Weierstraß–Institut
für Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.

On the existence of transition layers of spike type in reaction–diffusion–convection equations

Klaus R. Schneider[1] , Adelaida B. Vasil'eva[2]

submitted: 31st August 1995

[1] Weierstraß–Institut
for Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany

[2] Faculty of Physics
Department of Mathematics
Moscow State University
119899 Moscow
Russia

Preprint No. 170
Berlin 1995

1991 Mathematics Subject Classification. 35K55, 35B25, 34D15, 34E05.
Key words and phrases. Reaction–diffusion–convection equations, contrast structures of spike type, singular perturbations, asymptotic expansions, boundary layer functions, integral manifolds.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
e–mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
e–mail (Internet): preprint@wias-berlin.de

On the existence of transition layers of spike type in reaction–diffusion–convection equations

Klaus R. Schneider *
Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstraße 39
10117 Berlin

Adelaida B. Vasil'eva †
Faculty of Physics
Department of Mathematics
Moscow State University
119899 Moscow

Abstract. We investigate steady state solutions to a class of systems of reaction-diffusion-convection equations with small diffusion and small convection, and which depend on one space variable. Our main concern is to prove the existence of a solution with an interior layer of spike type for higher order systems without taking into consideration the influence of boundary conditions. To this end we combine two methods of the theory of singular perturbations: the method of integral manifolds and the method of boundary layer functions.

Key words: Reaction-diffusion-convection equations, contrast structures of spike type, singular perturbations, asymptotic expansions, boundary layer functions, integral manifolds.

AMS(MOS) subject classification: 35K55, 35B25, 34D15, 34E05

1 Introduction

The mathematical modeling of numerous nonlinear phenomena in biology, chemistry and other fields leads to reaction–diffusion equations and reaction–diffusion–convection equations. A necessary first step in the study of the longtime behavior of such systems is the investigation of stationary solutions. There are important phenomena where the influence of diffusion and convection is small compared with the reaction process. In such cases the investigation of stationary solutions leads to the study of differential equations with small parameters at the highest derivatives, that is, we have to consider singularly perturbed problems.

*e–mail: schneide@wias-berlin.de

†e–mail: ABV@R11523.phys.msu.su

Important contributions to the theory of singular perturbation were published at the beginning of the fifties [12, 13, 8, 14], a bibliography of asymptotic methods can be found in [15, 16, 2]. In the sequel we shall study the existence of stationary solutions with transition layers to the following class of reaction–diffusion–convection equations

$$\begin{aligned} \tilde{u}_t + \mu^2 \tilde{u}_{xx} &= \tilde{f}(\tilde{u}, \mu\tilde{u}_x, \tilde{v}, \varepsilon\tilde{v}_x, x) \\ \tilde{v}_t + \varepsilon^2 \tilde{v}_{xx} &= \tilde{g}(\tilde{u}, \mu\tilde{u}_x, \tilde{v}, \varepsilon\tilde{v}_x, x) \end{aligned} \tag{1.1}$$

where $\tilde{u} \in R, \tilde{v} \in R^k$ $x \in R$ μ and ε are small parameters. Denoting differentiation with respect to x by $'$ and setting $\mu\tilde{u}' = \tilde{u}_1$, $\varepsilon\tilde{v}' = \tilde{v}_1$ we get that any stationary solution of (1.1) satisfies

$$\begin{aligned} \mu\tilde{u}_1' &= \tilde{f}(\tilde{u}, \tilde{u}_1, \tilde{v}, \tilde{v}_1, x) \\ \mu\tilde{u}' &= \tilde{u}_1 \\ \varepsilon\tilde{v}_1' &= \tilde{g}(\tilde{u}, \tilde{u}_1, \tilde{v}, \tilde{v}_1, x) \\ \varepsilon\tilde{v}' &= \tilde{v}_1. \end{aligned} \tag{1.2}$$

A solution of system (1.2) can have so-called transition layers as ε and μ tend to zero. Transition layers can be divided into two groups: boundary layers (the transition takes place near the boundary of the interval where we consider the solution) and interior layers (the transition takes place at interior points). Solutions with an interior layer are called *contrast structures.* We distinguish interior layers of step type and of spike type. In this paper we study the existence of contrast structures of spike type without taking into consideration the influence of boundary conditions. Our approach consists in combining two methods of the theory of singular perturbations: the method of *integral manifolds* [17, 9, 11] and the method of *boundary layer functions* [15, 16, 3].

In section 2, for sufficiently small ε we reduce the given higher dimensional problem to a two–dimensional one by means of an integral manifold. In section 3 we derive sufficient conditions for the existence of a spike solution by using the method of boundary functions for interior transition layers. The obtained conditions are of Melnikov integral type such that the obtained results can be interpreted geometrically as transversality conditions for the stable and unstable manifolds.

2 Assumptions. Reduction of the problem

System (1.2) is a special case of the system

$$\begin{aligned} \mu w' &= f(u, w, v, x) \\ \mu u' &= w \\ \varepsilon v' &= g(u, w, v, x) \end{aligned} \tag{2.1}$$

where $u, w \in R, v \in R^n$. Our basic idea is to consider system (2.1) as a singularly perturbed system with respect to the parameter ε and to reduce it to a two-dimensional system (second order equation) by means of an integral manifold. Then we derive conditions ensuring the existence of a spike solution for the reduced problem. To be able to do so we study system (2.1) under the following conditions.

($\mathbf{H_1}$). $f : G \to R$, and $g : G \to R^n$ are sufficiently smooth and uniformly bounded where $G := R \times R \times R^n \times R$.

($\mathbf{H_2}$). The equation

$$g(u, w, v, x) = 0$$

has an isolated solution $v = h^0(u, w, x)$ defined on $R \times R \times R$, and with the same smoothness as g.

($\mathbf{H_3}$). The spectrum σ of the matrix $g_v(u, w, h^0(u, w, x), x)$ satisfies

$$Re\, \sigma \left\{ \frac{\partial g}{\partial v}(u, w, h^0(u, w, x), x) \right\} \neq 0$$

for all $(u, w, x) \in R \times R \times R$.

