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On a singularly perturbed initial value problem in case of a double root of the degenerate equation

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Abstract

We study the initial value problem of a singularly perturbed first order ordinary differential equation in case that the degenerate equation has a double root. We construct the formal asymptotic expansion of the solution such that the boundary layer functions decay exponentially. This requires a modification of the standard procedure. The asymptotic solution will be used to construct lower and upper solutions guaranteeing the existence of a unique solution and justifying its asymptotic expansion.

1 Introduction. Formulation of the problem

Consider the initial value problem for a scalar singularly perturbed first order ordinary differential equation

$$\varepsilon^2 \frac{dy}{dx} = f(y,x,\varepsilon)\,, \qquad x \in I := [0,X], \tag{1}$$

$$y(0,\varepsilon) = y^0\,, \tag{2}$$

where ε is a small positive parameter. The initial value problem (1), (2) is well studied in the case that the degenerate equation

$$f(y,x,0) = 0 \tag{3}$$

has a simple root $y = \varphi(x)$ for $x \in I$ (see [1]). Under the conditions that f and φ are continuously differentiable and satisfy $f_y(\varphi(x),x,0) < 0$ for $x \in I$, and that y^0 belongs to the region of attraction of φ, problem (1), (2) has a unique solution $y(x,\varepsilon)$ with the asymptotic representation

$$y(x,\varepsilon) = \varphi(x) + \Pi_0\Big(\frac{x}{\varepsilon^2}\Big) + O(\varepsilon) \quad \text{for} \quad x \in I, \tag{4}$$

where Π_0 is the boundary layer function defined by the initial value problem

$$\frac{d\Pi_0}{d\tau} = f\big(\varphi(0) + \Pi_0, 0, 0\big), \quad \tau \geq 0, \quad \Pi_0(0) = y^0 - \varphi(0).$$

We note (see [1]) that Π_0 satisfies the estimate

$$|\Pi_0(\tau)| \leq c\exp(-\kappa\tau) \quad \text{for} \quad \tau \geq 0. \tag{5}$$

Here, and in the sequel, c and κ denote suitable positive numbers not depending on ε (by this way we avoid to introduce indices for each estimate).

If the function f is sufficiently smooth, then for any natural number n there exists an asymptotic expansion of the solution $y(x, \varepsilon)$ with a reminder term of order $O(\varepsilon^n)$.
In this paper, we study the initial value problem (1) (2) under the condition that $\varphi(x)$ is a double root of equation (3) for $x \in I$. We note that under the same assumption, in the papers [2, 3] some boundary value problems have been studied for the differential equation

$$\varepsilon^2 \frac{d^2y}{dx^2} = f(y, x),$$

where the function f does not depend on ε. We emphasize that in our case the dependence of f on ε plays a fundamental role.

For the sequel we assume

(A_1). There are sufficiently smooth functions h, φ and f_1 such that f can be represented in the form

$$f(y, x, \varepsilon) = -h(x)(y - \varphi(x))^2 + \varepsilon f_1(y, x, \varepsilon), \tag{6}$$

where we additionally suppose $h(x) > 0$ for $x \in I$.

We will prove that under the assumption (A_1) and some additional conditions which will be introduced in the following, problem (1), (2) has for sufficiently small ε a unique boundary layer type solution $y(x, \varepsilon)$ whose asymptotic expansion depends essentially on the function f_1.
We distinguish two cases related to the following assumptions:

(A_2). $\bar{f}_1(x) := f_1(\varphi(x), x, 0) > 0$ for $x \in I$.

(A_2'). $\bar{f}_1(x) \equiv 0$ for $x \in I$, and $y = \varphi(x)$ is a simple root of the equation $f_1(y, x, 0) = 0$.

The first case is studied in sections 2 and 3, the second one is considered in sections 4 and 5.

2 Construction of an asymptotic solution in case $\bar{f}_1(x) > 0$

We suppose that the hypotheses (A_1) and (A_2) hold true. As in the case of a simple root, we use the following ansatz for the asymptotic expansion of the solution $y(x, \varepsilon)$ of (1), (2)

$$y(x, \varepsilon) \quad = \bar{y}(x, \varepsilon) + \Pi(\tau, \varepsilon) \tag{1}$$

with $\tau = x/\varepsilon^2$, where $\bar{y}(x, \varepsilon)$ is the regular part and $\Pi(\tau, \varepsilon)$ the boundary layer part of the solution $y(x, \varepsilon)$. If we substitute (1) into the differential equation (1) and use the representation

$$f(y(x, \varepsilon), x, \varepsilon) = f(\bar{y}(x, \varepsilon) + \Pi(\tau, \varepsilon), x, \varepsilon)$$

