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Global existence for a strongly coupled Cahn–Hilliard system with viscosity

In memory of Enrico Magenes

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Abstract

An existence result is proved for a nonlinear diffusion problem of phase-field type, consisting of a parabolic system of two partial differential equations, complemented by Neumann homogeneous boundary conditions and initial conditions. This system is meant to model two-species phase segregation on an atomic lattice under the presence of diffusion. A similar system has been recently introduced and analyzed in [3]. Both systems conform to the general theory developed in [5]: two parabolic PDEs, interpreted as balances of microforces and microenergy, are to be solved for the order parameter ρ and the chemical potential μ. In the system studied in this note, a phase-field equation in ρ fairly more general than in [3] is coupled with a highly nonlinear diffusion equation for μ, in which the conductivity coefficient is allowed to depend nonlinearly on both variables.

1 Introduction

In this paper, we prove an existence result for the following system in the unknown fields μ and ρ:

$$\big(1+2g(\rho)\big)\,\partial_t\mu+\mu\,g'(\rho)\,\partial_t\rho-\operatorname{div}\big(\kappa(\mu,\rho)\nabla\mu\big)=0, \tag{1.1}$$

$$\partial_t\rho-\Delta\rho+f'(\rho)=\mu\,g'(\rho), \tag{1.2}$$

$$\big(\kappa(\mu,\rho)\nabla\mu\big)\cdot\nu|_\Gamma=0 \quad\text{and}\quad \partial_\nu\rho|_\Gamma=0, \tag{1.3}$$

$$\mu(0)=\mu_0 \quad\text{and}\quad \rho(0)=\rho_0. \tag{1.4}$$

Each of the partial differential equations (1.1)–(1.2) is meant to hold in a three-dimensional bounded domain Ω, endowed with a smooth boundary Γ, and in some time interval $[0,T]$. Such a system generalizes the phase-field model of Cahn–Hilliard type studied recently in [3]. Both models are of the type proposed in [5], and aim to describe phase segregation of two species (atoms and vacancies, say) on a lattice in presence of diffusion. The state variables are the *order parameter* ρ, interpreted as the volume density of one of the two species, and the *chemical potential* μ. For physical reasons, μ is required to be nonnegative, while the phase parameter ρ must, as such, obey $0\le\rho\le1$. Here are the features of [3] that have been generalized.

Firstly, the nonlinearity f considered in [3] is a double-well potential defined in $(0,1)$, whose derivative f' diverges at the endpoints $\rho=0$ and $\rho=1$: e.g., for $f=f_1+f_2$ with f_2 smooth, one can take $f_1(\rho)=c\,(\rho\,\log(\rho)+(1-\rho)\,\log(1-\rho))$, with c a positive constant. In this paper, we let f_1 be a maximal monotone graph from $\mathbb{R}$ to $\mathbb{R}$. Consequently, equation (1.2) has to be read as a differential inclusion, in which the derivative of the convex part f_1 of f is replaced by the subdifferential $\beta:=\partial f_1$, i.e.,

$$\partial_t\rho-\Delta\rho+\xi+f_2'(\rho)=\mu g'(\rho) \quad\text{with}\quad \xi\in\beta(\rho); \tag{1.5}$$

moreover, since f_1 is not required to be smooth, its subdifferential may be multivalued; the selection of ξ in $\beta(\rho)$ is a further difficulty we face.

Secondly, while in [3] $g(\rho) = \rho$, here g is any nonnegative-valued smooth function, defined (at least) in the domain where f_1 and its derivative (or rather, its subdifferential) live.

Thirdly, and this is the most important novelty, conductivity κ is not anymore a constant, but rather a positive-valued, continuous, bounded, and possibly nonlinear, function of μ and ρ. For simplicity, we confine ourselves to study the existence of a solution under an assumption that guarantees uniform parabolicity, i.e., $\kappa \geq \kappa_* > 0$. We point out that in a recent study [4] we let κ depend only on μ and possibly degenerate somewhere.

Finally, relations (1.4) specify the initial conditions for μ and ρ, while (1.3) are nothing but homogeneous boundary conditions of Neumann type, involving precisely those boundary operators that match the elliptic differential operators in (1.1)–(1.2).

Our paper is organized as follows. In the next section, we state our assumptions and our results. The existence of a solution is proved in Section 3, making use of a time-delay approximation and of a number of *a priori estimates*, that allow us to pass to the limit by compactness and monotonicity techniques.

2 Results

In this section, we describe the mathematical problem under investigation, make our assumptions precise, and state our results. First of all, we assume Ω to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ (treating lower-dimensional cases would require only minor changes). Next, we fix a final time $T \in (0, +\infty)$ and set:

$$Q := \Omega \times (0,T), \quad \Sigma := \Gamma \times (0,T), \tag{2.1}$$
$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0 \text{ on } \Gamma\}. \tag{2.2}$$

We endow the spaces (2.2) with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$; for powers of these spaces, norms are denoted by the same symbols. We remark that the embeddings $W \subset V \subset H$ are compact, because Ω is bounded and smooth. Moreover, for $p \in [1, +\infty]$, we write $\|\cdot\|_p$ for the usual norm in $L^p(\Omega)$; as no confusion can arise, the symbol $\|\cdot\|_p$ is used for the norm in $L^p(Q)$ as well.

First of all, we present the structural assumptions we make. We require that:

$$\kappa : (m,r) \mapsto \kappa(m,r) \text{ is continuous from } [0,+\infty) \times \mathbb{R} \text{ to } \mathbb{R}, \tag{2.3}$$
$$\text{the partial derivatives } \partial_r \kappa \text{ and } \partial_r^2 \kappa \text{ exist and are continuous,} \tag{2.4}$$
$$\kappa_*, \kappa^* \in (0,+\infty), \tag{2.5}$$
$$\kappa_* \leq \kappa(m,r) \leq \kappa^*, \ \ |\partial_r \kappa(m,r)| \leq \kappa^*, \ \ |\partial_r^2 \kappa(m,r)| \leq \kappa^* \quad \text{for } m \geq 0 \text{ and } r \in \mathbb{R} \tag{2.6}$$
$$f = f_1 + f_2\,, \quad f_1 : \mathbb{R} \to [0,+\infty], \quad f_2 : \mathbb{R} \to \mathbb{R}, \quad g : \mathbb{R} \to [0,+\infty), \tag{2.7}$$
$$f_1 \text{ is convex, proper, l.s.c. and } f_2 \text{ and } g \text{ are } C^2 \text{ functions,} \tag{2.8}$$
$$f_2',\ g, \text{ and } g' \text{ are Lipschitz continuous.} \tag{2.9}$$