By means of the scaling $x = \mu\, \xi,\ \varepsilon = \mu^2$ we get from (2.1)

$$\begin{aligned} \frac{dw}{d\xi} &= f(u, w, v, x) \\ \frac{du}{d\xi} &= w \\ \frac{dx}{d\xi} &= \mu \\ \mu\, \frac{dv}{d\xi} &= g(u, w, v, x). \end{aligned} \tag{2.2}$$

For small μ, system (2.2) is a singularly perturbed system. According to the theory of integral manifolds for singularly perturbed differential systems [11], for sufficiently small μ, hypotheses $(H_1) - (H_3)$ imply the existence of an integral manifold M_μ to system (2.2) of the form

$$\mathcal{M}_\mu := \{(u, v, w, x) \in R^{n+3} : v = h(u, w, x, \mu)\} \tag{2.3}$$

possessing the asymptotic representation with respect to μ

$$v = h^0(u, w, x) + \mu h^1(u, w, x) + \mu^2 h^2(u, w, x) + \cdots \tag{2.4}$$

Let us recall the algorithm for the computation of the functions h^i in (2.4). By substituting (2.4) into the last equation of (2.2) we get

$$\begin{aligned}\mu & \left(\frac{\partial h^0}{\partial u}\frac{du}{d\xi}+\frac{\partial h^0}{\partial w}\frac{dw}{d\xi}+\frac{\partial h^0}{\partial x}\frac{dx}{d\xi}\right)+\mu^2\left(\frac{\partial h^1}{\partial u}\frac{du}{d\xi}+\frac{\partial h^1}{\partial w}\frac{dw}{d\xi}+\frac{\partial h^1}{\partial x}\frac{dx}{d\xi}\right)+\cdots \\ = & \; g(u,w,h^0+\mu h^1+\cdots,x).\end{aligned} \tag{2.5}$$

Taking into account the first three equations of (2.2) and using the expansion (e is a placeholder for f and g)

$$e(u,w,h^0+\mu h^1+\ldots,x)=e(u,w,h^0,x)+\mu\; e_{v0}+\mu^2\left(e_{v0}h^2+\mu^2 e_{vv0}h^1h^1\right)+\cdots$$

where here and in what follows the lower index zero indicates that v is replaced by $h^0(u,w,x)$, then we get from (2.5)

$$\begin{aligned}\mu & \left\{\frac{\partial h^0}{\partial u}w+\frac{\partial h^0}{\partial w}f_0\right\}+\mu^2\left\{\frac{\partial h^0}{\partial x}+\frac{\partial h^1}{\partial u}w+\frac{\partial h^1}{\partial w}f_0+\frac{\partial h^0}{\partial w}f_{v0}h^1\right\}+\cdots \\ = & \; g_0+\mu\; g_{v0}h^1+\mu^2\left\{g_{v0}h^2+\frac{1}{2}g_{vv0}h^1h^1\right\}+\cdots\end{aligned}$$

By equating the coefficients with the same power of μ we get for the μ^0-terms

$$0=g_0:=g(u,w,h^0(u,w,x),x).$$

This relation holds identically according to hypothesis (H_2). Concerning the coefficients multiplied by μ we obtain

$$\frac{\partial h^0}{\partial u}w+\frac{\partial h^0}{\partial w}f_0=\frac{\partial g}{\partial v}\left(u,w,h^0,x\right)h^1=:g_{v0}h^1. \tag{2.6}$$

By hypothesis (H_3), the function h^1 can be determined uniquely from (2.6)

$$h^1(u,w,x,\mu)=g_{v0}^{-1}\left[\frac{\partial h^0}{\partial u}w+\frac{\partial h^0}{\partial w}f_0\right]. \tag{2.7}$$

Analogously we obtain by equating the coefficients belonging to μ^2

$$\frac{\partial h^0}{\partial x}+\frac{\partial h^1}{\partial u}w+\frac{\partial h^1}{\partial w}f_0+\frac{\partial h^0}{\partial w}f_{v0}h^1=g_{v0}h^2+\frac{1}{2}g_{vv0}h^1h^1.$$

From this it follows

$$h^2=g_{v0}^{-1}\left\{\frac{\partial h^0}{\partial x}+\frac{\partial h^1}{\partial u}w+\frac{\partial h^1}{\partial w}f_0+\frac{\partial h^0}{\partial w}f_{v0}h^1-\frac{1}{2}g_{vv0}h^1h^1\right\} \tag{2.8}$$

where

$$
\begin{aligned}
\frac{\partial h^1}{\partial u} &= \left[(g_{v0}^{-1})_u + (g_{v0}^{-1})_v h_u^0\right]\left(h_u^0 w + h_w^0 f_0\right) + \\
&+ g_{v0}^{-1}\left[h_{uu}^0 w + h_{wu}^0 f_0 + h_w^0 (f_{0u} + f_{0v} h_u^0)\right], \\
\frac{\partial h^1}{\partial w} &= \left[(g_{v0}^{-1})_w + (g_{v0}^{-1})_v h_w^0\right]\left(h_u^0 w + h_w^0 f_0)\right) \\
&+ g_{v0}^{-1}\left[h_{uw}^0 w + h_u^0 + h_{ww}^0 f_0 + h_w^0 \left(f_{0_w} + f_{0_v} h_w^0\right)\right].
\end{aligned}
\tag{2.9}
$$

In the same way we may determine higher order terms. After substituting (2.7) – (2.9) into (2.4) we get an asymptotic expansion of the integral manifold $\mathcal{M}_\mu$ with respect to the parameter μ:

$$
\begin{aligned}
v = h(u, w, x, \mu) &= h^0 + \mu g_{v0}^{-1}(h_u^0 w + h_w^0 f_0) + \\
&+ \mu^2 g_{v0}^{-1}\Big\{h_x^0 \left[(g_{v0}^{-1})_u + (g_{v0}^{-1})_v h_u^0\right](h_u^0 w + h_w^0 f_0) w + \\
&+ g_{v0}^{-1}\left[h_{uu}^0 w + h_{wu}^0 f_0 + h_w^0 (f_{0u} + f_{0v} h_u^0\right] w + \\
&+ \left[(g_{v0}^{-1})_w + (g_{v0}^{-1})_v h_w^0\right](h_u^0 w + h_w^0 f_0) f_0 + \\
&+ g_{v0}^{-1}\left[h_{uw}^0 w + h_u^0 + h_{ww}^0 f_0 + (f_{0w} + f_{0u} h_w^0) h_w^0\right] f_0 + \\
&+ g_{v0}^{-1} h_w^0 f_{v0}(h_u^0 w + h_w^0 f_0) - \\
&- g_{v0}^{-1}\frac{g_{vv0}}{2}\left[h_u^0 w + h_w^0 f_0\right]^2 \Big\} + \cdots
\end{aligned}
\tag{2.10}
$$

On the integral manifold $\mathcal{M}_\mu$ system (2.2) reads as follows

$$
\begin{aligned}
\frac{dw}{d\xi} &= f(u, w, h(u, w, x, \mu), x) \\
\frac{du}{d\xi} &= w \\
\frac{dx}{d\xi} &= \mu
\end{aligned}
\tag{2.11}
$$

which is equivalent to the second order equation

$$
\mu^2 \frac{d^2 u}{dx^2} = f(u, \mu\frac{du}{dx}, h(u, \mu\frac{du}{dx}, x, \mu), x). \tag{2.12}
$$

In the next section we investigate the existence of contrast structures of spike type to equation (2.12).

3 Existence of an interior layer of spike type

As mentioned in the introduction, we denote a solution of system (1.2) exhibiting an interior layer as a contrast structure. In what follows we investigate the existence of a spike solution $u(x, \mu)$ of (2.12), that is, the existence of a contrast structure of the form as indicated in Fig. 1 (a definition is given below).