$$= f(\bar{y}(x,\varepsilon),x,\varepsilon) + f(\bar{y}(\varepsilon^2\tau,\varepsilon) + \Pi(\tau,\varepsilon),\varepsilon^2\tau,\varepsilon) - f(\bar{y}(\varepsilon^2\tau,\varepsilon),\varepsilon^2\tau,\varepsilon)$$

we get the following differential equations to determine $\bar{y}$ and Π:

$$\varepsilon^2 \frac{d\bar{y}}{dx} = f(\bar{y},x,\varepsilon), \tag{2}$$

$$\frac{d\Pi}{d\tau} = f(\bar{y}(\varepsilon^2\tau,\varepsilon) + \Pi,\varepsilon^2\tau,\varepsilon\big) - f\big(\bar{y}(\varepsilon^2\tau,\varepsilon),\varepsilon^2\tau,\varepsilon\big) =: \Pi f. \tag{3}$$

We assume that the regular part $\bar{y}(x,\varepsilon)$ has the asymptotic representation

$$\bar{y}(x,\varepsilon) \equiv \bar{y}_0(x) + \sqrt{\varepsilon}\,\bar{y}_1(x) + \varepsilon\,\bar{y}_2(x) + \cdots . \tag{4}$$

If we substitute (4) and (6) into (2) we get

$$\begin{aligned}\varepsilon^2 \frac{d}{dx}\big(\bar{y}_0 + \sqrt{\varepsilon}\,\bar{y}_1 + \cdots\big) \qquad &= -h(x)\big(\bar{y}_0 + \sqrt{\varepsilon}\,\bar{y}_1 + \cdots - \varphi(x)\big)^2 \\ &+\varepsilon\, f_1(\bar{y}_0 + \sqrt{\varepsilon}\,\bar{y}_1 + \cdots, x, \varepsilon).\end{aligned} \tag{5}$$

The standard procedure to determine the functions $\bar{y}_i$ for $i = 0, 1, \cdots$ consists in expanding the right hand side of (5) into a series in $\sqrt{\varepsilon}$ and in comparing the coefficients related to the same power of $\sqrt{\varepsilon}$ on the right and left hand sides of (5). It easy to verify that this procedure yields only algebraic equations for the determination of the functions $\bar{y}_i$.
For the determination of $\bar{y}_0$ we obtain the equation

$$-h(x)\big(\bar{y}_0(x) - \varphi(x)\big)^2 = 0,$$

which yields

$$\bar{y}_0(x) = \varphi(x). \tag{6}$$

Comparing the coefficients in (5) belonging to ε, we get the equation for $\bar{y}_1$

$$0 = -h(x)\bar{y}_1^2(x) + \bar{f}_1(x). \tag{7}$$

According to the assumptions (A_1) and (A_2), equation (7) has two solutions. We select the solution

$$\bar{y}_1(x) = \Big[h^{-1}(x)\bar{f}_1(x)\Big]^{1/2}$$

satisfying

$$\bar{y}_1(x) > 0 \quad \text{for} \quad x \in I. \tag{8}$$

This inequality will be used later.
By the same way, we obtain for the function $\bar{y}_2$

$$\bar{y}_2(x) = \frac{1}{2}\,h^{-1}(x)\frac{\partial \bar{f}_1}{\partial y}(\varphi(x),x,0) =: \frac{1}{2}\,h^{-1}(x)\bar{f}_{1y}(x).$$

Analogously we can determine the functions $\bar{y}_i$ for $i = 3, \cdots$.

If we assume that the boundary layer function $\Pi(\tau, \varepsilon)$ has the asymptotic representation

$$\Pi(\tau, \varepsilon) = \Pi_0(\tau) + \sqrt{\varepsilon}\,\Pi_1(\tau) + \varepsilon\,\Pi_2(\tau) + \cdots \tag{9}$$

and if we substitute (9) and (6) into (3) we get

$$\begin{aligned} &\frac{d}{d\tau}(\Pi_0 + \sqrt{\varepsilon}\,\Pi_1 + \cdots) = -h(\varepsilon^2\tau)\Big[\big(\Pi_0 + \sqrt{\varepsilon}\,\Pi_1 + \cdots\big)^2 \\ &+2\sqrt{\varepsilon}\,\big(\bar{y}_1(\varepsilon^2\tau) + \sqrt{\varepsilon}\,\bar{y}_2(\varepsilon^2\tau) + \cdots\big)\big(\Pi_0 + \sqrt{\varepsilon}\,\Pi_1 + \cdots\big)\Big] \\ &+\varepsilon\,\Pi f_1, \end{aligned} \tag{10}$$

where the expression Πf_1 is defined analogously to (3).

