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On existence and asymptotic stability of periodic solutions with an interior layer of reaction-advection-diffusion equations

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Abstract

We consider a singularly perturbed parabolic periodic boundary value problem for a reaction-advection-diffusion equation. We construct the interior layer type formal asymptotics and propose a modified procedure to get asymptotic lower and upper solutions. By using sufficiently precise lower and upper solutions, we prove the existence of a periodic solution with an interior layer and estimate the accuracy of its asymptotics. Moreover, we are able to establish the asymptotic stability of this solution with interior layer.

1 Statement of the problem. Construction of formal asymptotics

We consider the singularly perturbed periodic boundary value problem

$$
\begin{aligned}
&N_\varepsilon(u) := \varepsilon\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - A(u,x,t)\,\frac{\partial u}{\partial x} - B(u,x,t) = 0 \\
&\quad \text{for} \quad (x,t) \in \mathcal{D} := \{(x,t) \in R^2 : -1 < x < 1, t \in R\}, \\
&u(-1,t,\varepsilon) = u^{(-)}(t), \quad u(1,t,\varepsilon) = u^{(+)}(t) \quad \text{for} \quad t \in R, \\
&u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for} \quad t \in R, \quad -1 \le x \le 1
\end{aligned}
\tag{1.1}
$$

for $\varepsilon \in I_{\varepsilon_0} := \{0 < \varepsilon \le \varepsilon_0\},\ 0 < \varepsilon_0 \ll 1$. The functions A, B, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

Our goal is to establish the existence of a T-periodic solution of problem (1.1) with an interior layer with respect to x, and to determine the stability of this solution. For this purpose we construct sufficiently precise asymptotic lower and upper solutions and apply the results from [2] where we developed an approach to investigate the asymptotic stability of periodic solutions to singularly perturbed reaction-advection-diffusion equations by using the theorem of Krein-Rutman. The construction of lower and upper solutions is based on the construction of a formal asymptotic approximation of the solution to (1.1) and develops further the approach used in the papers [3, 4, 7] .

1.1 Assumptions

We consider problem (1.1) under the following assumptions

(A_0). A, B, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

If we put $\varepsilon = 0$ in equation (1.1) we get the so-called degenerate equation

$$A(u,x,t)\,\frac{\partial u}{\partial x} + B(u,x,t) = 0, \tag{1.2}$$

where t has to be considered as a parameter. Equation (1.2) is a first order ordinary differential equation and will be studied with one of the following initial conditions from problem (1.1)

$$u(-1,t) = u^{(-)}(t) \quad \text{for} \quad t \in R, \tag{1.3}$$

$$u(1,t) = u^{(+)}(t) \quad \text{for} \quad t \in R. \tag{1.4}$$

Concerning these initial value problems we assume

(A_1). The problems (1.2),(1.3) and (1.2),(1.4) have the solutions $u = \varphi^{(-)}(x,t)$ and $u = \varphi^{(+)}(x,t)$, respectively, which are defined for $(x,t) \in \overline{\mathcal{D}}$, are T-periodic in t and satisfy

$$\varphi^{(-)}(x,t) < \varphi^{(+)}(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}},$$

$$A(\varphi^{(+)}(x,t),x,t) < 0, \quad A(\varphi^{(-)}(x,t),x,t) > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}.$$

To formulate the next assumptions we introduce the function $I(x,t)$ by

$$I(x,t) := \int_{\varphi^{(-)}(x,t)}^{\varphi^{(+)}(x,t)} A(u,x,t)\,du$$

and suppose

(A_2). The equation

$$I(x,t) = 0 \tag{1.5}$$

has a smooth solution $x = x_0(t)$ which is T-periodic and obeys the conditions

$$-1 < x_0(t) < 1 \quad \text{for} \quad t \in R,$$

$$\int_{\varphi^{(-)}(x_0(t),t)}^{s} A(u,x_0(t),t)\,du > 0 \quad \text{for any } s \in \big(\varphi^{(-)}(x_0(t),t), \varphi^{(+)}(x_0(t),t)\big) \text{ and for } t \in R.$$

(A_3). The root $x_0(t)$ of equation (1.5) satisfies the condition

$$\frac{\partial I}{\partial x}(x_0(t),t) < 0 \quad \text{for} \quad t \in R,$$

that is, $x_0(t)$ is a simple root for all $t \in R$.

Remark 1.1 *Our goal is for sufficiently small ε to establish a solution to problem (1.1) with an interior layer near $x_0(t)$ that stays near $\varphi^{(-)}(x,t)$ for $x < x_0(t)$ and near $\varphi^{(+)}(x,t)$ for $x > x_0(t)$.*

Our main result is the following theorem.

Theorem 1.1 *Let the assumptions (A_0)–(A_3) be satisfied. Then, for sufficiently small ε, there exists a solution $u(x,t,\epsilon)$ of problem (1.1) such that for any small but fixed δ we have the limit relation*

$$\lim_{\epsilon\to 0} u(x,t,\epsilon) = \begin{cases} \varphi^{(-)}(x,t) & \text{for} \quad x\in[0,x_0(t)-\delta],\ t\in R, \\ \varphi^{(+)}(x,t) & \text{for} \quad x\in[x_0(t)+\delta,1],\ t\in R. \end{cases}$$

We get a more precise description of the solution with an interior layer in Section 3.

1.2 Construction of a formal asymptotic solution

In this section we describe the construction of a formal asymptotic solution of the periodic boundary value problem (1.1) with an interior layer near $x_0(t)$, where $x_0(t)$ is defined in assumption (A_2). Later on we will prove the existence of a solution to (1.1) near this formal asymptotic approximation.

To characterize the location $x_*(t,\varepsilon)$ of the interior layer of the formal asymptotic solution in the (x,t)-plane we make the ansatz

$$x_*(t,\varepsilon) = x_0(t) + \varepsilon\, x_1(t) + \ldots.., \tag{1.6}$$

where $x_k(t), k=1,2,...,$ are T-periodic functions to be determined. By this way we decompose the periodic boundary value problem (1.1) with an interior layer near $x_*(t,\varepsilon)$ into two boundary value problems with boundary layers near $x=x_*(t,\varepsilon)$.

For the following we introduce the notation

$$\xi := \frac{x-x_*(t,\varepsilon)}{\varepsilon},$$

$$\overline{\mathcal{D}}^{(-)} := \{(x,t)\in R^2 : -1\le x\le x_*(t,\varepsilon),\, t\in R\},$$

$$\overline{\mathcal{D}}^{(+)} := \{(x,t)\in R^2 : x_*(t,\varepsilon)\le x\le 1,\, t\in R\}.$$

First we consider for small ε in $\overline{\mathcal{D}}^{(-)}$ the boundary value problem

$$\begin{aligned} &\varepsilon\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - A(u,x,t)\,\frac{\partial u}{\partial x} - B(u,x,t) = 0 \quad \text{for} \quad (x,t)\in\mathcal{D}^{(-)}, \\ &u(-1,t,\varepsilon) = u^{(-)}(t), \quad u(x_*(t,\varepsilon),t,\varepsilon) = \varphi(x_*(t,\varepsilon),t) \quad \text{for} \quad t\in R, \\ &u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}^{(-)}, \end{aligned} \tag{1.7}$$

where φ is defined by

$$\varphi(x,t) := \frac{1}{2}\left(\varphi^{(-)}(x,t) + \varphi^{(+)}(x,t)\right),$$

the functions $\varphi^{(-)}$ and $\varphi^{(+)}$ are introduced in assumption (A_1).