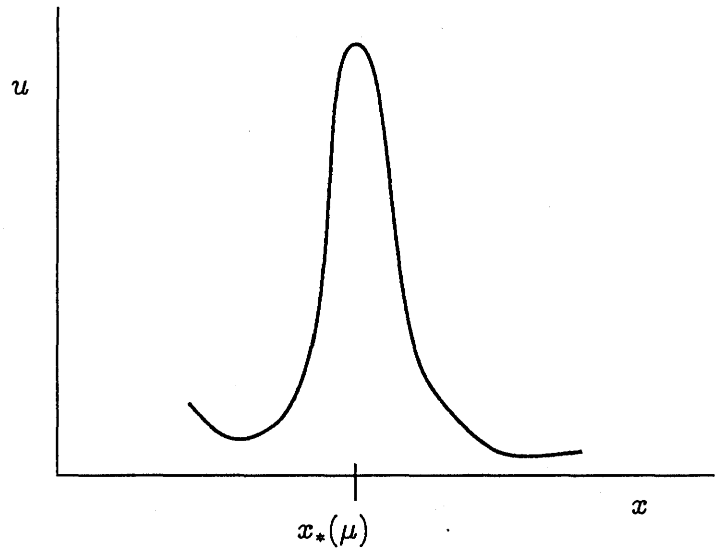

Fig. 1

In the case $\varepsilon = \mu^2$, and under the assumptions (H_1)–(H_3) the problem of existence of a contrast structure to the higher dimensional system (1.1) has been reduced in section 2 to the same problem for the second order equation (2.12) which is equivalent to the system

$$\begin{aligned} \mu \frac{dw}{dx} &= f(u, w, h(u, w, x, \mu), x) =: \hat{f}(u, w, x, \mu) \\ \mu \frac{du}{dx} &= w. \end{aligned} \tag{3.1}$$

If we replace x by $\mu\,\xi$ in the left-hand side of (3.1), fix x and put $\mu = 0$ in the right–hand side then we get the so-called associated system

$$\begin{aligned} \frac{dw}{d\xi} &= \hat{f}(u, w, x, 0) \\ \frac{du}{d\xi} &= w. \end{aligned} \tag{3.2}$$

Definition 3.1 *Let $\hat{f}$ be sufficiently smooth. Let there be an interval $I \subset R$ such that $\forall x \in I$ the equation $\hat{f}(u,0,x,0) = 0$ has a root $u = \rho(x)$ such that ρ is a smooth function, and $\hat{f}_u(\rho(x),0,x,0) > 0$, that is, $(0,\rho(x))$ is a saddle point of (3.2). If there are numbers $x_{0,i} \in I$ and $\rho_{0,i}$ with $\rho(x_{0,i}) \neq \rho_{0,i}$, $i = 1,\cdots,k$, then we call a solution $u(x,\mu)$ of (3.1) satisfying*

$$\lim_{\mu\to 0} u(x,\mu) = \begin{cases} \rho(x) & \text{for } x \neq x_{0,i} \\ \rho_{0,i} & \text{for } x = x_{0,i} \end{cases}$$

a spike–type solution.

Before we study equation (2.12) we consider the simpler case that f does not depend on du/dx. This case arises when (1.1) has no convection terms and leads to an equation of the form

$$\mu^2 \frac{d^2u}{dx^2} = \bar{f}(u,x,\mu). \tag{3.3}$$

Let $\bar{f}(u,x,0) := \tilde{f}(u,x)$. The associated system to (3.3) reads

$$\begin{aligned} \frac{dw}{d\xi} &= \tilde{f}(u,x) \\ \frac{du}{d\xi} &= w. \end{aligned} \tag{3.4}$$

The following hypothesis implies that (3.4) has a family of homoclinic orbits parametrized by x.

(**H**). *Let $\tilde{f}$ be sufficiently smooth and obey the following properties:*

(i) *For all $x \in R$ the equation $\tilde{f}(u,x) = 0$ has two roots $u = \varphi(x)$ and $u = \chi(x)$ such that*

$$\tilde{f}_u(\varphi(x),x) > 0, \qquad \tilde{f}_u(\chi(x),x) < 0.$$

(ii) *There exists a function $\psi(x)$ such that $\forall\, x \in R$*

$$\int_{\varphi(x)}^{\psi(x)} \tilde{f}(u,x)du = 0. \tag{3.5}$$

Without loss of generality we may assume $\varphi(x) < \chi(x) < \psi(x)$.

Hypothesis (H) implies that for fixed x, the graph of $\tilde{f}(u,x)$ has the form as represented in Fig. 2. According to the definition of $\psi(x)$ the dashed regions in Fig. 2 have the same area.

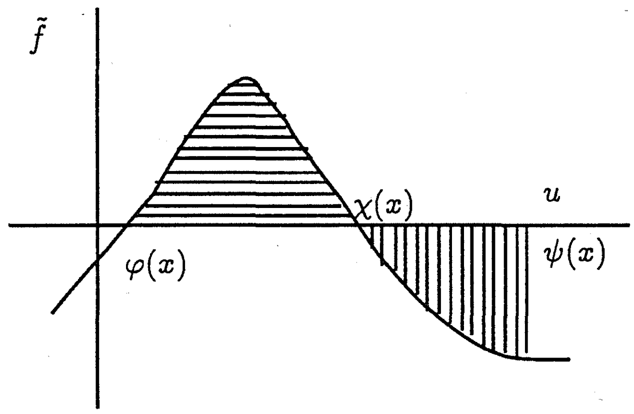

Fig. 2

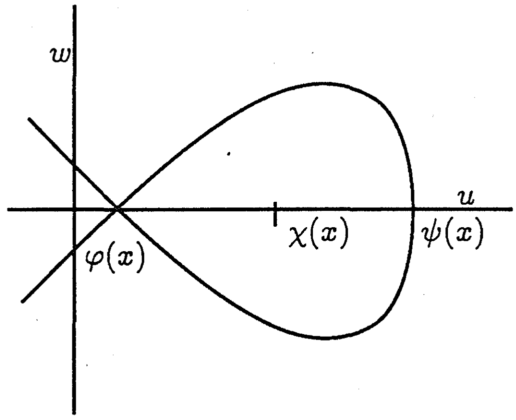

Fig. 3

From hypothesis (H) it follows that system (3.4) has a homoclinic orbit (see Fig. 3) the graph of which can be obtained by a simple quadrature for all $x \in R$

$$w(u, x) = \pm \left(2 \int_{\varphi(x)}^{u} \tilde{f}(\sigma, x) d\sigma \right)^{\frac{1}{2}}, \qquad \varphi(x) \leq u \leq \psi(x). \tag{3.6}$$

To verify this we note that $w(\varphi(x), x) = 0$ holds, and that according to (3.5) we have $w(\psi(x), x) = 0$. From

$$\frac{dw}{du} = \pm \frac{\tilde{f}(u, x)}{\left(2 \int_{\varphi(x)}^{u} \tilde{f}(\sigma, x) d\sigma \right)^{\frac{1}{2}}}$$

and from hypothesis (H) we get

$$\left. \frac{dw}{du} \right|_{u \to \varphi(x)} = \pm \sqrt{\tilde{f}_u(\varphi(x), x)}.$$

Taking (3.5) into account we obtain

$$\left. \frac{dw}{du} \right|_{u \to \psi(x)} \longrightarrow \pm \infty,$$

that is, at $u = \psi(x)$ we have a vertical slope.