For convenience, we set:

$$\kappa' := \partial_r \kappa, \quad \kappa'' := \partial_r^2 \kappa, \quad \beta := \partial f_1 \,, \quad \text{and} \quad \pi := f_2'; \tag{2.10}$$

$$K(m,r) := \int_0^m \kappa(s,r)\,ds,\ K_1(m,r) := \int_0^m \kappa'(s,r)\,ds,\ K_2(m,r) := \int_0^m \kappa''(s,r)\,ds$$
$$\text{for } m \geq 0 \text{ and } r \in \mathbb{R}; \tag{2.11}$$

and we write $D(f_1)$ and $D(\beta)$ for the effective domains of f_1 and β, respectively. Clearly, thanks to (2.6),

$$\max\{|K(m,r)|, |K_1(m,r)|, |K_2(m,r)|\} \leq \kappa^* m \quad \text{for every } m \geq 0 \text{ and } r \in \mathbb{R}. \tag{2.12}$$

We also note that the structural assumptions of [4] are fulfilled if κ only depends on m, and that a strong singularity in equations (1.2) for ρ is allowed. At variance with [4], equation (1.1) for μ is here uniformly parabolic, since g is nonnegative and κ is bounded away from zero.

Remark 2.1. Note that any convex, proper, l.s.c. function is bounded from below by an affine function (cf., e.g., [1, Prop. 2.1, p. 51]), so that the assumption $f_1 \geq 0$ looks reasonable, because one can suitably modify the smooth perturbation f_2. Moreover, let us point out that the other positivity condition, $g \geq 0$, is just needed on the set $D(\beta)$, while g can be extended outside of $D(\beta)$ accordingly.

As to initial data, we require that:

$$\mu_0 \in V \cap L^\infty(\Omega), \quad \rho_0 \in W, \quad \mu_0 \geq 0 \quad \text{and} \quad \rho_0 \in D(\beta) \quad \text{a.e. in } \Omega; \tag{2.13}$$
$$\text{there exists } \xi_0 \in H \text{ such that } \xi_0 \in \beta(\rho_0) \text{ a.e. in } \Omega. \tag{2.14}$$

Since f_1 is convex and f_2 smooth, the above assumptions imply that $f(\rho_0) \in L^1(\Omega)$.

As to the a priori regularity we require for any solution (μ, ρ, ξ), we begin to observe that, for any given μ, equation (1.5) has the form of a standard phase-field equation. Therefore, it is natural to look for pairs (ρ, ξ) that satisfy

$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.15}$$
$$\xi \in L^\infty(0,T;H). \tag{2.16}$$

Note that the Neumann boundary condition for ρ has been incorporated into (2.15) (cf. $(2.2)_3$). Next, as to μ, we require that

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^\infty(Q), \quad \mu \geq 0 \quad \text{a.e. in } Q, \tag{2.17}$$
$$\operatorname{div}\big(\kappa(\mu,\rho)\nabla\mu\big) \in L^2(0,T;H) \quad \text{and} \quad \big(\kappa(\mu,\rho)\nabla\mu\big)\cdot\nu = 0 \quad \text{a.e. on } \Sigma, \tag{2.18}$$

and note that we can expect that $\mu \in L^2(0,T;W)$ (from the regularity theory of elliptic equations) only if the function κ is smooth with respect to both variables. Nevertheless, (2.17) and the regularity of the divergence are sufficient to write the Neumann boundary condition as done in (2.18). We also observe that

$$\rho \in C^0([0,T];C^0(\overline{\Omega})) = C^0(\overline{Q}), \tag{2.19}$$

as a direct consequence of (2.15) and the compact embedding $W \subset C^0(\overline{\Omega})$ (see, e.g., [6, Sect. 8, Cor. 4]), whence $g'(\rho) \in C^0(\overline{Q})$. Thus, under all of the above requirements, we can write the system of equations and the initial condition in the following strong form

$$\big(1 + 2g(\rho)\big)\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu,\rho)\nabla\mu\big) = 0 \qquad \text{a.e. in } Q, \tag{2.20}$$

$$\partial_t\rho - \Delta\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \qquad \text{a.e. in } Q, \tag{2.21}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{2.22}$$

Here is our existence result.

Theorem 2.2. *Assume that both* (2.3)–(2.11) *and* (2.13)–(2.14) *hold. Then, there exists at least a triplet* (μ, ρ, ξ) *satisfying* (2.15)–(2.18) *and solving problem* (2.20)–(2.22).

This is the only result we prove in the present paper. We note, however, that the uniqueness result obtained in [4] still holds here, provided that κ is taken constant and μ_0 smoother. For the reader's convenience, we summarize the results of [4].

Theorem 2.3. *Assume that both* (2.3)–(2.11) *and* (2.13)–(2.14) *hold, and, moreover, that* $\mu_0 \in W$ *and* $\kappa = 1$. *Then, there is a unique triplet* (μ, ρ, ξ) *satisfying* (2.15)–(2.18) *and solving problem* (2.20)–(2.22) *and its component* μ *enjoys the following regularity property:*

$$\mu \in W^{1,p}(0,T;H) \cap L^p(0,T;W) \quad \textit{for every } p \in [1,+\infty). \tag{2.23}$$

Throughout the paper, we make use of some well-known embeddings of Sobolev type, namely, $V \subset L^p(\Omega)$ for $p \in [1,6]$, together with the related Sobolev inequality

$$\|v\|_p \le C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \le p \le 6, \tag{2.24}$$

and $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ for $p > 3$, together with

$$\|v\|_\infty \le C_p\|v\|_{W^{1,p}(\Omega)} \quad \text{for every } v \in W^{1,p}(\Omega) \text{ and } p > 3. \tag{2.25}$$

In (2.24), C depends only on Ω, while C_p in (2.25) depends also on p. In particular, the continuous embedding $W \subset W^{1,6}(\Omega) \subset C^0(\overline{\Omega})$ holds. Some of the previous embeddings are in fact compact. This is the case for $V \subset L^4(\Omega)$ and $W \subset C^0(\overline{\Omega})$. We also account for the corresponding inequality

$$\|v\|_4 \le \varepsilon\|\nabla v\|_H + C_\varepsilon\|v\|_H \quad \text{for every } v \in V \text{ and } \varepsilon > 0 \tag{2.26}$$

where C_ε depends on Ω and ε, only. Furthermore, we repeatedly make use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T], \tag{2.27}$$

of the well-known Hölder inequality, and of the elementary Young inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}\, b^2 \quad \text{for every } a, b \ge 0 \text{ and } \varepsilon > 0. \tag{2.28}$$

Finally, again throughout the paper, we use a small-case italic c for different constants, that may only depend on Ω, the final time T, the shape of the nonlinearities f and g, and the properties of the data involved in the statements at hand; a notation like c_ε signals a constant that depends also on the parameter ε. The reader should keep in mind that the meaning of c and c_ε might change from line to line and even in the same chain of inequalities, whereas those constants we need to refer to are always denoted by capital letters, just like C in (2.24).

