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Persistence of rogue waves in extended nonlinear Schrödinger equations: Integrable Sasa-Satsuma case

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Abstract

We present the lowest order rogue wave solution of the Sasa-Satsuma equation (SSE) which is one of the integrable extensions of the nonlinear Schrödinger equation (NLSE). In contrast to the Peregrine solution of the NLSE, it is significantly more involved and contains polynomials of fourth order rather than second order in the corresponding expressions. The correct limiting case of Peregrine solution appears when the extension parameter of the SSE is reduced to zero.

1 Introduction

Sasa-Satsuma equation (SSE) is one of the existing integrable extensions of the NLSE. Although with fixed relation between higher order terms, it contains the most essential contributions often found in important physical applications: dynamics of deep water waves [1, 2], pulse propagation in optical fibres [3, 4] and generally in dispersive nonlinear media [5]. Namely, it contains the terms describing the third order dispersion, the self-frequency shift and self-steepening in fixed proportions providing integrability. According to the original work of Sasa and Satsuma [6], the equation can be written as:

$$i\psi_\tau + \frac{\psi_{xx}}{2} + |\psi|^2\psi = i\epsilon\left[\psi_{xxx} + 3(|\psi|^2)_x\psi + 6|\psi|^2\psi_x\right]. \tag{1}$$

Here, an arbitrary real parameter ϵ scales the integrable perturbations of NLSE. When $\epsilon = 0$, Eq. (1) reduces to standard NLSE which has only the terms describing lowest order dispersion and self-phase modulation.

There is a number of publications dealing with the solutions of SSE [7, 8, 9, 10, 11, 12, 13]. The same form of equation, i.e. (1) has been used in the series of works by Mihalache et al. [7, 8, 9]. The form of the SS equation in the work by Wright III [10] is slightly different from the original version. Namely, for the focusing case he used the form

$$ip_t - p_{xx} - 2|p|^2p = i\delta\left[p_{xxx} - 3(|p|^2)_x p + 6(|p|^2p)_x\right]. \tag{2}$$

In addition to simple rescaling, in this form, the self-steepening (last) term is explicitly singled out.

The two equations (1) and (2) are equivalent under the following transformation:

$$\psi(x,t) = p^*(x,t), \quad \tau = 2t, \quad \delta = 2\epsilon. \tag{3}$$

This means that we can easily transform solutions of one equation into solutions of the other using (3). Also, changing the sign of ϵ or δ is equivalent to changing the direction in x. For convenience of comparison with previous rogue wave solutions of the NLSE [14, 15], here, we will work with the equation (1) rather than (2).

When the parameter ϵ is small, rogue wave solutions of (1) can be found as a perturbation of the Peregrine soliton of NLSE [16]. Another way of extending the NLSE rational solution is to consider integrable cases. Namely, the rational solutions of the Hirota equation (HE)

$$i\psi_\tau + \frac{\psi_{xx}}{2} + |\psi|^2\psi = i\epsilon\left[\psi_{xxx} + 6|\psi|^2\psi_x\right] \tag{4}$$

when only two additional terms are present in Eq.(1) have been presented in [17]. HE is a relatively simple case and its solutions can be obtained from the NLSE solutions just adding a velocity. In contrast to the case of HE, the solutions of Sasa-Satsuma equation are significantly more complicated. This complexity can be seen even considering relatively simple soliton solutions [7, 8, 9]. Finding rogue wave solutions is more complicated and generally requires special methods that are beyond ordinary inverse scattering technique [18, 19, 20].

Here, we present the lowest order rogue wave solution of SSE that we obtained based on direct spectral analysis of Wright III [10] which is different from the original Sasa-Satsuma version [6]. This method is not straightforward though because the technique developed by Wright III allows us to find only solutions that are singular in the limit of NLSE. This singularity must be avoided if we want the rogue wave solution to be an extension of the Peregrine solution of the NLSE. Indeed, the solution presented in our work has this important property.

Before presenting the rogue wave solution, let us give some introductory remarks. As we know, the rational solutions of the NLSE are always located on a background plane wave. They are tightly related to modulation instability of a plane wave and represent the infinite period limit of modulation instability breathers. Thus, in the first instance we have to consider the plane wave solution and its stability properties.