In what follows we construct a spike solution $u(x, \mu)$ of (3.1), that is, a solution satisfying

$$\lim_{\mu \to 0} u(x, \mu) = \begin{cases} \varphi(x) & \text{for all } x \neq x_0 \\ \psi(x_0) & \text{for } x = x_0. \end{cases}$$

Here, x_0 is the leading term in the asymptotic representation of the point $x_*(\mu)$,

$$x_*(\mu) = x_0 + \mu x_1 + \cdots \tag{3.7}$$

where $u(x,\mu)$ takes its maximum. By applying the method of boundary layer functions the following equation has been derived in [1] to determine the first coefficient x_0

$$\begin{aligned}\Phi(x) := \int_{\varphi(x)}^{\psi(x)} \int_{\psi(x)}^{u} \tilde{f}_x(u,x)\frac{1}{w(s,x)} ds\, du \quad &\equiv \\ \equiv \int_{\varphi(x)}^{\psi(x)} \tilde{f}_x(u,x) \int_{\psi(x)}^{u} \left(2\int_{\varphi(x)}^{s} \tilde{f}(\tau,x)d\tau\right)^{-\frac{1}{2}} ds\, du \quad &= \quad 0,\end{aligned} \tag{3.8}$$

at the same time the existence of a spike solution has been proved. Especially the following result is contained in [1].

Theorem 3.2 *Assume the hypothesis* (H) *is valid. If there exists a simple zero x_0 of the function Φ defined in* (3.8), *that is,*

$$\Phi(x_0) = 0\ ,\ \ \Phi'(x_0) \neq 0, \tag{3.9}$$

then system (3.3) *has a spike solution for sufficiently small μ.*

Following an idea in [10] we give a geometric interpretation of these conditions. By assumption (H) system (3.4) has a family of homoclinic orbits $\gamma(x)$ to the saddle point $(0, \varphi(x))$. We denote by $A(x)$ the area bounded by $\gamma(x)$. Then we have

Theorem 3.3 *Assume hypothesis* (H) *to be valid. If x_0 is a simple zero of Φ then A takes an extremal value at $x = x_0$.*

Proof. By (3.6) we have

$$\frac{A(x)}{2} = \int_{\varphi(x)}^{\psi(x)} w(u,x)du.$$

Using $w(\psi(x),x) \equiv w(\varphi(x),x) \equiv 0$ it follows

$$\begin{aligned}\frac{1}{2}A'(x) \quad &= \quad \int_{\varphi(x)}^{\psi(x)} w_x(u,x)du + w(\psi(x),x)\psi'(x) - w(\varphi(x),x)\varphi'(x) \\ &= \quad \int_{\varphi(x)}^{\psi(x)} w_x(u,x)du.\end{aligned}$$

From (3.6) we get

$$\begin{aligned}w_x(u,x) \quad &= \quad \frac{1}{w(u,x)}\int_{\varphi(x)}^{u} \tilde{f}_x(\sigma,x)d\sigma - \tilde{f}(\varphi(x),x)\varphi'(x) \\ &= \quad \frac{1}{w(u,x)}\int_{\varphi(x)}^{u} \tilde{f}_x(\sigma,x)d\sigma.\end{aligned}$$

By means of changing the order of integration we obtain

$$\begin{aligned}\int_{\varphi(x)}^{\psi(x)} w_x(u,x)du &= \int_{\varphi(x)}^{\psi(x)}\int_{\varphi(x)}^{u} \tilde{f}_x(\sigma,x)\frac{1}{w(u,x)}d\sigma\, du\\ &= \int_{\psi(x)}^{\varphi(x)}\int_{\psi(x)}^{\sigma} \tilde{f}_x(\sigma,x)\frac{1}{w(u,x)}du\, d\sigma = -\Phi(x),\end{aligned}$$

q.e.d.

In what follows we use Theorem 3.2 to prove the existence of a spike solution of (1.1) under the assumptions that $\varepsilon = \mu^2$ and that no convection term arises. Then any stationary solution of (1.1) obeys the system

$$\begin{aligned}\mu^2 u'' &= f(u,v,x)\\ \mu^4 v'' &= g(u,v,x)\end{aligned} \tag{3.10}$$

where $u \in R,\ v \in R^n$. Let us assume that f and g satisfy hypotheses (H_1)–(H_3). Then there is an integral manifold $v = h(u,x,\mu)$ on which system (3.10) reduces to the singularly perturbed second order equation (3.3). Hence, we have the following result

Theorem 3.4 *Let us assume that system* (1.1) *has no convection term and that the corresponding stationary system* (3.10) *is such that the hypotheses* (H_1) - (H_3) *are satisfied. Further we suppose that the corresponding reduced equation* (3.3) *fulfils hypothesis* (H) *and is such that the associated function* Φ *has a simple zero* x_0. *Then system* (1.1) *has a spike solution for sufficiently small* μ.

As an example we consider the well-known Belousov–Zhabotinskii reaction ([7], 179 ff.) which describes the catalyzed oxidation of citric acid by bromate. We use the Oregonator model and take into account the effect of small diffusion. If we look for a stationary solution of the corresponding reaction-diffusion equation we arrive at the system of ordinary differential equations

$$\begin{aligned}\mu^2 u'' &= qu + \zeta u - 2p\psi := f(u,\zeta,\psi,x),\\ \varepsilon^2 \zeta'' &= -qu + \zeta u + \zeta(\zeta - 1) =: g_1(u,\zeta,\psi,x),\\ \varepsilon^2 \psi'' &= \psi - \zeta =: g_2(u,\zeta,\psi,x)\end{aligned} \tag{3.11}$$

where $'$ means the differentiation with respect to the spatial variable x. u, ζ, ψ are normalized concentrations of $HBrO_2, Br^-$, and Ce^{4+}, respectively, ε and μ are small positive parameters characterizing the diffusivities. q, p are usually considered as positive constants. In what follows we suppose $0 < q << 1$ and consider p as a smooth positive function of x satisfying for all x

$$0 < 2p(x) < 1. \tag{3.12}$$

Under these conditions it is our aim to prove the existence of spike solutions to (3.11). System (3.11) can be rewritten in the form (2.1). Hypothesis (H_1) is satisfied if we appropriately modify the right hand side of (3.11) outside some compact region.