3 Existence

In this section, we prove Theorem 2.2, which ensures the existence of a solution. Although our proof follows the argument in [3] and [4] closely, we present the whole argument, and sometimes give some details, since the changes with respect to the quoted papers are spread over the whole exposition. Our starting point is an approximating problem, which is still based on introducing a time delay in the right-hand side of (2.21). Precisely, we define the translation operator $\mathcal{T}_\tau : L^1(0,T;H) \to L^1(0,T;H)$ depending on a time step $\tau > 0$ by setting, for $v \in L^1(0,T;H)$ and for a.a. $t \in (0,T)$,

$$(\mathcal{T}_\tau v)(t) := v(t-\tau) \quad \text{if } t > \tau \quad \text{and} \quad (\mathcal{T}_\tau v)(t) := \mu_0 \quad \text{if } t < \tau \tag{3.1}$$

(the same notation $\mathcal{T}_\tau v$ will be used also for a function v that is defined in some subinterval $[0,T']$ of $[0,T]$). At bottom, what we do is to replace μ by $\mathcal{T}_\tau \mu$ in (2.21). However, since it is not obvious that we can keep μ positive, we extend κ to a function $\bar\kappa : \mathbb{R} \to \mathbb{R}$ satisfying similar properties. Moreover, we assume that the analogue of (2.6) holds for $\bar\kappa$ and its derivatives, with the same constants κ_* and κ^* (we replace κ_* and κ^* by $2\kappa_*$ and $\kappa^*/2$ in the original (2.6) if necessary). So, the approximating problem consists of the equations

$$\big(1 + 2g(\rho_\tau)\big)\, \partial_t \mu_\tau + \mu_\tau\, g'(\rho_\tau)\, \partial_t \rho_\tau - \operatorname{div}\big(\bar\kappa(\mu_\tau, \rho_\tau) \nabla \mu_\tau\big) = 0 \qquad \text{a.e. in } Q, \tag{3.2}$$

$$\partial_t \rho_\tau - \Delta \rho_\tau + \xi_\tau + \pi(\rho_\tau) = (\mathcal{T}_\tau \mu_\tau)\, g'(\rho_\tau) \quad \text{and} \quad \xi_\tau \in \beta(\rho_\tau) \qquad \text{a.e. in } Q, \tag{3.3}$$

complemented by the initial and boundary conditions

$$\mu_\tau(0) = \mu_0\,, \quad \rho_\tau(0) = \rho_0\,, \quad \partial_\nu \rho_\tau|_\Sigma = 0, \quad (\bar\kappa(\mu_\tau, \rho_\tau) \nabla \mu_\tau \cdot \nu|_\Sigma = 0. \tag{3.4}$$

For convenience, we allow τ to take just discrete values, namely, $\tau = T/N$, where N is any positive integer.

Lemma 3.1. *The approximating problem has a solution $(\mu_\tau, \rho_\tau, \xi_\tau)$ satisfying the analogue of* (2.15)–(2.18)*.*

Proof. We confine ourselves to give a sketch. As in [3], we inductively solve N problems on the time intervals $I_n = [0, t_n] := [0, n\tau]$, $n = 1, \dots, N$, by constructing the solution directly on the whole of I_n at each step. Namely, given μ_{n-1}, which is defined in $\Omega \times I_{n-1}$, we note that $\mathcal{T}_\tau \mu_{n-1}$ is well defined and known in $\Omega \times I_n$ and solve the boundary value problem for ρ_n given by the phase field equations

$$\partial_t \rho_n - \Delta \rho_n + \xi_n + \pi(\rho_n) = (\mathcal{T}_\tau \mu_n)\, g'(\rho_n) \quad \text{and} \quad \xi_n \in \beta(\rho_n) \quad \text{in } \Omega \times I_n \tag{3.5}$$

complemented by the boundary and initial conditions just mentioned for ρ_τ. Such a problem is quite standard and has a unique solution ρ_n in a proper functional framework. Now, we observe that the function

$$\hat\kappa : (x,t,m) \mapsto \bar\kappa(m, \rho_n(x,t)), \quad (x,t) \in \Omega \times I_n, \quad m \in \mathbb{R}$$

is a Carathéodory function satisfying $\kappa_* \le \hat\kappa \le \kappa^*$ on its domain, so that the equation

$$\big(1 + 2g(\rho_n)\big)\, \partial_t \mu_n + \mu_n\, g'(\rho_n)\, \partial_t \rho_n - \operatorname{div}\big(\bar\kappa(\mu_n, \rho_n) \nabla \mu_n\big) = 0 \quad \text{in } \Omega \times I_n \tag{3.6}$$

in the unknown function μ_n is uniformly parabolic (let also recall that g is nonnegative). Thus, we can solve the problem obtained by complementing (3.6) with the boundary and initial conditions prescribed for μ_τ. Therefore, the problem to be solved has a unique solution in a proper space, provided that the coefficient $g'(\rho_n)\partial_t\rho_n$ is not too irregular. So, we should prove that, step by step, we get the right regularity for ρ_n and μ_n. This could be done by induction, as in [3], with some modifications due to our more general framework. We omit this detail and just observe that the needed a priori estimates are close (and even simpler, since τ is fixed here) to the ones we perform later on in order to let τ go to zero. The final point is $\mu_n \geq 0$. We give the proof in detail. We multiply equation (3.6) by $-\mu_n^- := -(-\mu_n)^+$, the negative part of μ_n, and integrate over Q_t with any $t \in I_n$. We observe that

$$\left[\left(1 + 2g(\rho_n(t))\right) \partial_t\mu_n + \mu_n\, g(\rho_n)\, \partial_t\rho_n\right] (-\mu_n^-) = \frac{1}{2}\, \partial_t\big((1 + 2g(\rho_n))\, |\mu_n^-|^2\big).$$

Hence, by using $\mu_0 \geq 0$ and owing to the boundary condition, we have

$$\frac{1}{2}\int_\Omega (1 + 2g(\rho_n(t)))\, |\mu_n^-(t)|^2 + \int_{Q_t} \bar\kappa(\mu_n, \rho_n)|\nabla\mu_n^-|^2 = 0 \quad \text{for every } t \in I_n.$$

