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Variational convergence of gradient flows and rate-independent evolutions in metric spaces

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ABSTRACT. We study the asymptotic behaviour of families of gradient flows in a general metric setting, when the metric-dissipation potentials degenerate in the limit to a dissipation with linear growth.

We present a general variational definition of BV solutions to metric evolutions, showing the different characterization of the solution in the absolutely continuous regime, on the singular Cantor part, and along the jump transitions. By using tools of metric analysis, BV functions and blow-up by time rescaling, we show that this variational notion is stable with respect to a wide class of perturbations involving energies, distances, and dissipation potentials.

As a particular application, we show that BV solutions to rate-independent problems arise naturally as a limit of p-gradient flows, $p > 1$, when the exponents p converge to 1.

1. INTRODUCTION

The aim of this paper is to study the asymptotic behaviour of the solutions to a sequence of gradient flows (in a suitable metric setting), when the governing energies and metric-dissipation potentials give raise in the limit to a rate-independent evolution or, more generally, to an evolution driven by a dissipation potential with linear growth.

A finite dimensional example: superlinear dissipation potentials and absolutely continuous gradient flows. In order to explain the problem, let us start from a simple example in a finite dimensional manifold $\mathscr{X}$ (see e.g. the motivating discussion in [13]). We fix a time interval $[0, T]$, we denote by $\mathscr{Q}$ the product space $\mathscr{Q} = [0, T] \times \mathscr{X}$, and we consider a sequence of smooth energies $\mathcal{E}_h : \mathscr{Q} \to \mathbb{R}$ indexed by $h \in \mathbb{N}$. We are also given a sequence of smooth dissipation potentials $\mathcal{R}_h : \mathrm{T}\mathscr{X} \to [0, \infty)$ of the form

$$\mathcal{R}_h(u, \dot u) = \psi_h(\|\dot u\|_{u,h}) \quad \text{where } \|\cdot\|_{u,h} \text{ are norms on the tangent space } \mathrm{T}_u\mathscr{X}$$

smoothly depending on $u \in \mathscr{X}$ and

$$\psi_h : [0, \infty) \to [0, \infty) \text{ are } \mathrm{C}^1 \text{ convex functions with superlinear growth.}$$

Typical examples are

$$\psi_h(v) = \frac{1}{p_h} v^{p_h} \qquad \text{with } p_h > 1, \qquad \psi_h(v) = v + \varepsilon_h v^p \qquad \text{with } p > 1,\ \varepsilon_h > 0. \tag{1.1}$$

For given initial data $\bar u_h \in \mathscr{X}$ we can consider the solutions $u_h : [0, T] \to \mathscr{X}$ of the Cauchy problem for the doubly nonlinear differential equations

$$\mathrm{D}_{\dot u}\mathcal{R}_h(u_h(t), \dot u_h(t)) + \mathrm{D}_u\mathcal{E}_h(t, u_h(t)) = 0 \quad \text{in } \mathrm{T}^*\mathscr{X},\ t \in [0, T]; \qquad u_h(0) = \bar u_h. \tag{1.2}$$

In (1.2) the parameter $h \in \mathbb{N}$ affects the limit behaviour of the initial data $\bar u_h$, of the energies $\mathcal{E}_h$ in $\mathscr{Q}$, of the norms $\|\cdot\|_{\cdot,h}$ on $\mathrm{T}\mathscr{X}$, and of the dissipation potentials ψ_h on $[0, \infty)$. Assuming that (in a suitable sense that we will describe later on) $\bar u_h \to \bar u$, $\mathcal{E}_h \to \mathcal{E}$, $\|\cdot\|_{u,h} \to \|\cdot\|_u$, $\psi_h \to \psi$ as $h \to \infty$, it is then natural to investigate if a limit curve u (possibly up to subsequence) of the solutions $(u_h)_h$ still satisfies the corresponding limit equation of (1.2).

BV solutions to rate-independent evolutions. We want to address here the singular situation when the limit dissipation potential ψ loses the superlinear growth; let us focus here on the 1-homogeneous case when

$$\lim_{h\to\infty}\psi_h(v)=\psi(v):=\mathsf{L}\,v,\quad\text{for some }\mathsf{L}>0 \tag{1.3}$$

corresponding e.g. to $\lim_{h\to\infty}p_h=1$, or $\lim_{h\to\infty}\varepsilon_h=0$ in (1.1) (in that cases $\mathsf{L}=1$). The limit problem is formally the differential inclusion

$$\mathsf{L}\,\partial_{\dot u}\mathcal{R}(u(t),\dot u(t))+\mathrm{D}_u\mathcal{E}(t,u(t))=0\quad\text{in }\mathrm{T}^*\mathscr{X},\ t\in[0,T],\qquad u(0)=\bar u, \tag{1.4}$$

where the presence of the subdifferential $\partial_{\dot u}\mathcal{R}$ is motivated by the lack of differentiability of the norm $\mathcal{R}(u,\dot u)=\|\dot u\|_u$ at $\dot u=0$. Since $\mathcal{R}(u,\cdot)$ is 1-positively homogeneous, (1.4) describes a rate-independent evolution and its solutions exhibit a different behavior with respect to the viscous flows (1.2). In particular, jumps can occur even for smooth energies $\mathcal{E}$ and various kinds of solutions have been proposed in the literature (we refer to the surveys [11], the overall presentation in [12] and the references therein). Here we focus on the notion of BV solution, proposed in [13, 14]: for the sake of simplicity, in this introductory section we consider the simplest case of a piecewise smooth curve u with a finite number of jump points $\mathrm{J}_u=\{t_1,t_2,\cdots t_n\}\subset[0,T]$; $u(t_{i\pm})$ will denote the left and the right limit of u at each t_i (see also [25] for an explicit characterization in a one-dimensional setting)

In this case a BV solution u can be characterized by two conditions:

(BV1) In each interval (t_{i-1},t_i) the velocity vector field $\dot u$ satisfies the differential inclusion (1.4), which yields in particular the local stability condition

$$\mathcal{F}(t,u(t))\le\mathsf{L}\quad\text{for every }t\in[0,T]\setminus\mathrm{J}_u \tag{1.5}$$

and the energy dissipation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t))+\mathcal{P}(t,u(t))=\mathsf{L}\,\|\dot u(t)\|_{u(t)}\quad\text{in }(t_{i-1},t_i),\quad\mathcal{P}(t,u):=\frac{\partial}{\partial t}\mathcal{E}(t,u), \tag{1.6}$$

where $\mathcal{F}$ denotes the dual norm of the (opposite) differential of the energy,

$$\mathcal{F}(t,u):=\|\mathrm{D}_u\mathcal{E}(t,u)\|_u^*=\sup\Big\{-\langle\mathrm{D}_u\mathcal{E}(t,u),v\rangle:\|v\|_u\le1\Big\}. \tag{1.7}$$

It turns out that in the smooth regime (1.5) and (1.6) are equivalent to (1.4).

(BV2) At each jump point t_i it is possible to find an optimal transition path $\vartheta_i:[r_{i-},r_{i+}]\to\mathscr{X}$, $r_{i-}\le0\le r_{i+}$, such that $\vartheta_i(r_{i\pm})=u(t_{i\pm})$, $\vartheta_i(0)=u(t_i)$, $\mathcal{F}(r,\vartheta_i(r))\ge\mathsf{L}$ in $[r_{i-},r_{i+}]$, and

$$\begin{aligned}\int_{r_{i-}}^{r_{i+}}\mathcal{F}(r,\vartheta_i(r))\|\dot\vartheta_i(r)\|_{\vartheta_i(r)}\,\mathrm{d}r&=\mathcal{E}(t_i,u(t_{i-}))-\mathcal{E}(t_i,u(t_{i+}))\\&=\min\Big\{\int_{r_{i-}}^{r_{i+}}(\mathcal{F}(r,\theta(r))\vee\mathsf{L})\|\dot\theta(r)\|_{\theta(r)}\,\mathrm{d}r:\theta(r_{i\pm})=u(t_{i\pm}),\ \theta(0)=u(t_i)\Big\}.\end{aligned} \tag{1.8}$$

Notice that the choice of the interval $[r_{i-},r_{i+}]$ is not essential, since the integrals in (1.8) are invariant with respect to monotone time rescaling. The minimum problem in (1.8) characterizes the minimal transition cost at each jump point t_i to connect in $u(t_{i-})$ with $u(t_{i+})$ passing through $u(t_i)$. Such a cost is influenced both by the norms $\|\cdot\|_u$ and by the slope $\mathcal{F}$ of the energy: we will denote it by $\Delta_{t_i}(u(t_{i-}), u(t_i), u(t_{i+}))$.

Energy-dissipation inequalities. It is a remarkable fact, highlighted in [13, 14], that the refined structure given in (BV1,BV2) can be captured by simply imposing the local stability condition (1.5) and a single energy-dissipation inequality, namely

(1.9)

$$\mathcal{E}(T,u(T)) + \int_0^T \|\dot u(t)\|_u \,\mathrm{d}t + \sum_{i=1}^n \Delta_{t_i}(u(t_{i-}), u(t_i), u(t_{i+})) \le \mathcal{E}(0,\bar u) + \int_0^T \mathcal{P}(t,u(t))\,\mathrm{d}t.$$

It turns out that (1.9) is in fact an identity, since the opposite inequality is always satisfied along *any* piecewise smooth curve u. If (1.9) holds, then u is forced to satisfy (1.6) along its smooth evolution, and the optimal transition paths obtained by solving the minimum problem in (1.8) provide the right energy balance between $u(t_{i\pm})$.

The link of (1.9) with the gradient flow (1.2) becomes more transparent if, following [1, 24, 23, 16], one notices that also (1.2) can be formulated as a energy-dissipation inequality. In fact, setting as before

$$\mathcal{F}_h(t,u) := \|\mathrm{D}_u\mathcal{E}_h(t,u)\|_{u,h}^*, \qquad \mathcal{P}_h(t,u) := \frac{\partial}{\partial t}\mathcal{E}_h(t,u), \tag{1.10}$$

it is not difficult to check (see the informal discussion in the next section) that a C^1 curve u_h with $u_h(0) = \bar u_h$ satisfies (1.2) if and only if the ψ_h energy-dissipation inequality holds

(1.11)

$$\mathcal{E}_h(T,u_h(T)) + \int_0^T \Big(\psi_h\big(\|\dot u_h(t)\|_{u_h,h}\big) + \psi_h^*\big(\mathcal{F}_h(t,u_h(t))\big)\Big)\,\mathrm{d}t \le \mathcal{E}_h(0,\bar u_h) + \int_0^T \mathcal{P}_h(t,u_h(t))\,\mathrm{d}t,$$

where ψ_h^* is the Legendre transform of ψ_h.

A more general formulation in metric spaces. Here we want to show that the metric-variational approach to gradient flows and rate-independent problems provides a natural framework to study this singular perturbation problem and suggests a robust and general strategy to pass to the limit in a much more general setting where

- $\mathscr{X}$ is a topological space endowed with a family of complete extended distances d_h,
- the terms like $\|\dot u_h\|_{u_h,h}$ are replaced by the *metric velocity* induced by d_h,
- the functions $\mathcal{F}_h, \mathcal{P}_h$ can be characterized as an *irreversible couple of upper gradients* in terms of the behaviour of the energies $\mathcal{E}_h$ along arbitrary absolutely continuous curves with values in $(\mathscr{X}, \mathsf{d}_h)$, and
- ψ is a general metric dissipation function with linear growth.

Postponing to the next two sections a more precise review of motivations and definitions, we just remark that whenever sufficiently strong a priori estimates are available to guarantee the pointwise convergence of u_h to some limit function $u \in \mathrm{BV}([0,T];(\mathscr{X},\mathsf{d}))$, then the heart of the problem consists in deriving (1.9) (in a suitably extended form allowing countably many jumps and Cantor-like terms in the metric velocity), starting from the viscous inequality (1.11). Assuming convergence in energy of the initial data, i.e. $\lim_{h\to\infty}\mathcal{E}_h(0,\bar u_h) = \mathcal{E}(0,\bar u)$, some lower-upper semicontinuity conditions on $(\mathcal{E}_h)_h$ and $(\mathcal{P}_h)_h$ along arbitrary sequences $(x_h)_h$ with equibounded energy and converging to x in a fixed reference topology σ of $\mathscr{X}$ are naturally suggested by the structure of the main inequalities (1.11) and (1.9):

(1.12) $$\liminf_{h\to\infty} \mathcal{E}_h(t,x_h) \ge \mathcal{E}(t,x), \qquad \limsup_{h\to\infty} \mathcal{P}_h(t,x_h) \le \mathcal{P}(t,x) \quad \text{whenever} \quad x_h \xrightarrow{\sigma} x \text{ in } \mathscr{X}.$$

The most challenging point is provided by the limit behaviour of the integral term

(1.13) $$\int_0^T \Big(\psi_h\big(\|\dot u_h(t)\|_{u_h,h}\big) + \psi_h^*\big(\mathcal{F}_h(t,u_h(t))\big)\Big)\,\mathrm{d}t,$$

which has been typically studied by a clever re-parametrization technique, introduced by [10] and then extended in various directions by [13, 18, 14]. This approach leads to the notion of the so-called *parametrized solutions* to the rate-independent evolution and the crucial assumption concerns the validity of the Γ-$\liminf$ space-time estimate for the slopes

(1.14) $$\liminf_{h\to\infty} \mathcal{F}_h(t_h,x_h) \ge \mathcal{F}(t,x) \quad \text{whenever} \quad t_h \to t,\ x_h \xrightarrow{\sigma} x.$$

In the present paper we propose a different technique, which avoids parametrized solutions and thus allows for more general non-homogenous dissipation potentials like

(1.15) $$\psi(v) := \int_0^v (r\wedge \mathsf{L})\,\mathrm{d}r = \begin{cases} \frac12 v^2 & \text{if } 0\le v\le \mathsf{L},\\ \mathsf{L}\,v - \frac12 \mathsf{L}^2 & \text{if } v \ge \mathsf{L},\end{cases} \qquad \psi(v) := (1+v^2)^{1/2}.$$

Our approach involves weak convergence of measures to deal with concentrations of the time derivative and blow-up around jump points of the limit solution to recover the variational structure of the transition. In this way, an easier rescaling is sufficient to construct the optimal transition paths (see (1.8)) from the converging family $(u_h)_h$ and to obtain the BV energy-dissipation inequality (1.9).