For $\varepsilon = 0$ we get from (3.11) the solution

$$\zeta = \psi = h^0(u) := \frac{1 - u + \sqrt{(1-u)^2 + 4qu}}{2} \tag{3.13}$$

which is positive for $q > 0$.

If we compute $g_{1\zeta}$ and $g_{2\psi}$ along this solution we obtain from (3.11)

$$\begin{aligned} \frac{\partial g_1}{\partial \zeta} &= u + 2\zeta - 1 = \sqrt{(1-u)^2 + 4qu} > 0, \\ \frac{\partial g_2}{\partial \psi} &= 1 > 0, \end{aligned} \tag{3.14}$$

that is, the hypotheses (H_2) and (H_3) are valid. Thus, for $\varepsilon = \mu^2$ in some compact region of the phase space there is an integral manifold of (3.11) of the form

$$\begin{aligned} \zeta &= h(u,\mu) = h^0(u) + \mu\, h^1(u) + \cdots \\ \psi &= \bar{h}(u,\mu) = h^0(u) + \mu\, \bar{h}^1(u) + \cdots \end{aligned}$$

where $h^0(u)$ is defined in (3.13). On this integral manifold system (3.11) can be represented in the form

$$\mu^2 u'' = qu + (u - 2p)h^0(u) + \mu f_1(u, x, \mu). \tag{3.15}$$

Now we consider the limit case $q = 0$ in equation (3.15) which leads to the system

$$\begin{aligned} \mu\,\frac{dw}{dx} &= (1-u)(u - 2p(x)) + \mu f_1(u, x, \mu) \\ \mu\,\frac{du}{dx} &= w. \end{aligned} \tag{3.16}$$

The corresponding associated system reads

$$\begin{aligned} \frac{dw}{d\xi} &= (1-u)(u - 2p(x)) := \tilde{f}(u, x) \\ \frac{du}{d\xi} &= w. \end{aligned} \tag{3.17}$$

The equation $\tilde{f}(u,x) = 0$ has the roots

$$u = \varphi(x) \equiv 2p(x) \text{ and } u = \chi(x) \equiv 1.$$

Under the condition (3.12) the derivative of $\tilde{f}(u,x)$ with respect to u at these roots satisfies

$$\tilde{f}_u(u,x)\Big|_{u=\varphi(x)} = 1-2p(x) > 0, \qquad \tilde{f}_u(u,x)\Big|_{u=\chi(x)} = 2p(x)-1 < 0,$$

that is, $(0, 2p(x))$ is a saddle point, and $(0,1)$ is a center of system (3.17).

Let $\psi(x) := 3/2 - p(x)$. By (3.12) we have $\psi(x) > 1 \equiv \chi(x)$. The following relation can be verified

$$\int_{\varphi(x)}^{\psi(x)} (1-u)(u-2p(x))du \equiv 0.$$

Thus, hypothesis (H) is fulfilled for system (3.17). From (3.8) we get

$$\Phi(x) \equiv -p'(x) \int_{2p(x)}^{3/2-p(x)} (1-u) \int_{3/2-p(x)}^{u} \left(\int_{2p(x)}^{s} (1-\tau)(\tau - 2p(x))d\tau \right)^{-1/2} ds\ du.$$

If $\Phi(x)$ has a simple zero at x_0 then by Theorem 3.4 system (3.16), and consequently (3.11) too, has a spike solution. Thus we have the result:

Theorem 3.5 *Let x_0 be a simple zero of $p'(x)$ and such that*

$$\int_{2p(x_0)}^{3/2-p(x_0)} (1-u) \int_{3/2-p(x_0)}^{u} \left(\int_{2p(x_0)}^{s} (1-\tau)(\tau - 2p(x_0))d\tau \right)^{-1/2} ds\ du \neq 0.$$

Then system (3.11) *has a spike solution.*

Let us return now to the general case (3.1). Under our smoothness assumptions we can rewrite system (3.1) in the form

$$\begin{aligned} \mu \frac{dw}{dx} &= F_0(u,w,x) + \mu\ F_1(u,w,x) + \cdots \\ \mu \frac{du}{dx} &= w \end{aligned} \tag{3.18}$$

where $F_0(u,w,x) \equiv f(u,w,h^0(u,w,x),x)$.

In what follows we assume

$$\textbf{(H}_4\textbf{)} \qquad F_0(u,w,x) = F(u,w^2,x). \tag{3.19}$$

Then the associated system to (3.1) reads

$$\begin{aligned} \frac{dw}{d\tau} &= F(u,w^2,x) \\ \frac{du}{d\tau} &= w \end{aligned} \tag{3.20}$$

and is equivalent to the differential equation

$$\frac{1}{2}\frac{d(w^2)}{du} = F(u, w^2, x). \tag{3.21}$$

In order to get the same qualitative behavior of system (3.20) as in case of system (3.4) we replace hypothesis (H) by the following assumption.

$(\tilde{\mathbf{H}})$. Let $F(u, w^2, x)$ be sufficiently smooth and let there be an interval I_0 such that for $x \in I_0$ system (3.20) has the properties:

(i) There are differentiable functions $\varphi : R \to R$ and $\chi : R \to R$ such that $(0, \varphi(x))$ is a saddle point and $(0, \chi(x))$ is a center point of (3.20). Without loss of generality we may assume $\varphi(x) < \chi(x)$.

(ii) System (3.20) has a homoclinic orbit to the saddle point $(0, \varphi(x))$.

The following examples show that the set of systems (3.20) satisfying hypothesis $(\tilde{H})$ is not empty.

Example 3.1. Let us assume that equation (3.21) has the form $(w^2 = z)$

$$\frac{dz}{du} = -u(u + x) + uz. \tag{3.22}$$

In this case we have $\varphi(x) = -x, \chi(x) = 0$, and the integration of equation (3.22) yields

$$z(u) = -e^{\frac{u^2}{2}} \int_{-x}^{u} v(v + x)e^{-\frac{v^2}{2}} dv.$$

It holds $z(-x) = w^2(-x) = 0$. For $-x < u < 0$ the integrand $v(v + x)$ is negative and we have

$$-e^{\frac{u^2}{2}} \int_{-x}^{u} v(v + x)e^{-\frac{v^2}{2}} dv > 0.$$

Thus, $z(0)$ is positive and $z(u)$ stays positive for $0 < u < \psi(x)$ where $\psi(x)$ is defined by

$$\int_{-x}^{\psi(x)} v(v + x)e^{-\frac{v^2}{2}} dv = 0. \tag{3.23}$$

At $u = \psi(x)$ we have

$$z\,|_{u=\psi(x)} = 0, \tag{3.24}$$

$$\left.\frac{dz}{du}\right|_{u=\psi(x)} = -\psi(x)(\psi(x) + x) < 0, \qquad \left.\frac{dw}{du}\right|_{u\to\psi(x)} \longrightarrow \infty. \tag{3.25}$$

In this way, we get the picture as in Fig. 3.