We look for a formal asymptotic solution $U^{(-)}(x,t,\varepsilon)$ of this problem in the form

$$U^{(-)}(x,t,\varepsilon) = \overline{U}^{(-)}(x,t,\varepsilon) + Q^{(-)}(\xi,t,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\left(\overline{U}_i^{(-)}(x,t) + Q_i^{(-)}(\xi,t)\right), \tag{1.8}$$

where $\overline{U}^{(-)}$ and $Q^{(-)}$ denote the regular and the boundary layer parts, respectively, of the formal asymptotic solution $U^{(-)}$ in the region $\overline{\mathcal{D}}^{(-)}$.

Analogously, we consider in $\overline{\mathcal{D}}^{(+)}$ the boundary value problem

$$\begin{aligned} &\varepsilon\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)-A(u,x,t)\,\frac{\partial u}{\partial x}-B(u,x,t)=0 \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}^{(+)},\\ &u(x_*(t,\varepsilon),t,\varepsilon)=\varphi(x_*(t,\varepsilon),t), \quad u(1,t,\varepsilon)=u^{(+)}(t) \quad \text{for} \quad t\in R,\\ &u(x,t,\varepsilon)=u(x,t+T,\varepsilon) \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}^{(+)} \end{aligned} \tag{1.9}$$

and a formal asymptotic solution $U^{(+)}(x,t,\varepsilon)$

$$U^{(+)}(x,t,\varepsilon)=\overline{U}^{(+)}(x,t,\varepsilon)+Q^{(+)}(\xi,t,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\left(\overline{U}_i^{(+)}(x,t)+Q_i^{(+)}(\xi,t)\right), \tag{1.10}$$

where $\overline{U}^{(+)}$ and $Q^{(+)}$ denote the regular and the boundary layer parts, respectively.

It follows from (1.7) and (1.9) that $U^{(-)}$ and $U^{(+)}$ are matched continuously at $x=x_*(t,\varepsilon)$. In order to find the terms $x_i(t)$ of the expansion (1.6) we use the C^1–matching condition of the i-th order in ε of the expression

$$\varepsilon\,\frac{\partial U^{(-)}}{\partial x}(x_*(t,\varepsilon),t,\varepsilon)=\varepsilon\,\frac{\partial U^{(+)}}{\partial x}(x_*(t,\varepsilon),t,\varepsilon) \quad \text{for} \quad t\in R. \tag{1.11}$$

To determine the terms in the expansions (1.8) and (1.10) we use the standard procedure proposed by Vasil'eva (see e.g. [6]): we represent the nonlinear functions $A(u,x,t)$ and $B(u,x,t)$ in a form which is similar to (1.8) and (1.10). For example, $A(u,x,t)$ has to be represented in $\overline{\mathcal{D}}^{(\pm)}$ in the form

$$A(u,x,t)=\Big[A\Big(\overline{U}^{(\pm)}(x,t,\varepsilon)+Q^{(\pm)}(\xi,t,\varepsilon),x,t\Big)-A\big(\overline{U}^{(\pm)}(x,t,\varepsilon),x,t\big)\Big]_{|x=x_*(t,\varepsilon)+\varepsilon\xi}$$
$$+A(\overline{U}^{(\pm)}(x,t,\varepsilon),x,t).$$

We use a similar representation for $B(u,x,t)$. We also represent the differential operator

$$L_\varepsilon=\varepsilon\,\frac{\partial^2}{\partial x^2}-\varepsilon\,\frac{\partial}{\partial t}$$

when it acts on the boundary layer functions by using the stretched variable ξ in the form:

$$L_\varepsilon=\frac{1}{\varepsilon}\,\frac{\partial^2}{\partial\xi^2}+x_*'(t,\varepsilon)\,\frac{\partial}{\partial\xi}-\varepsilon\,\frac{\partial}{\partial t}.$$

Substituting the representations for A, B and L_ε into equation (1.1), and equating separately the parts depending on x and on ξ we get the relations, which serve to determine the terms of the asymptotic expansions (1.8) and (1.10).

For the regular parts we have

$$\begin{aligned} &\varepsilon\left(\frac{\partial^2\overline{U}^{(\pm)}}{\partial x^2}-\frac{\partial\overline{U}^{(\pm)}}{\partial t}\right)-A(\overline{U}^{(\pm)},x,t)\,\frac{\partial\overline{U}^{(\pm)}}{\partial x}-B(\overline{U}^{(\pm)},x,t)=0\\ &\text{for} \quad (x,t)\in\mathcal{D}^{(\pm)}, \quad u(\pm1,t,\varepsilon)=u^{(\pm)}(t), \quad \text{for} \quad t\in R. \end{aligned} \tag{1.12}$$

It is clear that for $k = 0$ we get from (1.12) the degenerate problems (1.2), (1.3) and (1.2), (1.4) and therefore by hypothesis (A_1) we have

$$\overline{U}_0^{(\pm)}(x,t) = \varphi^{(\pm)}(x,t) \quad \text{for} \quad (x,t) \in \mathcal{D}^{(\pm)}.$$

We can use (1.12) to derive first order linear differential equations to determine $\overline{U}_k^{(\pm)}(x,t)$ for $k = 1, 2, ...$ by means of the corresponding initial value problems:

$$\begin{aligned}&A(\varphi^{(\pm)}(x,t),x,t)\frac{\partial \overline{U}_k^{(\pm)}}{\partial x} + \Big(A_u(\varphi^{(\pm)}(x,t),x,t)\frac{\partial \overline{U}_0^{(\pm)}}{\partial x} + B_u(\varphi^{(\pm)}(x,t),x,t)\Big)\overline{U}_k^{(\pm)} = \\ &f_k^{(\pm)}(x,t) \quad \text{for} \quad (x,t) \in \mathcal{D}^{(\pm)}, \quad \overline{U}_k^{(\pm)}(\pm 1, t) = 0 \quad \text{for} \quad t \in R.\end{aligned} \tag{1.13}$$

The functions $f_k^{(\pm)}(x,t)$ are determined by the functions $\overline{U}_j^{(\pm)}(x,t)$ with $j < k$, in particular we have

$$f_1^{(\pm)}(x,t) = \frac{\partial^2 \overline{U}_0^{(\pm)}}{\partial x^2}(x,t) - \frac{\partial \overline{U}_0^{(\pm)}}{\partial t}(x,t).$$

The initial value problems (1.13) are linear and their solutions can be given explicitly.