Particular cases. Let us remark that various particular cases of the present setting are interesting by themselves and have been considered from many different points of view.

(i) A first important case for applications is when $\mathscr{X}$ is a Hilbert space, $\psi_h(v) = \frac12 v^2$, and the norms $\|\cdot\|_{u,h}$ are independent of h and coincide with the norm $\|\cdot\|$ of $\mathscr{X}$. In this case we are dealing with the convergence of gradient flows and a typical situation arises when $\mathcal{E}_h(t,u) = \mathcal{E}_h(u) - \langle \ell(t), u\rangle$. It is well known, since the pioneering contributions of [28, 29, 9], that convexity (or λ-convexity for some $\lambda \in \mathbb{R}$ independent of h) of the energies makes it possible to reduce (1.14) to the simpler Mosco-convergence [19] of $\mathcal{E}_h$ (see e.g. [4] or [5] for the connection with the graph-convergence of the differential operators). The link between Γ-convergence of the energies and convergence of the gradient flows in a metric setting has been considered in [1, 2, 8].

(ii) Another relevant situation is when both the energies and the distances depend on h: in the quadratic case a convergence result can be deduced by a joint Γ-convergence, see e.g. [26, 22, 21]. The role of the Γ-$\liminf$ condition on the slopes as in (1.14) in general non-convex setting has been clarified in [20, 27]. A very general stability result has been given in [16]. An interesting example where the limit of gradient flows gives raise to a singular limit in a new geometry is discussed in [3].

(iii) The particular case when the h-dependence affects only the dissipation potential ψ and gives raise to a rate-independent problem in the limit has been studied in [13, 14, 15]. The Γ-limit of rate-independent evolutions, in the framework of energetic solutions, has been studied in [17].

Plan of the paper. In the next section we give more details on the simple finite-dimensional example we introduced before, in order to motivate the abstract metric approach, whose setting is explained in Section 3.

Section 4 contains our main results, concerning compactness (Theorem 4.1) and convergence (Theorem 4.2) of gradient flows in a general setting. A few examples are briefly presented at the end of the paper.

2. THE METRIC FORMULATION OF GRADIENT FLOWS IN A SMOOTH SETTING

Let $\mathscr{X}$ be the finite-dimensional differentiable manifold discussed in the Introduction.

Length and metric derivative. Let us first recall that the Finsler structure $\|\cdot\|_{u,h}$ on $\mathrm{T}\mathscr{X}$ allows us to define the length of a smooth curve $\mathsf{u} \in \mathrm{C}^1([r_0,r_1];\mathscr{X})$ by

$$\mathrm{Length}_h[\mathsf{u}] := \int_{r_0}^{r_1} \|\dot{\mathsf{u}}(r)\|_{\mathsf{u}(r),h}\,\mathrm{d}r \tag{2.1}$$

and a distance

$$\mathsf{d}_h(u_0,u_1) := \inf\Big\{\mathrm{Length}_h[\mathsf{u}] : \mathsf{u} \in \mathrm{C}^1([r_0,r_1];\mathscr{X}),\ \mathsf{u}(r_i) = u_i\Big\}, \tag{2.2}$$

which still retains the information of the norms $\|\cdot\|_{u,h}$, since

$$\|\dot{\mathsf{u}}(r)\|_{\mathsf{u}(r),h} = \lim_{s\to r}\frac{\mathsf{d}_h(\mathsf{u}(s),\mathsf{u}(r))}{|s-r|} \quad \text{for every } \mathsf{u} \in \mathrm{C}^1([r_0,r_1];\mathscr{X}). \tag{2.3}$$

The limit in (2.3) can be extended to the general setting of absolutely continuous curves in metric spaces: it is denoted by $|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r)$ and it is called *metric derivative* of the curve u, see Definition 3.1.

Chain rule and irreversible upper gradients. A second crucial quantity is the dual norm of the opposite differential of the energy

$$\|\mathrm{D}_u\mathcal{E}_h(t,u)\|^*_{u,h} = \sup\Big\{-\langle \mathrm{D}_u\mathcal{E}_h(t,u),v\rangle : \|v\|_{u,h} \le 1\Big\}. \tag{2.4}$$

Observe that the quantity in (2.4) also has a nice characterization in terms of curves, since the function $(t,u) \mapsto \|\mathrm{D}_u\mathcal{E}_h(t,u)\|^*_{u,h}$ is minimal among the functions $\mathcal{F}_h : \mathscr{Q} \to [0,\infty)$ satisfying the chain-rule inequality

$$-\frac{\partial}{\partial r}\mathcal{E}_h(t,\mathsf{u}(r)) \le \mathcal{F}_h(t,\mathsf{u}(r))|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r) \tag{2.5}$$

along arbitrary curves $\mathsf{u} \in \mathrm{C}^1([r_0,r_1];\mathscr{X})$. If one wants to allow for time variation of the energy, it is natural to introduce the partial time derivative $\frac{\partial}{\partial t}\mathcal{E}_h(t,u)$, so that (2.5) is in fact equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{E}_h(t(r),\mathsf{u}(r)) + \frac{\partial}{\partial t}\mathcal{E}_h(\mathsf{t}(r),\mathsf{u}(r))\dot{\mathsf{t}}(r) \le \mathcal{F}_h(\mathsf{t}(r),\mathsf{u}(r))|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r) \tag{2.6}$$

along arbitrary regular curves $r \mapsto (\mathsf{t}(r),\mathsf{u}(r)) \in \mathscr{Q}$. If we only consider nondecreasing time parametrizations $r \mapsto t(r)$, and we integrate (2.6) along arbitrary intervals $[r_0,r_1]$, we see that the map $(t,u) \mapsto$

$\frac{\partial}{\partial t}\mathcal{E}(t,u)$ is maximal among all the functions $\mathcal{P}_h : \mathscr{Q} \to \mathbb{R}$ satisfying
(2.7)

$$\mathcal{E}_h(\mathsf{t}(r_0),\mathsf{u}(r_0)) + \int_{r_0}^{r_1} \mathcal{P}_h(\mathsf{t}(r),\mathsf{u}(r))\dot{\mathsf{t}}(r)\,\mathrm{d}r \le \mathcal{E}_h(\mathsf{t}(r_1),\mathsf{u}(r_1)) + \int_{r_0}^{r_1} \mathcal{F}_h(\mathsf{t}(r),\mathsf{u}(r))|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r)\,\mathrm{d}r.$$

In fact, let us suppose that $\mathcal{F}_h, \mathcal{P}_h$ are continuous functions satisfying (2.7) along arbitrary regular curves with $\dot{\mathsf{t}}(r) \ge 0$: it would not be difficult to check that this property is equivalent to

$$\mathcal{P}_h(t,u) \le \frac{\partial}{\partial t}\mathcal{E}(t,u), \quad \mathcal{F}_h(t,u) \ge \|\mathrm{D}_u\mathcal{E}_h(t,u)\|_{u,h}^* \quad \text{for every } (t,u) \in \mathscr{Q}. \tag{2.8}$$

If (2.8) holds, we say that the couple $(\mathcal{F}_h, \mathcal{P}_h)$ is an *irreversible upper gradient* for the energy $\mathcal{E}_h$ with respect to the distance d_h, see Definition 3.2, and $(\mathscr{X}, \mathsf{d}_h, \mathcal{E}_h, \mathcal{F}_h, \mathcal{P}_h)$ is an upper-gradient evolution system.

This definition is the natural adaptation to time-dependent functionals of the well-known notion of upper-gradient in the frame of analysis in metric spaces (see [7, 1]); the interesting fact is that (2.7) only involves the notion of absolutely continuous curves in $(\mathscr{X}, \mathsf{d}_h)$.

ψ-gradient flows and energy-dissipation inequality. The distinguished role of gradient flows with respect to (2.6) can be easily seen by recalling the Fenchel duality

$$-\mathrm{D}_v\mathcal{R}_h(u,v) = f \quad \Leftrightarrow \quad -\langle f, v\rangle = \|v\|_{u,h}\|f\|_{u,h}^* = \psi_h(\|v\|_{u,h}) + \psi_h^*(\|f\|_{u,h}^*), \tag{2.9}$$

where ψ^* is the Legendre transform of ψ:

$$\psi^*(f) = \sup_{v\ge 0}\Big(f\,v - \psi(v)\Big). \tag{2.10}$$

A crucial feature of Fenchel duality is that for *every* couple $(v,f) \in \mathrm{T}_u\mathscr{X} \times \mathrm{T}_u^*\mathscr{X}$ one has the inequality

$$-\langle f, v\rangle \le \|v\|_{u,h}\|f\|_{u,h}^* \le \psi_h(\|v\|_{u,h}) + \psi_h^*(\|f\|_{u,h}^*), \tag{2.11}$$

so that in order to check the identity $-\mathrm{D}_v\mathcal{R}_h(u,v) = f$ it is sufficient to prove the opposite inequality, i.e.

$$-\langle f, v\rangle \ge \psi_h(\|v\|_{u,h}) + \psi_h^*(\|f\|_{u,h}^*) \quad \Rightarrow \quad -\mathrm{D}_v\mathcal{R}_h(u,v) = f. \tag{2.12}$$

Taking into account these remarks and observing that we have the chain rule

$$-\langle \mathrm{D}_u\mathcal{E}_h(t,u_h(t)), \dot{u}_h(t)\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_h(t,u_h(t)) + \frac{\partial}{\partial t}\mathcal{E}_h(t,u_h(t)),$$

we deduce that u_h solves (1.2) if and only if

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_h(t,u_h(t)) + \frac{\partial}{\partial t}\mathcal{E}_h(t,u_h(t)) \ge \psi_h\big(\|\dot{u}_h(t)\|_{u_h,h}\big) + \psi_h^*\big(\|\mathrm{D}_u\mathcal{E}_h(t,u_h(t))\|_{u_h,h}^*\big). \tag{2.13}$$

Since, as we already noticed in (2.11), the opposite inequality is always true, we immediately see that it is sufficient to impose the integrated version of (2.13) in $(0,T)$:

$$\begin{aligned}\mathcal{E}_h(T,u_h(T))+\int_0^T\Big(\psi_h\big(\|\dot u_h(t)\|_{u_h,h}\big)+\psi_h^*\big(\|\mathrm{D}_u\mathcal{E}_h(t,u_h(t))\|^*_{u_h,h}\big)\Big)\,\mathrm{d}t\\ \le\mathcal{E}_h(0,\bar u_h)+\int_0^T\frac{\partial}{\partial t}\mathcal{E}_h(t,u_h(t))\,\mathrm{d}t.\end{aligned}\tag{2.14}$$

Now we can make the last step: instead of looking for curves satisfying (2.14), we reinforce it by replacing $\partial_t\mathcal{E}_h$ and $\|\mathrm{D}_u\mathcal{E}_h\|^*_{u_h,h}$ with a couple $\mathcal{P}_h,\mathcal{F}_h$ of irreversible upper gradients satisfying (2.7). Since ψ and ψ^* are nondecreasing maps and (2.8) holds, it is immediate to see that if a curve $u\in\mathrm{C}^1([0,T];\mathscr{X})$ satisfies the ψ-ψ^* energy-dissipation inequality

(2.15)
$$\mathcal{E}_h(T,u_h(T))+\int_0^T\Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}(t)\big)+\psi_h^*\big(\mathcal{F}_h(t,u_h(t))\big)\Big)\,\mathrm{d}t\le\mathcal{E}_h(0,\bar u_h)+\int_0^T\mathcal{P}_h(t,u_h(t))\,\mathrm{d}t$$

(see also the next Definition 3.3), then it also satisfies (2.14) and by the argument above it satisfies (1.2); moreover, along the curve we find a posteriori

$$\|\mathrm{D}_u\mathcal{E}_h(t,u_h(t))\|^*_{u_h,h}=\mathcal{F}_h(t,u_h(t)),\qquad\frac{\partial}{\partial t}\mathcal{E}_h(t,u_h(t))=\mathcal{P}_h(t,u_h(t))\qquad t\in[0,T].$$

We thus have seen that (2.15) for a couple $(\mathcal{F}_h,\mathcal{P}_h)$ of irreversible upper gradients provides a natural metric definition of ψ-gradient flow, which can be immediately extended to a metric framework.

Marginal functionals and conditional time derivative of the energy. In order to motivate the even more general definition of evolution system considered in Section 3.2, where the power functional $\mathcal{P}$ *can also depend on a further variable* F *satisfying the constraint* $\mathsf{F}(t)\ge\mathcal{F}(t,u(t))$, let us consider a non smooth situation, where $\mathcal{E}$ is a *marginal functional*: it means that $\mathcal{E}$ results from a minimization of the form

$$\mathcal{E}(t,u):=\min_\eta\big\{\mathcal{J}(t,u,\eta):\eta\in\mathscr{K}\big\},\tag{2.16}$$

where $\mathscr{K}$ is a compact topological space and $\mathcal{J}:\mathscr{Q}\times\mathscr{K}\to\mathbb{R}$ is a continuous function such that $\mathcal{J}(\cdot,\cdot,\eta)\in\mathrm{C}^1(\mathscr{Q})$ for every $\eta\in\mathscr{K}$ with uniformly continuous derivatives.