If we expand the right hand side of (10) into a series in $\sqrt{\varepsilon}$ and compare the expressions belonging to the same power of $\sqrt{\varepsilon}$ on the left and right hand sides in (10) we get differential equations to determine the functions $\Pi_i, i = 0, 1, \cdots$.
In order to derive the corresponding initial conditions we substitute (1) into (2) using the expansions (4) and (9). We get

$$\bar{y}_0(0) + \sqrt{\varepsilon}\,\bar{y}_1(0) + \cdots + \Pi_0(0) + \sqrt{\varepsilon}\,\Pi_1(0) + \cdots = y^0.$$

Comparing the coefficients for the same power of $\sqrt{\varepsilon}$, we obtain the initial conditions for $\Pi_i, i = 0, 1, \cdots$:

$$\begin{aligned} &\Pi_0(0) = y^0 - \bar{y}_0(0) = y^0 - \varphi(0) =: \Pi^0, \\ &\Pi_j(0) = -\bar{y}_j(0), \quad j = 1, \cdots. \end{aligned} \tag{11}$$

Using the standard procedure, we get from (10) and (11) for the function Π_0 the initial value problem

$$\frac{d\Pi_0}{d\tau} = -h(0)\,\Pi_0^2, \quad \tau > 0, \quad \Pi_0(0) = \Pi^0. \tag{12}$$

Its solution reads

$$\Pi_0(\tau) = \frac{\Pi^0}{1 + h(0)\Pi^0\,\tau}. \tag{13}$$

We recall that $h(0)$ is positive by assumption (A_1). In order to guarantee that Π_0 is a boundary layer function, that is, it decays to zero as τ tends to ∞, we have to suppose that the initial value Π^0 is not negative. For the sequel, we introduce the assumption

(A_3). $\quad \Pi^0 > 0.$

Under that assumption, the function Π_0 decays to zero of order $O(\frac{1}{\tau})$ as $\tau \to \infty$, that is, the decay has power character. We note that in the case that φ is a simple root of equation (3), the decay of the boundary layer function is of exponential character (see (5)). As we will prove later, the boundary layer functions $\Pi_i, i = 0, 1, \cdots$, decay in our case also exponentially. Therefore, we have to modify the standard procedure for deriving the differential equations for the functions Π_i. We do it in the following way: if we compare the coefficients belonging to $\sqrt{\varepsilon}^{\,i}$, we take into account on the right hand side of (10) also the higher order term $-2h(0)\bar{y}_1(0)\Pi_i(\sqrt{\varepsilon})^{i+1}$. Of course, this term will then be neglected in deriving the differential equation for Π_{i+1}. A consequence of this approach is that the functions Π_i will depend also on ε, that is, we get an asymptotic representation for the solution of (1), (2) in the form

$$\begin{aligned} y(x,\varepsilon) \quad &= \bar{y}_0(x) + \sqrt{\varepsilon}\,\bar{y}_1(x) + \varepsilon\,\bar{y}_2(x) + \cdots \\ &+\Pi_0(\tau,\varepsilon) + \sqrt{\varepsilon}\,\Pi_1(\tau,\varepsilon) + \varepsilon\,\Pi_2(\tau,\varepsilon) + \cdots . \end{aligned} \tag{14}$$

In case $i = 0$, we get instead of (12) the initial value problem

$$\frac{d\Pi_0}{d\tau} = -h(0)\Big(\Pi_0 + 2\sqrt{\varepsilon}\bar{y}_1(0)\Big)\Pi_0,\ \tau \geq 0,\ \Pi_0(0,\varepsilon) = \Pi^0. \tag{15}$$

Its solution reads

$$\Pi_0(\tau,\varepsilon) = \frac{2\sqrt{\varepsilon}\bar{y}_1(0)\Pi^0 \exp\big(-2\sqrt{\varepsilon}\bar{y}_1(0)h(0)\tau\big)}{2\sqrt{\varepsilon}\bar{y}_1(0) + \Pi^0\Big(1 - \exp\big(-2\sqrt{\varepsilon}\bar{y}_1(0)h(0)\tau\big)\Big)}. \tag{16}$$

By (8) and (A_3) we have

$$\Pi_0(\tau,\varepsilon) > 0, \quad \frac{\partial\Pi_0(\tau,\varepsilon)}{\partial\tau} < 0 \quad \text{for} \quad \tau \geq 0,\ \varepsilon > 0. \tag{17}$$

Moreover, it can be shown that for given $\varepsilon > 0$ the function Π_0 satisfies

$$\Pi_0(\tau,\varepsilon) = O\Big(\exp(-\sqrt{\varepsilon}\kappa\tau)\Big) \quad \text{as } \tau \text{ tends to } \infty, \tag{18}$$

$$\Pi_0(\tau,\varepsilon) = O\Big(\sqrt{\varepsilon}\exp(-\sqrt{\varepsilon}\kappa\tau)\Big) \quad \text{for} \quad \tau \geq \frac{1}{\sqrt{\varepsilon}} \quad \text{and} \quad \tau \to \infty. \tag{19}$$

We emphasize that the exponential decay of Π_0 as $\tau \to \infty$ is a consequence of taking into account the term $-2\sqrt{\varepsilon}h(0)\bar{y}_1(0)$, where $h(0)\bar{y}_1(0) > 0$. We note that this decay is not so fast as in the case of a simple root $\varphi(x)$ (see(5)).