As both g and $\bar\kappa$ are nonnegative, this implies $\mu_n^- = 0$, that is, $\mu_n \geq 0$ a.e. in $\Omega \times I_n$. Once all this is checked, the finite sequence (μ_n, ρ_n, ξ_n), $n = 1, \dots, N$, is constructed and it is clear that a solution to the approximating problem we are looking for is simply obained by taking $n = N$. $\square$

Thus, we fix a solution $(\mu_\tau, \rho_\tau, \xi_\tau)$ for each τ. We note that, a posteriori, we can replace $\bar\kappa$ by κ in (3.2), since the component μ_τ of our solution is nonnegative. Our aim is to let τ go to zero, so as to obtain a solution as stated in Theorem 2.2. Our proof uses compactness arguments and thus relies on a number of uniform (with respect to τ) a priori estimates. In order to make the formulas to come more readable, we write μ and ρ rather than μ_τ and ρ_τ in the calculations.

Remark 3.2. Sometimes, when deriving our a priori estimates, we proceed formally. However, our procedures can be made rigorous. For instance, one can get more regularity for the approximating problem by regularizing $\bar\kappa$ and the initial data, if necessary. Moreover, local regularization is often sufficient. Consider, e.g., equation (3.3) and rewrite it in the form

$$-\Delta\rho + \rho + \beta(\rho) \ni h := \rho - \partial_t\rho - \pi(\rho) + (\mathfrak{T}_\tau\mu)g'(\rho). \tag{3.7}$$

Now, (here t is just a parameter) the elliptic equation:

$$-\Delta u + u + w = h \quad \text{and} \quad w \in \beta(u),$$

complemented by homogeneous Neumann boundary condition, yields a well-posed problem; as is well known, its solution (u, w) is the limit of the much smoother pair $(u_\varepsilon, \beta_\varepsilon(u_\varepsilon))$, where β_ε is a regularization of β of Yosida type (see, e.g., [2, p. 28]; see also the proof of Lemma 3.1 of [3] for a further regularization) and u_ε is the solution of the analogous boundary value problem for

$$-\Delta u_\varepsilon + u_\varepsilon + \beta_\varepsilon(u_\varepsilon) = h.$$

On the other hand, we have $(u,w) = (\rho,\xi)$ by (3.7). Therefore, it is essentially correct to regard β as if it were a smooth function in the original equation (3.3), and treat such equation like a more regular one (e.g., by differentiating it or taking irregular functions as test functions).

First a priori estimate. We test (3.2) by μ_τ and observe that

$$\big[\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho\big]\mu = \frac{1}{2}\,\partial_t\big[(1+2g(\rho))\mu^2\big].$$

Thus, by integrating over $(0,t)$, where $t\in[0,T]$ is arbitrary, we obtain

$$\int_\Omega \big(1+2g(\rho(t))\big)|\mu(t)|^2 + \int_{Q_t}\kappa(\mu,\rho)|\nabla\mu|^2 = \int_\Omega (1+2g(\rho_0))\mu_0^2\,.$$

Hence, we recall that $g\geq 0$ and $\bar\kappa \geq \kappa_* > 0$, and conclude that

$$\|\mu_\tau\|_{L^\infty(0,T;H)} + \|\mu_\tau\|_{L^2(0,T;V)} \leq c. \tag{3.8}$$

Consequence. The Sobolev inequality (2.24), estimate (3.8), and (2.12), imply that

$$\|\mu_\tau\|_{L^2(0,T;L^6(\Omega))} + \|\psi(\mu_\tau,\rho_\tau)\|_{L^\infty(0,T;H)\cap L^2(0,T;L^6(\Omega))} \leq c \quad \text{with } \psi = K, K_1, K_2. \tag{3.9}$$

Another implication of (3.8), along with (3.1) and (2.13), is

$$\|\mathfrak{T}_\tau\mu_\tau\|_{L^\infty(0,T;H)} + \|\mathfrak{T}_\tau\mu_\tau\|_{L^2(0,T;V)} \leq c. \tag{3.10}$$

Second a priori estimate. We add ρ_τ to both sides of (3.3) and test by $\partial_t\rho_\tau$. We obtain:

$$\begin{aligned}
&\int_{Q_t}|\partial_t\rho|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t))\\
&= \frac{1}{2}\int_\Omega |\nabla\rho_0|^2 + \int_\Omega f(\rho_0) + \frac{1}{2}\int_\Omega \big(\rho^2(t) + 2f_2(\rho(t))\big) + \int_{Q_t} g'(\rho)(\mathfrak{T}_\tau\mu)\partial_t\rho\\
&\leq c + c\int_\Omega |\rho(t)|^2 + \frac{1}{4}\int_{Q_t}|\partial_t\rho|^2 + c\|\mathfrak{T}_\tau\mu\|^2_{L^\infty(0,T;H)},
\end{aligned}$$

for every $t\in[0,T]$. Thanks to the chain rule and the Young inequality (2.28), we have:

$$c\int_\Omega |\rho(t)|^2 \leq c\int_\Omega |\rho_0|^2 + \frac{1}{4}\int_{Q_t}|\partial_t\rho|^2 + c\int_0^t \|\rho(s)\|_H^2\,ds.$$

Then, as f_1 is nonnegative, by accounting for (3.8), with the help of the Gronwall lemma we infer that

$$\int_{Q_t}|\partial_t\rho|^2 + \|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t)) \leq c.$$

Thus, we conclude that

$$\|\rho_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c \quad \text{and} \quad \|f(\rho_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \tag{3.11}$$

Third a priori estimate. We proceed formally (see Remark 3.2). We rewrite (3.3) as

$$-\Delta\rho + \beta(\rho) = h := -\partial_t\rho - \pi(\rho) + (\mathfrak{T}_\tau\mu)g'(\rho), \tag{3.12}$$

and multiply this relation by either $-\Delta\rho$ or $\beta(\rho)$. By doing that, we derive an estimate for both $\Delta\rho$ and $\beta(\rho)$ and we can use the regularity theory for elliptic equations. We conclude that

$$\|\rho_\tau\|_{L^2(0,T;W)} \le c \quad \text{and} \quad \|\xi_\tau\|_{L^2(0,T;H)} \le c. \tag{3.13}$$