For the boundary layer parts we have

$$\begin{aligned}&\frac{1}{\varepsilon}\frac{\partial^2 Q^{(\pm)}}{\partial \xi^2} + \frac{\partial x_*(t,\varepsilon)}{\partial t}\frac{\partial Q^{(\pm)}}{\partial \xi} - \varepsilon\frac{\partial Q^{(\pm)}}{\partial t} \\ &= \frac{1}{\varepsilon}\Big[A\Big(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon) + Q^{(\pm)}, x_*(t,\varepsilon)+\varepsilon\xi,t\Big)\frac{\partial}{\partial \xi}\Big(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon) \\ &+Q^{(\pm)}\Big) - A\Big(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon), x_*(t,\varepsilon)+\varepsilon\xi,t\Big)\frac{\partial}{\partial \xi}(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon)\Big] \\ &+\Big[B\Big(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon) + Q^{(\pm)}, x_*(t,\varepsilon)+\varepsilon\xi,t\Big) \\ &-B\Big(\overline{U}^{(\pm)}(x_*(t,\varepsilon)+\varepsilon\xi,t,\varepsilon), x_*(t,\varepsilon)+\varepsilon\xi,t\Big)\Big], \\ &Q^{(\pm)}(0,t,\varepsilon) + \overline{U}^{(\pm)}(x_*(t,\varepsilon),t,\varepsilon) = \varphi(x_*(t,\varepsilon),t).\end{aligned} \tag{1.14}$$

For $Q_k^{(\pm)}(\xi,t)$ we use the additional condition at $\pm\infty$:

$$Q_k^{(\pm)}(\pm\infty,t) = 0.$$

From (1.14) we get the problems to determine the functions $Q_k^{(\pm)}(\xi,t)$ in the asymptotic expansions (1.8) and (1.10). For the zero-th order boundary layer functions $Q_0^{(-)}$ and $Q_0^{(+)}$ we obtain the boundary value problems

$$\frac{\partial^2 Q_0^{(-)}}{\partial \xi^2} = A\Big(\varphi^{(-)}(x_0(t),t) + Q_0^{(-)}, x_0(t),t\Big)\frac{\partial Q_0^{(-)}}{\partial \xi} \quad \text{for} \quad \xi < 0,\ t \in R, \tag{1.15}$$

$$Q_0^{(-)}(-\infty,t)=0,\quad Q_0^{(-)}(0,t)=\varphi(x_0(t),t)-\varphi^{(-)}(x_0(t),t)\quad \text{for}\quad t\in R \tag{1.16}$$

and

$$\frac{\partial^2 Q_0^{(+)}}{\partial\xi^2}=A\Big(\varphi^{(+)}(x_0(t),t)+Q_0^{(+)},x_0(t),t\Big)\,\frac{\partial Q_0^{(+)}}{\partial\xi}\quad \text{for}\quad \xi>0,t\in R, \tag{1.17}$$

$$Q_0^{(+)}(\infty,t)=0,\quad Q_0^{(+)}(0,t)=\varphi(x_0(t),t)-\varphi^{(+)}(x_0(t),t)\quad \text{for}\quad t\in R. \tag{1.18}$$

In order to investigate the problems (1.15)–(1.16) and (1.17)–(1.18) we introduce the function

$$\tilde u(\xi,x_0):=\begin{cases}\varphi^{(-)}(x_0(t),t)+Q_0^{(-)}(\xi,t), & \xi<0,\ t\in R,\\ \varphi(x_0(t),t), & \xi=0,\ t\in R,\\ \varphi^{(+)}(x_0(t),t)+Q_0^{(+)}(\xi,t), & \xi>0,\ t\in R.\end{cases}$$

Now we can rewrite problems (1.15)–(1.16) and (1.17)–(1.18) in the form

$$\begin{gathered}\frac{\partial^2\tilde u}{\partial\xi^2}=A(\tilde u,x_0(t),t))\frac{\partial\tilde u}{\partial\xi},\quad \xi\in R,\ t\in R\\ \tilde u(0,t)=\varphi(x_0(t),t),\quad \tilde u(-\infty,t)=\varphi^{(+)}(x_0(t),t),\quad \tilde u(+\infty,t)=\varphi^{(+)}(x_0(t),t).\end{gathered} \tag{1.19}$$

The differential equation in (1.19) is a second order autonomous ordinary differential equation (t is a parameter) , which can be analyzed in the phase plain ($\tilde u,\tilde u'$), where we have

$$\frac{\partial\tilde u}{\partial\xi}=\int_{\varphi^{(-)}(x_0(t),t)}^{\tilde u}A(u,x_0(t),t)\,du. \tag{1.20}$$

From (1.20) and assumptions (A_1) and (A_2) we get that problem (1.19) has a unique solution satisfying

$$|\tilde u(\xi,t)-\varphi^{(\pm)}(x_0(t),t)|\le c\,exp(-\kappa|\xi|)\quad \text{for}\quad \xi\in R^{(\pm)},\ t\in R,$$

where κ and c are some positive numbers.

Therefore, we get that the boundary value problems (1.15), (1.16) and (1.17), (1.18) have unique solutions satisfying the estimate

$$|Q_0^{(\pm)}(\xi,t)|\le c\,exp(-\kappa|\xi|)\quad \text{for}\quad \xi\in R^{(\pm)},\ t\in R.$$

We also note that the zero-th order C^1-matching condition (1.11) which implies

$$\frac{\partial Q_0^{(-)}}{\partial\xi}(0,t)=\frac{\partial Q_0^{(+)}}{\partial\xi}(0,t)\quad \text{for}\quad t\in R \tag{1.21}$$

is satisfied since we have

$$\frac{\partial Q_0^{(\pm)}}{\partial\xi}(0,t)=\int_{\varphi^{\pm}(x_0(t),t)}^{\varphi(x_0(t),t)}A(u,x_0(t),t)\,du,$$

and therefore according to assumption (A_2)

$$\frac{\partial Q_0^{(-)}}{\partial \xi}(0,t) - \frac{\partial Q_0^{(+)}}{\partial \xi}(0,t) = \int_{\varphi^{(-)}(x_0(t),t)}^{\varphi^{(+)}(x_0(t),t)} A(u, x_0(t), t)\, du = 0.$$

Using (1.14) we get that the boundary layer functions $Q_1^{(\pm)}(\xi, t)$ can be determined from the equations

$$\begin{aligned}
&\frac{\partial^2 Q_1^{(\pm)}}{\partial \xi^2} - \frac{\partial}{\partial \xi}\left[A\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big) Q_1^{(\pm)}\right] \\
&= A_u\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big) \frac{\partial Q_0^{(\pm)}(\xi,t)}{\partial \xi} \overline{U}_1^{(\pm)}(x_0(t),t) \\
&+ x_1(t)\Bigg[A_u\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\, \frac{\partial \varphi^{(\pm)}}{\partial x}(x_0(t),t) \\
&+ A_x\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\Bigg] \frac{\partial Q_0^{(\pm)}(\xi,t)}{\partial \xi} + q_1^{(\pm)}(\xi,t) := r_1^{(\pm)}(\xi,t)
\end{aligned} \tag{1.22}$$

and the additional conditions

$$Q_1^{(-)}(0,t) = -\overline{U}_1^{(-)}(x_0(t),t) - x_1(t)\frac{\partial \varphi^{(-)}}{\partial x}(x_0(t),t) \equiv p_1^{(-)}(\xi,t),\ Q_1^{(-)}(-\infty,t) = 0,\ t \in R, \tag{1.23}$$

$$Q_1^{(+)}(0,t) = -\overline{U}_1^{(+)}(x_0(t),t) - x_1(t)\frac{\partial \varphi^{(+)}}{\partial x}(x_0(t),t) \equiv p_1^{(+)}(\xi,t),\ Q_1^{(+)}(\infty,t) = 0,\ t \in R, \tag{1.24}$$

where

$$\begin{aligned}
q_1^{(\pm)}(\xi,t) = \Bigg[&A\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\, \frac{\partial \varphi^{(\pm)}}{\partial x}(x_0(t),t) \\
&+ A_x\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\Bigg] \frac{\partial Q_0^{(\pm)}}{\partial \xi}(x_0(t),t)\xi \\
+\Bigg[&A\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}, x_0(t), t\Big) - A\Big(\varphi^{(\pm)}(x_0(t), t, x_0(t), t\Big)\Bigg] \frac{\partial \varphi^{(\pm)}}{\partial x}(x_0(t),t) \\
+&B\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}, x_0(t), t\Big) - B\Big(\varphi^{(\pm)}(x_0(t),t), x_0(t), t\Big) + x_0'(t)\frac{\partial Q_0^{(\pm)}}{\partial \xi}.
\end{aligned}$$