Indeed, Sasa-Satsuma equation (1) allows for plane-wave solutions in the form

$$\psi_0(x,\tau) \quad = \quad -\frac{c}{2\epsilon}\,\exp\left[i\left(\frac{k}{2\epsilon}x + \frac{\omega}{8\epsilon^2}\tau\right)\right] \tag{5}$$

where the amplitude c, the wavenumber k and the frequency ω are coupled through the dispersion relation

$$\omega = 2c^2 - k^2 + k(6c^2 - k^2). \tag{6}$$

The plane wave (5) serves as a background for rogue waves. Despite it looks singular with respect to the parameter ϵ, we can always adjust c, k and ω to be of the order of ϵ. When taking the NLSE limit, $\epsilon \to 0$, we can take c and k to be directly proportional to ϵ while taking $\omega \sim \epsilon^2$ to eliminate the singularity.

The plane wave solution (5) can be unstable to small amplitude modulations. Indeed, taking one of the Fourier-modes of perturbation in the form

$$\Delta = A\exp[i\kappa(x - \Omega\tau)] + B^*\exp[-i\kappa(x - \Omega^*\tau)],$$

where A and B are small amplitudes of perturbation, substituting $\psi = \psi_0(1 + \Delta)$ into the Sasa-Satsuma equation (1) and linearising around the plane wave solution (5) we find that it is unstable with the growth rate γ given by the imaginary part of $\kappa\Omega$:

$$\gamma = \frac{\kappa}{4\epsilon}\sqrt{4(3k+1)^2\left(c^2 - \kappa^2\epsilon^2\right) - 9c^4}. \tag{7}$$

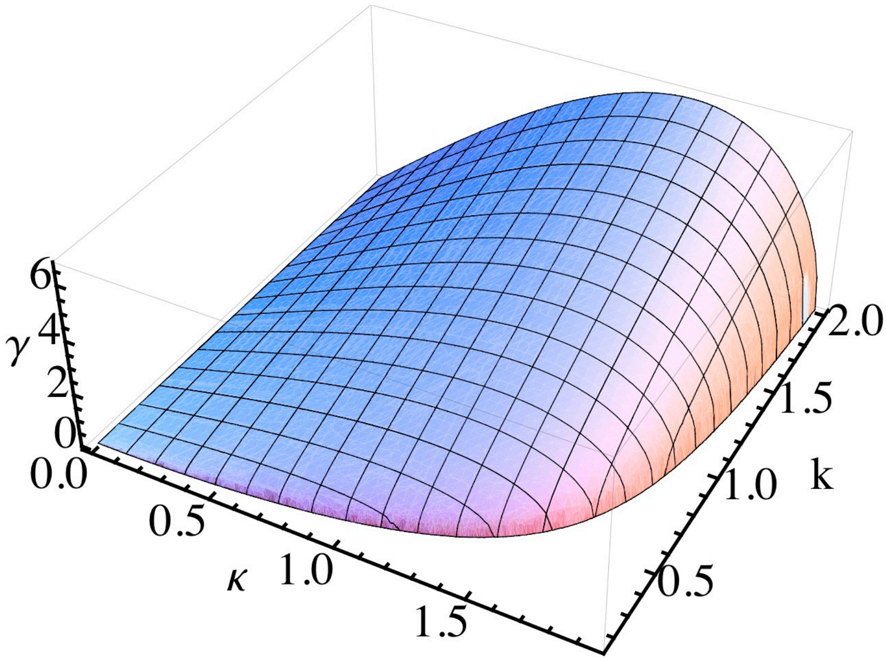

Figure 1: (a) Growth rate of instability vs. perturbation frequency κ and the plane wave wavenumber k. Here parameters $c = 1$ and $\epsilon = 0.5$.

The dependence of the growth rate γ on κ and k is shown in Fig.1. The plot is shown only for positive values of κ and k. The growth rate is real in a range of frequencies κ within the curves

$$\kappa = \pm\frac{c\sqrt{4(3k+1)^2 - 9c^2}}{2(3k+1)\epsilon}. \tag{8}$$

The growth rate is also zero at $\kappa = 0$ where the period of modulation increases to infinity. In this limit, only one maximum of modulation remains and this is the limit we are interested in. Here, k also has to satisfy the condition

$$4(1+3k)^2 > 9c^2. \tag{9}$$

In the NLSE case, the plane wave wavenumber k can be easily introduced through the Galilean boost [21]. Thus, any solution can be parametrized with this variable through an easy transformation. In case of SSE, this is not trivial and we have to keep this variable in all calculations.