Example 3.2. Suppose that F has the form $F(u, w^2, x) \equiv \frac{1}{2}[uxe^{w^2} - u^2]$ such that we get from (3.21)

$$\frac{dz}{du} = uxe^z - u^2. \tag{3.26}$$

In this case we have $\varphi(x) = 0$, $\chi(x) = x$. The solution of equation (3.26) with $z(0) = 0$ reads

$$z = \ln \frac{1 - \int_0^u v^2 e^{-v^3/3} dv}{1 - \int_0^u x v e^{-v^3/3} dv}.$$

In order to determine a positive function $\psi(x)$ such that $z(\psi(x))$ vanishes we get the equation

$$\int_0^{\psi(x)} (xv - v^2) e^{-v^3/3} dv \quad = \quad 0. \tag{3.27}$$

This situation is analogous to Example 3.1: For $v > x$ the integrand is negative, and there is some $\psi(x) > x$ satisfying equation (3.27). For $0 < u < \psi(x)$ the argument of the logarithm is greater than 1. Thus, z is positive and w exists. For $u > \psi(x)$ the function z is negative and w does not exist. For $u < 0$ the function w also exists since z is positive.

In the sequel we shall derive a condition sufficient for the existence of a spike solution for systems (3.18) satisfying (3.19). To this end we first construct an asymptotic expansion of a solution of (3.18). A crucial point in this construction is to find the first coefficient x_0 in the asymptotic expansion (3.7) of $x_*(\mu)$ where the spike takes its maximum, that is,

$$u'(x_*(\mu), \mu) \equiv 0. \tag{3.28}$$

By an asymptotic expansion of a spike–type solution $(u(x,\mu), w(x,\mu))$ of (3.18) we mean a representation of $u(x,\mu)$ and $w(x,\mu)$ in the form

$$z_a(x,\mu) = Rz(x,\mu) + \Pi z(\xi,\mu) \tag{3.29}$$

where z is a placeholder for u and w respectively, $Rz(x,\mu)$ is the regular part of the asymptotics, that is,

$$Rz(x,\mu) := \sum_{i=0}^{\infty} \mu^i R_i z(x), \tag{3.30}$$

and $\Pi z(\xi,\mu)$ is the boundary layer correction near $x = x_*(\mu)$,

$$\Pi z(\xi,\mu) := \sum_{i=0}^{\infty} \mu^i \Pi_i z(\xi) \tag{3.31}$$

where ξ is the stretched variable $\xi = (x - x_*(\mu))/\mu$.

Let H be some function defined on $R^k \times R \times R$. By means of the representation (3.29) we may rewrite $H(z_a(x,\mu), x, \mu)$ in the form

$$\begin{aligned} H(z_a(x,\mu), x, \mu) \quad = \quad & H(Rz(x,\mu), x, \mu) + H(z_a(\xi\mu + x_*(\mu), \mu), \xi\mu + x_*(\mu), \mu) \\ & - H(Rz(\xi\mu + x_*(\mu), \mu), \xi\mu + x_*(\mu), \mu) =: RH + \Pi H \end{aligned} \tag{3.32}$$

where

$$RH := H(Rz(x,\mu),x,\mu),$$
$$\Pi H := H(z_a(\xi\mu + x_*(\mu),\mu),\xi\mu + x_*(\mu),\mu) - H(Rz(\xi\mu + x_*(\mu),\mu),\xi\mu + x_*(\mu),\mu). \tag{3.33}$$

In order to compute the coefficients $R_i z(x)$ and $\Pi_i z(\xi)$ we substitute (3.29) – (3.31) into (3.18) and use the representation (3.32), (3.33) where H is a placeholder for $F_0, F_1, \ldots$ By equating expressions with the same power of μ (separately for x and ξ) and taking into account Definition 3.1 and Hypothesis ($\tilde{\mathrm{H}}$) we obtain equations which let us determine the unknown coefficients of the asymptotic expansion. This way we get

$$\begin{array}{rclc} Ru_0(x) &=& \varphi(x), & Rw_0 = 0, \\ Ru_1(x) &=& -\frac{F_1(\varphi(x),0,x)}{F_u(\varphi(x),0,x)}, & Rw_1 = \varphi'(x), \ldots \end{array} \tag{3.34}$$

The functions $\Pi_0 u$ and $\Pi_0 w$ satisfy the differential system

$$\begin{aligned} \frac{d}{d\xi}\Pi_0 w &= F(\varphi(x_0) + \Pi_0 u, (\Pi_0 w)^2, x_0) \\ \frac{d}{d\xi}\Pi_0 u &= \Pi_0 w \end{aligned} \tag{3.35}$$

and obey the boundary conditions

$$\Pi_0 u(\xi)\mid_{\pm\infty} = 0, \qquad \Pi_0 w(0) = 0 \tag{3.36}$$

where the last condition follows from (3.28).

The coefficient x_0 is not defined by these equations. The equations for the next approximation read as follows

$$\begin{aligned} \frac{d}{d\xi}\Pi_1 w &= F_u(\xi)\Pi_1 u + 2F_{w^2}\Pi_0 w \Pi_1 w + G_1(\xi) \\ \frac{d}{d\xi}\Pi_1 u &= \Pi_1 w, \end{aligned} \tag{3.37}$$

$$\Pi_1 u\mid_{\pm\infty} = 0, \qquad \Pi_1 w(0) = -\varphi'(0) \tag{3.38}$$

where

$$\begin{aligned} F_u(\xi) &= F_u(\varphi(x_0) + \Pi_0 u(\xi), (\Pi_0 w(\xi))^2, x_0), \\ F_{w^2}(\xi) &= F_{w^2}(\varphi(x_0) + \Pi_0 u(\xi), (\Pi_0 w(\xi))^2, x_0), \\ G_1(\xi) &= [F_u(\xi)\varphi'(x_0) + F_x(\xi)](x_1 + \xi) + \\ &+ F_u(\xi)\bar{u}_1(x_0) + F_1(\varphi(x_0) + \Pi_0 u(\xi), \Pi_0 w(\xi), x_0). \end{aligned}$$

The homogeneous system to (3.37) obviously has the nontrivial solution $(\Pi_0 u)', (\Pi_0 w)'$. This can be verified by differentiating (3.35) with respect to ξ. To get a solution of the inhomogeneous system (3.37) we need that the orthogonality condition must be satisfied. This condition yields the equation defining x_0. This equation is more complicated than the corresponding formula (15) in [1] since the differential operator

$$\frac{d^2}{d\xi^2}\Pi_1 u \quad = \quad F_u(\xi)\Pi_1 u + 2F_{w^2}(\xi)\frac{d}{d\xi}\Pi_1 u$$

is not selfadjoint.
Let us introduce the notation

$$p_1(\xi) := 2F_{w^2}(\xi)\frac{d}{d\xi}\Pi_0 u(\xi) \tag{3.39}$$

and rewrite system (3.37) in the selfadjoint form

$$\frac{d}{d\xi}\left(e^{-\int_0^\xi p_1(\sigma)d\sigma}\frac{d}{d\xi}\Pi_1 u\right) = F_u(\xi)e^{-\int_0^\xi p_1(\sigma)d\sigma}\Pi_1 u \ + \ G_1(\xi)e^{-\int_0^\xi p_1(\sigma)d\sigma}. \tag{3.40}$$