The problems (1.22), (1.23) and (1.22), (1.24) are linear inhomogeneous problems with exponentially decaying inhomogeneous terms. The solutions of these problems can be given explicitly

$$Q_1^{(\pm)}(\xi,t) = z(\xi,t)\left\{p_1^{(\pm)}(t) - \int_0^{\xi} \frac{1}{z(\eta,t)}\left[\int_{\eta}^{\pm\infty} q_1^{(\pm)}(\chi,t)d\chi\right] d\eta\right\}, \tag{1.25}$$

where

$$z(\xi,t) \equiv \frac{\partial \tilde{u}}{\partial \xi}(\xi,t) \left(\frac{\partial \tilde{u}}{\partial \xi}(0,t) \right)^{-1}.$$

From the representation for $q_1^{(\pm)}(\xi,t)$ it follows that $|q_1^{(\pm)}(\xi,t)| \leq c\, exp(-\kappa|\xi|)$, therefore from (1.25) we get that the functions $Q_1^{(\pm)}$ satisfy the exponential estimate

$$|Q_1^{(\pm)}(\xi,t)| \leq c\, exp(-\kappa|\xi|) \quad \text{for} \quad \xi \in R^{(\pm)}, t \in R,$$

where c and κ are some positive numbers.

The first order C^1–matching condition reads

$$\frac{\partial Q_1^{(-)}}{\partial \xi}(0,t) + \frac{\partial \varphi^{(-)}}{\partial x}(x_0(t),t) = \frac{\partial Q_1^{(+)}}{\partial \xi}(0,t) + \frac{\partial \varphi^{(+)}}{\partial x}(x_0(t),t). \tag{1.26}$$

Using the expressions for $\frac{\partial Q_1^{(\pm)}}{\partial \xi}(0,t)$ in (1.25) we can show that $x_1(t)$ is uniquely determined by the equation

$$I_x(x_0(t),t)\, x_1(t) = \Phi_1(t), \tag{1.27}$$

where $\Phi_1(t)$ is the known smooth periodic function:

$$\Phi_1(t) := \frac{\partial \varphi^{(+)}}{\partial x}(x_0(t),t) - \frac{\partial \varphi^{(-)}}{\partial x}(x_0(t),t)$$

$$-x_0'(t)\Big(\frac{\partial \varphi^{(+)}}{\partial t}(x_0(t),t) - \frac{\partial \varphi^{(-)}}{\partial t}(x_0(t),t) + \frac{\partial Q_0^{(+)}}{\partial t}(x_0(t),t) - \frac{\partial Q_0^{(-)}}{\partial t}(x_0(t),t) \Big)$$

$$-A\Big(\varphi^{(+)}(x_0(t),t), x_0(t), t\Big)\overline{U}_1^{(+)}(x_0(t),t) + A\Big(\varphi^{(-)}(x_0(t),t), x_0(t), t\Big)\overline{U}_1^{(-)}(x_0(t),t)$$

$$+\int_{-\infty}^{0} q_1^{(-)}(\xi,t)d\xi + \int_{0}^{+\infty} q_1^{(+)}(\xi,t)d\xi.$$

From assumption (A_3) it follows that (1.27) can be solved uniquely for $x_1(t)$. The higher order terms $Q_k^{(\pm)}$ can be determined by problems, which have the same structure as (1.22) (index 1 has to be replaced by index k and $q_k^{(\pm)}$ is a known function). The k-th order C^1-matching condition leads to an equation similar to (1.27):

$$I_x(x_0(t),t)\, x_k(t) = \Phi_k(t), \tag{1.28}$$

where $\Phi_k(t)$ is a known smooth periodic function.

Since $A, B, u^{(\pm)}$ are sufficiently smooth, the formal asymptotics can be constructed to any order n. From these constructions it follows that the corresponding approximations satisfy (1.1) up to order ε^n.

2 Existence results

2.1 Main theorem

Let $\mathcal{D}_n^{(-)}$ and $\mathcal{D}_n^{(+)}$ be the domains

$$\mathcal{D}_n^{(-)} := \{(x,t) \in R^2 : -1 \le x \le \sum_{i=0}^{n+1} x_i(t)\varepsilon^i,\ t \in R\},$$

$$\mathcal{D}_n^{(+)} := \{(x,t) \in R^2 : \sum_{i=0}^{n+1} x_i(t)\varepsilon^i \le x \le 1,\ t \in R\},$$

and let $U_n^{(\pm)}$ be the partial sums of order n of the expansions (1.8) and (1.10), respectively, where ξ is replaced by $\left(x - \sum_{i=0}^{n+1} x_i(t)\varepsilon^i\right)/\varepsilon$.
We introduce the notation

$$U_n(x,t,\varepsilon) \ := \ \begin{cases} U_n^{(-)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(-)}, \\ U_n^{(+)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(+)}. \end{cases}$$

Then we have the following existence theorem:

Theorem 2.1 *Suppose the assumptions $(A_0) - (A_3)$ to be valid. Then, for sufficiently small ε, there exists a solution $u(x,t,\varepsilon)$ of (1.1) which has an interior layer near $x_0(t)$, i.e.*

$$\lim_{\varepsilon \to 0} u(x,t,\varepsilon) = \begin{cases} \varphi^{(-)}(x,t) & \text{for} \quad -1 \le x < x_0(t),\ t \in R, \\ \varphi^{(+)}(x,t) & \text{for} \quad x_0(t) < x \le 1,\ t \in R \end{cases}$$

and satisfies

$$|u(x,t,\varepsilon) - U_n(x,t,\varepsilon)| \ \le \ c_n\varepsilon^n \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}},$$

where the positive constant c_n does not depend on ε.

2.2 Construction of upper and lower solutions

The proof of this theorem is based on the technique of lower and upper solutions. For the convenience of the reader we recall the definition of these functions.

Definition 2.1 *We say the functions $\alpha, \beta : \overline{\mathcal{D}} \times \overline{I}_{\varepsilon_0} \to R$ have the smoothness property S, if they are continuous, twice continuously differentiable in x, continuously differentiable in t and T-periodic in t. The functios α and β are called ordered lower and upper solutions of* (1.1) *for $\varepsilon \in I_{\varepsilon_0}$, if they have the smoothness property S and satisfy for $\varepsilon \in I_{\varepsilon_0}$ the following conditions:*

$$1^\circ \quad \alpha(x,t,\varepsilon) \le \beta(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{2.1}$$

$$2^\circ \quad N_\varepsilon(\alpha) \ge 0 \ge N_\varepsilon(\beta) \ \text{for}\ (x,t) \in \mathcal{D}, \tag{2.2}$$

$$3^\circ \quad \alpha(-1,t,\varepsilon) \le u^{(-)}(t) \le \beta(-1,t,\varepsilon), \tag{2.3}$$

$$\alpha(1,t,\varepsilon) \le u^{(+)}(t) \le \beta(1,t,\varepsilon) \ \text{for}\ t \in R.$$