Within the limitations given above, the solution describing the rogue wave of SSE (1) is given by:

$$\psi(x,\tau) = -\frac{c}{2\epsilon}\left(1 - \frac{\zeta-\zeta^*}{c}G\right)\exp\left[i\left(\frac{k}{2\epsilon}x + \frac{\omega}{8\epsilon^2}\tau\right)\right], \tag{10}$$

with $\omega = 2c^2 - k^2 + (6kc^2 - k^3)$, i.e. the same as (6), and

$$G = \frac{|u|^2 \mathrm{Re}[\zeta](\zeta u^* g + \zeta^* u h^*) + (\zeta |g|^2 + \zeta^* |h|^2)(\zeta^* u^* g + \zeta u h^*)}{|\zeta|^2 \left(|u|^2 + |g|^2 + |h|^2\right)^2 - |u^2 + 2hg|^2 \mathrm{Im}[\zeta]^2} \tag{11}$$

where

$$\begin{aligned}
\zeta &= \pm \frac{i\sqrt{9c^2\left(9c^2 + 10K^2\right) + 3c\left(9c^2 - 4K^2\right)^{3/2} - 2K^4}}{3\sqrt{2}K}, \\
u &= \left(\frac{v_{21}}{2}\,\tau - 2\epsilon x\right), \\
h &= 3c\left(\frac{u}{M_1} + i\frac{12\epsilon^2}{{M_1}^2}\right), \\
g &= 3c\left(\frac{u}{M_2} - i\frac{12\epsilon^2}{{M_2}^2}\right), \\
M_1 &= K + d - \zeta, \\
M_2 &= K - d + \zeta, \\
d &= \left(\frac{b}{2} + \frac{2\left(K^2 + 18c^2 + 3\zeta^2\right)}{3b}\right), \\
b &= \left(-1 + i\sqrt{3}\right)\left[\left(K^2 - 9c^2 - \zeta^2\right)\zeta\right]^{1/3}, \\
v_{21} &= \frac{9\left(a - 6c^2\right)\zeta^4 + 3a\left(a - 1 - 18c^2\right)\zeta^2 + a^3}{3\left(2\zeta^2 + d\zeta + a\right)^2}, \\
a &= \frac{\left(K^2 - 3 - 36c^2\right)}{3}.
\end{aligned}$$

The solution depends on variables x and τ, as well as on three real parameters ϵ, $K = 1 + 3k$, and c which is proportional to the background plane wave amplitude $c/(2\epsilon)$.

Comparison of this rogue wave solution to the one of the NLSE [14] or Hirota [17] equations shows that SSE rogue wave has significantly more complicated structure. In particular, it involves polynomials of fourth order rather than second order as in the two previous cases. This can be seen from the structure of the expression (11) with the nominator and denominator being of fourth power of u. These complications are related to the fact that the spectral problem for SSE involves 3×3 matrices rather than 2×2 as for the two other cases. Consequently, soliton, rogue wave or any other similar solutions take significantly more complex forms.

Rogue waves do exist provided that $9c^2 - 4K^2 < 0$. This follows from the requirement for the eigenvalue ζ of the spectral problem to have a nonzero real part. This happens when $|1+3k| > 3c/2$. For positive c, this is equivalent to (9). Explicitly, the condition is either

$$k > \frac{c}{2} - \frac{1}{3} \quad \text{or} \quad k < -\frac{c}{2} - \frac{1}{3}. \tag{12}$$

Thus, the wavenumber k can be zero only when $c < 2/3$. Otherwise, the plane wave propagation direction has to be skewed for the rogue wave to exist.