Using this modified procedure we obtain from (10) and (11) for the determination of Π_1 the initial value problem

$$\frac{d\Pi_1}{d\tau} = -2h(0)\Big(\Pi_0(\tau,\varepsilon) + \sqrt{\varepsilon}\bar{y}_1(0)\Big)\Pi_1, \quad \Pi_1(0,\varepsilon) = -\bar{y}_1(0).$$

Its solution reads

$$\Pi_1(\tau,\varepsilon) = -\bar{y}_1(0)\exp\Big(-2h(0)\int_0^\tau \big(\Pi_0(s,\varepsilon)+\sqrt{\varepsilon}\bar{y}_1(0)\big)ds\Big). \tag{20}$$

If we represent $\Pi_0(\tau,\varepsilon)$ in the form (see (15))

$$\Pi_0(\tau,\varepsilon) = \Pi^0\exp\Big(-h(0)\int_0^\tau \big(\Pi_0(s,\varepsilon)+2\sqrt{\varepsilon}\bar{y}_1(0)\big)ds\Big)$$

and compare $\Pi_0(\tau,\varepsilon)$ and $\Pi_1(\tau,\varepsilon)$, we obtain by taking into account $h(0)\bar{y}_1(0) > 0$ the estimate

$$|\Pi_1(\tau,\varepsilon)| \le c\,\Pi_0(\tau,\varepsilon) \quad \text{for} \quad \tau\ge 0, \varepsilon > 0. \tag{21}$$

According to the modified procedure, for the determination of the function Π_2 we get from (10) and (11) the initial value problem

$$\frac{d\Pi_2}{d\tau} = -2h(0)\Big(\Pi_0(\tau,\varepsilon)+\sqrt{\varepsilon}\bar{y}_1(0)\Big)\Pi_2 + \pi_2(\tau,\varepsilon), \quad \Pi_2(0,\varepsilon) = -\bar{y}_2(0), \tag{22}$$

where

$$\pi_2(\tau,\varepsilon) := -h(0)\Big(2\bar{y}_2(0)\Pi_0(\tau,\varepsilon)+\Pi_1^2(\tau,\varepsilon)\Big) + \Pi_0 f_1(\tau,\varepsilon),$$
$$\Pi_0 f_1(\tau,\varepsilon) := f_1(\varphi(0)+\Pi_0(\tau,\varepsilon),0,0) - f_1(\varphi(0),0,0).$$

As we have $|\Pi_0 f_1(\tau,\varepsilon)| \le c\Pi_0(\tau,\varepsilon)$ and since Π_1 satisfies the estimate (21) we have

$$|\pi_2(\tau,\varepsilon)| \le c\,\Pi_0(\tau,\varepsilon). \tag{23}$$

The solution of the initial value problem (22) can be represented in the form

$$\begin{aligned}\Pi_2(\tau,\varepsilon) &= -\bar{y}_2(0)\exp\Big(-2h(0)\int_0^\tau \big(\Pi_0(s,\varepsilon)+\sqrt{\varepsilon}\bar{y}_1(0)\big)ds\Big)\\ &+\int_0^\tau \exp\Big(-2h(0)\int_{\tau_0}^\tau \big(\Pi_0(s,\varepsilon)+\sqrt{\varepsilon}\bar{y}_1(0)\big)ds\Big)\pi_2(\tau_0,\varepsilon)d\tau_0.\end{aligned} \tag{24}$$

Taking into account (23) we obtain from (24) the estimate

$$|\Pi_2(\tau,\varepsilon)| \le c(1+\tau)\Pi_0(\tau,\varepsilon). \tag{25}$$

Analogously, we can formulate the initial value problem for Π_3 and obtain

$$|\Pi_3(\tau,\varepsilon)| \le c(1+\tau)\Pi_0(\tau,\varepsilon).$$

Now we estimate the product $\tau\Pi_0(\tau,\varepsilon)$ by exploiting the relation (16). Using the notation $z := 2h(0)\sqrt{\varepsilon}\bar{y}_1(0)\tau$ we get from (16)

$$\tau\Pi_0(\tau,\varepsilon) \le \frac{1}{h(0)}\frac{z\exp(-z)}{1-\exp(-z)}.$$

Taking into account the inequality

$$\frac{z}{1-\exp(-z)} \le (1+z) \quad \text{for} \quad z>0,$$

it holds

$$\tau\Pi_0(\tau,\varepsilon) \le \tilde{c}(1+z)\exp(-z) \le c\exp(-\sqrt{\varepsilon}\kappa\tau). \tag{26}$$