In case that there exists in $\mathcal{D}$ some smooth curve $x = \bar{x}(t), t \in R$, periodic in t and deviding $\mathcal{D}$ into two subregions $\mathcal{D}^+$ and $\mathcal{D}^-$ such that α and β have the smoothness property S only in $\mathcal{D}^+$ and $\mathcal{D}^-$, then α and β are called ordered lower and upper solutions of (1.1) *for $\varepsilon \in I_{\varepsilon_0}$, if they satisfy the relations (2.1) and (2.2) in $\mathcal{D}^+$ and $\mathcal{D}^-$, the relation (2.3) and the inequalities*

$$\frac{\partial\alpha}{\partial x}(\bar{x}(t)+0,t,\varepsilon) \geq \frac{\partial\alpha}{\partial x}(\bar{x}(t)-0,t,\varepsilon), \tag{2.4}$$

$$\frac{\partial\beta}{\partial x}(\bar{x}(t)+0,t,\varepsilon) \leq \frac{\partial\beta}{\partial x}(\bar{x}(t)-0,t,\varepsilon). \tag{2.5}$$

Remark 1. *It is known (see, e.g., [1]) that the existence of ordered lower and upper solutions implies the existence of a unique solution $u(x,t,\varepsilon)$ of (1.1) satisfying*

$$\alpha(x,t,\varepsilon) \leq u(x,t,\varepsilon) \leq \beta(x,t,\varepsilon) \quad \textit{for} \quad (x,t) \in \mathcal{D} \quad \textit{and} \quad \varepsilon \in I_{\varepsilon_0}.$$

In what follows we describe a method to construct upper and lower solutions by some modification of the formal asymptotic expansion of the solution to (1.1). For this purpose we introduce T-periodic functions x_β and x_α as the $(n+1)$-th partial sums of the asymptotic expansion of $x_*(t,\varepsilon)$ with a small shift in the last term

$$x_\beta(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + ... + \varepsilon^{n+1}(x_{n+1}(t) - \delta),$$
$$x_\alpha(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + ... + \varepsilon^{n+1}(x_{n+1}(t) + \delta),$$

where $\delta > 0$ is a small number independent of ε. The curves $x = x_\beta(t,\varepsilon)$ and $x = x_\alpha(t,\varepsilon)$ divide the domain $\overline{\mathcal{D}}$ into two subdomains $\overline{\mathcal{D}}_\beta^{(-)}$, $\overline{\mathcal{D}}_\beta^{(+)}$ and $\overline{\mathcal{D}}_\alpha^{(-)}$, $\overline{\mathcal{D}}_\alpha^{(+)}$, where

$$\overline{\mathcal{D}}_\beta^{(-)} := \{(x,t) \in R^2 : -1 \leq x \leq x_\beta(t,\varepsilon),\ t \in R\},$$
$$\overline{\mathcal{D}}_\beta^{(+)} := \{(x,t) \in R^2 : x_\beta(t,\varepsilon) \leq x \leq 1,\ t \in R\}.$$

The domains $\overline{\mathcal{D}}_\alpha^{(\pm)}$ are defined similarly.

Now we can define an upper solution $\beta(x,t,\varepsilon) = \beta_n(x,t,\varepsilon)$ and a lower solution $\alpha(x,t,\varepsilon) = \alpha_n(x,t,\varepsilon)$ for $\varepsilon \in I_{\varepsilon_0}$ in $\mathcal{D}$ by the expressions

$$\begin{aligned}
&\beta_n(x,t,\varepsilon) = \beta_n^{(\pm)}(x,t,\varepsilon) = \bar{U}_0^{(\pm)}(x,t) + \varepsilon\bar{U}_1^{(\pm)}(x,t) + ... + \varepsilon^{n+1}\bar{U}_{n+1}^{(\pm)}(x,t)\\
&+ Q_0^{(\pm)}(\xi_\beta,t) + \varepsilon Q_1^{(\pm)}(\xi_\beta,t) + ... + \varepsilon^{n+1}Q_{(n+1)}^{(\pm)}(\xi_\beta,t)\\
&+ \varepsilon^{n+1}\Big(v^{(\pm)}(x) + q_\beta^{(\pm)}(\xi_\beta,t)\Big) + \varepsilon^{n+2}Q_{(n+2),\beta}^{(\pm)}(\xi_\beta,t,\varepsilon)\\
&=: U_{n+1,\beta}^{(\pm)}(x,t,\varepsilon) + \varepsilon^{n+2}Q_{(n+2),\beta}^{(\pm)}(\xi_\beta,t,\varepsilon)
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\alpha_n(x,t,\varepsilon) = \alpha_n^{(\pm)}(x,t,\varepsilon) = \bar{U}_0^{(\pm)}(x,t) + \varepsilon\bar{U}_1^{(\pm)}(x,t) + ... + \varepsilon^{n+1}\bar{U}_{n+1}^{(\pm)}(x,t)\\
&+ Q_0^{(\pm)}(\xi_\alpha,t) + \varepsilon Q_1^{(\pm)}(\xi_\alpha,t) + ... + \varepsilon^{n+1}Q_{(n+1)}^{(\pm)}(\xi_\alpha,t)\\
&+ \varepsilon^{n+1}\Big(q_\alpha^{(\pm)}(\xi_\alpha,t) - v^{(\pm)}(x)\Big) + \varepsilon^{n+2}Q_{(n+2),\alpha}^{(\pm)}(\xi_\alpha,t,\varepsilon)\\
&=: U_{n+1,\alpha}^{(\pm)}(x,t,\varepsilon) + \varepsilon^{n+2}Q_{(n+2),\alpha}^{(\pm)}(\xi_\alpha,t,\varepsilon).
\end{aligned} \tag{2.7}$$

Here, $v^{(\pm)}(x) := e^{\pm mx}$ with $m > 0$ sufficiently large and independent of ε, $\xi_\beta = (x - x_\beta(t,\varepsilon))/\varepsilon$, $\quad \xi_\alpha = (x - x_\alpha(t,\varepsilon))/\varepsilon$.
The functions $Q_i^{(\pm)}(\xi_\beta, t)$ and $Q_i^{(\pm)}(\xi_\alpha, t)$, $i = 0, ..., n+1$ are defined by replacing ξ by ξ_β and ξ by ξ_α in the terms of the asymptotics $Q_i^{(\pm)}(\xi, t)$.
The functions $q_\alpha(\xi_\alpha, t)$ and $q_\beta(\xi_\beta, t)$ are introduced into the interior layer part to compensate the changes which are produced by the modification of the $(n+1)$-th order term of the regular part of the asymptotic expansion and the interior layer curve expansion. The functions $q_\beta^{(\pm)}(\xi_\beta, t)$ are determined by the problems

$$\begin{aligned} \frac{\partial^2 q_\beta^{(\pm)}}{\partial \xi^2} &= \frac{\partial}{\partial \xi}\left[A\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\Big(q_\beta^{(\pm)} + v^{(\pm)}(x_0(t))\Big)\right] \\ &+ \delta\Bigg[A_u\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big)\frac{\partial \varphi^{(\pm)}}{\partial x}(x_0(t),t) \\ &+ A_x\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t)\Big)\Bigg]\frac{\partial Q_0^{(\pm)}(\xi,t)}{\partial \xi} \quad \text{for} \quad \xi \in R^{(\pm)}, t \in R \end{aligned} \tag{2.8}$$

and the additional conditions

$$q_\beta^{(-)}(0,t) + v^{(-)}(x_0(t)) + \delta\frac{\partial \varphi^{(-)}}{\partial x}(x_0(t),t) = 0,\ q_\beta^{(-)}(-\infty,t) = 0 \quad \text{for} \quad t \in R, \tag{2.9}$$

$$q_\beta^{(+)}(0,t) + v^{(+)}(x_0(t)) + \delta\frac{\partial \varphi^{(+)}}{\partial x}(x_0(t),t) = 0,\ q_\beta^{(+)}(\infty,t) = 0 \quad \text{for} \quad t \in R. \tag{2.10}$$