Even if each single functional $\mathcal{J}(\cdot,\eta)$ is regular, $\mathcal{E}$ is not C^1 in general. Referring to [16] for a more detailed discussion, we recall here that setting

$$M(t,u):=\operatorname{argmin}\mathcal{J}(t,u,\cdot)=\Big\{\eta\in\mathscr{K}:\mathcal{J}(t,u,\eta)=\mathcal{E}(t,u)\Big\}\tag{2.17}$$

it is natural to replace the smooth differential equation (1.2) with the differential inclusion

$$\mathrm{D}_{\dot u}\mathcal{R}_h(u_h(t),\dot u_h(t))+\mathrm{D}^{\mathrm{m}}_u\mathcal{E}(t,u_h(t))\ni 0\quad\text{in }\mathrm{T}^*\mathscr{X},\ t\in[0,T],\qquad u_h(0)=\bar u_h,\tag{2.18}$$

where, just for the purposes of this section, $\mathrm{D}^{\mathrm{m}}\mathcal{E}$ denotes the so-called *marginal differential* of $\mathcal{E}_h$, i.e.

$$\mathrm{D}^{\mathrm{m}}\mathcal{E}(t,u):=\Big\{(\mathsf{p},\mathsf{w})\in\mathbb{R}\times\mathrm{T}^*_u\mathscr{X}:\mathsf{p}=\partial_t\mathcal{J}(t,u,\eta),\ \mathsf{w}=\mathrm{D}_u\mathcal{J}(t,u,\eta)\text{ for some }\eta\in M(t,u)\Big\}$$

and $\mathrm{D}^{\mathrm{m}}_u\mathcal{E}$ is its projection onto the second component,

$$\mathrm{D}^{\mathrm{m}}_u\mathcal{E}(t,u):=\Big\{w\in\mathrm{T}^*_u\mathscr{X}:w=\mathrm{D}_u\mathcal{J}(t,u,\eta)\quad\text{for some }\eta\in M(t,u)\Big\}.\tag{2.19}$$

If we want to differentiate the energy along a regular curve $r \mapsto (\mathsf{t}(r), \mathsf{u}(r))$ as in (2.6) we get for a.a. r

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mathsf{t}(r), \mathsf{u}(r)) + \mathsf{p}(r)\,\dot{\mathsf{t}}(r) = -\langle \mathsf{w}(r), \dot{\mathsf{u}}(r)\rangle \le \|\mathsf{w}(r)\|^*_{\mathsf{u},h}\,|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r), \tag{2.20}$$

where $(\mathsf{p}(r), \mathsf{w}(r))$ is an arbitrary selection in $\mathrm{D}^{\mathrm{m}}\mathcal{E}(\mathsf{t}(r), \mathsf{u}(r))$. Setting

$$\mathcal{F}_h(t,u) := \min\Big\{\|\mathsf{w}\|_{u,h} : \mathsf{w} \in \mathrm{D}^{\mathrm{m}}_u(t,u)\Big\}, \tag{2.21}$$

$$\mathcal{P}_h(t,u,f) := \max\Big\{\mathsf{p} : (\mathsf{p},\mathsf{w}) \in \mathrm{D}^{\mathrm{m}}\mathcal{E}(t,u),\ \|\mathsf{w}\|_{u,h} \le f\Big\}, \tag{2.22}$$

it is easy to check that for every $\mathsf{F}(r) \ge \mathcal{F}_h(\mathsf{t}(r), \mathsf{u}(r))$ we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mathsf{t}(r), \mathsf{u}(r)) + \mathcal{P}_h(\mathsf{t}(r), \mathsf{u}(r), \mathsf{F}(r))\dot{\mathsf{t}}(r) \le \mathsf{F}(r)\,|\dot{\mathsf{u}}|_{\mathsf{d}_h}(r). \tag{2.23}$$

Conversely, if a curve $[0,T] \ni t \mapsto u_h$ satisfies the ψ_h energy-dissipation inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) + \mathcal{P}_h(t,u(t),\mathsf{F}_h(t)) \ge \psi_h\big(|\dot u|_{\mathsf{d}_h}(t)\big) + \psi_h^*\big(\mathsf{F}_h(t)\big) \quad \text{for a.a. } t \in (0,T) \tag{2.24}$$

for some $\mathsf{F}_h(t) \ge \mathcal{F}_h(t, u_h(t))$, we get by (2.20) and (2.23)

$$-\langle \mathsf{w}, \dot u_h(t)\rangle - \mathsf{p} + \mathcal{P}_h(t,u(t),\mathsf{F}_h(t)) \ge \psi_h\big(|\dot u|_{\mathsf{d}_h}(t)\big) + \psi_h^*\big(\mathsf{F}_h(t)\big) \quad \text{for every } (\mathsf{p},\mathsf{w}) \in \mathrm{D}^{\mathrm{m}}\mathcal{E}(t,u_h(t)).$$

Choosing in particular a couple $(\bar{\mathsf{p}}, \bar{\mathsf{w}})$ attaining the maximum in (2.22) for $f := \mathsf{F}_h(t)$ we obtain

$$-\langle \bar{\mathsf{w}}, \dot u_h(t)\rangle \ge \psi_h\big(|\dot u|_{\mathsf{d}_h}(t)\big) + \psi_h^*\big(\mathsf{F}_h(t)\big) \ge \psi_h\big(|\dot u|_{\mathsf{d}_h}(t)\big) + \psi_h^*\big(\bar{\mathsf{w}}\big),$$

which eventually yields by (2.9)

$$-\mathrm{D}_{\dot u}\mathcal{R}_h(u_h(t), \dot u_h(t)) = \bar{\mathsf{w}} \in \mathrm{D}^{\mathrm{m}}_u\mathcal{E}(t,u_h(t)), \quad \mathsf{F}_h(t) = \|\bar{\mathsf{w}}\|_{u_h,h},$$

so that u_h solves (2.18).

Towards a general form of chain rule and energy-dissipation inequalities. Notice that we were able to formulate the non-smooth differential inclusion (2.18) in a metric variational form by looking for a chain-rule inequality with the more general structure given by (2.23): this will be reflected in the definition 3.2 of irreversible upper gradients.

The differential inclusion is then characterized by the ψ_h energy-dissipation inequality (2.24): it metric metric formulation will be considered in Definition 3.3 in the superlinear case and in Definition 3.6 in the case of a metric dissipation ψ with linear growth.

It is then natural to investigate the stability of inequality (2.14) with respect to perturbations of the parameter h. One of the most difficult points is to guess how to state (2.15) when the metric dissipation functional ψ has only a linear growth, and therefore one expects a solution in $\mathrm{BV}([0,T];(\mathscr{X},\mathsf{d}))$. We have already discussed in the introduction the case of a piecewise smooth curve, but a robust theory should allow for general BV curves, possibly exhibiting countably many jumps and a metric derivative with a singular Cantor part. The correct treatment of this case will be discussed in the next section.

3. THE METRIC SETTING AND PRELIMINARY RESULTS

Complete extended distances. Let $\mathscr{X}$ be a given set; an *extended distance* on $\mathscr{X}$ is a map d : $\mathscr{X} \times \mathscr{X} \to [0, \infty]$ satisfying

$$\begin{aligned} &\mathsf{d}(x,y) = 0 && \text{if and only if } x = y, \\ &\mathsf{d}(x,y) = \mathsf{d}(y,x) && \text{for every } x, y \in \mathscr{X}, \\ &\mathsf{d}(x,z) \le \mathsf{d}(x,y) + \mathsf{d}(y,z) && \text{for every } x, y, z \in \mathscr{X}. \end{aligned}$$

We say that $(\mathscr{X}, \mathsf{d})$ is an extended metric space. Most of the definitions concerning metric spaces generalize verbatim to extended metric spaces, in particular it makes perfectly sense to speak about a complete extended metric space.

3.1. BV, absolutely continuous curves, and metric derivative.

Let $(\mathscr{X}, \mathsf{d})$ be an extended metric space.

Definition 3.1 (Absolutely continuous curves and metric derivatives)**.** *We say that a curve* $u : [a, b] \to \mathscr{X}$ *is absolutely continuous (a.c. for short) and belongs to* $\mathrm{AC}(a, b; (\mathscr{X}, \mathsf{d}))$ *if there exists* $m \in L^1(a, b)$ *such that*

$$\mathsf{d}(u(s), u(t)) \le \int_s^t m(r)\,\mathrm{d}r \quad \text{for every } a \le s < t \le b. \tag{3.1}$$

If $u \in \mathrm{AC}(a, b; (\mathscr{X}, \mathsf{d}))$ *then the limit*

$$|\dot u|_{\mathsf{d}}(t) := \lim_{\tau \downarrow 0} \frac{\mathsf{d}(u(t+\tau), u(t))}{|\tau|} \quad \text{exists for } \mathscr{L}^1\text{-a.a. } t \in (a, b), \tag{3.2}$$

it satisfies $|\dot u|_{\mathsf{d}} \le m$ $\mathscr{L}^1$*-a.e. in* (a, b)*, belongs to* $L^1(a, b)$*, and it is called* metric derivative of u; $|\dot u|_{\mathsf{d}}$ *provides the minimal function* m *such that* (3.1) *holds.*

The (pointwise) d-variation of $u : [a, b] \to \mathscr{X}$ in an interval $[\alpha, \beta] \subset [a, b]$ is defined by

$$\mathrm{Var}_{\mathsf{d}}(u; [\alpha, \beta]) := \sup\Big\{ \sum_{j=1}^n \mathsf{d}(u(t_j), u(t_{j-1})) : \alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \Big\}. \tag{3.3}$$

We say that $u \in \mathrm{BV}([a, b]; (\mathscr{X}, \mathsf{d}))$ if $\mathrm{Var}_{\mathsf{d}}(u; [a, b]) < \infty$ and u takes values in a complete subset of $(\mathscr{X}, \mathsf{d})$; in this case, u admits left and right limits (denoted by $u(t_-)$ and $u(t_+)$) at every point of $[a, b]$ and we adopt the convention to extend u to $\mathbb{R} \setminus [a, b]$ by setting

$$u(t) := \begin{cases} u(a) & \text{if } t < a, \\ u(b) & \text{if } t > b, \end{cases} \quad \text{so that} \quad u(a_-) := u(a),\ u(b_+) := u(b). \tag{3.4}$$

The *pointwise jump set* of u is

$$\mathrm{J}_u := \big\{ t \in [a, b] : u(t) \ne u(t_-) \text{ or } u(t) \ne u(t_+) \big\} \supset \text{ess-}\mathrm{J}_u := \big\{ t \in [a, b] : u(t_-) \ne u(t_+) \big\}. \tag{3.5}$$

If $u \in \mathrm{BV}([a,b];(\mathscr{X},\mathsf{d}))$ we denote by $\mathrm{V}_u : \mathbb{R} \to [0,\infty)$ the bounded monotone function

$$\mathrm{V}_u(t) := \begin{cases} 0 & \text{if } t < a, \\ \mathrm{Var}_{\mathsf{d}}(u;[a,t]) & \text{if } t \in [a,b], \\ \mathrm{Var}_{\mathsf{d}}(u;[a,b]) & \text{if } t > b, \end{cases} \tag{3.6}$$

and by $|\mathrm{d}u|_{\mathsf{d}}$ its distributional derivative: $|\mathrm{d}u|_{\mathsf{d}}$ is a finite measure in $\mathbb{R}$ supported in $[a,b]$, and we can decompose it as the sum of a diffuse part and a jump part

$$|\mathrm{d}u|_{\mathsf{d}} = |u'|_{\mathsf{d}} + |\mathrm{J}u|_{\mathsf{d}}, \quad |\mathrm{J}u|_{\mathsf{d}} = |\mathrm{d}u|_{\mathsf{d}} \llcorner \mathrm{J}_u, \quad |u'|_{\mathsf{d}}(\{t\}) = 0 \quad \text{for every } t \in \mathbb{R}, \tag{3.7}$$

where $\llcorner$ denotes the restriction of a measure to a Borel set; thus $|\mathrm{J}u|_{\mathsf{d}}$ is concentrated on the (at most) countable jump set J_u and

$$|\mathrm{J}u|_{\mathsf{d}}(\{t\}) = \mathsf{d}(u(t_-),u(t)) + \mathsf{d}(u(t),u(t_+)) \quad \text{for every } y \in \mathrm{J}_u. \tag{3.8}$$

The Lebesgue decomposition of the diffuse part $|u'|_{\mathsf{d}}$ can be written as

$$|u'|_{\mathsf{d}} = |\dot{u}|_{\mathsf{d}} \mathscr{L}^1 + |\mathrm{C}u|_{\mathsf{d}}, \quad \text{with } |\dot{u}|_{\mathsf{d}} \text{ given by (3.2) and the Cantor part } |\mathrm{C}u|_{\mathsf{d}} \perp \mathscr{L}^1. \tag{3.9}$$

We obtain

$$\mathrm{Var}_{\mathsf{d}}(u;[\alpha,\beta]) = \int_\alpha^\beta \mathrm{d}\,|u'|_{\mathsf{d}} + \mathrm{Jmp}_{\mathsf{d}}(u;[\alpha,\beta]) \tag{3.10}$$

$$= \int_\alpha^\beta |\dot{u}|_{\mathsf{d}}(t)\,\mathrm{d}t + \int_\alpha^\beta \mathrm{d}\,|\mathrm{C}u|_{\mathsf{d}} + \mathrm{Jmp}_{\mathsf{d}}(u;[\alpha,\beta]) \tag{3.11}$$

where for every subinterval $[\alpha,\beta] \subset [a,b]$

$$\mathrm{Jmp}_{\mathsf{d}}(u;[\alpha,\beta]) := \mathsf{d}(u(\alpha_-),u(\alpha)) + \sum_{t \in \mathrm{J}_u \cap (a,b)} |\mathrm{J}u|_{\mathsf{d}}(\{t\}) + \mathsf{d}(u(\beta_-),u(\beta)). \tag{3.12}$$

3.2. **Metric evolution systems, irreversible upper gradients and ψ-gradient flows.** Let $(\mathscr{X},\mathsf{d})$ be a complete extended metric space and $[0,T]$ a fixed time interval of $\mathbb{R}$. We denote by $\mathscr{Q}$ the product space $[0,T] \times \mathscr{X}$ and we say that an a.c. curve $\mathsf{q} = (\mathsf{t},\mathsf{u}) : [\alpha,\beta] \to \mathscr{Q}$ is *time-ordered* if t is non decreasing.

If I is some interval of $\mathbb{R}$, $\mathrm{B}_+(I)$ (resp. $\mathrm{M}_+(I)$) will denote the collections of Borel (resp. $\mathscr{L}^1$-measurable) maps defined in I with values in $[0,+\infty]$. We say that a map $G : \mathscr{Q} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is measurable along time-ordered a.c. curves if for every time-ordered a.c. curve q in $\mathscr{Q}$ the composition $G \circ \mathsf{q}$ is Lebesgue measurable. We denote by $\mathrm{M}(\mathscr{Q})$ the collection of all such a functions.