Hence, all constructed boundary layer functions $\Pi_i(\tau,\varepsilon)$ satisfy the estimate

$$|\Pi_i(\tau,\varepsilon)| \le c\exp(-\sqrt{\varepsilon}\kappa\tau) \quad \text{for} \quad \tau \ge 0, i=0,1,2,3.$$

For $i \ge 4$, the corresponding estimates are not so sharp, e.g. it holds

$$|\Pi_4(\tau,\varepsilon)| \le c(1+\tau)^2\Pi_0 \le \frac{c}{\sqrt{\varepsilon}}\exp(-\sqrt{\varepsilon}\kappa\tau).$$

In the sequel we restrict ourselves to an asymptotic approximation of the solution $y(x,\varepsilon)$ containing three regular terms $\bar{y}_i(x)$ and four boundary layer terms $\Pi_k(\tau,\varepsilon)$.

3 Justification of the asymptotics in case $\bar{f}_1(x) > 0$

We define the function Y_3 by

$$Y_3(x,\varepsilon) := \sum_{i=0}^{3} \varepsilon^{i/2}\big(\bar{y}_i(x)+\Pi_i(\tau,\varepsilon)\big) + \varepsilon^2\Pi_4(\tau,\varepsilon). \tag{1}$$

By means of this function we will construct lower and upper solutions for the initial value problem (1), (2).
A function $\underline{Y}$ with the properties

1^0. $\underline{Y}(0,\varepsilon) \le y^0$,

2^0. The expression

$$L_\varepsilon\underline{Y}(x,\varepsilon) := \varepsilon^2\frac{d\underline{Y}(x,\varepsilon)}{dx} + h(x)\big(\underline{Y}(x,\varepsilon)-\varphi(x)\big)^2 - \varepsilon f_1(\underline{Y}(x,\varepsilon),x,\varepsilon)$$

satisfies

$$L_\varepsilon\underline{Y}(x,\varepsilon) \le 0 \quad \text{for} \quad x \in I$$

is called a lower solution to the initial value problem (1), (2).

For $\underline{Y}(x,\varepsilon)$ we make the ansatz

$$\underline{Y}(x,\varepsilon) = Y_3(x,\varepsilon) - A\varepsilon^2, \tag{2}$$

where A is some positive number. We will prove that for some sufficiently large A (not depending on ε) and for sufficiently small ε, the function $\underline{Y}(x,\varepsilon)$ satisfies the conditions 1^0 and 2^0.
From (2), (1) and (11) we get

$$\underline{Y}(0,\varepsilon) = y^0 - (A + \bar{y}_4(0))\varepsilon^2,$$

which is less than y^0 for sufficiently large A, that is, condition 1^0 is fulfilled.
Concerning the expression $L_\varepsilon \underline{Y}(x,\varepsilon)$ we have

$$L_\varepsilon \underline{Y}(x,\varepsilon) = \varepsilon^2 \frac{d}{dx} \sum_{i=0}^{3} (\sqrt{\varepsilon})^i \bar{y}_i(x) + \frac{d}{d\tau} \sum_{i=0}^{4} (\sqrt{\varepsilon})^i \Pi_i(\tau,\varepsilon)$$

$$+h(x)\Big[\sum_{i=1}^{3} (\sqrt{\varepsilon})^i \bar{y}_i(x) - A\varepsilon^2\Big]^2$$

$$h(\varepsilon^2\tau)\Big[\Big(\sum_{i=0}^{4} (\sqrt{\varepsilon})^i \Pi_i(\tau,\varepsilon)\Big)^2 + 2\Big(\sum_{i=0}^{3} (\sqrt{\varepsilon})^i \bar{y}_i(\varepsilon^2\tau) - A\varepsilon^2\Big) \sum_{i=0}^{4} (\sqrt{\varepsilon})^i \Pi_i(\tau,\varepsilon)\Big]$$

$$-\varepsilon f_1\Big(\sum_{i=0}^{3} (\sqrt{\varepsilon})^i \bar{y}_i(x) - A\varepsilon^2, x, \varepsilon\Big)$$

$$-\varepsilon\Big[f_1\Big(\sum_{i=0}^{3} (\sqrt{\varepsilon})^i \big(\bar{y}_i(\varepsilon^2\tau) + \Pi_i(\tau,\varepsilon)\big) + \varepsilon^2 \Pi_4(\tau,\varepsilon) - A\varepsilon^2, \varepsilon^2\tau, \varepsilon\Big)$$

$$-f_1\Big(\sum_{i=0}^{3} (\sqrt{\varepsilon})^i \bar{y}_i(\varepsilon^2\tau) - A\varepsilon^2, \varepsilon^2\tau, \varepsilon\Big)\Big].$$