Problems (2.8),(2.9) and (2.8),(2.10) can be investigated analogously as the problems (1.22),(1.23) and (1.22),(1.24) and have a unique exponentially decaying solution for all real δ.
The differential equations to determine the functions $Q_{(n+2),\beta}^{(\pm)}$ and $Q_{(n+2),\alpha}^{(\pm)}$ are slightly different from the equation for $Q_{n+1}^{(\pm)}$. One can check that the functions $U_{n+1,\beta}(x,t,\varepsilon)$

$$U_{n+1,\beta}(x,t,\varepsilon) \quad := \quad \begin{cases} U_{n+1,\beta}^{(-)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(-)}, \\ U_{n+1,\beta}^{(+)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(+)} \end{cases}$$

and $U_{n+1,\alpha}(x,t,\varepsilon)$

$$U_{n+1,\alpha}(x,t,\varepsilon) \quad := \quad \begin{cases} U_{n+1,\alpha}^{(-)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(-)}, \\ U_{n+1,\alpha}^{(+)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in \mathcal{D}_n^{(+)} \end{cases}$$

are discontinuous at the curves $x_\beta(t,\varepsilon)$ and $x_\alpha(t,\varepsilon)$, respectively. Particularly, we get from (2.7), (2.6), (2.8) - (2.10)

$$U_{n+1,\beta}^{(+)}(x_\beta(t,\varepsilon),t,\varepsilon) - U_{n+1,\beta}^{(-)}(x_\beta(t,\varepsilon),t,\varepsilon) = \varepsilon^{n+1}[q_\beta^{(+)}(0,t) + v^{(+)}(x_0(t)) + \delta\frac{\partial \varphi^{(+)}}{\partial x}(x_0(t),t)$$

$$-(q_\beta^{(-)}(0,t) + v^{(-)}(x_0(t)) + \delta\frac{\partial \varphi^{(-)}}{\partial x}(x_0(t),t)) + O(\varepsilon)] = O(\varepsilon^{n+2}).$$

The functions $Q^{(\pm)}_{(n+2),\beta}$ in (2.6) are introduced to get upper solutions which are continuous also at the curve $x_\beta(t,\varepsilon)$ and to satisfy the differential equation (1.14) up to the terms of order $(n+1)$ at the interior layer. They are defined as the solutions of the problems

$$\frac{\partial^2 Q^{(\pm)}_{(n+2),\beta}}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left[A\Big(\varphi^{(\pm)}(x_0(t),t) + Q_0^{(\pm)}(\xi,t), x_0(t), t\Big) Q^{(\pm)}_{(n+2),\beta}\right] + q^{(\pm)}_{n+2,\beta} \tag{2.11}$$

with the additional conditions

$$Q^{(-)}_{(n+2),\beta}(0,t) = 0,\ Q^{(-)}_{(n+2),\beta}(-\infty,t) = 0 \quad \text{for} \quad t \in R, \tag{2.12}$$

$$\begin{aligned} Q^{(+)}_{(n+2),\beta}(0,t) = -\Big(U^{(+)}_{n+1,\beta}(x_\beta(t,\varepsilon),t,\varepsilon) - U^{(-)}_{n+1,\beta}(x_\beta(t,\varepsilon),t,\varepsilon)\Big),\\ Q^{(+)}_{(n+2),\beta}(-\infty,t,\varepsilon) = 0 \quad \text{for} \quad t \in R, \end{aligned} \tag{2.13}$$

where $\xi = \xi_\beta$, the term $q^{(\pm)}_{n+2,\beta}$ has the same structure as $q^{(\pm)}_{n+2}$ in the corresponding problems for $Q^{(\pm)}_{(n+2)}$ with the replacements of $Q^{(\pm)}_{(n+1)}$ by $Q^{(\pm)}_{(n+1)}+q^{(\pm)}_\beta$, $\bar{U}^{(\pm)}_{n+1}(x_0(t),t)$ by $(\bar{U}^{(\pm)}_{n+1}(x_0(t),t) + v^{(\pm)}(x_0(t)))$, $x_{n+1}(t)$ by $(x_{n+1}(t)-\delta)$, $x_{n+2}(t)$ by 0.

Corresponding results hold for α_n. It is clear that the functions $Q^{(\pm)}_{(n+2),\beta}$ and $Q^{(\pm)}_{(n+2),\alpha}$ will not influence the estimates of the lower and upper solutions.

To verify that the introduced functions α_n and β_n are upper and lower solutions we substitute the expressions for α_n and β_n into the operator $N_\varepsilon(u)$ defined in (1.1). We get

$$\begin{aligned} N_\varepsilon(\beta_n(x,t,\varepsilon)) = \varepsilon^{n+1}\Big[A(\varphi^{(\pm)}(x,t),x,t)\frac{\partial v^{(\pm)}(x)}{\partial x} + \Big(A_u(\varphi^{(\pm)}(x,t),x,t)\frac{\partial \overline{U}_0^{(\pm)}}{\partial x}\\ + B_u(\varphi^{(\pm)}(x,t),x,t)\Big)v^{(\pm)}(x)\Big] + O(\varepsilon^{n+2}) \quad \text{for} \quad (x,t) \in \mathcal{D}^{(\pm)}. \end{aligned} \tag{2.14}$$

It can be easily verified that by choosing $v^{(\pm)}(x) = e^{\pm mx}$, where m is sufficiently large, we get that the coefficient of ε^{n+1} in (2.14) is negative and therefore we have for sufficiently small ε

$$N_\varepsilon(\beta_n(x,t,\varepsilon)) \le -\gamma\varepsilon^{n+1},$$

where γ is a positive number. Similarly we obtain

$$N_\varepsilon(\alpha_n(x,t,\varepsilon)) \ge \gamma\varepsilon^{n+1}.$$

In order to prove that $\alpha_n(x,t,\varepsilon)$ and $\beta_n(x,t,\varepsilon)$ are ordered we use the approach proposed in [3]. We divide the domain $\overline{\mathcal{D}}$ into three parts:

$$\overline{\mathcal{D}_1} := \{(x,t) \in R^2 : -1 \le x \le x_\beta(t,\varepsilon),\ t \in R\},$$

$$\overline{\mathcal{D}_2} := \{(x,t) \in R^2 : x_\beta(t,\varepsilon) \le x \le x_\alpha(t,\varepsilon),\ t \in R\},$$

$$\overline{\mathcal{D}_3} := \{(x,t) \in R^2 : x_\alpha(t,\varepsilon) \le x \le 1,\ t \in R\}.$$

In $\overline{\mathcal{D}_2}$ we obtain from (2.7),(2.6)

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = \varepsilon^n 2\delta \frac{\partial Q_0}{\partial \xi}(0,t) + O(\varepsilon^{n+1}) \quad \text{for} \quad t \in R.$$

Using $\frac{\partial Q_0}{\partial \xi}(0,t) > 0$ we get for sufficiently small ε

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}_2}.$$

In the domain $\overline{\mathcal{D}_1}$ we apply the mean value theorem and get

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = \varepsilon^n 2\delta \tfrac{\partial Q_0^{(-)}}{\partial \xi}(\zeta_1,t)$$
$$+\varepsilon^{n+1}\Big[2\delta\Big(\tfrac{\partial Q_1^{(-)}}{\partial \xi}(\zeta_2,t) + \tfrac{\partial q_\beta^{(-)}}{\partial \xi}(\zeta_3,t)\Big) + 2v^{(-)}(x)\Big] + O(\varepsilon^{n+2}),$$

where ζ_1, ζ_2 and ζ_3 are some values in the interval $\Big[(x - x_\beta(t,\varepsilon))/\varepsilon, (x - x_\alpha(t,\varepsilon))/\varepsilon\Big]$.

