting on vector-valued functions. Here, for each component a different Dirichlet boundary might be prescribed. To be more precise, we assume the following

(A1) $\Omega \subset \mathbb{R}^d$ is a bounded domain and for $1 \le i \le m$ the sets $D_i \subset \partial\Omega$ are closed $(d-1)$-sets. Let $D := \bigcap_{i=1}^m D_i$ and $\Gamma := \partial\Omega \setminus D$. It is assumed that Ω and Γ satisfy Assumption 2.2.

For $p \in [1, \infty)$ we introduce the space

$$\mathbb{W}_D^{1,p}(\Omega) = \prod_{i=1}^m W_{D_i}^{1,p}(\Omega)$$

and its dual $\mathbb{W}_D^{-1,p'}(\Omega)$ for $\frac{1}{p} + \frac{1}{p'} = 1$, Furthermore, we define the operator $\mathcal{L}_p : \mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega; \mathbb{C}^m \times \mathbb{C}^{m \times d})$ by $\mathcal{L}_p(u) = (u, \nabla u)$. Given a complex valued coefficient function $\mathbb{A} \in L^\infty(\Omega; \mathrm{Lin}\,(\mathbb{C}^m \times \mathbb{C}^{m\times d}, \mathbb{C}^m \times \mathbb{C}^{m \times d}))$, we investigate differential operators of the type

$$\mathcal{A} : \mathbb{W}_D^{1,p}(\Omega) \to \mathbb{W}_D^{-1,p}(\Omega), \quad \mathcal{A} = \mathcal{L}_{p'}^* \mathbb{A} \mathcal{L}_p.$$

The corresponding weak formulation on $\mathbb{W}^{1,2}_D(\Omega)$ reads $\langle \mathcal{A}(u), v\rangle = \int_\Omega \mathbb{A}\left(\begin{smallmatrix} u \\ \nabla u \end{smallmatrix}\right) : \overline{\left(\begin{smallmatrix} v \\ \nabla v \end{smallmatrix}\right)}\,\mathrm{dx}$ for $u, v \in \mathbb{W}^{1,2}_D(\Omega)$, where

$$(b_1, B_1) : (b_2, B_2) = \sum_{i=1}^{m} b_1^i b_2^i + \sum_{j=1}^{m}\sum_{k=1}^{d} B_1^{jk} B_2^{jk}$$

for $(b_1, B_1), (b_2, B_2) \in \mathbb{C}^m \times \mathbb{C}^{m\times d}$. It is assumed that the operator $\mathcal{A}$ is elliptic. More precisely, we assume that

(A2) There is a constant $\kappa > 0$ such that for all $v \in \mathbb{W}^{1,2}_D(\Omega)$ it holds $\Re\langle \mathcal{A}v, v\rangle \geq \kappa\|v\|^2_{\mathbb{W}^{1,2}(\Omega)}$.

Remark 5.1. We recall that in case of systems of partial differential equations the positivity property formulated in (A2) in general does not imply that the coefficient tensor belonging to the principle part of the differential operator is positive definite. In general, this coefficient tensor only satisfies the weaker Legendre-Hadamard condition, cf. [16]: Assume that (A2) is satisfied for $\mathbb{A} = \left(\begin{smallmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{smallmatrix}\right)$, where $\mathbb{A}_{22} \in \mathrm{Lin}\left(\mathbb{C}^{m\times d}, \mathbb{C}^{m\times d}\right)$ corresponds to the principle part of the operator $\mathcal{A}$. Then there exists a constant $c_\kappa > 0$ such that for all $\xi \in \mathbb{C}^m$, $\eta \in \mathbb{C}^d$ it holds

$$\Re\big(\mathbb{A}_{22}\xi \otimes \eta : \overline{\xi \otimes \eta}\big) \geq c_\kappa |\xi|^2 |\eta|^2. \tag{5.1}$$

Theorem 5.2. *Let (A1) and (A2) be satisfied. Then there exists an open interval $J \ni 2$, such that for all $q \in J$ the operator $\mathcal{A} : \mathbb{W}^{1,q}_D(\Omega) \to \mathbb{W}^{-1,q}_D(\Omega)$ is a topological isomorphism.*

Proof. Exactly the same arguments as in the proof of Theorem 4.7 can be applied. ☐

If in addition the operator $\mathcal{A}$ satisfies a certain symmetry relation, then the interval J can be determined uniformly for classes of coefficient tensors satisfying uniform upper bounds and ellipticity properties.