After some cumbersome calculations and using the equations for $\bar{y}_i(x)$ and $\Pi_i(\tau,\varepsilon)$ we get

$$L_\varepsilon \underline{Y}(x,\varepsilon) = O(\varepsilon^{5/2}) - 2h(x)\bar{y}_1(x)A\big(1 + O(\sqrt{\varepsilon})\big)\varepsilon^{5/2}$$

$$+O(\varepsilon^4)A^2 + O(\varepsilon^2)\Pi_0(\tau,\varepsilon) - 2h(0)A\Pi_0(\tau,\varepsilon)\big(1 + O(\sqrt{\varepsilon})\big)\varepsilon^2,$$

where the expressions $O(\varepsilon^{i/2}), i = 1, 4, 5, 8$ do not depend on A. Hence, we have

$$L_\varepsilon \underline{Y}(x,\varepsilon) = \varepsilon^{5/2}\Big[-2h(x)\bar{y}_1(x)A\big(1 + O(\sqrt{\varepsilon}A)\big) + O(1)\Big] \tag{3}$$

$$+\varepsilon^2 \Pi_0(\tau,\varepsilon)\Big\{-2h(0)A\big(1 + O(\sqrt{\varepsilon})\big) + O(1)\Big\}.$$

Because of $h(x) > 0, \bar{y}_1(x) > 0$ and $\Pi_0(\tau,\varepsilon) > 0$, it follows from (3) that for sufficiently large A and sufficiently small ε the inequality $L_\varepsilon \underline{Y}(x,\varepsilon) < 0$ is valid, that is, the condition 2^0 is satisfied.
Therefore, the function $\underline{Y}(x,\varepsilon)$ defined in (2) is a lower solution of the initial value problem (1), (2).

Remark 3.1 *If $Y_3(x,\varepsilon)$ would not contain the term $\varepsilon^2\Pi_4(\tau,\varepsilon)$, then the last term in the curly bracket in (3) would read $O(1+\tau)$ instead of $O(1)$. In that case, the term $-2h(0)A$ could not guarantee that the expression in the curly bracket is negative for large τ.*

Analogously can be proved that the function $\overline{Y}(x,\varepsilon)$

$$\overline{Y}(x,\varepsilon) := Y_3(x,\varepsilon) + A\varepsilon^2 \tag{4}$$

is an upper solution of (1), (2) for sufficiently large A and sufficiently small ε.
It is well-known [4] that the existence of a lower solution and of an upper solution to the initial value problem (1), (2) implies the existence of a unique solution $y(x,\varepsilon)$ satisfying

$$\underline{Y}(x,\varepsilon) \le y(x,\varepsilon) \le \overline{Y}(x,\varepsilon) \quad \text{for} \quad x \in I.$$

Furthermore, we obtain from (2) and (4) the relation

$$y(x,\varepsilon) = Y_3(x,\varepsilon) + O(\varepsilon^2) \quad \text{for} \quad x \in I. \tag{5}$$

Therefore, we have proved the following result

Theorem 3.2 *Suppose the assumptions $(A_1),(A_2),(A_3)$ are satisfied. Then for sufficiently small ε the initial value problem (1), (2) has a unique solution $y(x,\varepsilon)$ with the asymptotic representation (5).*

Corollary 3.3 *From the relation (5) we obtain*

$$y(x,\varepsilon) = Y_k(x,\varepsilon) + O(\varepsilon^{\frac{k+1}{2}}) \quad \textit{for} \quad x \in I,$$

where

$$Y_k(x,\varepsilon) = \sum_{i=0}^{k} \varepsilon^{i/2}\big(\bar{y}_i(x) + \Pi_i(\tau,\varepsilon)\big), \ k = 0,1,2.$$

Remark 3.4 *It can be easily shown that outside the boundary layer, that is in the interval $[\delta, X]$, where δ is any small positive fixed number, the solution $y(x,\varepsilon)$ has for any natural number k the asymptotic representation*

$$y(x,\varepsilon) = \sum_{i=0}^{k} (\sqrt{\varepsilon})^i \bar{y}_i(x) + O(\varepsilon^{\frac{k+1}{2}}).$$

4 Construction of the asymptotic solution in case $\bar{f}_1(x) \equiv 0$

We keep the hypotheses (A_1) and (A_3), but the assumption (A_2') will be replaced by the stronger condition

(A_2'').