Fourth a priori estimate. Our aim is to improve estimates (3.11) and (3.13). To do that, we proceed formally, at least at the beginning (our procedure could be made completely rigorous, as sketched in Remark 3.2). We start from an estimate coming from the theory of maximal monotone operators [2], namely,

$$\|\partial_t u(0)\|_H \le \|\psi(0) + \Delta\rho_0\|_H + \min_{\eta\in\beta(\rho_0)}\|\eta\|_H, \tag{3.14}$$

for the unique solution (u,ω) to the equations (cf. (3.3))

$$\partial_t u - \Delta u + \omega = \psi := g'(\rho)\mathfrak{T}_\tau\mu - \pi(\rho) \quad \text{and} \quad \omega \in \beta(u),$$

complemented by the same initial and boundary conditions prescribed for ρ. Note that in (3.14) β is understood as the induced maximal monotone operator from H to H. By observing that $(u,\omega) = (\rho,\xi)$, applying (3.14), and combining with our assumptions on ρ_0 (see (2.14), in particular), we obtain:

$$\|\partial_t\rho_\tau(0)\|_H \le c\big(\|\mu_0\|_H + \|\rho_0\|_W + 1 + \|\xi_0\|_H\big) = c. \tag{3.15}$$

We use (3.15) in the calculation we are about to start: once again, we proceed formally, and write $\xi = \beta(\rho)$ as if β were a smooth function. We differentiate (3.3) with respect to time and test the equation obtained for $\partial_t\rho$. We find:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t}|\nabla\partial_t\rho|^2 + \int_{Q_t}\beta'(\rho)|\partial_t\rho|^2 \\
&= \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 - \int_{Q_t}\pi'(\rho)|\partial_t\rho|^2 + \int_{Q_t} g''(\rho)(\mathfrak{T}_\tau\mu)|\partial_t\rho|^2 \\
&\quad + \int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho \\
&\le \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 + c\int_{Q_t}(1+\mathfrak{T}_\tau\mu)|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho.
\end{aligned} \tag{3.16}$$

We treat each term on the right-hand side, separately. The first one is estimated by (3.15). In order to deal with the second one, we account for the Hölder inequality, (3.10), the compact embedding $V \subset L^4(\Omega)$ (see (2.26)), and (3.11). We obtain:

$$\begin{aligned}
&\int_{Q_t}(1+\mathfrak{T}_\tau\mu)|\partial_t\rho|^2 \le \int_0^t \|1 + (\mathfrak{T}_\tau\mu)(s)\|_H \|\partial_t\rho(s)\|_4^2\,ds \\
&\le c\int_0^t \|\partial_t\rho(s)\|_4^2\,ds \le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_{Q_t}|\partial_t\rho|^2 \\
&\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\,,
\end{aligned} \tag{3.17}$$

for every $\varepsilon > 0$. Let us come to the last term of (3.16). Firstly, on recalling that $\partial_t \mathcal{T}_\tau \mu = 0$ in $(0, \tau)$ by the definition of $\mathcal{T}_\tau$, we compute $\partial_t \mu$ from (3.2). Then, we integrate by parts and have repeated recourse to Hölder, Sobolev, and Young inequalities. We deduce that

$$\int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho = \int_0^{t-\tau}\!\!\int_\Omega \partial_t\mu(s)\,\partial_t g(\rho(s+\tau))\,ds$$
$$= \int_0^{t-\tau}\!\!\int_\Omega \frac{1}{1+2g(\rho(s))}\big[\mathrm{div}\big(\kappa(\mu(s),\rho(s))\nabla\mu(s)\big) - \mu(s)g'(\rho(s))\partial_t\rho(s)\big]\partial_t g(\rho(s+\tau))\,ds$$
$$= \int_0^{t-\tau}\!\!\int_\Omega \kappa(\mu(s),\rho(s))\nabla\mu(s)\cdot\nabla\,\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds$$
$$- \int_0^{t-\tau}\!\!\int_\Omega \frac{g'(\rho(s))g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds\,. \tag{3.18}$$

We treat the last two integrals separately, by using our structural assumptions.

As to the former, we have:

$$\int_0^{t-\tau}\!\!\int_\Omega \kappa(\mu(s),\rho(s))\nabla\mu(s)\cdot\nabla\,\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds$$
$$= \int_0^{t-\tau}\!\!\int_\Omega \kappa(\mu(s),\rho(s))\nabla\mu(s)\cdot\nabla\,\frac{g'(\rho(s+\tau))\partial_t\rho(s+\tau)}{1+2g(\rho(s))}\,ds$$
$$\le c\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\partial_t\rho(s+\tau)|\,ds$$
$$+ c\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds$$
$$+ c\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\rho(s+\tau)|\,|\partial_t\rho(s+\tau)|\,ds. \tag{3.19}$$

Moreover, thanks to (3.8), we infer:

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\partial_t\rho(s+\tau)|\,ds \le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_{Q_t}|\nabla\mu|^2$$
$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\,, \tag{3.20}$$

for every $\varepsilon \in (0,1)$. On the other hand, we also have:

$$\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds$$
$$\le \int_0^{t-\tau} \|\nabla\mu(s)\|_2\|\nabla\rho(s)\|_4\|\partial_t\rho(s+\tau)\|_4\,ds$$
$$\le \varepsilon\int_0^t \|\partial_t\rho(s)\|_V^2\,ds + c_\varepsilon\int_0^{t-\tau}\|\nabla\mu(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds$$
$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c\int_{Q_t}|\partial_t\rho|^2 + c_\varepsilon\int_0^{t-\tau}\|\nabla\mu(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds$$
$$\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c + c_\varepsilon\int_0^{t-\tau}\|\nabla\mu(s)\|_H^2\|\nabla\rho(s)\|_V^2\,ds \tag{3.21}$$

(in the last inequality, (3.11) has been used). Now, we improve the estimate just obtained by owing to the regularity theory for linear elliptic equations, as well as to estimates (3.8) and (3.11). For each fixed $s \in (0, T)$, we have

$$\begin{aligned}&\|\nabla\rho(s)\|_V^2 \le c\big(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\big)\\&\le c + c\|{-\partial_t\rho(s)} - \pi(\rho(s)) + g'(\rho(s))\mathfrak{T}_\tau\mu(s)\|_H^2 \le \|\partial_t\rho(s)\|_H^2 + c.\end{aligned}$$

Therefore, the above estimate becomes

$$\begin{aligned}&\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\rho(s)|\,|\partial_t\rho(s+\tau)|\,ds\\&\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t \|\nabla\mu(s)\|_2^2\,\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,.\end{aligned}\tag{3.22}$$

Analogously, one shows that

$$\begin{aligned}&\int_0^{t-\tau}\!\!\int_\Omega |\nabla\mu(s)|\,|\nabla\rho(s+\tau)|\,|\partial_t\rho(s+\tau)|\,ds\\&\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t \|\nabla(\mathfrak{T}_\tau\mu)(s)\|_2^2\,\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,.\end{aligned}\tag{3.23}$$

Thus, by collecting (3.20) and (3.22)–(3.23), we deduce that (3.19) yields:

$$\begin{aligned}&\int_0^{t-\tau}\!\!\int_\Omega \kappa(\mu(s),\rho(s))\nabla\mu(s)\cdot\nabla\,\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds\\&\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t \Big(\|\nabla\mu(s)\|_H^2 + \|\nabla(\mathfrak{T}_\tau\mu)(s)\|_H^2\Big)\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,,\end{aligned}\tag{3.24}$$

for every $\varepsilon > 0$.