Then the orthogonality condition (the determining equation for x_0) has the form

$$\int_{-\infty}^{+\infty} G_1(\xi)e^{-\int_0^\xi p_1(\sigma)d\sigma}\Pi_0 w d\xi = 0. \tag{3.41}$$

Let us rewrite equation (3.41). From (3.35), (3.36) it follows that $\Pi_0 u(\xi)$ is an even function and $\Pi_0 w(\xi)$ is an odd function of ξ. Thus $(\Pi_0 w)^2$ and $F(\varphi(x_0)+\Pi_0 u, (\Pi_0 w)^2, x_0) =: F(\xi), F_u(\xi), F_{w^2}(\xi)$, and $F_x(\xi)$ are even function of ξ . According to (3.39) the function $p_1(\xi)$ is odd, hence $\int_0^\xi p_1(\sigma)d\sigma$ is even. Therefore, in equation (3.41) only the odd part $\tilde{G}_1(\xi)$ of the function $G_1(\xi)$ plays a role,

$$\tilde{G}_1 := [F_u\varphi'(x_0) + F_x]\xi + \tilde{F}_1(\varphi(x_0) + \Pi_0 u, \Pi_0 w, x_0), \tag{3.42}$$

and equation (3.41) can be written in the form

$$\tilde{\Phi}(x_0) = 0$$

where $\tilde{\Phi}$ is defined by

$$\tilde{\Phi}(x) := \int_0^\infty \left\{[F_u(\xi)\varphi'(x) + F_x(\xi)]\xi + \tilde{F}_1(\varphi(x) + \Pi_0 u, \Pi_0 w, x)\right\} e^{-\int_0^\xi p_1(\sigma)d\sigma}\Pi_0 w d\xi. \tag{3.43}$$

If F_0 and F_1 in (3.18) do not depend on w then $p_1(\xi)$ vanishes and it can be shown that $\tilde{\Phi}(x)$ and $\Phi(x)$ coincide.

Now we are able to formulate the following result.

Theorem 3.6 *Suppose the function $\hat{f}$ in* (3.1) *to be sufficiently smooth. Further we assume that the function F_0 in* (3.18) *satisfies hypothesis* (H$_4$) *and is such that hypothesis* ($\tilde{H}$) *is fulfilled. Finally, we assume that the function $\tilde{\Phi}$ defined in* (3.43) *has a simple zero x_0. Then system* (3.1) *has a spike solution for sufficiently small μ.*

Proof. The proof is similar to the corresponding one in [1]. So we recall only the main idea. Let $x_*^N(\mu)$ be the following asymptotic approximation of $x_*(\mu)$ where the spike solution $u(x,\mu)$ takes its maximum

$$x_*^N(\mu) := x_0 + \mu\, x_1 + \cdots + \mu^N\, x_N.$$

Let δ be a sufficiently small positive number such that there is a small positive μ^* with the property that there are numbers $\underline{x}$ and $\overline{x}$ such that for $0 \le \mu \le \mu^*$

$$\underline{x} < x_*^N(\mu) - \delta\mu^N < x_*^N(\mu) + \delta\mu^N < \overline{x}.$$

Now we look for solutions of (3.18) satisfying the boundary conditions

$$\begin{aligned}
u(\underline{x},\mu) = \varphi(\underline{x}),\quad & u'(x_*^N(\mu) - \delta\mu^N,\mu) = 0,\\
u(\overline{x},\mu) = \varphi(\overline{x}),\quad & u'(x_*^N(\mu) - \delta\mu^N,\mu) = 0,\\
u(\underline{x},\mu) = \varphi(\underline{x}),\quad & u'(x_*^N(\mu) + \delta\mu^N,\mu) = 0,\\
u(\overline{x},\mu) = \varphi(\overline{x}),\quad & u'(x_*^N(\mu) + \delta\mu^N,\mu) = 0.
\end{aligned}$$

To each of these boundary value problems there is a unique solution $\underline{u}_{-\delta}(x,\mu), \overline{u}_{-\delta}(x,\mu)$, $\underline{u}_{+\delta}(x,\mu), \overline{u}_{+\delta}(x,\mu)$ for sufficiently small μ. In general, we have $\underline{u}_{-\delta}(x_*^N(\mu) - \delta\mu^N,\mu) \neq \overline{u}_\delta(x_*^N(\mu) - \delta\mu^N,\mu)$, that is, from $\underline{u}_{-\delta}$ and $\overline{u}_{-\delta}$ we do not get a spike solution. The same is valid for $\underline{u}_{+\delta}$ and $\overline{u}_{+\delta}$. Let $\underline{u}_{\mp\delta}^{N+1}$ and $\overline{u}_{\mp\delta}^{N+1}$ be asymptotic expansions of order $N+1$ to these solutions, that is,

$$\begin{aligned}
\underline{u}_{\mp\delta}(x,\mu) &= \underline{u}_{\mp\delta}^{N+1}(x,\mu) + O(\mu^{N+2}),\\
\overline{u}_{\mp\delta}(x,\mu) &= \overline{u}_{\mp\delta}^{N+1}(x,\mu) + O(\mu^{N+2}).
\end{aligned}$$

Let

$$\begin{aligned}
\Delta_{-\delta,\mu} &:= \overline{u}_{-\delta}^{N+1}(x_*^N(\mu) - \delta\mu^N,\mu) - \underline{u}_{-\delta}^{N+1}(x_*^N(\mu) - \delta\mu^N,\mu),\\
\Delta_{+\delta,\mu} &:= \overline{u}_{+\delta}^{N+1}(x_*^N(\mu) + \delta\mu^N,\mu) - \underline{u}_{+\delta}^{N+1}(x_*^N(\mu) + \delta\mu^N,\mu).
\end{aligned}$$

It can be verified that $\Delta_{-\delta,\mu}$ and $\Delta_{+\delta,\mu}$ have different sign for given δ and for sufficiently small μ. Therefore, there exists a unique $x_*(\mu)$ located between $x_*^N(\mu) - \delta\mu^N$ and $x_*^N(\mu) + \delta\mu^N$ such that the solutions $\underline{u}(x,\mu)$ and $\overline{u}(x,\mu)$ of the boundary value problems

$$\begin{aligned}
u(\underline{x},\mu) = \varphi(\underline{x}),\quad & u'(x_*(\mu),\mu) = 0,\\
u(\overline{x},\mu) = \varphi(\overline{x}),\quad & u'(x_*(\mu),\mu) = 0
\end{aligned}$$

satisfy $\underline{u}(x_*(\mu),\mu) = \overline{u}(x_*(\mu),\mu)$. Consequently, there exists a spike solution $u_s(x,\mu)$ of (3.18) with the first order asymptotic approximation

$$\begin{aligned}
u_s(x,\mu) &= \varphi(x) + \Pi_0 u\left(\frac{x - x_*(\mu)}{\mu}\right) + O(\mu),\\
w_s(x,\mu) &= \Pi_0 w\left(\frac{x - x_*(\mu)}{\mu}\right) + O(\mu),
\end{aligned}$$

q.e.d.