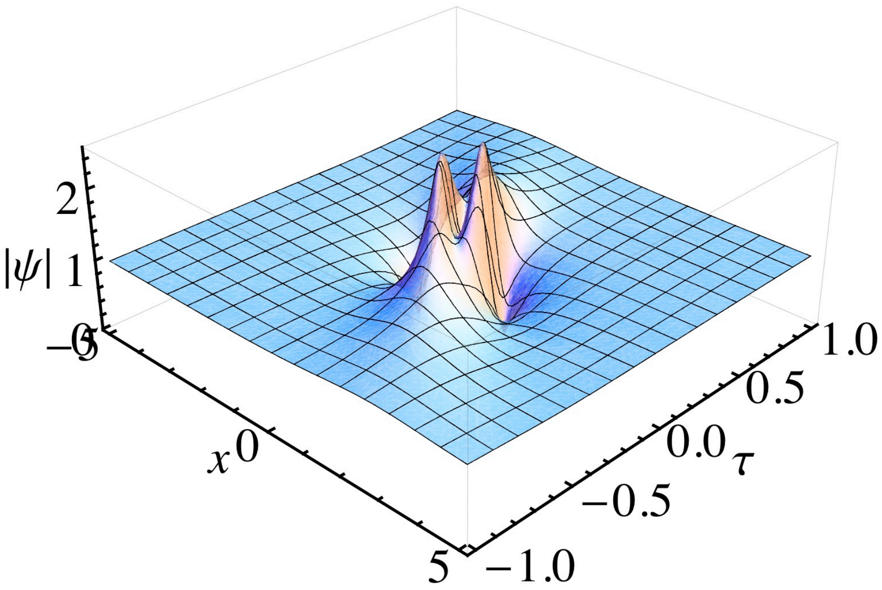

Figure 2: Rogue wave of the Sasa-Satsuma equation when $c = 1$, $\epsilon = 0.5$, and plane wave wavenumber $k = 0.8$.

The solution (10) is illustrated in Fig.2 for the values of parameters $c = 1$, $\epsilon = 0.5$ and $k = 0.8$. It exhibits a double peak and has a maximum amplitude of around 2.5. The possibility of having a double peak structure by the rogue waves of extended NLSE has already been noticed in [16]. The background amplitude is $c/(2\epsilon)$, which is equal to 1 here. For $c = 1$ and any ϵ, the wavenumber k has to be larger than 1/6. Thus, the plane wave propagates at an angle to the τ axis. The solution itself is also tilted. The solution keeps double peak structure at all values of k in the interval $1/6 < k \lesssim 2$. However, at larger values of k the two maxima merge and the solution has a single peak. An example is shown in Fig.3 where $k = 2$.

Despite seemingly singular structure of the solution with ϵ being in the denominator, the amplitude of the background for the rogue wave is finite and equal to $\psi_0 = c/(2\epsilon)$ just as in (5). We can keep it to be constant having the ratio c/ϵ to be a constant. When taking the limit $\epsilon \longrightarrow 0$, we should simultaneously take the limit of $c \longrightarrow 0$. Then k and ω should also be considered in the same zero limit. This should be done carefully, to keep k in the limits (12).

When reducing the parameter ϵ to zero, and having the plane wave wavenumber $k = 0$, the limit is rather complicated but admits factorisation in the nominator and denominator. After cancellation of the same polynomial expression $4^8(2\tau^2 + 2x(x+1)+1)$ of the second order in the nominator and denominator, the degree of polynomials is reduced and the solution is simplified to the rogue wave of the NLSE

$$\psi = -\left[1 + \frac{4i(\tau + ix)(i\tau + x + 1)}{2\tau^2 + 2x(x+1) + 1}\right] e^{i\tau}. \tag{13}$$

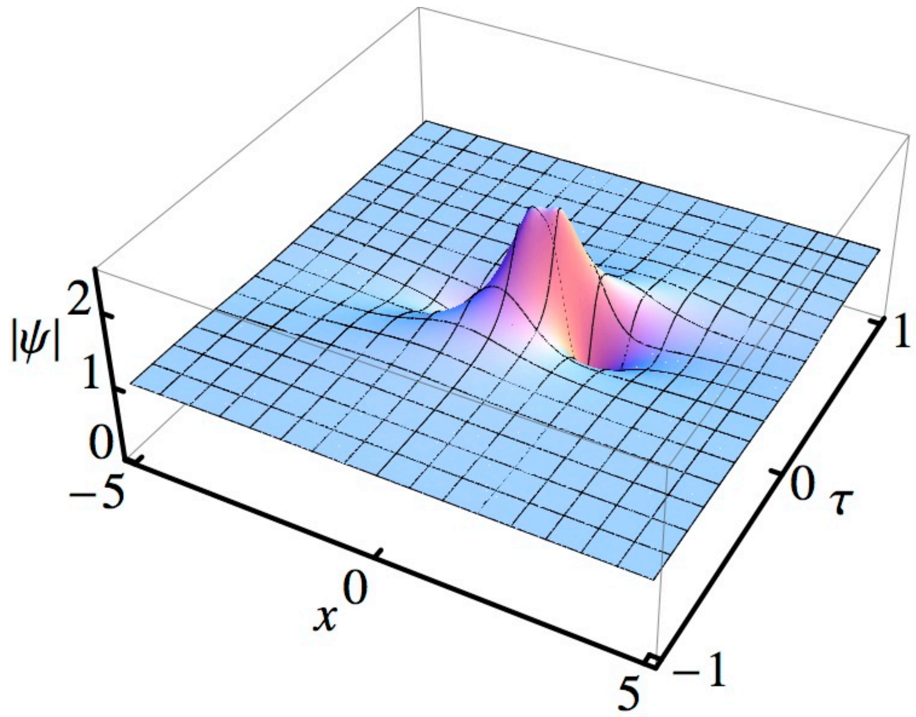

Figure 3: Rogue wave of the Sasa-Satsuma equation when $c = 1$, $\epsilon = 0.5$, and plane wave wavenumber $k = 2$.