We now take up the last integral in (3.18). By using the compactness inequality (2.26) and (3.8), we have:

$$\begin{aligned}&-\int_0^{t-\tau}\!\!\int_\Omega \frac{g'(\rho(s))g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds\\&\le c\int_0^{t-\tau}\|\mu(s)\|_4\|\partial_t\rho(s+\tau)\|_4\|\partial_t\rho(s)\|_2\,ds\\&\le \varepsilon\int_0^{t-\tau}\|\partial_t\rho(s+\tau)\|_V^2\,ds + c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds\\&\le \varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c + c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds.\end{aligned}\tag{3.25}$$

Therefore, due to (3.24) and (3.25), (3.18) becomes:

$$\begin{aligned}\int_{Q_t} g'(\rho)\partial_t(\mathfrak{T}_\tau\mu)\,\partial_t\rho \le 2\varepsilon\int_{Q_t}|\nabla\partial_t\rho|^2 + c_\varepsilon\int_0^t\|\nabla\mu(s)\|_H^2\|\partial_t\rho(s)\|_H^2\,ds&\\+ c_\varepsilon\int_0^t\|\mu(s)\|_4^2\|\partial_t\rho(s)\|_H^2\,ds + c_\varepsilon\,.&\end{aligned}\tag{3.26}$$

At this point, we combine (3.15), (3.17), and (3.26) with (3.16), and we choose ε small enough. Since the last integral on the left-hand side of (3.16) is nonnegative, because f_1 is convex, we obtain:

$$\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 \le c\int_0^t \phi(s)\|\partial_t\rho(s)\|_H^2\,ds + c,$$
$$\text{where}\quad \phi(s) := \|\nabla\mu(s)\|_H^2 + \|\nabla(\mathfrak{T}_\tau\mu)(s)\|_H^2 + \|\mu(s)\|_4^2\,.$$

As $\phi \in L^1(0,T)$ by (3.8)–(3.10), we can apply the Gronwall lemma and conclude that

$$\|\partial_t\rho_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{3.27}$$

Consequence. Note that $-\Delta\rho_\tau + \xi_\tau = -\partial_t\rho_\tau + g'(\rho_\tau)\mathfrak{T}_\tau\mu_\tau$ is bounded in $L^\infty(0,T;H)$, due to (3.8) and (3.27). Therefore, by a standard argument (multiply formally by ξ_τ), we deduce that both $-\Delta\rho_\tau$ and ξ_τ are bounded in the same space, whence by elliptic regularity

$$\|\rho_\tau\|_{L^\infty(0,T;W)} \le c \quad\text{and}\quad \|\xi_\tau\|_{L^\infty(0,T;H)} \le c; \tag{3.28}$$

moreover,

$$\|\rho_\tau\|_{L^\infty(Q)} + \|\psi(\rho_\tau))\|_{L^\infty(Q)} \le c \quad\text{with } \psi = g, g', \pi, \tag{3.29}$$

due to the continuous embedding $W \subset L^\infty(\Omega)$ and the continuity of such ψ's.

Fifth a priori estimate. To prove an L^∞ estimate rather than just a boundedness property, we borrow the argument in [3]. We observe that the approximating solution satisfies:

$$\frac{1}{2}\,\partial_t\big[\big(1+2g(\rho)\big)\,|(\mu-k)^+|^2\big] = \big[\big(1+2g(\rho)\big)\partial_t\mu + (\mu-k)g'(\rho)\partial_t\rho\big](\mu-k)^+\,, \tag{3.30}$$

for every $k \in \mathbb{R}$. Hence, by assuming that $k \ge \mu_0^* := \|u_0\|_\infty$ and by testing (3.2) with $(\mu-k)^+$, we obtain:

$$\frac{1}{2}\int_\Omega (1+2g(\rho(t)))|(\mu(t)-k)^+|^2 + \int_{Q_t} \kappa(\mu,\rho)|\nabla(\mu-k)^+|^2 = -k\int_{Q_t} \partial_t g(\rho)\,(\mu-k)^+;$$

with this, on recalling that $g \ge 0$ and $\kappa \ge \kappa_*$, we deduce the inequality

$$\frac{1}{2}\int_\Omega |(\mu(t)-k)^+|^2 + \kappa_*\int_{Q_t} |\nabla(\mu-k)^+|^2 = -k\int_{Q_t} \partial_t g(\rho)\,(\mu-k)^+.$$

In [3], for $\varepsilon = 1$, we have that $g(r) = r$ and $\kappa = 1$; the corresponding inequality is:

$$\frac{1}{2}\int_\Omega |(\mu(t)-k)^+|^2 + \int_{Q_t} |\nabla(\mu-k)^+|^2 = -k\int_{Q_t} \partial_t\rho\,(\mu-k)^+. \tag{3.31}$$

Therefore, the argument used in that paper can be repeated here essentially without changes. As a matter of fact, the analogue of (2.21) is never used in [3], the whole proof being based just on (3.31), the regularity $\partial_t\rho \in L^\infty(0,T;H) \cap L^2(0,T;V)$, and an upper bound, say C_0, for the corresponding norm; moreover, the upper bound for μ, that is constructed explicitely, depends only on Ω, T, μ_0^*, and C_0. In the present case, we have to use the same regularity for $\partial_t g(\rho)$ and estimate (3.29). In conclusion, we obtain:

$$\|\mu_\tau\|_{L^\infty(Q)} \le c. \tag{3.32}$$

Sixth a priori estimate. We proceed formally, as done for the third a priori estimate, by writing $\xi = \beta(\rho)$ as if β were a smooth function (see Remark 3.2). We test by $(\xi(t))^5$ (3.3), written in the form (3.12), at (almost) any fixed time $t \in (0,T)$. We obtain:

$$5\int_\Omega (\xi(t))^4 \beta'(\rho(t))|\nabla\rho(t)|^2 + \int_\Omega |\xi(t)|^6 = \int_\Omega h(t)\,(\xi(t))^5.$$