An *evolution system* $(\mathscr{X},\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$ consists of

1 a complete extended metric space $(\mathscr{X},\mathsf{d})$,
2 an energy functional $\mathcal{E} : \mathscr{Q} \to \mathbb{R} \cup \{+\infty\}$ in $\mathrm{M}(\mathscr{Q})$,
3 a slope functional $\mathcal{F} : \mathscr{Q} \to [0,\infty]$ in $\mathrm{M}(\mathscr{Q})$,
4 a power functional $\mathcal{P} : \mathscr{Q} \times [0,\infty] \to \overline{\mathbb{R}}$ such that for every $(q,f) \in \mathscr{Q} \times [0,\infty)$ the map $\mathcal{P}(\cdot,f)$ belongs to $\mathrm{M}(\mathscr{Q})$ and the map $\mathcal{P}(q,\cdot)$ is nondecreasing and upper semicontinuous.

Notice that if $\mathsf{q} = (\mathsf{t},\mathsf{u}) : [\alpha,\beta] \to \mathscr{Q}$ is an time-ordered a.c. curve and $\mathsf{F} \in \mathrm{M}_+([\alpha,\beta])$, the composition $s \mapsto \mathcal{P}(\mathsf{q}(s),\mathsf{F}(s))$ is measurable.

The essential feature of this structure is captured by the following definition:

Definition 3.2 (Irreversible upper gradients for time-dependent functionals)**.** *We say that* $(\mathscr{X},\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$ *is an (irreversible) upper gradient system if for every time-ordered a.c. curve* $[\alpha,\beta] \ni s \mapsto \mathsf{q}(s) = (\mathsf{t}(s),\mathsf{u}(s)) \in \mathscr{Q}$ *and every* $\mathsf{F} \in \mathrm{M}_+([\alpha,\beta])$ *satisfying*

$$\mathcal{E}(\mathsf{t}(\alpha),\mathsf{u}(\alpha)) < \infty, \quad \mathsf{F} \geq \mathcal{F}\circ\mathsf{q} \quad \textit{in } [\alpha,\beta], \quad \int_\alpha^\beta \big[\mathcal{P}(\mathsf{q}(s),\mathsf{F}(s))\big]_-\dot{\mathsf{t}}(s)\,\mathrm{d}s < \infty \tag{3.13}$$

there holds

$$\mathcal{E}(\mathsf{q}(\alpha)) + \int_\alpha^\beta \mathcal{P}(\mathsf{q}(s),\mathsf{F}(s))\dot{\mathsf{t}}(s)\,\mathrm{d}s \leq \mathcal{E}(\mathsf{q}(\beta)) + \int_\alpha^\beta \mathsf{F}(s)\,|\dot{\mathsf{u}}|(s)\,\mathrm{d}s. \tag{3.14}$$

A *metric dissipation function* is a

$$\text{convex function} \quad \psi : [0,\infty) \to [0,\infty) \quad \text{with } \psi(0) = 0, \quad \mathsf{L} = \lim_{v\to+\infty}\frac{\psi(v)}{v} > 0. \tag{3.15}$$

We say that ψ has L-linear growth if $\mathsf{L} < +\infty$ and that ψ is *superlinear* if $\mathsf{L} = +\infty$. Its dual $\psi^* : [0,\infty) \to [0,\infty]$ is defined as

$$\psi^*(f) = \sup_{v\geq 0}\Big(f\,v - \psi(v)\Big), \tag{3.16}$$

and it is a convex and superlinear function with $\psi^*(0) = 0$ as well, satisfying the Fenchel duality

$$\begin{aligned} \psi(v) + \psi^*(f) &\geq f\,v \quad \text{for every } v, f \in [0,\infty); \\ \psi(v) + \psi^*(f) = f\,v \quad &\Leftrightarrow \quad f \in \partial\psi(v) \quad \Leftrightarrow \quad \psi'(v_-) \leq f \leq \psi'(v_+) \end{aligned} \tag{3.17}$$

where $\partial\psi(v) = [\psi'(v_-),\psi'(v_+)]$ denotes the convex subdifferential of ψ. Notice that at $v = 0$ we have

$$\partial\psi(0) = [0,\psi'(0_+)]$$

so that $\partial\psi(0)$ is single valued only when the right derivative of ψ at 0 vanishes.

The proper domain $D(\psi^*) := \{f \in [0,\infty) : \psi^*(f) < \infty\}$ is related to L by the relation

$$\mathsf{L} = \sup\{f : \psi^*(f) < \infty\}, \tag{3.18}$$

so that ψ^* is finite in $[0,\infty)$ if and only if ψ is superlinear. The typical examples are

$$\psi(v) = \frac{1}{p}v^p, \quad \psi^*(f) = \frac{1}{p^*}f^{p^*}, \quad \partial\psi(v) = v^{p-1}; \qquad p > 1, \quad \frac{1}{p} + \frac{1}{p^*} = 1; \tag{3.19}$$

$$\psi(v) = \mathsf{L}v, \quad \psi^*(f) = \begin{cases} 0 & \text{if } f \leq \mathsf{L}, \\ +\infty & \text{if } f > \mathsf{L}. \end{cases} \tag{3.20}$$

The ψ-gradient flows associated with an evolution system can be characterized by a simple family of dissipation inequalities:

Definition 3.3 (Energy-dissipation inequality)**.** *Let* $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ *an evolution system and let* ψ *be a metric dissipation function. A curve* $u \in \mathrm{AC}([0,T]; (\mathscr{X}, \mathsf{d}))$ *with* $\mathcal{E}(0, u(0)) < \infty$ *satisfies the* ψ*-*ψ^* *energy-dissipation inequality if there exists a measurable map* $\mathsf{F} \in \mathrm{M}_+([0,T])$ *satisfying*

$$\mathsf{F}(t) \geq \mathcal{F}(t, u(t)) \quad \text{in } [0,T], \qquad \int_0^T \big[\mathcal{P}(t, u(t), \mathsf{F}(t))\big]_+ \,\mathrm{d}t < \infty, \tag{3.21}$$

and for every $t \in [0,T]$

$$\mathcal{E}(t, u(t)) + \int_0^t \Big(\psi\big(|\dot{u}|_{\mathsf{d}}(r)\big) + \psi^*\big(\mathsf{F}(r)\big)\Big)\,\mathrm{d}r \leq \mathcal{E}(0, u(0)) + \int_0^t \mathcal{P}(r, u(r), \mathsf{F}(r))\,\mathrm{d}r. \tag{3.22}$$

It is immediate to see that if $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ is an *upper gradient* evolution system, then by (3.14) and (3.17) the integral characterization (3.22) is equivalent to the following two properties:

$$t \mapsto \mathcal{E}(t, u(t)) \quad \text{is absolutely continuous in } [0,T], \quad \mathsf{F}(t) \geq \mathcal{F}(t, u(t)) \quad \text{a.a. } t \text{ in } (0,T), \tag{3.23}$$

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) + \mathcal{P}(t, u(t), \mathsf{F}(t)) &= |\dot{u}|_{\mathsf{d}}(t)\,\mathsf{F}(t) \\ &= \psi\big(|\dot{u}|_{\mathsf{d}}(t)\big) + \psi^*\big(\mathsf{F}(t)\big) \end{aligned} \qquad \text{for a.e. } t \in (0,T). \tag{3.24}$$

Notice that (3.24) and (3.17) yields the velocity-slope relation

$$\mathsf{F}(t) \in \partial\psi\big(|\dot{u}|_{\mathsf{d}}(t)\big) \quad \text{for a.a. } t \in (0,T), \tag{3.25}$$

and, by (3.14), $\mathsf{F}(t)$ realizes the minimal selection property

$$|\dot{u}|_{\mathsf{d}}(t)\,\mathsf{F}(t) \; - \mathcal{P}(t, u(t), \mathsf{F}(t)) = \min\Big\{f \geq \mathcal{F}(t, u(t)) : |\dot{u}|_{\mathsf{d}}(t)\,f - \mathcal{P}(t, u(t), f)\Big\} \tag{3.26}$$

for a.a. $t \in (0,T)$. In particular, (3.26) yields

$$\mathsf{F}(t) = \mathcal{F}(t, u(t)) \quad |\dot{u}|_{\mathsf{d}}\,\mathscr{L}^1\text{-a.e. in } (0,T) \quad \text{when } \mathcal{P} \text{ is independent of } \mathsf{F}, \tag{3.27}$$

and (3.27) holds $\mathscr{L}^1$-a.e. when $\psi'(0+) = 0$.

Definition 3.4 (ψ-gradient flows)**.** *Let* $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ *be an upper gradient evolution system, and let* ψ *a metric dissipation function as in* (3.15)*. A curve* $u \in \mathrm{AC}(a,b; (\mathscr{X}, \mathsf{d}))$ *is a* ψ*-gradient flow of the system if it satisfies* (3.21) *and* (3.22) *at* $t = T$*, or, equivalently,* (3.23) *and* (3.24)*.*

3.3. **BV solutions to evolution systems.** Let us now consider the case of a dissipation potential ψ with linear growth, corresponding to $\mathsf{L} < \infty$ in (3.15). In this case, absolutely continuous solutions to (3.23), (3.24) often do not exist, even in the smooth and finite-dimensional setting of Section and therefore we have to extend the previous definitions to the BV setting.

As before, we fix the time interval $[0,T]$ and we denote by $\mathscr{Q}$ the product space $[0,T] \times \mathscr{X}$ and we consider a function $\mathfrak{f} : \mathscr{Q} \to [0,\infty]$ measurable along absolutely continuous curves. Relevant examples will be $\mathfrak{f} := \mathcal{F}$ and

$$\mathfrak{f}(q) := \mathcal{F}(q) \vee \mathsf{L} \quad \text{for every } q \in \mathscr{Q}. \tag{3.28}$$

We interpret $\mathfrak{f}(t,\cdot)$ as a conformal factor that induces a modified geometry in $\mathscr{X}$: the corresponding length of a curve $\vartheta \in \mathrm{AC}(r_0, r_1; (\mathscr{X}, \mathsf{d}))$ (notice that the curve is parametrized by a different variable r, and t remains fixed) is

$$\mathrm{Length}_{\mathsf{d};\mathfrak{f},t}[\vartheta] := \int_{r_0}^{r_1} \mathfrak{f}(t,\vartheta(r))\, |\dot\vartheta(r)|_{\mathsf{d}}\,\mathrm{d}r \tag{3.29}$$

and the cost of a transition from u_0 to u_1 in $\mathscr{X}$ at the time $t \in [0,T]$ is then defined by

$$\Delta_{\mathsf{d};\mathfrak{f},t}(u_0,u_1) = \inf\Big\{\mathrm{Length}_{\mathsf{d},\mathfrak{f},t}[\vartheta] : \vartheta \in \mathrm{AC}(r_0,r_1;(\mathscr{X},\mathsf{d})),\ \vartheta(r_i)=u_i\Big\}. \tag{3.30}$$

We also set
(3.31)

$$\begin{aligned}&\Delta_{\mathsf{d},\mathfrak{f},t}(u_0,u,u_1) := \Delta_{\mathsf{d},\mathfrak{f},t}(u_0,u) + \Delta_{\mathsf{d},\mathfrak{f},t}(u,u_1)\\ &= \inf\Big\{\mathrm{Length}_{\mathsf{d},\mathfrak{f},t}[\vartheta] : \vartheta \in \mathrm{AC}(r_0,r_1;(\mathscr{X},\mathsf{d})),\ \vartheta(r_i)=u_i,\ \vartheta(r)=u \text{ for some } r\in[r_0,r_1]\Big\}.\end{aligned}$$

We can thus consider a modified Jump functional
(3.32)

$$\mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[\alpha,\beta]) = \Delta_{\mathsf{d},\mathfrak{f},\alpha}(u(\alpha),u(\alpha_+)) + \sum_{t\in \mathrm{J}_u}\Delta_{\mathsf{d},\mathfrak{f},t}(u(t_-),u(t),u(t_+)) + \Delta_{\mathsf{d},\mathfrak{f},\beta}(u(\beta_-),u(\beta)).$$

The previous quantities will be quite useful to extend the chain-rule inequality (3.14) to the BV setting. Notice that we are assuming that F is a Borel map (instead of Lebesgue measurable as in Definition 3.2), since an integration with respect to the possibly singular measure $|u'|_{\mathsf{d}}$ occurs in (3.34).

Proposition 3.5. *Let $(\mathscr{X},\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$ be an upper gradient evolution system, $\mathfrak{f} := \mathcal{F}\vee \mathsf{L}$ for some $\mathsf{L}>0$, and let $u \in \mathrm{BV}([0,T];(\mathscr{X},\mathsf{d}))$ satisfy*

$$\mathcal{E}(0,u(0)) < \infty, \quad \int_0^T (\mathcal{P}(t,u(t),\mathsf{F}(t)))_-\,\mathrm{d}t < \infty, \quad \mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[0,T]) < \infty, \tag{3.33}$$

for some Borel map $\mathsf{F} \in \mathrm{B}_+([0,T])$ with $\mathsf{F}(t) \ge \mathcal{F}(t,u(t))$ in $[0,T]$. Then

$$\mathcal{E}(0,u(0)) + \int_0^t \mathcal{P}(r,u(r),\mathsf{F}(r))\,\mathrm{d}r \le \mathcal{E}(t,u(t)) + \int_0^t \mathsf{F}(r)\,\mathrm{d}|u'|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathcal{F}}(u;[0,t]). \tag{3.34}$$

Proof. It is not restrictive to assume $t=T$. Let us denote by $(t_n)_n$ the jump set J_u of u and let us first set

$$\begin{gathered}\mathsf{s}(t) := t + \mathrm{V}_u(t), \quad \mathsf{S} := T + \mathrm{V}_u(T), \quad I_n := (\mathsf{s}(t_{n-}),\mathsf{s}(t_{n+})), \quad I := \cup_n I_n,\\ D := [0,\mathsf{S}]\setminus I, \quad \mathsf{t} := \mathsf{s}^{-1} : D \to [0,T], \quad \mathsf{u} := u\circ \mathsf{t} : D \to \mathscr{X}.\end{gathered} \tag{3.35}$$