(A3) For all $u, v \in \mathbb{W}^{1,2}_D(\Omega)$ it holds $\langle \mathcal{A}u, v\rangle = \overline{\langle \mathcal{A}v, u\rangle}$.

Theorem 5.3. *Let (A1) be satisfied and let $\mathcal{M}$ be a set of coefficient tensors fulfilling (A2) and (A3) with a uniform upper L^∞-bound and a common lower bound for the ellipticity constant κ in (A2). Then, there exists an open interval $J_\mathcal{M} \ni 2$ such that for all $p \in J_\mathcal{M}$ and all $\mathbb{A} \in \mathcal{M}$ the corresponding operator $\mathcal{A}$ is a topological isomorphism between $\mathbb{W}^{1,p}_D(\Omega)$ and $\mathbb{W}^{-1,p}_D(\Omega)$. Moreover, there exists a constant $c_\mathcal{M} > 0$ such that for all $f \in \mathbb{W}^{-1,p}_D(\Omega)$ we have*

$$\sup\left\{\|\mathcal{A}^{-1}(f)\|_{\mathbb{W}^{1,p}(\Omega)} ; \, p \in J_\mathcal{M}, \, \mathbb{A} \in \mathcal{M}\right\} \leq c_\mathcal{M}\|f\|_{\mathbb{W}^{-1,p}_D(\Omega)}. \tag{5.2}$$

Remark 5.4. In the case of scalar equations, i.e. $m = 1$, the previous theorem is also valid for operators $\mathcal{A}$ which do not satisfy (A3), (see Remark 4.8). Similar arguments as in the scalar case can be applied to the vectorial case without assuming (A3) provided that the coefficient tensor $\mathbb{A}_{22}$ satisfies (5.1) for all $B \in \mathbb{C}^{m\times d}$ and not only for $B = \xi \otimes \eta$. In this case, the proof of the uniform bound (5.2) relies on certain estimates that are derived using the positivity of the coefficient-tensors (see [24]). In the general non-symmetric vector valued case, we do not see how the proof can be generalised, if only the weaker positivity (5.1) is assumed. In the case studied in Theorem 5.3 we derive estimates for the corresponding operators directly (and not pointwise for the coefficients) and use the fact that for self-adjoint operators on a Hilbert space $\mathbb{H}$ the operator norm is given by $\|T\|_{\mathrm{op}} = \sup\left\{|\langle Ta, a\rangle| ; \, a \in \mathbb{H}, \, \|a\| \leq 1\right\}$.

Proof. Let $\mathcal{P} : \mathbb{W}_D^{1,2}(\Omega) \to \mathbb{W}_D^{-1,2}(\Omega)$ be defined by $\mathcal{P} = \mathcal{L}^*\mathcal{L}$. Due to Theorem 5.2 there exist $q_0^* < 2 < q_1^*$ such that for all $p \in [q_0^*, q_1^*]$ the operator $\mathcal{P}$ is a topological isomorphism between $\mathbb{W}_D^{1,p}(\Omega)$ and $\mathbb{W}_D^{-1,p}(\Omega)$. This implies that for all $t > 0$ and $p \in [q_0^*, q_1^*]$ the operator $\mathcal{Q}_t$, given by $\mathcal{Q}_t = \mathcal{P}^{-1}(\mathcal{P} - t\mathcal{A})$ is a bounded linear operator from $\mathbb{W}_D^{1,p}(\Omega)$ to $\mathbb{W}_D^{1,p}(\Omega)$. In a first step, we will show that there exist $t_0 > 0$ and $q_0, q_1 \in [q_0^*, q_1^*]$ with $q_0 < 2 < q_1$, such that

$$\sup_{p\in[q_0,q_1]} \|\mathcal{Q}_{t_0}\|_{\mathsf{op},p} \le \rho < 1, \tag{5.3}$$

where $\|\mathcal{Q}_{t_0}\|_{\mathsf{op},p}$ denotes the operator norm with respect to the space $\mathbb{W}_D^{1,p}(\Omega)$. In the second step, the uniform estimate (5.2) is derived from (5.3).