As the first term on the left-hand side is nonnegative, by the Hölder inequality we deduce that

$$\|\xi(t)\|_6^6 \le \|h(t)\|_6\,\|(\xi(t))^5\|_{6/5} = \|h(t)\|_6\,\|\xi(t)\|_6^5,$$

whence he have immediately that $\|\xi(t)\|_6 \le \|h(t)\|_6$. We infer that

$$\|\Delta\rho(t)\|_{L^6(\Omega)} \le c\|h(t)\|_{L^6(\Omega)} \quad \text{and} \quad \|\rho(t)\|_{W^{2,6}(\Omega)} \le c\|h(t)\|_{L^6(\Omega)},$$

first by comparison in (3.12) and then by the standard regularity theory of linear elliptic equations. As $W^{1,6}(\Omega)$ is continuously embedded in $C^0(\overline{\Omega})$ (see (2.25)), and as the above inequalities hold for a.a. $t \in (0,T)$, we deduce that

$$\|\nabla\rho\|_{L^2(0,T;L^\infty(\Omega))} \le c\|h\|_{L^2(0,T;L^6(\Omega))}.$$

Now, we observe that h is bounded in $L^2(0,T;L^6(\Omega))$, thanks to (3.27), (3.29), (3.32), and the Sobolev inequality. Therefore, we conclude that

$$\|\nabla\rho_\tau\|_{L^2(0,T;L^\infty(\Omega))} \le c. \tag{3.33}$$

A byproduct of our proof is that $\|\rho_\tau\|_{L^2(0,T;W^{2,6}(\Omega))} \le c$.

Seventh a priori estimate. On recalling (2.10)–(2.11), the following preparatory identities hold for the approximating solution:

$$\nabla K(\mu,\rho) = \kappa(\mu,\rho)\nabla\mu + K_1(\mu,\rho)\nabla\rho, \tag{3.34}$$

$$\partial_t K(\mu,\rho) = \kappa(\mu,\rho)\partial_t\mu + K_1(\mu,\rho)\partial_t\rho, \tag{3.35}$$

$$\partial_t K_1(\mu,\rho) = \kappa'(\mu,\rho)\partial_t\mu + K_2(\mu,\rho)\partial_t\rho. \tag{3.36}$$

Moreover, we notice that

$$\|\psi(\mu,\rho)\|_{L^\infty(Q)} \le c \quad \text{with } \psi = \kappa, \kappa', K, K_1, \text{or } K_2, \tag{3.37}$$

due to our structural assumptions (2.6) and to (3.32), (2.12). Now, we formally test (3.2) by $\partial_t K(\mu,\rho)$ and get:

$$\int_{Q_t} (1+2g(\rho))\,\partial_t\mu\,\partial_t K(\mu,\rho) + \int_{Q_t} \mu\,\partial_t g(\rho)\,\partial_t K(\mu,\rho) + \int_{Q_t} \kappa(\mu,\rho)\nabla\mu\cdot\nabla\partial_t K(\mu,\rho) = 0. \tag{3.38}$$

With the help of (3.34)–(3.35), we rewrite the first two integral as follows:

$$\begin{aligned}
&\int_{Q_t} (1+2g(\rho))\,\partial_t\mu\,\partial_t K(\mu,\rho) \\
&= \int_{Q_t} (1+2g(\rho))\,\kappa(\mu,\rho)\,|\partial_t\mu|^2 + \int_{Q_t} (1+2g(\rho))\,K_1(\mu,\rho)\,\partial_t\mu\,\partial_t\rho \\
&\int_{Q_t} \mu\,\partial_t g(\rho)\,\partial_t K(\mu,\rho) = \int_{Q_t} \mu\,\partial_t g(\rho)\,\kappa(\mu,\rho)\,\partial_t\mu + \int_{Q_t} \mu\,\partial_t g(\rho)\,K_1(\mu,\rho)\,\partial_t\rho.
\end{aligned}$$

In the third integral of (3.38), we also integrate by parts and use (3.36). We get:

$$\begin{aligned}
&\int_{Q_t} \kappa(\mu,\rho)\nabla\mu\cdot\nabla\partial_t K(\mu,\rho) = \int_{Q_t} \bigl(\nabla K(\mu,\rho) - K_1(\mu,\rho)\nabla\rho\bigr)\cdot\nabla\partial_t K(\mu,\rho) \\
&= \frac12\int_\Omega |\nabla K(\mu(t),\rho(t))|^2 - \int_\Omega K_1(\mu(t),\rho(t))\nabla\rho(t)\cdot\nabla K(\mu(t),\rho(t)) - c \\
&\quad + \int_{Q_t} \nabla K(\mu,\rho)\cdot\partial_t\bigl(K_1(\mu,\rho)\nabla\rho\bigr) \\
&= \frac12\int_\Omega |\nabla K(\mu(t),\rho(t))|^2 - \int_\Omega K_1(\mu(t),\rho(t))\nabla\rho(t)\cdot\nabla K(\mu(t),\rho(t)) - c \\
&\quad + \int_{Q_t} \nabla K(\mu,\rho)\cdot\nabla\rho\,\bigl(\kappa'(\mu,\rho)\partial_t\mu + K_2(\mu,\rho)\partial_t\rho\bigr) + \int_{Q_t} K_1(\mu,\rho)\nabla K(\mu,\rho)\cdot\nabla\partial_t\rho.
\end{aligned}$$

If we insert these identities in (3.38), on keeping just the positive terms on the left-hand side, on recalling that $g \geq 0$ and $\kappa \geq \kappa_*$, and on and accounting for estimates (3.32) and (3.37), then we deduce that

$$\begin{aligned}
&\kappa_*\int_{Q_t} |\partial_t\mu|^2 + \frac12\int_\Omega |\nabla K(\mu(t),\rho(t))|^2 \\
&\leq c\int_{Q_t} |\partial_t\mu|\,|\partial_t\rho| + c\int_{Q_t} |\partial_t\rho|^2 + c\int_\Omega |\nabla\rho(t)|\,|\nabla K(\mu(t),\rho(t))| \\
&\quad + c\int_{Q_t} |\nabla K(\mu,\rho)|\,|\nabla\rho|\,\bigl(|\partial_t\mu| + |\partial_t\rho|\bigr) + c\int_{Q_t} |\nabla K(\mu,\rho)|\,|\nabla\partial_t\rho| + c.
\end{aligned} \tag{3.39}$$