$$f_1(y,x,\varepsilon) = (y-\varphi(x))g(y,x) + \varepsilon f_2(y,x,\varepsilon), \tag{1}$$

where g and f_2 are sufficiently smooth functions satisfying

$$\bar{g}(x) := g(\varphi(x),x) > 0 \quad \text{for} \quad x \in I.$$

In that case, it turns out that the asymptotics of the solution of the initial value problem (1), (2) also exhibits boundary layer character, but the asymptotic expansion takes place in powers with respect to ε and not in powers with respect to $\sqrt{\varepsilon}$ as in the case before, that is, we have

$$y(x,\varepsilon) = \bar{y}_0(x) + \varepsilon\bar{y}_1(x) + \cdots + \Pi_0(\tau,\varepsilon) + \varepsilon\Pi_0(\tau,\varepsilon) + \cdots, \tag{2}$$

where $\tau = x/\varepsilon^2$. By means of the standard procedure for the determination of the coefficient functions we obtain $\bar{y}_0(x) = \varphi(x)$, and the function $\bar{y}_1(x)$ satisfies the quadratic equation

$$\varphi'(x) = -h(x)\bar{y}_1^2 + \bar{g}(x)\bar{y}_1 + \bar{f}_2(x), \tag{3}$$

where $\bar{f}_2(x) := f_2(\varphi(x),x,0)$.
For the sequel we assume

(A_4). The equation (3) has two different real solutions $\bar{y}_{11}(x)$ and $\bar{y}_{12}(x)$, where

$$\bar{y}_{11}(x) < \bar{y}_{12}(x) \quad \text{for} \quad x \in I. \tag{4}$$

For the following we choose

$$\bar{y}_1(x) = \bar{y}_{12}(x).$$

We note that the sum of the roots of equation (3) is $h^{-1}(x)\bar{g}(x)$. Hence, by the assumptions (A_1) and (A_2''), we have $\bar{y}_{11}(x) + \bar{y}_{12}(x) > 0$. Taking into account (4), it holds $\bar{y}_1(x) > 0$ for $x \in I$ which we need later.
In the sequel we also use the relation

$$\begin{aligned} a(x) &:= 2\bar{y}_1(x) - h^{-1}(x)\bar{g}(x) = 2\bar{y}_{12}(x) - (\bar{y}_{11}(x) + \bar{y}_{12}(x)) \\ &= \bar{y}_{12}(x) - \bar{y}_{11}(x) = \bar{y}_1(x) - \bar{y}_{11}(x) > 0 \quad \text{for} \quad x \in I, \end{aligned} \tag{5}$$

which follows from (4).
By the procedure described above we may determine successively further regular coefficient functions, especially we get

$$\bar{y}_2(x) = [h(x)a(x)]^{-1}\big(\bar{g}_y(x)\bar{y}_1^2(x) + \bar{f}_{2y}(x)\bar{y}_1(x) + \bar{f}_{2\varepsilon}(x) - \bar{y}_1'(x)\big),$$

where the function a is defined in (5).

The equations for the boundary layer functions $\Pi_i(\tau,\varepsilon)$ can be obtained from an equation of the type (10). Taking into account (1) and (2), this equation reads in our case

$$\begin{aligned}
&\frac{d}{d\tau}(\Pi_0+\varepsilon\Pi_1+\cdots)=-h(\varepsilon^2\tau)\Big[\big(\Pi_0+\varepsilon\Pi_1+\cdots\big)^2\\
&+2\big(\varepsilon\bar{y}_1(\varepsilon^2\tau)+\cdots\big)\big(\Pi_0+\varepsilon\Pi_1+\cdots\big)\Big]\\
&+\varepsilon(\Pi_0+\varepsilon\Pi_1+\cdots)g(\varphi(\varepsilon^2\tau)+\varepsilon\bar{y}_1(\varepsilon^2\tau)+\cdots+\Pi_0+\varepsilon\Pi_1+\cdots,\varepsilon^2\tau)\\
&\quad+\varepsilon\big(\varepsilon\bar{y}_1(\varepsilon^2\tau)+\cdots\big)\Pi g+\varepsilon^2\Pi f_2.
\end{aligned} \tag{6}$$

According to our modified standard procedure we get from (6) for Π_0 the differential equation

$$\frac{d\Pi_0}{d\tau}=-h(0)\big(\Pi_0+2\varepsilon\bar{y}_1(0)\big)\Pi_0, \tag{7}$$

which is an analog to equation (15). The solution of this equation satisfying the condition (11) can be represented by means of the expression (16) if we replace there $\sqrt{\varepsilon}$ by ε.
The differential equation to determine Π_1 reads

$$\frac{d\Pi_1}{d\tau}=-2h(0)\big(\Pi_0(\tau,\varepsilon)+\varepsilon\bar{y}_1(0)\big)\Pi_1+g(\varphi(0)+\Pi_0(\tau,\varepsilon),0)\Pi_0(\tau,\varepsilon). \tag{8}$$