We start the investigation with $p = 2$. Observe that the standard inner product on $\mathbb{W}_D^{1,2}(\Omega)$ satisfies $(u, v)_{1,2} = (\mathcal{L}u, \mathcal{L}v)_{0,2} = \langle \mathcal{P}(u), v\rangle$. Hence, by (A3) the following identities are valid for $u, v \in \mathbb{W}_D^{1,2}(\Omega)$:

$$(\mathcal{Q}_t u, v)_{1,2} = \langle (\mathcal{P} - t\mathcal{A})u, v\rangle = \overline{\langle (\mathcal{P} - t\mathcal{A})v, u\rangle} = \overline{(\mathcal{P}^{-1}(\mathcal{P} - t\mathcal{A})v, u)_{1,2}} = (u, \mathcal{Q}_t v)_{1,2}.$$

This shows that $\mathcal{Q}_t$ is self adjoint on $\mathbb{W}_D^{1,2}(\Omega)$. Moreover, taking into account the upper bound M of the coefficient matrix $\mathbb{A}$ and the uniform ellipticity property, the following estimates are valid for all $u \in \mathbb{W}_D^{1,2}(\Omega)$:

$$\begin{aligned}(\mathcal{Q}_t u, u)_{1,2} &= \langle (\mathcal{P} - t\mathcal{A})u, u\rangle \ge (1 - tM)\|u\|^2_{\mathbb{W}^{1,2}(\Omega)}\\ (\mathcal{Q}_t u, u)_{1,2} &\le (1 - t\kappa)\|u\|^2_{\mathbb{W}^{1,2}(\Omega)}.\end{aligned}$$

Thus, the operator norm $\|\mathcal{Q}_t\|_{\mathsf{op},2}$ with respect to $\mathbb{W}_D^{1,2}(\Omega)$ can be estimated as

$$\begin{aligned}\|\mathcal{Q}_t\|_{\mathsf{op},2} &= \sup\{|\langle \mathcal{Q}_t u, u\rangle| \,;\, u \in \mathbb{W}_D^{1,2}(\Omega),\ \|u\|_{\mathbb{W}^{1,2}(\Omega)} \le 1\}\\ &\le \max\{|1 - tM|, |1 - t\kappa|\}.\end{aligned}$$

Hence, the operator $\mathcal{Q}_t$ is a strict contraction provided that $t \in \,]0, 2/M[$. We choose now $t_0 = 2/(\kappa + M)$ and define $\widetilde{\rho} = 1 - t_0\kappa = (M - \kappa)/(M + \kappa)$. With this, we have $\|\mathcal{Q}_{t_0}\|_{\mathsf{op},2} \le \widetilde{\rho} < 1$.

For $p \in [2, q_1^*]$, interpolation theory gives the estimate $\|\mathcal{Q}_{t_0}\|_{\mathsf{op},p} \le \widetilde{\rho}^{1-\theta}\|\mathcal{Q}_{t_0}\|^{\theta}_{\mathsf{op},q_1^*}$, where $1/p = (1-\theta)/2 + \theta/q_1^*$. Hence, there exist $\rho_1 \in \,]0, 1[$ and $q_1 \in \,]2, q_1^*]$ such that for all $p \in [2, q_1]$ it holds $\|\mathcal{Q}_{t_0}\|_{\mathsf{op},p} \le \rho_1$. Similar arguments applied to the interval $[q_0^*, 2]$ finally imply (5.3).