Using the known estimates

$$\frac{\partial Q_0^{(-)}}{\partial \xi}(\xi,t) \geq C_1 \exp(\nu_1 \xi)$$

and

$$\Big|\frac{\partial Q_1^{(-)}}{\partial \xi}(\xi,t) + \frac{\partial q_\beta^{(-)}}{\partial \xi}(\xi,t)\Big| \leq C_2 \exp(\nu_2 \xi),$$

where C_1, C_2, ν_1 and ν_2 are some positive numbers, one can easily calculate that

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}_1.$$

A similar approach can be used in the region $\overline{\mathcal{D}_3}$.

We summarize the results of our construction of upper and lower solutions in the following lemma.

Lemma 2.1. *The functions $\beta_n(x,t,\varepsilon)$ and $\alpha_n(x,t,\varepsilon)$ defined by the expressions (2.6) and (2.7), respectively, satisfy the Definition 2.1, and therefore they are upper and lower solutions of problem (1.1). Moreover, they obey the following relations:*

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = O(\varepsilon^n) \quad \textit{for} \quad x \in [0,1],\ t \in R, \tag{2.15}$$

$$\beta_n(x,t,\varepsilon) - U_n(x,t,\varepsilon) = O(\varepsilon^n) \quad \textit{for} \quad x \in [0,1],\ t \in R, \tag{2.16}$$

$$\frac{\partial \alpha_n}{\partial x} = \frac{\partial U_n}{\partial x} + O(\varepsilon^{n-1}), \quad \frac{\partial \beta_n}{\partial x} = \frac{\partial U_n}{\partial x} + O(\varepsilon^{n-1}) \quad \textit{for} \quad x \in [0,1],\ t \in R, \tag{2.17}$$

$$N_\varepsilon(\beta_n(x,t,\varepsilon)) \leq -\gamma\varepsilon^{n+1},\ N_\varepsilon(\alpha_n(x,t,\varepsilon)) \geq \gamma\varepsilon^{n+1}. \tag{2.18}$$

Therefore, we get that problem (1.1) has a solution which satisfies (see Remark 1)

$$\alpha_n(x,t,\varepsilon) \leq u(x,t,\varepsilon) \leq \beta_n(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \mathcal{D} \quad \text{and} \quad \varepsilon \in I_{\varepsilon_0}.$$

The statements of Theorem 2.1 follow from the estimates (2.15), (2.16) of Lemma 2.1.

In our approach to investigate the stability (see Sect. 3) we need the following relations.

Lemma 2.2. *The functions $\beta_n(x,t,\varepsilon)$ and $\alpha_n(x,t,\varepsilon)$ satisfy for $(x,t) \in \overline{\mathcal{D}}$ the following relations:*

$$\frac{\partial \alpha_n}{\partial x} = \frac{\partial u}{\partial x} + O(\varepsilon^{n-1}), \quad \frac{\partial \beta_n}{\partial x} = \frac{\partial u}{\partial x} + O(\varepsilon^{n-1}), \tag{2.19}$$

where $u = u(x,t,\varepsilon)$ is the periodic interior layer solution of problem (1.1), stated in Theorem 2.1.

Proof. The proof of Lemma 2.2 is based on the estimate for the difference $z_n(x,t,\varepsilon) \equiv u(x,t,\varepsilon) - U_n(x,t,\varepsilon)$; estimate (2.19) then trivially follows from estimate (2.17).

The function $z_n(x,t,\varepsilon)$ satisfies the equation

$$\begin{aligned} &\varepsilon\left(\frac{\partial^2 z_n}{\partial x^2} - \frac{\partial z_n}{\partial t}\right) - \left[A(u,x,t)\,\frac{\partial u}{\partial x} - A(U_n,x,t)\,\frac{\partial U_n}{\partial x}\right] \\ &- \left[B(u,x,t) - B(U_n,x,t)\right] = \varepsilon^{n+1}\psi(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \mathcal{D} \end{aligned} \tag{2.20}$$

with zero boundary conditions, where $|\psi(x,t,\varepsilon)| \le c_1$. From Theorem 2.1 we get

$$z_n(x,t,\varepsilon) \equiv u(x,t,\varepsilon) - U_n(x,t,\varepsilon) \le c\varepsilon^n, \tag{2.21}$$

and therefore

$$|r_1| := |B(u,x,t) - B(U_n,x,t)| \le c\varepsilon^n.$$

The second term of equation (2.20) can be represented in the form

$$A(u,x,t)\,\frac{\partial u}{\partial x} - A(U_n,x,t)\,\frac{\partial U_n}{\partial x} = \frac{\partial}{\partial x}\int_{U_n}^{u} A(s,x,t)ds - \int_{U_n}^{u} A_x(s,x,t)ds.$$

From (2.21) it follows

$$|r_2| := |\int_{U_n}^{u} A_x(s,x,t)ds| \le c\varepsilon^n.$$

We can rewrite equation (2.20) in the following form

$$\begin{aligned} &\frac{\partial^2 z_n}{\partial x^2} - \frac{\partial z_n}{\partial t} - kz_n = -kz_n + \frac{1}{\varepsilon}\frac{\partial}{\partial x}\int_{U_n}^{u} A(s,x,t)ds \\ &+ \frac{1}{\varepsilon}\Big[r_1(x,t,\varepsilon) - r_2(x,t,\varepsilon) + \varepsilon^{n+1}\psi(x,t,\varepsilon)\Big] \quad \text{for} \quad (x,t) \in \mathcal{D}. \end{aligned} \tag{2.22}$$

Now we can define

$$r(x,t,\varepsilon) := \frac{1}{\varepsilon}\Big[r_1(x,t,\varepsilon) - r_2(x,t,\varepsilon) + \varepsilon^{n+1}\psi(x,t,\varepsilon)\Big].$$

From the estimates above we obtain

$$|r(x,t,\varepsilon)| \le c\varepsilon^{n-1}.$$

Using the Green function for the parabolic operator of the left hand side of (2.22) we get the following representation for z_n (see, for example, [5]).

$$
\begin{aligned}
z_n = &\int_{-1}^{1} G(x,t,\xi,t_0) z_n(\xi,t_0) d\xi - \int_{t_0}^{t} d\tau \int_{-1}^{1} G(x,t,\xi,\tau) \\
&\left(-k z_n(\xi,\tau) + r(\xi,\tau,\varepsilon) + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi} \int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau) ds \right) d\xi.
\end{aligned} \tag{2.23}
$$