Since $\mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[0,T]) < \infty$ it is not difficult to check that t,u are Lipschitz maps (if we only know $\mathit{Jmp}_{\mathsf{d},\mathcal{F}}(u;[0,T]) < \infty$, it would not clear how to derive a uniform upper bound on the total variation of the functionu). We easily extend t to $[0,\mathsf{S}]$ by setting

$$\mathsf{t}(s) \equiv t_n \quad \text{if } s \in I_n. \tag{3.36}$$

In order to extend u, we fix $\varepsilon > 0$ and for every interval I_n we consider two curves $\vartheta_n, \zeta_n : [\mathsf{s}(t_{n-}), \mathsf{s}(t_{n+})] \to \mathscr{X}$ satisfying $\vartheta_n(\mathsf{s}(t_{n\pm})) = \zeta_n(\mathsf{s}(t_{n\pm})) = u(t_{n\pm})$, taking the value $u(t_n)$ at some point in I_n, and fulfilling

$$\begin{aligned}\int_{I_n} \mathcal{F}(t_n, \vartheta_n(s))|\dot\vartheta_n|_{\mathsf{d}}(s)\,\mathrm{d}s &\le \Delta_{\mathsf{d},\mathcal{F},t_n}(u(t_{n-}), u(t_n), u(t_{n+})) + \varepsilon 2^{-n},\\ \int_{I_n} \mathfrak{f}(t_n, \zeta_n(s))|\dot\zeta_n|_{\mathsf{d}}(s)\,\mathrm{d}s &\le \Delta_{\mathsf{d},\mathfrak{f},t_n}(u(t_{n-}), u(t_n), u(t_{n+})) + \varepsilon 2^{-n}.\end{aligned} \tag{3.37}$$

For $N \in \mathbb{N}$ we define

$$\mathsf{u}_N(s) := \begin{cases} u(s) & \text{if } s \in [0, \mathsf{S}] \setminus I,\\ \vartheta_n(s) & \text{if } s \in I_n,\ n \le N,\\ \zeta_n(s) & \text{if } s \in I_n,\ n > N.\end{cases} \tag{3.38}$$

It is not difficult to check that u_N is absolutely continuous, so that (3.14) yields (see [23, Lemma 4.1])

$$\begin{aligned}\mathcal{E}(0, u(0)) &+ \int_0^T \mathcal{P}(t, u(t), \mathsf{F}(t))\,\mathrm{d}t = \mathcal{E}(\mathsf{t}(0), \mathsf{u}_N(0)) + \int_0^{\mathsf{S}} \mathcal{P}(\mathsf{t}(s), \mathsf{u}(s), \mathsf{F}(\mathsf{t}(s)))\,\dot{\mathsf{t}}(s)\,\mathrm{d}s\\ &\le \mathcal{E}(\mathsf{t}(\mathsf{S}), \mathsf{u}_N(\mathsf{S})) + \int_0^{\mathsf{S}} \mathsf{F}(\mathsf{t}(s))|\dot{\mathsf{u}}_N|_{\mathsf{d}}(s)\,\mathrm{d}s\\ &= \mathcal{E}(T, u(T)) + \int_D \mathsf{F}(\mathsf{t}(s))|\dot{\mathsf{u}}_N|_{\mathsf{d}}(s)\,\mathrm{d}s\\ &\qquad + \sum_{n=1}^N \int_{I_n} \mathcal{F}(t_n, \vartheta_n(s))|\dot\vartheta_n|_{\mathsf{d}}(s)\,\mathrm{d}s + \sum_{n>N} \int_{I_n} \mathfrak{f}(t_n, \zeta_n(s))|\dot\zeta_n|_{\mathsf{d}}(s)\,\mathrm{d}s\\ &\le \mathcal{E}(T, u(T)) + \int_0^T \mathsf{F}(t)\,\mathrm{d}|u'|\\ &\qquad + \varepsilon + \sum_{n=1}^N \Delta_{\mathsf{d},\mathcal{F},t_n}(u(t_{n-}), u(t_n), u(t_{n+})) + \sum_{n>N} \Delta_{\mathsf{d},\mathfrak{f},t_n}(u(t_{n-}), u(t_n), u(t_{n+})).\end{aligned}$$

Passing first to the limit as $N \uparrow \infty$ (notice that the last term vanishes as $N \uparrow \infty$ since *Jmp*$_{\mathsf{d},\mathfrak{f}}(u; [0, T])$ is finite) and then as $\varepsilon \downarrow 0$ we obtain (3.34). □

Definition 3.6 (Energy-dissipation inequality for BV functions)**.** *Let* $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ *be an evolution system in the time interval* $[0, T]$*, let* ψ *be a metric dissipation function with* L*-linear growth, and let* $\mathfrak{f} := \mathcal{F} \vee \mathsf{L}$*. A curve* $u \in \mathrm{BV}([0, T]; (\mathscr{X}, \mathsf{d}))$ *with* $\mathcal{E}(0, u(0)) < \infty$ *satisfies the* ψ*-*ψ^* *energy dissipation inequality if there exists a Borel map* $\mathsf{F} \in \mathrm{M}_+([0, T])$ *satisfying* (3.21)*,*

$$\mathsf{F}(t) \ge \mathcal{F}(t, u(t)) \quad \textit{in } [0, T], \qquad \int_0^T \big[\mathcal{P}(t, u(t), \mathsf{F}(t))\big]_+\,\mathrm{d}t < \infty, \tag{3.39}$$

the stability condition on the Cantor part

$$\mathsf{F}(t) \le \mathsf{L} \quad \textit{for } |\mathrm{C}u|_{\mathsf{d}}\textit{-a.a. } t \in [0, T] \tag{3.40}$$

and

$$
\text{(ED)} \qquad \begin{aligned} &\mathcal{E}(t,u(t)) + \int_0^t \Big(\psi\big(|\dot u|_{\mathsf{d}}(r)\big) + \psi^*\big(\mathsf{F}(r)\big)\Big)\,\mathrm{d}r + \mathsf{L}\int_0^t \mathrm{d}|\mathrm{C}u|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[0,t]) \\ &\qquad \le \mathcal{E}(0,u(0)) + \int_0^t \mathcal{P}(r,u(r),\mathsf{F}(r))\,\mathrm{d}r \quad \textit{for every } t\in[0,T]. \end{aligned}
$$

Since $\psi^*(f)=\infty$ if $f>\mathsf{L}$, (ED) yields in fact a stronger version of the local stability condition

$$
(\mathrm{S_{loc}}) \qquad \mathsf{F}(t)\le \mathsf{L} \quad \text{for } (\mathscr{L}^1+|\mathrm{C}u|_{\mathsf{d}})\text{-a.a. } t\in[0,T].
$$

In the rate-independent case $\psi(v)=\mathsf{L}v$, (3.40) and (ED) are thus equivalent to $(\mathrm{S_{loc}})$ and

$$
\text{(EDRI)} \qquad \mathcal{E}(t,u(t)) + \mathsf{L}\int_0^t \mathrm{d}\,|u'|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[0,t]) \le \mathcal{E}(0,u(0)) + \int_0^t \mathcal{P}(r,u(r),\mathsf{F}(r))\,\mathrm{d}r
$$

for every $t\in[0,T]$.

Definition 3.7 (BV solutions to evolution systems and rate-independent flows)**.** *Let* $(\mathscr{X},\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$ *be an upper gradient system in the time interval* $[0,T]$*, let* ψ *be a metric dissipation function with* L*-linear growth, and let* $\mathfrak{f}:=\mathcal{F}\vee\mathsf{L}$*.*
A curve $u\in\mathrm{BV}([0,T];(\mathscr{X},\mathsf{d}))$ *with* $\mathcal{E}(0,u(0))<\infty$ *is a* BV solution *of the corresponding evolution if there exists* $\mathsf{F}\in\mathrm{B}_+([0,T])$ *satisfying* (3.39)*, the* local stability condition $(\mathrm{S_{loc}})$ *and the* energy balance

$$
\text{(EB)} \qquad \begin{aligned} &\mathcal{E}(t,u(t_2)) + \int_{t_1}^{t_2} \Big(\psi\big(|\dot u|_{\mathsf{d}}(r)\big) + \psi^*\big(\mathsf{F}(r)\big)\Big)\,\mathrm{d}r + \mathsf{L}\int_{t_1}^{t_2} \mathrm{d}|\mathrm{C}u|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[t_1,t_2]) \\ &\qquad = \mathcal{E}(t_1,u(t_1)) + \int_{t_1}^{t_2} \mathcal{P}(r,u(r),\mathsf{F}(r))\,\mathrm{d}r \qquad \textit{for every } [t_1,t_2]\subset[0,T], \end{aligned}
$$

Notice that (EB) holds if and only if the curve $t\mapsto e(t):=\mathcal{E}(t,u(t))$ (extended to $\mathbb{R}$ as in (3.4)) is of bounded variation, $\mathrm{J}_e=\mathrm{J}_u$, and its distributional time derivative $\frac{\mathrm{d}}{\mathrm{d}t}e$ satisfies

$$
-\frac{\mathrm{d}}{\mathrm{d}t}e + \mathcal{P}(\cdot,u,\mathsf{F}) = \Big(\psi\big(|\dot u|_{\mathsf{d}}\big)+\psi^*\big(\mathsf{F}\big)\Big)\mathscr{L}^1 + \mathsf{L}|\mathrm{C}u|_{\mathsf{d}} - \mathrm{J}e \quad \text{in } \mathbb{R}, \tag{3.41}
$$

where at each jump point $t\in\mathrm{J}_u$ we have $e(t_\pm)=\mathcal{E}(t,u(t_\pm))$ and the jump part $\mathrm{J}e$ is
(3.42)

$$
-\mathrm{J}e(\{t\}) = \mathcal{E}(t,u(t_-)) - \mathcal{E}(t,u(t_+)), \qquad \begin{cases} \mathcal{E}(t,u(t_-)) - \mathcal{E}(t,u(t)) &= \Delta_{\mathsf{d},\mathfrak{f}}(u(t_-),u(t)), \\ \mathcal{E}(t,u(t)) - \mathcal{E}(t,u(t_+)) &= \Delta_{\mathsf{d},\mathfrak{f}}(u(t),u(t_+)). \end{cases}
$$

As for gradient flows, thanks to Proposition 3.5 it is immediate to see that whenever (3.40) holds the energy balance (EB) is equivalent to the energy-dissipation inequality (ED) at the final point $t=T$. Moreover, a BV solution u satisfies

$$
\mathsf{F}(t)\in\partial\psi\big(|\dot u|_{\mathsf{d}}(t)\big) \qquad \text{for } \mathscr{L}^1\text{-a.a. } t\in(0,T), \tag{3.43}
$$

$$
\mathsf{F}(t)=\mathcal{F}(t,u(t))=\mathsf{L} \qquad \text{for } |\mathrm{C}u|_{\mathsf{d}}\text{-a.a. } t\in(0,T), \tag{3.44}
$$

and the minimal selection principle

$$|\dot{u}|_{\mathsf{d}}(t)\,\mathsf{F}(t) - \mathcal{P}(t,u(t),\mathsf{F}(t)) = \min\Big\{f \geq \mathcal{F}(t,u(t)) : |\dot{u}|_{\mathsf{d}}(t)\,f - \mathcal{P}(t,u(t),f)\Big\} \tag{3.45}$$

for $\mathscr{L}^1$-a.a. $t \in (0,T)$. In the rate-independent case $\psi(v) = \mathsf{L}v$, a BV solution is equivalently characterized by the local stability $(\mathrm{S_{loc}})$ and the energy balance

$$\mathcal{E}(t_2,u(t_2)) + \mathsf{L}\int_{t_1}^{t_2} \mathrm{d}\,|u'|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathfrak{f}}(u;[t_1,t_2]) = \mathcal{E}(t_1,u(t_1)) + \int_{t_1}^{t_2} \mathcal{P}(r,u(r),\mathsf{F}(r))\,\mathrm{d}r \tag{EBRI}$$

for every $0 \leq t_1 \leq t_2 \leq T$.

4. Compactness and convergence for families of Gradient Flows

In this section we will state and prove our main results. For the sake of clarity, we distinguish between the compactness (Theorem 4.1) and the stability (Theorem 4.2) issues.

We also take care to highlight the role of the energy-dissipation inequality for general metric-evolution systems $(\mathscr{X},\mathsf{d}_h,\mathcal{E}_h,\mathcal{F}_h,\mathcal{P}_h)$, even if they do not satisfy the irreversible upper gradient condition 3.2. Therefore, compactness and stability of the energy-dissipation inequality always hold whenever suitable topological properties (see the next (C1,2,3,4) assumptions) are satisfied.

In order to recover a ψ-gradient flow or a BV solution in the limit, we will ask that $(X,\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$ is an upper gradient system.

We also notice that our theorems can also be extremely useful to prove existence results for solutions to the limit evolution system: in this case one could think that $(\mathscr{X},\mathsf{d}_h,\mathcal{E}_h,\mathcal{F}_h,\mathcal{P}_h)$ is a family of suitably regularized problems (e.g. with smooth superlinear dissipations and better energies) for which existence is already known (see [16]).

Let $(\mathscr{X},\sigma)$ be a topological space, and let $\mathscr{Q} = [0,T]\times\mathscr{X}$ with the standard product topology, which we will denote by π. We consider a family of evolution systems $(\mathscr{X},\mathsf{d}_h,\mathcal{E}_h,\mathcal{F}_h,\mathcal{P}_h)$ in the time interval $[0,T]$ indexed by the parameter $h \in \mathbb{N}$, a sequence $\bar{u}_h$ of initial points, and metric dissipation functions ψ_h,ψ such that

$$\lim_{h\to\infty} \psi_h(v) = \psi(v) \quad \text{for every } v \in [0,\infty), \tag{4.1}$$

Since each function ψ_h is monotone, (4.1) is equivalent to

$$\Gamma\text{-}\lim_{h\to\infty} \psi_h(v) = \psi(v) \quad \text{for every } v \in [0,\infty), \tag{4.2}$$

and also to the following property, valid for arbitrary sequences $(w_h)_h \subset [0,\infty)$:

$$w_h \to w \quad \Rightarrow \quad \liminf_{h\to\infty} \psi_h(w_h) \geq \psi(w), \quad \liminf_{h\to\infty} \psi_h^*(w_h) \geq \psi^*(w) \tag{4.3}$$

Typical examples are given in (1.1) and (1.3). We want to study the limit of absolutely continuous ψ_h-gradient flows $u_h \in \mathrm{AC}(0,T;(\mathscr{X},\mathsf{d}_h))$ of the systems $(\mathscr{X},\mathsf{d}_h,\mathcal{E}_h,\mathcal{F}_h,\mathcal{P}_h)$ with $u_h(0) = \bar{u}_h$ as $h \to \infty$, assuming that they "converge" (in a variational sense that we are going to make precise) to a limit system $(\mathscr{X},\mathsf{d},\mathcal{E},\mathcal{F},\mathcal{P})$. The most interesting case is when ψ has L-linear growth, so that we expect a function of bounded variation in the limit.