Now, we proceed analogously to the arguments in the proof of Theorem 1 in [23]: Since the operator $\mathcal{Q}_{t_0}$ is a contraction on $\mathbb{W}_D^{1,p}(\Omega)$, for every $f \in \mathbb{W}_D^{-1,p}(\Omega)$ the operator $v \mapsto \mathcal{Q}_{t_0}(v) + t_0\mathcal{P}^{-1}f$ has a unique fixed point u_f. Observe that u_f satisfies $\mathcal{A}u_f = f$. Hence, for all $p \in [p_0, p_1]$ the operator $\mathcal{A}$ is a topological isomorphism with respect to $\mathbb{W}_D^{1,p}(\Omega)$. Finally, since

$$\|u_f\|_{\mathbb{W}^{1,p}(\Omega)} = \|\mathcal{Q}_{t_0}u_f + t_0\mathcal{P}^{-1}f\|_{\mathbb{W}^{1,p}(\Omega)} \le \rho\|u_f\|_{\mathbb{W}^{1,p}(\Omega)} + t_0 c_{q_0^*,q_1^*}\|f\|_{\mathbb{W}_D^{-1,p}(\Omega)},$$

the operator norm of $\mathcal{A}^{-1}$ is uniformly bounded on $[q_0, q_1]$, which is (5.2). □

Example 5.5. The equations of linear elasticity as well as the Cosserat-model fit into this framework. In the case of linear elasticity, the vector-function $u : \Omega \to \mathbb{R}^d$ (i.e. $m = d$) denotes the displacement field. Typically, the Dirichlet-boundary is the same for all components of u. Hence, we define $\mathbb{W}_D^{1,p}(\Omega) = \prod_{i=1}^d W_D^{1,p}(\Omega)$, where $D \subset \partial\Omega$ is a closed $(d-1)$-set. The operator of linear elasticity is defined through the form $\langle \mathcal{A}u, v\rangle = \int_\Omega \mathbf{C}e(u) : e(v)\ \mathrm{d}x$ for $u, v \in \mathbb{W}_D^{1,2}(\Omega)$, Here, $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the symmetrised gradient and $\mathbf{C} \in L^\infty(\Omega; \mathsf{Lin}(\mathbb{R}^{d\times d}_{\mathsf{sym}}, \mathbb{R}^{d\times d}_{\mathsf{sym}}))$ denotes the fourth order elasticity tensor. It is assumed that $\mathbf{C}$ is symmetric and positive definite on the symmetric matrices: for all $F_1, F_2 \in \mathbb{R}^{d\times d}_{\text{sym}}$ it holds

$$\mathbf{C}F_1 : F_2 = \mathbf{C}F_2 : F_1, \quad \mathbf{C}F_1 : F_1 \geq c_\kappa |F_1|^2.$$

In order to have Korn's second inequality at our disposal, in addition to (A1) we assume that Ω is a Lipschitz domain. Korn's second inequality states that the standard norm in $\mathbb{W}^{1,2}_D(\Omega)$ and the norm $|||u||| := \|u\|_{L^2(\Omega)} + \|e(u)\|_{L^2(\Omega)}$ are equivalent, cf. [15] and the references therein. Moreover, if $\operatorname{meas}(D) > 0$, then standard arguments relying on the compact embedding of $\mathbb{W}^{1,2}_D(\Omega)$ in $L^2(\Omega)$ show that also Korn's first inequality is valid and assumption (A2) is satisfied. Hence, Theorems 5.2 and 5.3 are applicable.