Using integration by parts and the boundary conditions for G one can transform the last term in (2.23) as follows

$$
\begin{aligned}
&\int_{t_0}^{t} d\tau \int_{-1}^{1} G(x,t,\xi,\tau) \frac{1}{\varepsilon}\frac{\partial}{\partial \xi} \int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau) ds d\xi = \\
&- \int_{t_0}^{t} d\tau \int_{-1}^{1} G_{\xi}(x,t,\xi,\tau) \frac{1}{\varepsilon} \int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau) ds d\xi.
\end{aligned} \tag{2.24}
$$

Using (2.24) we get from (2.23) the following representation for the derivative $\frac{\partial z_n}{\partial x}$:

$$
\begin{aligned}
\frac{\partial z_n}{\partial x} = &\int_{-1}^{1} G_x(x,t,\xi,t_0) z_n(\xi,t_0) d\xi - \int_{t_0}^{t} d\tau \int_{-1}^{1} G_x(x,t,\xi,\tau) \\
&\left(-k z_n(\xi,\tau) + r(\xi,\tau,\varepsilon) \right) d\xi + \int_{t_0}^{t} d\tau \int_{-1}^{1} G_{\xi x}(x,t,\xi,\tau) \frac{1}{\varepsilon} \int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau) ds d\xi.
\end{aligned} \tag{2.25}
$$

The validity of the representation (2.25) follows from the estimates

$$
\Big| \int_{-1}^{1} G_x(x,t,\xi,t_0) d\xi \Big| \leq C, \ \Big| \int_{t_0}^{t} d\tau \int_{-1}^{1} G_x(x,t,\xi,\tau) d\xi \Big| \leq C.
$$

We get that the first and second terms of the representation (2.25) have the estimates $O(\varepsilon^n)$ and $O(\varepsilon^{n-1})$, respectively. From the estimates for $G_{\xi x}(x,t,\xi,\tau)$ it also follows that the last term in the representation (2.25) can be estimated by

$$
\frac{1}{\varepsilon} \Big| \int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau) ds d\xi \Big|.
$$

Using these estimates, finally we get from (2.25)

$$
\frac{\partial z_n}{\partial x}(x,t,\varepsilon) = O(\varepsilon^{n-1}) \quad \text{for} \quad (x,t) \in \mathcal{D}. \tag{2.26}
$$

This completes the proof of Lemma 2.2.

3 Stability results

In this section we investigate the stability (in the sense of Lyapunov) of the periodic solution $u(x,t,\varepsilon)$ with the interior layer established in Theorem 2.1 by applying Theorem 4.4. from [2].

Recall that we denote the lower and upper solutions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ as asymptotic lower and upper solutions of problem (1.1) of order q if they satisfy the corresponding differential inequality with the residual term of order ε^q.

For convenience we restate the mentioned theorem which we apply to establish the asymptotic stability of our periodic solution from Theorem 2.1 as the following lemma.

Lemma 3.1 *Let $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ be lower and upper solutions of order $q>0$ to (1.1), let $u(x,t,\varepsilon)$ be the corresponding periodic solution to (1.1). Suppose that for sufficiently small ε, and all t and x it holds*

$$|u(x,t,\varepsilon)|+|\alpha(x,t,\varepsilon)|+|\beta(x,t,\varepsilon)|\le\kappa_1,$$

$$|\beta(x,t,\varepsilon)-u(x,t,\varepsilon)|+|\alpha(x,t,\varepsilon)-u(x,t,\varepsilon)|\le\kappa_2\varepsilon^{\frac{p+1}{2}},$$

$$|\partial_x\beta(x,t,\varepsilon)-\partial_x u(x,t,\varepsilon)|+|\partial_x\alpha(x,t,\varepsilon)-\partial_x u(x,t,\varepsilon)|\le c_2\varepsilon^{\frac{p-1}{2}},$$

where κ_1, κ_2 and $p>q$ are constants. Then, for sufficiently small $\varepsilon>0$, the solution $u(x,t,\varepsilon)$ to (1.1) is asymptotically stable in the sense of Lyapunov (see the definition, for example, in [1]).

In our case, it follows from Lemma 2.1 and Lemma 2.2 that $q=n+1$, and $\frac{p+1}{2}=n$, and therefore $p=2n-1$. The condition of Lemma 3.1 $p>q$ leads to the condition for the order of our lower and upper solution. We get

$$n>2.$$

Therefore applying Lemma 3.1 we can state the following theorem on the stability of the periodic solution

Theorem 3.1 *Suppose the assumptions $(A_0)-(A_3)$ to be satisfied. Then for sufficiently small ε the periodic solution of problem (2.1) with interior layer is asymptotically stable with the domain of attraction $\alpha_3(x,t,\varepsilon)\le u\le\beta_3(x,t,\varepsilon)$.*

4 Example

We consider problem (1.1) in the special case

$$A(u,x,t)\equiv -u,\ B(u,x,t)\equiv u,\ u^{(-)}(t)\equiv -4+k\sin t,\ u^{(-)}(t)\equiv 3,$$

where k satisfies

$$-1<k<1.$$

Thus, the boundary value problem (1.1) reads

$$\begin{aligned}
&\varepsilon\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)+u\,\frac{\partial u}{\partial x}-u=0\\
&\quad\text{for}\quad (x,t)\in\mathcal{D}:=\{(x,t)\in R^2:-1<x<1,t\in R\},\\
&u(-1,t,\varepsilon)=-4+k\sin t,\quad u(1,t,\varepsilon)=3\quad\text{for}\quad t\in R,\\
&u(x,t,\varepsilon)=u(x,t+2\pi,\varepsilon)\quad\text{for}\quad t\in R\quad -1\le x\le 1
\end{aligned}\tag{4.1}$$

and the assumption (A_0) is fulfilled. The corresponding degenerate equation (1.2) can be written as

$$-u\,(\frac{\partial u}{\partial x}+1)=0.$$

The solutions $u=\varphi^{(-)}(x,t)$ and $u=\varphi^{(+)}(x,t)$ of this equation satisfying the initial values

$$u(-1,t)=-4+k\sin t, \quad u(1,t)=3$$

can be given explicitly

$$u=\varphi^{(-)}(x,t)=x-3+k\sin t, \quad u=\varphi^{(+)}(x,t)=x+2.$$

Thus, we have

$$\varphi^{(-)}(x,t)<0<\varphi^{(+)}(x,t) \quad \text{for} \quad (x,t)\in\mathcal{D},$$

$$A(\varphi^{(+)}(x,t),x,t)=-\varphi^{(+)}(x,t)<0<-\varphi^{(-)}(x,t)=A(\varphi^{(+)}(x,t),x,t) \quad \text{for} \quad (x,t)\in\mathcal{D}.$$

Hence, assumption (A_1) is satisfied. Furthermore, we have

$$I(x,t)=\int_{x-3+k\sin t}^{x+2} -u\,du=-\frac{1}{2}(5-k\sin t)(2x-1+k\sin t)=0.$$

Thus, the equation $I(x,t)=0$ has the smooth solution $x=x_0(t)=\frac{1-k\sin t}{2}$ which 2π-periodic. Finally, it holds

$$\frac{\partial I}{\partial x}(x_0(t),t)=-(5-k\sin t)<0 \quad \text{for} \quad t\in R.$$

Therefore, all assumptions (A_3) of Theorem 3.1 are satisfied and we have the result

Theorem 4.1 *The boundary value problem (4.1) has for sufficiently sall ε and $-1<k<1$ a solution $u(x,t)$ with the properties*
(i) $u(x,t)$ has an interior boundary layer near $x_0(t)=(1-k\sin t)/2$.
(ii) $u(x,t)$ is asymptotically stable.

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