After simple translation $x \to x - 1/2$ along the x-axis, we obtain the standard expression for the Peregrine solution:

$$\psi = \left[1 - 4\frac{1 + 2i\tau}{1 + 4\tau^2 + 4x^2}\right] e^{i\tau}. \tag{14}$$

It is shown in Fig.4. In normalised form, it does not have any free parameters.

For the sake of completeness, we also give here the lowest order rogue wave solution of the Hirota equation (4) presented earlier in [17]:

$$\psi(x, \tau) = -\left[1 - 4\frac{1 + 2i\tau}{1 + 4\,(x + 6\epsilon\tau)^2 + 4\tau^2}\right] e^{i\tau}. \tag{15}$$

As we can see, the only difference of (15) from the Peregrine solution (14) of the NLSE is the "velocity"term $6\epsilon\tau$. Clearly, extending the equation from the NLSE and Hirota cases to Sasa-Satsuma version makes the dramatic increase in the complexity of their solutions.

From physical point of view, two additional terms in Hirota equation responsible for third order dispersion and self-frequency shift are perfectly balanced. As a result, the Peregrine solution is having only trivial velocity shift. The presence of the third term in SSE ruins this delicate balance thus resulting in more complicated structure of the rogue wave. Revealing this influence of additional terms on rogue waves within and beyond the integrable cases deserves further study.

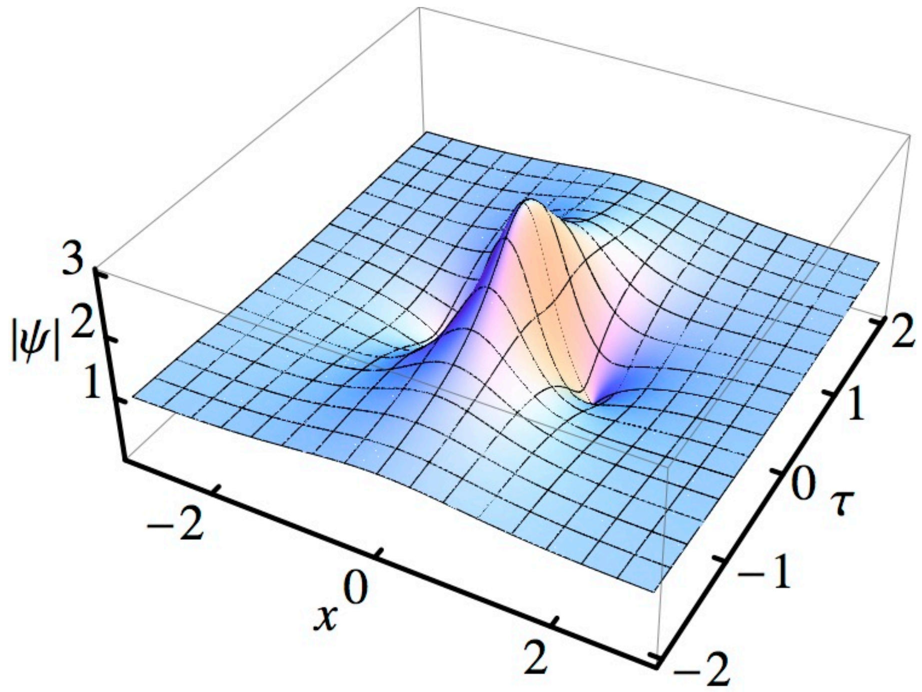

Figure 4: Rogue wave of the NLSE, according to (14).

In conclusion, we presented, for the first time, the lowest order rogue wave solution of the Sasa-Satsuma equation. Despite the fact that SSE is one of the integrable extensions of the nonlinear Schrödinger equation, its solutions are significantly more complicated than the corresponding NLSE counterparts. In particular, the rational solution that we have found is given by the polynomials of the fourth order rather than polynomials of the second order in the NLSE limit. We illustrated the solution for various values of three real parameters: background amplitude of the plane wave, its transverse wavenumber, and the parameter of the SSE which is responsible for deviation of the equation from the NLSE case.

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