In the Cosserat models, additionally to the displacement fields the skew symmetric Cosserat-microrotation-tensor $R \in \mathbb{R}^{3\times 3}_{\text{skew}}$ plays a role. Via the relation

$$\operatorname{axl} R := \operatorname{axl} \begin{pmatrix} 0 & r_1 & r_2 \\ -r_1 & 0 & r_3 \\ -r2 & -r3 & 0 \end{pmatrix} := \begin{pmatrix} -r_3 \\ r_2 \\ -r_1 \end{pmatrix},$$

$\mathbb{R}^{3\times 3}_{\text{skew}}$ is identified with $\mathbb{R}^3$. Assume that $D_{\text{el}}, D_{\text{R}} \subset \partial\Omega$ are nonempty, closed 2-sets describing the Dirichlet boundary for the displacements and the tensor R, respectively. The state space is defined as $\mathbb{W}^{1,p}_D(\Omega) = \prod_{i=1}^3 W^{1,p}_{D_{\text{el}}}(\Omega) \times \prod_{i=1}^3 W^{1,p}_{D_{\text{R}}}(\Omega)$. A typical differential operator occurring in the theory of Cosserat models is given by the following weak form for $(u, R), (v, Q) \in \mathbb{W}^{1,2}_D(\Omega)$:

$$\begin{aligned}\langle \mathcal{A} \left(\begin{smallmatrix} u \\ R \end{smallmatrix} \right), \left(\begin{smallmatrix} v \\ Q \end{smallmatrix} \right) \rangle = \int_\Omega & 2\mu e(u) : e(v) + \lambda \operatorname{div} u \operatorname{div} v \\ & + 2\mu_c \operatorname{skew}(\nabla u - R) : \operatorname{skew}(\nabla v - Q) + \gamma \nabla \operatorname{axl} R : \nabla \operatorname{axl} Q \, \mathrm{d}x.\end{aligned}$$

If in addition to (A1) the domain is a Lipschitz domain and if for the Lamé-constants λ, μ, the Cosserat-couple modulus μ_c and the parameter γ it holds $\mu > 0$, $2\mu + 3\lambda > 0$, $\mu_c \geq 0$ and $\gamma > 0$, then condition (A2) is satisfied, see [48, 38], where also more general situations are discussed. Obviously, (A3) is satisfied as well, and hence Theorems 5.2 and 5.3 are applicable.

Remark 5.6. We finally remark that on the basis of the previous example the results from [30] for nonlinear elasticity models can be extended to the more general geometric situation discussed here by repeating the arguments in [30, Section 3].

6. Applications

In this chapter we intend to indicate possible applications, which were the original motivation for this work.

It is more or less clear that the results of this paper cry for applications primarily in spatially two-dimensional elliptic problems. We suggest that in almost all applications resting on [23] the geometric conditions can be relaxed to those of this paper, and the results still hold, (see e.g. [41], [8], [34], [46], [7], [12], [18], [14], [35], [31] to name only a few).

Moreover, the generator property for an analytic semigroup gives the opportunity to deal also with parabolic problems. When employing the main result from [10] and then applying the classical semilinear theory, see e.g. [29, Ch. 3], one should be able to treat also semilinear ones. Generally, the $W^{-1,q}_D$-calculus allows for right hand sides of the equations which contain distributional objects as e.g. surface densities which still belong to the space $W^{-1,q}_D(\Omega)$. In particular, in the $2d$-case one may even admit functions in time which take their values in the space of Borel measures, since the space of these measures then continuously embeds into any space $W_D^{-1,q}(\Omega)$ with $q < 2$, compare also [3].

Moreover, the elliptic regularity result enables a simpler treatment of problems which include quadratic gradient terms: the a priori knowledge $\nabla u \in L^q$ with $q > 2$ improves the standard information $|\nabla u|^2 \in L^1$ to $|\nabla u|^2 \in L^r$ with $r > 1$ – what makes the analysis of such problems easier, compare [31, 39].