References

[1] V.F. Butuzov and A.B. Vasil'eva, On the asymptotics of a solution of contrast structure type. (in Russian) Mat. Zam. 42, 831–841 (1987).

[2] A.B. Vasil'eva, V.F. Butuzov and L.V. Kalachev, The boundary function method for singular perturbation problems. SIAM Studies in Applied Mathematics, 1995.

[3] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations. J. Diff. Equs. 31, 53–98 (1979).

[4] L. Flatto and N. Levinson, Periodic solutions of singularly perturbed systems, J. Rat. Mech. Anal. 4, 943–950 (1955).

[5] K.O. Friedrichs and W. Wasow, Singular perturbations of nonlinear oscillations. Duke Math. J. 13, 367–381 (1946).

[6] F. Hoppensteadt, Properties of solutions of ordinary differential equations with small parameters. Comm. Pure Appl. Math. 24, 807–840 (1971).

[7] J.D. Murray, Mathematical Biology. Springer–Verlag, 1989.

[8] E.F. Mishchenko and L.S. Pontryagin, Periodic solutions of differential systems near discontinuous ones. (in Russian) Dokl. Akad. Nauk SSSR 102, 889–891 (1955).

[9] Yu.A. Mitropol'skij and O.B. Lykova, On an integral manifold of nonlinear differential equations containing slow and fast variables. (in Russian) Ukr. Mat. Zh. 16, 157 –163 (1964).

[10] H. Oka, Singular perturbations of autonomous ordinary differential equations and heteroclinic bifurcations. In: Dynamical Systems, Pitman Res. Notes Math. Ser. 285, 159 - 194 (1993).

[11] V.V. Strygin and V.A. Sobolev, Separation of motion by means of integral manifolds. (in Russian) Moscow. Nauka, 1988.

[12] A.N. Tikhonov, On dependence of solutions of differential equations on a small parameter. (in Russian) Mat. Sb. 22 (64), 193–204 (1948).

[13] A.N. Tikhonov, Systems of differential equations containing small parameters. (in Russian) Mat. Sb. 31 (73), 575–586 (1951).

[14] M.I. Vishik and L.A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with a small parameter. (in Russian) Usp. Mat. Nauk 12, no. 5, 3–122 (1957).

[15] A.B. Vasil'eva and V.F. Butuzov, Asymptotic Expansions of Solutions of Singularly Perturbed Equations. (in Russian) Moscow, Nauka, 1973.

[16] A.B. Vasil'eva and V.F. Butuzov, Asymptotic Methods in the Theory of Singular Perturbations. (in Russian) Moscow, Vysshaja Shkola, 1990.

[17] K.V. Zadiraka, On an integral manifold of differential equations containing a small parameter. (in Russian) Dokl. Akad. Nauk SSSR 115, 646–647 (1957).

Recent publications of the
Weierstraß–Institut für Angewandte Analysis und Stochastik

Preprints 1995

141. Ralf Kornhuber: A posteriori error estimates for elliptic variational inequalities.

142. Johannes Elschner, Youngmok Jeon, Ian H. Sloan, Ernst P. Stephan: The collocation method for mixed boundary value problems on domains with curved polygonal boundaries.

143. Johannes Elschner, Ernst P. Stephan: A discrete collocation method for Symm's integral equation on curves with corners.

144. Dietmar Hömberg: A numerical simulation of the Jominy end–quench test.

145. Sabine Hengst: On the existence of classical solutions for a two phase flow through saturated porous media.

146. Anton Bovier, Véronique Gayrard: An almost sure large deviation principle for the Hopfield model.

147. Hans Babovsky: Limit theorems for deterministic Knudsen flows between two plates.

148. Björn Sandstede: Stability of multiple–pulse solutions.

149. Björn Sandstede: Constructing dynamical systems possessing homoclinic bifurcation points of codimension two.

150. Boris N. Khoromskij, Siegfried Prössdorf: Multilevel preconditioning on the refined interface and optimal boundary solvers for the Laplace equation.

151. Anton Bovier, Christof Külske: There are no nice interfaces in 2+1 dimensional SOS–models in random media.

152. Ilja Schmelzer: Covariant geometry description.

153. Alexander Korostelev, Michael Nussbaum: Density estimation in the uniform norm and white noise approximation.

154. Peter Hall, Michael Nussbaum, Steven E. Stern: On the estimation of a support curve of indeterminate sharpness.

155. Lev D. Pustyl'nikov: On the stability of solutions and absence of Arnol'd diffusion in a nonintegrable Hamiltonian system of a general form with three degrees of freedom.

156. Ralf Kornhuber: Adaptive monotone multigrid methods for some non–smooth optimization problems.

157. Sergej Rjasanow, Wolfgang Wagner: A generalized collision mechanism for stochastic particle schemes approximating Boltzmann type equations.

158. Nikolai Nefedov, Klaus Schneider: Singularly perturbed systems: Case of exchange of stability.

159. Rainer Dahlhaus, Michael H. Neumann, Rainer von Sachs: Nonlinear wavelet estimation of time–varying autoregressive processes.

160. Henri Schurz: Numerical regularization for SDEs: Construction of nonnegative solutions.

161. Anton Bovier, Véronique Gayrard: The retrieval phase of the Hopfield model: A rigorous analysis of the overlap distribution.

162. Boris N. Khoromskij, Gunther Schmidt: A fast interface solver for the biharmonic Dirichlet problem on polygonal domains.

163. Michael H. Neumann: Optimal change–point estimation in inverse problems.

164. Dmitry Ioffe: A note on the extremality of the disordered state for the Ising model on the Bethe lattice.

165. Donald A. Dawson, Klaus Fleischmann: A continuous super–Brownian motion in a super–Brownian medium.

166. Norbert Hofmann, Peter Mathé: On quasi–Monte Carlo simulation of stochastic differential equations.

167. Henri Schurz: Modelling, analysis and simulation of stochastic innovation diffusion.

168. Annegret Glitzky, Rolf Hünlich: Energetic estimates and asymptotics for electro–reaction–diffusion systems.

169. Pierluigi Colli, Jürgen Sprekels: Remarks on the existence for the one–dimensional Frémond model of shape memory alloys.