Here and in the following we identify diverging subsequences in $\mathbb{N}$ with subsets $H \subset \mathbb{N}$ with $\sup H = \infty$ and we write $\lim_{h\in H}$ for $\lim_{h\to\infty, h\in H}$.

We will assume that:

(C1) There exist constants $\mathsf{a} < \mathsf{L}, \mathsf{b} \geq 0$ such that

$$\tilde{\mathcal{E}}_h(t,u) := \mathcal{E}_h(t,u) + \mathsf{a}\mathsf{d}_h(u,\bar{u}_h) + \mathsf{b} \geq 0 \quad \text{for every } (t,u) \in \mathscr{Q}. \tag{4.4}$$

(C2) The energies $(\tilde{\mathcal{E}}_h)_{h\in\mathbb{N}}$ are equi-coercive in $\mathscr{Q}$: for every sequence $(q_h)_{h\in H} \subset \mathscr{Q}$ with $\sup_{h\in H} \tilde{\mathcal{E}}_h(q_h) < \infty$ there exists a subsequence $H' \subset H$ such that $\lim_{h\in H'} q_h = q$ in the π-topology.

(C3) If two sequences $q_h^i = (t_h^i, x_h^i) \subset \mathscr{Q}$, $h \in H$, $i = 1,2$, satisfy $\sup_{h\in H} \tilde{\mathcal{E}}_h(q_h^i) < \infty$ and π-converge to q^i, then we have

$$\liminf_{h\in H} \mathsf{d}_h(x_h^1, x_h^2) \geq \mathsf{d}(x^1, x^2) \tag{4.5}$$

where $\mathsf{d}(\cdot,\cdot)$ is a complete extended distance on $\mathscr{X}$.

Theorem 4.1 (A priori bounds and compactness)**.** *Let us suppose that* (C1) *holds and let* $u_h \in \mathrm{AC}(0,T;(\mathscr{X},\mathsf{d}_h))$, $\mathsf{F}_h \in \mathrm{B}_+([0,T])$ *be sequence of curves satisfying* $u_h(0) = \bar{u}_h$ *and the corresponding energy-dissipation inequalities* (3.21) *and* (3.22) *for every* $h \in \mathbb{N}$*, namely*

$$\mathsf{F}_h(t) \geq \mathcal{F}_h(t,u(t)) \quad \textit{in } [0,T], \qquad \int_0^T \big[\mathcal{P}_h(t,u(t),\mathsf{F}(t))\big]_+ \,\mathrm{d}t < \infty, \tag{4.6}$$

and

$$\mathcal{E}_h(t,u_h(t)) + \int_0^t \Big(\psi_h\big(|\dot{u}_h|_{\mathsf{d}_h}(r)\big) + \psi_h^*\big(\mathsf{F}_h(r)\big)\Big)\,\mathrm{d}r \leq \mathcal{E}_h(0,\bar{u}_h) + \int_0^t \mathcal{P}_h(r,u_h(r),\mathsf{F}_h(r))\,\mathrm{d}r \tag{4.7}$$

for every $t \in [0,T]$*. If there exists a constant* $A \geq 0$ *such that*

$$\mathcal{E}_h(0,\bar{u}_h) + \int_0^t \mathcal{P}_h\big(r,u_h(r),\mathsf{F}_h(r)\big)\,\mathrm{d}r \leq A \quad \textit{for every } h \in \mathbb{N},\ t \in [0,T], \tag{4.8}$$

then there exists a constant $C > 0$ *such that for every* $t \in [0,T]$ *and* $h \in \mathbb{N}$

$$\mathcal{E}_h(t,u_h(t)) \leq C, \quad \int_0^T \Big(\psi_h\big(|\dot{u}_h|_{\mathsf{d}_h}(r)\big)\,\mathrm{d}r + \psi_h^*\big(\mathsf{F}_h(r)\big)\Big)\,\mathrm{d}r \leq C, \tag{4.9}$$

so that

$$\lim_{h\to\infty} \mathscr{L}^1\{t \in (0,T) : \mathsf{F}_h(t) \geq f\} = 0 \quad \textit{for every } f > \mathsf{L}. \tag{4.10}$$

If moreover (C2,3) *hold, then for every subsequence* $H \subset \mathbb{N}$ *there exists a further subsequence* $H' \subset H$ *such that*

$$\lim_{h\in H'} u_h(t) = u(t) \quad \textit{in } (\mathscr{X},\sigma) \textit{ for every } t \in [0,T], \quad \textit{with} \quad u \in \mathrm{BV}([0,T];(\mathscr{X},\mathsf{d})), \tag{4.11}$$

$$\lim_{h\in H'} u_h(t_h) = u(t) \quad \textit{for every } (t_h)_h \subset [0,T] \textit{ converging to } t \in [0,T] \setminus \mathrm{J}_u, \tag{4.12}$$

and, for every interval $[t_1, t_2] \subset [0, T]$,

$$\liminf_{h\in H'} \int_{t_1}^{t_2} \psi_h\big(|\dot u_h|_{\mathsf{d}_h}(r)\big)\,\mathrm{d}r \ge \int_{t_1}^{t_2} \psi\big(|\dot u|_{\mathsf{d}}(r)\big)\,\mathrm{d}r + \mathsf{L}|\mathrm{C}u|_{\mathsf{d}}([\alpha,\beta]) \tag{4.13}$$

Proof. Let us first prove (4.9) and (4.10).

First of all we show that $\mathsf{d}_h(u_h(t), \bar u_h)$ is uniformly bounded, we choose $\bar{\mathsf{a}} \in (\mathsf{a}, \mathsf{L})$, and we observe that (4.3), the monotonicity of ψ_h, and the continuity of ψ^* in $[0, \mathsf{L})$ yield $\lim_{h\to\infty} \psi_h^*(\bar{\mathsf{a}}) = \psi^*(\bar{\mathsf{a}}) < \infty$ so that $\mathsf{c} := \sup_h \psi_h^*(\bar{\mathsf{a}}) < \infty$. It follows that

$$\psi_h(v) \ge \bar{\mathsf{a}}\, v - \mathsf{c} \quad \text{for every } v \ge 0,\ h \in \mathbb{N},$$

and therefore (4.7) and (4.4) yield

$$\begin{aligned}\bar{\mathsf{a}}\,\mathsf{d}_h(u_h(t), \bar u_h) &\le \bar{\mathsf{a}} \int_0^t |\dot u_h|_{\mathsf{d}_h}(r)\,\mathrm{d}r \le \int_0^t \psi_h\big(|\dot u_h|_{\mathsf{d}_h}(r)\big)\,\mathrm{d}r + \mathsf{c} \le A - \mathcal{E}_h(t, u_h(t)) + \mathsf{c}\\ &\le A + \mathsf{b} + \mathsf{a}\mathsf{d}_h(u_h(t), \bar u_h) + \mathsf{c}\end{aligned}$$

so that

$$\mathsf{d}_h(u_h(t), \bar u_h) \le (\bar{\mathsf{a}} - \mathsf{a})^{-1}\big(A + \mathsf{b} + \mathsf{c}\big) \quad \text{for every } t \in [0, T],\ h \in \mathbb{N}. \tag{4.14}$$

Combining (C1), (4.8) and (4.14) we conclude that there exists a constant $B \ge 0$ such that

$$-B \le \mathcal{E}_h(t, u_h(t)) \le A \quad \text{for every } t \in [0, T],\ h \in \mathbb{N}. \tag{4.15}$$

Therefore (4.15) and (4.7) yield

$$\int_0^T \psi_h\big(|\dot u_h|_{\mathsf{d}_h}(t)\big)\,\mathrm{d}t \le A + B, \quad \int_0^T \psi_h^*\big(\mathsf{F}_h(t)\big)\,\mathrm{d}t \le A + B. \tag{4.16}$$

We eventually obtain (4.10), since by the monotonicity of ψ_h^* we get from the second of (4.16)

$$\psi_h^*(f)\,\mathscr{L}^1\{t \in (0, T) : \mathcal{F}_h(t, u_h(t)) \ge f\} \le A + B \quad \text{for every } f \ge 0,$$

and $\lim_{h\to\infty} \psi_h^*(f) = \infty$ when $f > \mathsf{L}$ by (4.3) and (3.18).

The proof of (4.11) and (4.12) can be easily obtained by adapting the argument of the extended Ascoli-Arzelà-Helly type result [1, Proposition 3.3.1].

By (C2) and the bound (4.9), for every $t \in [0, T]$ the sequence $(u_h(t))_{h\in H}$ admits a σ-converging subsequence (possibly depending on t). For every $f \in [0, \mathsf{L})$ and $0 \le t_0 < t_1 \le T$ we recall the bound

$$f\mathsf{d}(u_h(t_1), u_h(t_0)) \le \int_{t_0}^{t_1} f|\dot u_h|_{\mathsf{d}_h}\,\mathrm{d}t \le \int_{t_0}^{t_1} \Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}\big) + \psi_h^*(f)\Big)\,\mathrm{d}t. \tag{4.17}$$

We consider the nonnegative finite measures on $[0, T]$

$$\nu_{h,f} := \Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}\big) + \psi_h^*(f)\Big)\mathscr{L}^1, \qquad f \in [0, \mathsf{L}) \tag{4.18}$$

on $[0,T]$; up to extracting a suitable subsequence, we can suppose that they weakly* converge to a finite measure $\nu_f = \nu_0 + \psi^*(f)\mathscr{L}^1$ in the duality with continuous function on $[0,T]$, so that

$$f \limsup_{h\to\infty} \mathsf{d}_h(u_h(t_1),u_h(t_2)) \le \limsup_{h\to\infty} \nu_{h,f}([t_1,t_2]) \le \nu_f([t_1,t_2]) \tag{4.19}$$

for every $0 \le t_1 \le t_2 \le T$. Denoting by $J := \{t \in [0,T] : \nu_0(\{t\}) > 0\}$ and considering a countable set $I \supset J$ dense in $[0,T]$, by a standard diagonal argument we can find a subsequence $H' \subset H$ such that $u_h(t) \xrightarrow{\sigma} u(t)$ for every $t \in I$ as $h \to \infty$, $h \in H'$. By (C3) we have

$$f\,\mathsf{d}(u(t_1),u(t_2)) \le \nu_f([t_1,t_2]) \quad \text{for every } t_1,t_2 \in I. \tag{4.20}$$

Since $(\mathscr{X},\mathsf{d})$ is complete, the curve $I \ni t \mapsto u(t)$ can be uniquely extended to a continuous curve in $[0,T]\setminus J$, which we still denote by u. In order to prove (4.12) we argue by contradiction and we find a sequence $H'' \subset H'$, points $t_h \to t \in [0,T]\setminus J$ and a σ-neighborhood U of $u(t)$ such that $(u_h(t_h)) \notin U$ for every $h \in H''$. Up to extracting a further subsequence (still denoted by H'') we can assume that $u_h(t_h) \xrightarrow{\sigma} \tilde u \ne u(t)$ so that by (C3)

$$f\mathsf{d}(u(t),\tilde u) \le \liminf_{h\in H''} f\mathsf{d}_h(u_h(t),u_h(t_h)) \le \limsup_{h\in H''} \nu_{h,f}([t,t_h]) = \nu_f(\{t\}) = 0.$$

This yields in particular that $u_h(t)$ converges pointwise to $u(t)$ as $h \to \infty$, $h \in H'$; (4.20) then holds for every $t_1,t_2 \in [0,T]$ and shows that $u \in \mathrm{BV}([0,T];(\mathscr{X},\mathsf{d}))$.

Since for an arbitrary subdivision $t_0 = \alpha < t_1 < \cdots < t_{n-1} < t_n = \beta$ there holds

$$\sum_{i=1}^n \mathsf{d}_h(u_h(t_i),u_h(t_{i-1})) \le \int_\alpha^\beta |\dot u_h|_{\mathsf{d}_h}(r)\,\mathrm{d}r \tag{4.21}$$

it is easy to check that

$$f\sum_{i=1}^n \mathsf{d}(u(t_i),u(t_{i-1})) \le \limsup_{h\in H''} \int_\alpha^\beta |\dot u_h|_{\mathsf{d}_h}(r)\,\mathrm{d}r \le \nu_0([\alpha,\beta]) + \psi^*(f)(\beta-\alpha), \tag{4.22}$$

so that

$$f\mathrm{Var}_{\mathsf{d}}(u;[\alpha,\beta]) \le \nu_0([\alpha,\beta]) + \psi^*(f)(\beta-\alpha). \tag{4.23}$$

Therefore, the duality formula $\psi(v) = \sup_{0<f<\mathsf{L}} \big(fv - \psi^*(f)\big)$ yields

$$\psi\Big((\beta-\alpha)^{-1}\mathrm{Var}_{\mathsf{d}}(u;[\alpha,\beta])\Big) \le (\beta-\alpha)^{-1}\nu_0([\alpha,\beta]). \tag{4.24}$$

From (4.23) we immediately get

$$f\big(|\mathrm{C}u|_{\mathsf{d}} + |\mathrm{J}u|_{\mathsf{d}}\big) \le \nu_0 + \psi^*(f)\mathscr{L}^1 \quad \text{for every } f < \mathsf{L}. \tag{4.25}$$

Since $|\mathrm{C}u|_{\mathsf{d}}$ and $|\mathrm{J}u|_{\mathsf{d}}$ are concentrated in a $\mathscr{L}^1$-negligible set, we deduce

$$\mathsf{L}\big(|\mathrm{C}u|_{\mathsf{d}} + |\mathrm{J}u|_{\mathsf{d}}\big) \le \nu_0, \tag{4.26}$$

When ψ is superlinear we conclude that u is absolutely continuous, since in this case $\mathsf{L} = \infty$ and (4.26) yields $|\mathrm{C}u|_{=}0, |\mathrm{J}u|_{=}0$.