Let us at the end sketch an idea how one can exploit the gain in elliptic regularity in a rather unexpected direction: Let $q > 2$ be a number such that (4.4) is a topological isomorphism and (4.4) is also a topological isomorphism if μ is there replaced by the adjoint coefficient function, then providing the adjoint operator in $L^2(\Omega)$. We abbreviate $A := \nabla \cdot \mu \nabla|_{L^2(\Omega)}$. It is known, see [10], that the semigroup operators e^{tA} possess kernels which admit upper Gaussian estimates. Obviously, these kernels are bounded, and, consequently, all semigroup operators are Hilbert-Schmidt and even nuclear. Consequently, $e^{\frac{t}{3}A} : L^2(\Omega) \to L^2(\Omega)$ admits a representation

$$e^{\frac{t}{3}A}\psi = \sum_j \lambda_j \langle \psi, f_j \rangle_{L^2(\Omega)} \; g_j$$

with $\|f_j\|_{L^2(\Omega)} = \|g_j\|_{L^2(\Omega)} = 1$ and $\sum_j |\lambda_j| < \infty$, see [37, Thm. 1.b.3]. Hence, e^{tA} admits the following representation via an integral kernel.

$$e^{tA} = \sum_j \lambda_j \langle e^{\frac{t}{3}A} \cdot, f_j \rangle_{L^2(\Omega)} \; e^{\frac{t}{3}A} g_j = \sum_j \lambda_j \; e^{\frac{t}{3}A} g_j \otimes \overline{e^{\frac{t}{3}A^\star} f_j}. \tag{6.1}$$

Let us estimate the $W^{1,q}$-norm of the elements $e^{\frac{t}{3}A} g_j$ and $e^{\frac{t}{3}A^\star} f_j$, respectively:

$$\left\| e^{\frac{t}{3}A} g_j \right\|_{W_D^{1,q}(\Omega)} \le \|(-A+1)^{-1}\|_{\mathcal{L}(L^q(\Omega);W_D^{1,q}(\Omega))} \left\| e^{\frac{t}{6}A} \right\|_{\mathcal{L}(L^2(\Omega);L^q(\Omega))} \left\| (-A+1) e^{\frac{t}{6}A} \right\|_{\mathcal{L}(L^2(\Omega))},$$

since $\|g_j\|_{L^2(\Omega)} = 1$. Let us discuss the factors on the right hand side: the first is finite due to our supposition on q and the embedding $L^q(\Omega) \hookrightarrow W_D^{-1,q}(\Omega)$. The second is finite because the semigroup operators are integral operators with bounded kernels. The third factor is bounded because A generates an analytic semigroup on $L^2(\Omega)$, what is well known.

The estimate for $e^{\frac{t}{3}A^\star} f_j$ is quite analogous, this time investing the continuity of $(-A^* + 1)^{-1} : L^q(\Omega) \to W_D^{1,q}(\Omega)$. Bringing now into play the summability of the series $\sum_j |\lambda_j|$, one obtains the convergence of the series $\sum_j \lambda_j \; e^{\frac{t}{3}A} g_j \otimes \overline{e^{\frac{t}{3}A^\star} f_j}$ in the space $W^{1,q}(\Omega \times \Omega)$. Thus, the semigroup operators have kernels which are even from $W^{1,q}(\Omega \times \Omega)$ with $q > 2$. We will discuss the consequences of this in a forthcoming paper.

7. Concluding remarks

Remark 7.1. i) As the example in the above figure suggests, admissible constellations for the domain Ω are finite unions of (suitable) Lipschitz domains, the closures of which have nonempty intersections. Thus, generically, the boundary is the finite union of $(d-1)$-dimensional Lipschitz manifolds with the corresponding boundary measures.

ii) The $W^{1,p}$-regularity result is also of use for the analysis of four-dimensional elliptic equations with right hand side from $W_D^{-1,p}(\Omega)$, $p > 4$. Namely, the information that the solution a priori belongs to a space $W_D^{1,q}$ with $q > 2$, allows to localise the elliptic problem within the same class of right hand sides, cf. [26].

iii) The regularity results on the spaces $W_D^{1,p}(\Omega)$ in case of $p < 2$ provide a frame where spatially two dimensional elliptic and parabolic equations with measure-valued right hand sides can be treated. This rests on the fact that in case of two space dimensions the space of bounded Radon measures on $\overline{\Omega}$ continuously embeds into any space $W_D^{-1,p}(\Omega)$ if only $p < 2$, compare also [3].

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