As the first three terms on the right-hand side can be trivially dealt with by accounting for (3.8), (3.11), and the elementary Young inequality, we concentrate on the last two integrals. For every $\varepsilon > 0$, we deduce that

$$\begin{aligned}
&\int_{Q_t} |\nabla K(\mu,\rho)|\,|\nabla\rho|\,\bigl(|\partial_t\mu| + |\partial_t\rho|\bigr) + \int_{Q_t} |\nabla K(\mu,\rho)|\,|\nabla\partial_t\rho| \\
&\leq \varepsilon\int_{Q_t} |\partial_t\mu|^2 + c_\varepsilon\int_{Q_t} |\nabla\rho|^2\,|\nabla K(\mu,\rho)|^2 \\
&\quad + \int_{Q_t} |\nabla K(\mu,\rho)|^2 + \frac12\int_{Q_t} |\nabla\rho|^2\,|\partial_t\rho|^2 + \frac12\int_{Q_t} |\nabla\partial_t\rho|^2.
\end{aligned}$$

On the other hand, we have:

$$\int_{Q_t} |\nabla\rho|^2 \, |\nabla K(\mu,\rho)|^2 + \int_{Q_t} |\nabla K(\mu,\rho)|^2 + \int_{Q_t} |\nabla\rho|^2 \, |\partial_t\rho|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2$$
$$\leq \int_0^t \phi(s) \, \|\nabla K(\mu(s),\rho(s))\|_2^2 \, ds + \int_0^t \|\nabla\rho(s)\|_\infty^2 \, \|\partial_t\rho(s)\|_2^2 \, ds + \int_{Q_t} |\nabla\partial_t\rho|^2$$
$$\leq \int_0^t \phi(s) \, \|\nabla K(\mu(s),\rho(s))\|_2^2 \, ds + \|\nabla\rho\|^2_{L^2(0,T;L^\infty(\Omega))} \|\partial_t\rho\|^2_{L^\infty(0,T;H)}$$
$$+ \|\nabla\partial_t\rho\|^2_{L^2(0,T;H)} \, ,$$

where $\phi(s) := \|\nabla\rho(s)\|_\infty^2 + 1$. As $\phi \in L^1(0,T)$ thanks to (3.33), and as the last norms in the above inequality are bounded by (3.27) and (3.33), we can choose ε small enough and apply the Gronwall lemma. We conclude that

$$\|\partial_t\mu_\tau\|_{L^2(0,T;H)} + \|K(\mu_\tau,\rho_\tau)\|_{L^\infty(0,T;V)} \leq c. \tag{3.40}$$

Consequence. By combining (3.40), (3.34), and $\kappa \geq \kappa_*$, we derive that

$$\|\nabla\mu_\tau\|_{L^\infty(0,T;H)} \leq c, \quad \text{whence} \quad \|\mu_\tau\|_{L^\infty(0,T;V)} \leq c. \tag{3.41}$$

Furthermore, by comparison in (3.2), we also deduce that

$$\|\mathrm{div}(\kappa(\mu_\tau,\rho_\tau)\nabla\mu_\tau)\|_{L^2(0,T;H)} \leq c. \tag{3.42}$$

Limit and conclusion. By the above estimates, there are a triplet (μ,ρ,ξ), with $\mu \geq 0$ a.e. in Q, and two functions k and ζ such that

$$\mu_\tau \to \mu \qquad \text{weakly star in } H^1(0,T;H) \cap L^2(0,T;V) \cap L^\infty(Q), \tag{3.43}$$
$$\rho_\tau \to \rho \qquad \text{weakly star in } L^\infty(0,T;W), \tag{3.44}$$
$$\partial_t\rho_\tau \to \partial_t\rho \qquad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.45}$$
$$\xi_\tau \to \xi \qquad \text{weakly star in } L^\infty(0,T;H), \tag{3.46}$$
$$K(\mu_\tau,\rho_\tau) \to k \qquad \text{weakly star in } L^\infty(0,T;V), \tag{3.47}$$
$$\mathrm{div}(\kappa(\mu_\tau,\rho_\tau)\nabla\mu_\tau) \to \zeta \qquad \text{weakly in } L^2(0,T;H), \tag{3.48}$$

at least for a susequence $\tau = \tau_i \searrow 0$. By the weak convergence of time derivatives, the Cauchy conditions (2.22) hold for the limit pair (μ,ρ). By (3.43)–(3.45) and the compact embeddings $W \subset C^0(\overline{\Omega})$ and $V \subset H$, we can apply well-known strong compactness results (see, e.g., [6, Sect. 8, Cor. 4]) and, possibly taking another subsequence, we have that

$$\mu_\tau \to \mu \qquad \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q \tag{3.49}$$
$$\rho_\tau \to \rho \qquad \text{strongly in } C^0(\overline{Q}). \tag{3.50}$$

The weak convergence (3.46), together with (3.50), implies that $\xi \in \beta(\rho)$ a.e. in Q (see, e.g., [2, Prop. 2.5, p. 27]), due to the maximal monotonicity of the operator induced by β on $L^2(Q)$.

Furthermore, the convergence stated in (3.49)–(3.50) entails that $\phi(\rho_\tau) \to \phi(\rho)$ uniformly in Q for $\phi = g, g', \pi$ and $\psi(\mu_\tau, \rho_\tau) \to \psi(\mu, \rho)$ a.e. in Q for $\psi = \kappa, K$, whence, in particular, $k = K(\mu, \rho)$. As all the above functions $\psi(\mu_\tau, \rho_\tau)$ are uniformly bounded, we deduce that the convergence is in fact strong in $L^p(Q)$ for every $p < +\infty$ and weak star in $L^\infty(Q)$ in each case. This shows that the limits of the products

$$(1 + 2g(\rho_\tau))\, \partial_t \mu_\tau, \quad \mu_\tau\, g'(\rho_\tau)\, \partial_t \rho_\tau, \quad \kappa(\mu_\tau, \rho_\tau)\, \nabla \mu_\tau, \quad \text{and} \quad \mu_\tau\, g'(\rho_\tau)$$

that appear in equations (3.2)–(3.3) can be identified as the products of the corresponding limits. In particular, by using also (3.48), we derive that $\operatorname{div}(\kappa(\mu, \rho) \nabla \mu)$ equals ζ and belongs to $L^2(Q)$. All this implies both (2.20) for the limit (μ, ρ) and the convergence of the normal trace $\kappa(\mu_\tau, \rho_\tau)\, \nabla \mu_\tau \cdot \nu$. Thus, the expected Neumann condition also holds in the limit, and the proof is complete.

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