Let us eventually prove (4.13).
(4.24) and the monotonicity of ψ yield, for $\alpha = t$ and $\beta = t + \varepsilon$

$$\psi\Big(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |\dot u|_{\mathsf d}(r)\,\mathrm{d}r\Big) \le \varepsilon^{-1}\,\nu_0([t, t+\varepsilon]). \tag{4.27}$$

Integrating this inequality from t_0 to $t_1 - \varepsilon$ with respect to t we obtain

$$\begin{aligned}\int_{t_0}^{t_1-\varepsilon} \psi\Big(\varepsilon^{-1}\int_t^{t+\varepsilon} |\dot u|_{\mathsf d}(r)\,\mathrm{d}r\Big)\,\mathrm{d}t &\le \varepsilon^{-1}\int_{t_0}^{t_1-\varepsilon} \nu_0([t, t+\varepsilon])\,\mathrm{d}t\\ &\le \varepsilon^{-1}(\mathscr{L}^1 \times \nu_0)(\{(t,s) \in [t_0, t_1]^2 : t \le s \le t+\varepsilon\})\\ &\le \varepsilon^{-1}\int_{t_0}^{t_1} \mathscr{L}^1([s-\varepsilon, s])\,\mathrm{d}\nu_0(s) = \nu_0([t_0, t_1])\end{aligned}$$

so that, passing to the limit as $\varepsilon \downarrow 0$ in the above inequality and applying Fatou's Lemma and Lebesgue's differentiation Theorem we get

$$\int_{t_0}^{t_1} \psi\big(|\dot u|_{\mathsf d}(r)\big)\,\mathrm{d}r\,\mathrm{d}t \le \nu_0([t_0, t_1]). \tag{4.28}$$

Since t_0 and t_1 are arbitrary, we conclude that $\nu_0 \ge \psi\big(|\dot u|_{\mathsf d}\big)\mathscr{L}^1$. Since $\mathscr{L}^1$ is singular with respect to $|\mathrm{C}u|_{\mathsf d}$ and $|\mathrm{J}u|_{\mathsf d}$ we eventually get

$$\nu_0 \ge \psi\big(|\dot u|_{\mathsf d}\big)\mathscr{L}^1 + \mathsf{L}\big(|\mathrm{C}u|_{\mathsf d} + |\mathrm{J}u|_{\mathsf d}\big), \tag{4.29}$$

which in particular yields (4.13), since

$$\liminf_{h\in H'} \int_{t_1}^{t_2} \psi_h\big(|\dot u_h|_{\mathsf d_h}(r)\big)\,\mathrm{d}r = \liminf_{h \in H''} \nu_{h,0}\big((t_1,t_2)\big) \ge \nu_0\big((t_1,t_2)\big)$$

and $\mathscr{L}^1$ and $|\mathrm{C}u|_{\mathsf d}$ are diffuse. □

We want to study now the properties of the limit function u; we will suppose that the following lower semicontinuity properties hold:

(C4) If a sequence $(u_h)_{h\in H} \subset \mathscr{X}$ σ-converges to u with $\sup_{h\in H} \tilde{\mathcal{E}}(t, u_h) < \infty$ for some $t \in [0,T]$ then

$$\liminf_{h\in H} \mathcal{E}_h(t, u_h) \ge \mathcal{E}(t,u), \tag{C4$_E$}$$

$$f_h \ge \mathcal{F}_h(t, u_h),\ f_h \to f \quad \Rightarrow \quad f \ge \mathcal{F}(t,u), \quad \limsup_{h\in H} \mathcal{P}_h(t, u_h, f_h) \le \mathcal{P}(t,u,f), \tag{C4$_{FP}$}$$

$$\text{if } \lim_{h\in H} t_h = t,\ \sup_{h\in H} \mathcal{E}(t_h, u_h) < \infty \quad \text{then} \quad \liminf_{h\in H} \mathcal{F}_h(t_h, u_h) \ge \mathcal{F}(t,u). \tag{C4$_F$}$$

Notice that in (C4$_F$) we allow for an h-dependence of t in the Γ-$\liminf$ inequality for $\mathcal{F}$, whereas t is independent of h in (C4$_{FP}$). (C4$_F$) will not be required for the convergence result in the superlinear case, see Theorem 4.4.

The next statement is the main result of our paper: it states that the energy-dissipation inequality is preserved in the limit for arbitrary evolution systems fulfilling (C1,2,3,4). When the limit $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ is also an upper gradient evolution system, then we recover a BV solution.

Theorem 4.2 (Stability of the energy-dissipation inequality and convergence)**.** *Let us assume that ψ has L-linear growth and for $h \in \mathbb{N}$ let $(\mathscr{X}, \mathsf{d}_h, \mathcal{E}_h, \mathcal{F}_h, \mathcal{P}_h)$ be a family of evolution systems satisfying* (C1,2,3,4) *with respect to a sequence $\bar{u}_h \in \mathscr{X}$ σ-converging to $\bar{u}$. Let $u_h \in \mathrm{AC}(0,T;(\mathscr{X}, \mathsf{d}_h))$, $\mathsf{F}_h \in \mathrm{B}_+([0,T])$, $h \in H$, be sequences of curves satisfying the ψ_h-energy dissipation inequality* (4.7) *pointwise converging to u in $(\mathscr{X}, \sigma)$, with $u_h(0) = \bar{u}_h$, and satisfying*

$$\lim_{h\in H} \mathcal{E}_h(0,\bar{u}_h) = \mathcal{E}(0,\bar{u}), \quad r \mapsto \big[\mathcal{P}_h(r,u_h(r),\mathsf{F}_h(r))\big]_+ \text{ are equi-integrable in } (0,T). \tag{4.30}$$

Then $u \in \mathrm{BV}([0,T];(\mathscr{X}, \mathsf{d}))$ satisfies the the local stability condition

$$\mathcal{F}(t,u(t)) \le \mathsf{L} \quad \text{for every } t \in [0,T] \setminus \mathrm{J}_u. \tag{4.31}$$

If

$$\int_0^T \mathcal{P}(t,u(t),\mathsf{L})\,\mathrm{d}t < \infty \tag{4.32}$$

then u satisfies the BV energy-dissipation inequality in the formulation of (ED)*,* $(\mathrm{S}_{\mathrm{loc}})$ *for some $\mathsf{F} \in \mathrm{B}_+([0,T])$.*

In particular, if $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ is an upper gradient evolution system, then u satisfies (4.32) *and therefore is a BV-solution of the corresponding rate-independent evolution and we have*

$$\lim_{h\in H} \mathcal{E}_h(t,u_h(t)) = \mathcal{E}(t,u(t)) \quad \text{for every } t \in [0,T]. \tag{4.33}$$

Proof. We denote by I the set $[0,T] \setminus \mathrm{J}_u$.

Let us first notice that (4.30) implies (4.8), so that (4.9), (4.10) and (4.12) holds.

Fatou's Lemma yields that

$$\liminf_{h\in H} \mathsf{F}_h(t) \le \mathsf{L}. \tag{4.34}$$

By a diagonal argument, combining the convergence (4.12) and the l.s.c. property (C4$_F$) we obtain

$$\tilde{\mathsf{F}}(t) = \inf\big\{ \liminf_{h\in H} \mathsf{F}_h(t_h) : t_h \to t\big\} \le \mathsf{L} \quad \mathcal{F}(t,u(t)) \le \tilde{\mathsf{F}}(t) \quad \text{for every } t \in \mathrm{I}, \tag{4.35}$$

hence we get (4.31).

Let us now assume (4.32) and let us prove the BV-dissipation inequality (ED); it is not restrictive to prove the inequality for $t = T$.

We consider the function $A : \mathrm{I} \times [0,\mathsf{L}] \to \mathbb{R} \cup \{\infty\}$

$$A(t,f) := \psi^*(f) - \mathcal{P}(t,u(t),f) \ge -\mathcal{P}(t,u(t),\mathsf{L}). \tag{4.36}$$

Denoting by $\mathcal{L}$ the Lebesgue-measurable subsets of I and by $\mathcal{B}$ the Borel sets of $\mathbb{R}^2$, it is easy to check that A is $\mathcal{L}\otimes\mathcal{B}$-measurable, thanks to the monotonicity and upper semicontinuity of $f \mapsto \mathcal{P}(t,u(t),f)$.

By (4.32), it is the immediate to see that for a.a. $t \in \mathrm{I}$

$$a(t) := \min\Big\{A(t,f) : \mathcal{F}(t,u(t)) \le f \le \mathsf{L}\Big\} > -\infty \tag{4.37}$$

Applying [6, Lemma III.39] we find a Lebesgue measurable map $\mathsf{F} \to \mathbb{R}$ such that

$$\mathcal{F}(t,u(t)) \le \mathsf{F}(t) \le \mathsf{L}, \quad \psi^*(\mathsf{F}(t)) - \mathcal{P}(t,u(t),\mathsf{F}(t)) = a(t) \qquad \text{for } \mathscr{L}^1\text{-a.a. } t \in \mathrm{I}. \tag{4.38}$$

and, up to a modification of F in a negligible set, it is not restrictive to assume F Borel and $\mathsf{F}(t) = \mathsf{L}$ on a Borel set containing J_u and where $|\mathrm{C}u|_{\mathsf{d}}$ is concentrated.

Setting $a_h(t) := \psi_h^*(\mathsf{F}_h(t)) - \mathcal{P}_h(t,u_h(t),\mathsf{F}_h(t))$, by (C4$_{FP}$) is immediate to see that

$$\liminf_{h\in H} a_h(t) \ge a(t). \tag{4.39}$$

Since (4.7) and (C1) also yield

$$\sup_h \int_0^T \big(\mathcal{P}_h(t,u_h(t),\mathsf{F}_h(t))\big)_-\,\mathrm{d}t < \infty, \tag{4.40}$$

Fatou's lemma and (4.39) imply

$$\int_0^T \big(a(t)\big)_+\,\mathrm{d}t \le \liminf_{h\in H}\int_0^T \big(a_h(t)\big)_+\,\mathrm{d}t < \infty, \tag{4.41}$$

and the inequality $a(t) \ge -\mathcal{P}(t,u(t),\mathsf{L})$ shows that $a \in L^1(0,T)$. (4.30) and Fatou's Lemma then yield

$$\liminf_{h\in H}\int_0^T \Big(\psi_h^*(\mathsf{F}_h(t)) - \mathcal{P}_h(t,u(t),\mathsf{F}_h(t))\Big)\,\mathrm{d}t \ge \int_0^T \Big(\psi^*(\mathsf{F}(t)) - \mathcal{P}(t,u(t),\mathsf{F}(t))\Big)\,\mathrm{d}t. \tag{4.42}$$

Recalling (4.18), we consider now the measures

$$\begin{aligned}\eta_h &:= \Big(\psi_h^*\big(\mathsf{F}_h\big) - \mathcal{P}_h(\cdot,u_h,\mathsf{F}_h)\Big)\mathscr{L}^1,\\ \mu_h &:= \Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}\big) + \psi_h^*\big(\mathsf{F}_h\big) - \mathcal{P}_h(\cdot,u_h,\mathsf{F}_h)\Big)\mathscr{L}^1 = \nu_{h,0} + \eta_h;\end{aligned} \tag{4.43}$$

up to extracting a further subsequence, it is not restrictive to assume that $\eta_h \rightharpoonup^* \eta$, $\mu_h \rightharpoonup^* \mu \ge \nu_0 + \eta$ in the duality with continuous functions. Fatou's Lemma and the previous arguments easily imply

$$\eta \ge \Big(\psi^*\big(\mathsf{F}\big) - \mathcal{P}\big(\cdot,u,\mathsf{F}\big)\Big)\mathscr{L}^1. \tag{4.44}$$

Since $\lim_{h\in H}\mu_h([0,T]) = \mu([0,T])$, combining (C4$_E$) and (4.42) inequality (ED) for $t = T$ follows if we show that

$$\mu([0,T]) \ge \int_0^T \Big(\psi\big(|\dot u|_{\mathsf{d}}(r)\big) + \psi^*\big(\mathsf{F}\big) - \mathcal{P}(\cdot,u,\mathsf{F})\Big)\,\mathrm{d}r + \mathsf{L}\int_0^T \mathrm{d}|\mathrm{C}u|_{\mathsf{d}} + \mathit{Jmp}_{\mathsf{d},\mathsf{f}}(u;[0,T]). \tag{4.45}$$

Now, (4.29) and (4.44) imply that

$$\mu \ge \nu_0 + \eta \ge \Big(\psi\big(|\dot u|_{\mathsf{d}}\big) + \psi^*\big(\mathsf{F}\big) - \mathcal{P}(\cdot,u,\mathsf{F})\Big)\mathscr{L}^1 + \mathsf{L}|\mathrm{C}u|_{\mathsf{d}}. \tag{4.46}$$

Since the atomic and the diffuse part of a measure are mutually singular, (4.45) ensues if for every $t \in [0,T]$ with $\mu(\{t\}) > 0$ we have

$$\mu(\{t\}) \geq \Delta_{\mathsf{d},\mathfrak{f},t}(u(t_-), u(t), u(t_+)). \tag{4.47}$$

In order to prove (4.47) we just take two sequence $r_h^- < t < r_h^+$, $h \in \mathbb{N}$, converging monotonically to t such that

$$u_h(r_h^-) \xrightarrow{\sigma} u(t_-), \quad u_h(r_h^+) \xrightarrow{\sigma} u(t_+), \tag{4.48}$$

and we set

$$\mathsf{s}_h(r) := r + \int_t^r \Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}(\tau)\big) + \psi_h^*\big(\mathcal{F}_h(\tau, u_h(\tau))\big)\Big)\,\mathrm{d}\tau, \quad \mathsf{s}_h^\pm := \mathsf{s}_h(r_h^\pm). \tag{4.49}$$

Since $\mathcal{F}_h(\tau, u_h(\tau)) \leq \mathsf{F}_h(\tau)$ and

$$\limsup_{h\in H} \int_{t_{h\,-}}^{t_{h\,+}} \mathcal{P}_h(t, u_h(t), \mathsf{F}_h(t))\,\mathrm{d}t = 0 \tag{4.50}$$

by (4.30), taking into account the definition (4.43) of μ_h we obtain

$$\limsup_{h\in H''}(\mathsf{s}_h^+ - \mathsf{s}_h^-) \leq \limsup_{h\in H''} \mu_h([t_h^-, t_h^+]) \leq \mu(\{t\}),$$

and up to extracting a suitable subsequence we can assume that $\mathsf{s}_h^\pm \to \mathsf{s}^\pm$ as $h \to \infty$. We denote by $\mathsf{r}_h := \mathsf{s}_h^{-1}$ the inverse map of s_h: r_h is 1-Lipschitz, monotone, and maps $[\mathsf{s}_h^-, \mathsf{s}_h^+]$ onto $[r_h^-, r_h^+]$. We also set

$$\vartheta_h(s) := \begin{cases} u_h(\mathsf{r}_h(s)) & \text{if } s \in [\mathsf{s}_h^-, \mathsf{s}_h^+], \\ u_h(r_h^+) & \text{if } s \geq \mathsf{s}_h^+, \\ u_h(r_h^-) & \text{if } s \leq \mathsf{s}_h^-, \end{cases} \tag{4.51}$$

so that, in particular, we have

$$\vartheta_h(\mathsf{s}_h^\pm) = u_h(r_h^\pm), \qquad \vartheta_h(t) = u_h(t).$$

We observe that $\tilde{\mathcal{E}}_h(\mathsf{r}_h(s), \vartheta_h(s)) \leq C$ and that the functions ϑ_h are uniformly d_h-Lipschitz: to show this fact, we choose $f \in [0, \mathsf{L})$ in such a way that $\sup_h \psi_h^*(f) \leq 1$; the inequality $\psi_h(v) \geq fv - \psi_h^*(f)$ yields

$$\dot{\mathsf{s}}_h(r) \geq 1 + \psi_h(|\dot u_h|_{\mathsf{d}_h}(r)) \geq 1 + f\,|\dot u_h|_{\mathsf{d}_h}(r) - \psi_h^*(f) \geq f\,|\dot u_h|_{\mathsf{d}_h}(r)$$

so that

$$\begin{aligned} f\mathsf{d}_h(\vartheta_h(\alpha), \vartheta_h(\beta)) = f\mathsf{d}_h(u_h(\mathsf{r}_h(\alpha)), u_h(\mathsf{r}_h(\beta)) &\leq \int_{\mathsf{r}_h(\alpha)}^{\mathsf{r}_h(\beta)} f\,|\dot u_h|_{\mathsf{d}_h}(r)\,\mathrm{d}r \\ &\leq \int_{\mathsf{r}_h(\alpha)}^{\mathsf{r}_h(\beta)} \dot{\mathsf{s}}_h(r)\,\mathrm{d}r \leq \mathsf{s}_h(\mathsf{r}_h(\beta)) - \mathsf{s}_h(\mathsf{r}_h(\alpha)) = \beta - \alpha, \end{aligned} \tag{4.52}$$

whence the uniform d_h-Lipschitz continuity of ϑ_h.

By arguing as in the proof of Theorem 4.1, we can find a compact interval I containing all the intervals $[\mathsf{s}_h^-, \mathsf{s}_h^+]$ and a further subsequence such that $\vartheta_h(s) \to \vartheta(s)$ for every $s \in I$: it follows from (4.52) that ϑ is f^{-1}-Lipschitz with respect to d, $\vartheta(\mathsf{s}^\pm) = u(t_\pm)$, $\vartheta(t) = u(t)$, and

$$|\dot\vartheta_h|_{\mathsf{d}_h} \rightharpoonup^* m \quad \text{in } L^\infty(I) \quad \text{with } m \geq |\dot\vartheta|_{\mathsf{d}}. \tag{4.53}$$

Moreover, by using the elementary inequality

$$\psi_h(v) + \psi_h^*(f) \geq (f \vee a)v - \psi_h^*(a) \quad \text{for every } a \in [0, \mathsf{L}), \tag{4.54}$$

and recalling that $r_h^+ - r_h^- \to 0$, $\psi_h^*(a) \to \psi^*(a) < \infty$, and (4.50), we have

$$\begin{aligned}
\mu(\{t\}) &\geq \limsup_{h\to\infty} \mu_h([r_h^-, r_h^+]) \overset{(4.50)}{=} \limsup_{h\in H} \int_{r_h^-}^{r_h^+} \Big(\psi_h\big(|\dot u_h|_{\mathsf{d}_h}(\tau)\big) + \psi_h^*\big(\mathcal{F}_h(\tau, u_h(\tau))\big)\Big)\,\mathrm{d}\tau \\
&\overset{(4.54)}{\geq} \liminf_{h\in H} \int_{r_h^-}^{r_h^+} \Big((\mathcal{F}_h(\tau, u_h(\tau)) \vee a)|\dot u_h|_{\mathsf{d}_h}(\tau) - \psi_h^*(a)\Big)\,\mathrm{d}\tau \\
&\geq \liminf_{h\in H} \int_{\mathsf{s}_h^-}^{\mathsf{s}_h^+} \Big((\mathcal{F}_h(\mathsf{r}_h(s), \vartheta_h(s)) \vee a)|\dot\vartheta_h|_{\mathsf{d}_h}(s)\Big)\,\mathrm{d}s - (r_h^+ - r_h^-)\psi_h^*(a) \\
&\geq \liminf_{h\in H} \int_I \Big((\mathcal{F}_h(\mathsf{r}_h(s), \vartheta_h(s)) \vee a)|\dot\vartheta_h|_{\mathsf{d}_h}(s)\Big)\,\mathrm{d}s \\
&\geq \int_I \Big((\mathcal{F}(t, \vartheta(s)) \vee a)m(s)\Big)\,\mathrm{d}s \qquad (4.55) \\
&\geq \int_{\mathsf{s}^-}^{\mathsf{s}^+} \Big((\mathcal{F}(t, \vartheta(s)) \vee a)|\dot\vartheta|_{\mathsf{d}}(s)\Big)\,\mathrm{d}s \geq \Delta_{\mathsf{d},\mathsf{f},t}(u(t_-), u(t), u(t_+)),
\end{aligned}$$

viz. the desired (4.47). For the last $\liminf$ inequality in (4.55) we used a result proved in the next lemma. □

Lemma 4.3. *Let I be a bounded interval in $\mathbb{R}$, $F, m,\ F_h, m_h : I \to [0, \infty)$, $h \in \mathbb{N}$, be measurable functions satisfying*

$$\liminf_{h\to\infty} F_h(s) \geq F(s) \quad \text{for } \mathscr{L}^1\text{-a.a. } s \in I, \quad m_h \rightharpoonup m \quad \text{in } L^1(I). \tag{4.56}$$

Then

$$\liminf_{h\to\infty} \int_I F_h(s)\, m_h(s)\,\mathrm{d}s \geq \int_I F(s)\, m(s)\,\mathrm{d}s \tag{4.57}$$

Proof. Let us set $G_k(s) := \inf_{h\geq k} F_h(s) \wedge k$, $k \in \mathbb{N}$. Since $G_k(s) \leq F_h(s)$ for every $h \geq k$ and $G_k \in L^\infty(I)$ we have for every $k \in \mathbb{N}$

$$\liminf_{h\to\infty} \int_I F_h(s)\, m_h(s)\,\mathrm{d}s \geq \liminf_{h\to\infty} \int_I G_k(s)\, m_h(s)\,\mathrm{d}s = \int_I G_k(s)\, m(s)\,\mathrm{d}s.$$

On the other hand, for $\mathscr{L}^1$-a.a. $s \in I$, $k \mapsto G_k(s)$ is a nondecreasing sequence converging to $\liminf_{h\to\infty} F_h(s)$, so that by monotone convergence

$$\lim_{k\to\infty} \int_I G_k(s)\, m(s)\, \mathrm{d}s \ge \int_I F(s)\, m(s)\, \mathrm{d}s.$$

□

In the case of a limit metric dissipation function ψ with superlinear growth we have a completely analogous result, which can be compared with [16, Theorem 4.8]. We omit the similar proof: it can be carried out without assuming (C4$_F$), which has been used only to characterize the contribution of jump part in the energy dissipation inequality (ED).

Theorem 4.4 (Convergence in the superlinear case)**.** *Let us assume that ψ is superlinear and for $h \in \mathbb{N}$ let $(\mathscr{X}, \mathsf{d}_h, \mathcal{E}_h, \mathcal{F}_h, \mathcal{P}_h)$ be a family of evolution systems satisfying* (C1,2,3) and (C4$_{E,FP}$) *with respect to a sequence $\bar u_h \in \mathscr{X}$ σ-converging to $\bar u$ and let $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ be an upper gradient evolution system.*
Let $u_h \in \mathrm{AC}(0,T;(\mathscr{X},\mathsf{d}_h))$, $\mathsf{F}_h \in \mathrm{B}_+([0,T])$, $h \in H$, be sequences of curves satisfying the ψ_h-energy dissipation inequality (4.7)*, pointwise converging to u in $(\mathscr{X}, \sigma)$, with $u_h(0) = \bar u_h$, and satisfying* (4.30)*. Then $u \in \mathrm{AC}(0,T;(\mathscr{X},\mathsf{d}))$ is a ψ-gradient flow of the limit system and*

$$\lim_{h\in H} \mathcal{E}_h(t, u_h(t)) = \mathcal{E}(t, u(t)) \quad \textit{for every } t \in [0,T]. \tag{4.58}$$

Examples.

λ-convex energies. Let $(\mathscr{X}_h, \|\cdot\|_h)$, $(\mathscr{X}, \|\cdot\|)$ be a family of Banach spaces such that $\mathscr{X}_0 \subset \mathscr{X}_h \subset \mathscr{X}$ with continuous and dense inclusions. Setting $\|u\|_h = \infty$ if $u \in \mathscr{X} \setminus \mathscr{X}_h$ we suppose that $\Gamma(\mathscr{X})\text{-}\lim_{h\to\infty} \|\cdot\|_h = \|\cdot\|$. Let $\mathcal{E} : \mathscr{X}_0 \to (-\infty, +\infty]$ be a proper, λ-convex functional, i.e. satisfying for every $u_0, u_1 \in \mathscr{X}_0$

$$\mathcal{E}((1-\theta)u_0 + \theta u_1) \le (1-\theta)\mathcal{E}(u_0) + \theta\mathcal{E}(u_1) - \frac{\lambda}{2}\theta(1-\theta)\|u_1 - u_0\| \quad \text{for every } \theta \in [0,1].$$

We consider a family $\ell \in \mathrm{C}^1([0,T]; \mathscr{X}_0')$, we suppose that $\mathcal{E}(t,u) := \mathcal{E}(u) - \langle \ell(t), u\rangle$ has compact sublevels on $\mathscr{X}_0$, and we extend it to $\mathscr{X}$ by setting $\mathcal{E}(t,u) = \infty$ if $u \in \mathscr{X} \setminus \mathscr{X}_0$.

We set $\mathcal{P}(t,u) = \partial_t \mathcal{E}(t,u) = \langle \ell'(t), u\rangle$ and

$$\mathcal{F}_h(t,u) := \min\left\{ \|\xi - \ell(t)\|_h^* : \xi \in \partial_h \mathcal{E}(u) \right\}$$

where ∂_h is the Frechét subdifferential of the restriction of $\mathcal{E}$ in $\mathscr{X}_h$. Notice that $\partial_h \mathcal{E} \subset \mathscr{X}_0'$ and it is not difficult to check (see [23, 16]) that $(\mathscr{X}, \mathsf{d}_h, \mathcal{E}, \mathcal{F}_h, \mathcal{P})$ and $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ are upper gradient evolution systems and all the assumptions (C1,2,3,4) are satisfied.

Dirichlet energy and double-well potentials. Here is a concrete example of the above setting. Consider a bounded open set $\Omega \subset \mathbb{R}^d$, a function $W \in \mathrm{C}^2(\mathbb{R})$ with $\inf_{\mathbb{R}} W > -\infty$, $\inf_{\mathbb{R}} W'' > -\infty$, and

$\ell \in \mathrm{C}^1([0,T];L^2(\Omega))$. In the space $\mathscr{X} := L^2(\Omega)$ endowed with the strong L^2-topology we set

$$\mathcal{E}(t,u) := \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + W(u) - \ell(t)\,u\Big)\,\mathrm{d}x \quad \text{if } u \in H^1_0(\Omega),\ W\circ u \in L^1(\Omega),$$
$$\mathcal{E}(t,u) := +\infty \quad \text{otherwise.}$$

We choose a sequence of exponents $p_h > 1$ converging to 1 as $h \to \infty$, and initial data $\bar u_h \in H^1_0(\Omega)$ with $\mathcal{E}(0,\bar u_h) < \infty$ strongly converging to $\bar u$ in $H^1_0(\Omega)$ with $W \circ u_h \to W\circ u$ in $L^1(\Omega)$.

We let d_h be the distance induced by the $L^{p_h}(\Omega)$ norm, d the $L^1(\Omega)$-distance, $\psi_h(v) := \frac{1}{p_h}v^{p_h}$, and

$$\mathcal{F}_h(t,u) := \| - \Delta u + W'(u) - \ell(t)\|_{L^{p_h^*}(\Omega)}, \quad \mathcal{P}(t,u) = \int_\Omega \ell'(t)\,u\,\mathrm{d}x, \tag{4.59}$$

with $\mathcal{F}_h(t,u) = +\infty$ if $-\Delta u + W'(u) - \ell(t) \notin L^{p_h}(\Omega)$; $\mathcal{F}$ has the analogous expression in $L^\infty(\Omega)$.

Applying the results of [23, §7.2] we see that $(\mathscr{X}, \mathsf{d}_h, \mathcal{E}, \mathcal{F}_h, \mathcal{P})$ and $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ are upper gradient evolution systems and for every $h \in \mathbb{N}$ there exists a solution u_h of the ψ_h-gradient flow. It is also easy to check that all the assumptions (C1,2,3,4) are satisfied so that, up to subsequences, $u_h(t,\cdot)$ converge to $u(t,\cdot)$ in $L^2(\Omega)$ at every time t with convergence of the energies $\mathcal{E}(t,\cdot)$ and u is a BV solution of the rate-independent evolution governed by $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$.

Marginal functionals. In the finite dimensional setting described in Section 2 (here we also assume that the norms $\|\cdot\|_u$ are independent of h), let us consider the marginal functional (2.16) and the couple $(\mathcal{F}, \mathcal{P})$ given by (2.21) and (2.22).

It is not difficult to see that $(\mathscr{X}, \mathsf{d}, \mathcal{E}, \mathcal{F}, \mathcal{P})$ is an upper gradient evolution system. ψ_h-gradient flows in the superlinear case can be obtained by applying the results of [16]: they in particular solve the differential inclusions (2.18). Existence of a BV solution and convergence of the ψ_h gradient flows can thus be obtained by applying Theorems 4.1 and 4.2.

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