From the estimate

$$|g(\varphi(0)+\Pi_0(\tau,\varepsilon),0)\Pi_0(\tau,\varepsilon)|\le c\Pi_0(\tau,\varepsilon)$$

we get that the solution of (8) with the initial condition $\Pi_1(0,\varepsilon)=-\bar{y}_1(0)$ satisfies

$$|\Pi_1(\tau,\varepsilon)|\le c(1+\tau)\Pi_0(\tau,\varepsilon). \tag{9}$$

If we compare this estimate with that one in (21) we see that in (9) the additional term $c\tau\Pi_0(\tau,\varepsilon)$ occurs, that is, the estimate (9) is weaker than the corresponding one in (21). This fact implies some difficulties for the construction of lower and upper solutions. To overcome these problems we proceed as follows. We add the term $g(\varphi(0)+\Pi_0(\tau,\varepsilon),0)\Pi_0(\tau,\varepsilon)$ arising in the right hand side of (8) to the right hand side in (7) taking into account that this term occurs in (6) with the factor ε. Thus, we obtain instead of (7) the equation

$$\frac{d\Pi_0}{d\tau}=-h(0)\big(\Pi_0^2+2\varepsilon\bar{y}_1(0)\Pi_0\big)+\varepsilon g(\varphi(0)+\Pi_0,0)\Pi_0. \tag{10}$$

Taking into account the relation

$$g(\varphi(0)+\Pi_0(\tau,\varepsilon),0)=\bar{g}(0)+\Pi_0 g,$$

where

$$\bar{g}(0)=g(\varphi(0),0),\Pi_0 g=g(\varphi(0)+\Pi_0(\tau,\varepsilon),0)-g(\varphi(0),0)=g_u^*\Pi_0(\tau,\varepsilon)$$

(g_u^* denotes the derivative taken at some intermediate point), then equation (10) can be rewritten in the form

$$\frac{d\Pi_0}{d\tau}=-h(0)\Big[(1+\varepsilon g_u^*)\Pi_0^2+\varepsilon a(0)\Pi_0\Big], \tag{11}$$

where $a(0) = 2\bar{y}_1(0) - h^{-1}(0)\bar{g}(0) > 0$ (see (5)).
Since g_u^* depends on Π_0, we are not able to give the exact expression for the solution of (11) satisfying the initial condition (11). But because of the same structure of the right hand sides of the differential equations (11) and (7), we can conclude that the behavior of the solution $\tilde{\Pi}(\tau,\varepsilon)$ of the initial value problem (11), (11) is qualitatively the same as that of the solution $\Pi_0(\tau,\varepsilon)$ of the problem (7), (11). It is easy to derive the estimate

$$|\tilde{\Pi}_0(\tau,\varepsilon) - \Pi_0(\tau,\varepsilon)| \leq c\varepsilon \exp(-\varepsilon\kappa\tau).$$

Thus, as the main part of the boundary layer asymptotics we can take the solution $\tilde{\Pi}_0(\tau,\varepsilon)$ of the problem (11), (11), and the problem to determine the function $\Pi_1(\tau,\varepsilon)$ takes the form

$$\frac{d\Pi_1}{d\tau} = -h(0)\Big[\big(2+\varepsilon k(\tau,\varepsilon)\big)\tilde{\Pi}_0(\tau,\varepsilon) + \varepsilon a(0)\Big]\Pi_1, \quad \Pi_1(0,\varepsilon) = -\bar{y}_1(0), \tag{12}$$

where

$$k(\tau,\varepsilon) = h^{-1}(0)\Big(g_y\big(\varphi(0)+\tilde{\Pi}_0(\tau,\varepsilon),0\big) + g_y\big(\varphi(0)+\theta\tilde{\Pi}_0(\tau,\varepsilon),0\big)\Big),$$

$0 < \theta < 1$. We denote the solution of the initial value problem (12) by $\tilde{\Pi}_1(\tau,\varepsilon)$. It holds

$$\tilde{\Pi}_1(\tau,\varepsilon) = -\bar{y}_1(0)\exp\Big(-h(0)\int_0^\tau \Big[\big(2+\varepsilon k(s,\varepsilon)\big)\tilde{\Pi}_0(s,\varepsilon) + \varepsilon a(0)\Big]ds\Big). \tag{13}$$

If we rewrite $\tilde{\Pi}_0(\tau,\varepsilon)$ in the form

$$\tilde{\Pi}_0(\tau,\varepsilon) = \Pi_0 \exp\Big(-h(0)\int_0^\tau \big[(1+\varepsilon g_u^*)\Pi_0(s,\varepsilon) + \varepsilon a(0)\big]ds\Big)$$

and compare this expression with the expression (13) for $\tilde{\Pi}_1(\tau,\varepsilon)$, we obtain the estimate

$$|\tilde{\Pi}_1(\tau,\varepsilon)| \leq c\tilde{\Pi}_0(\tau,\varepsilon),$$

that is, after replacing the equations (7) and (8) by the equations (11) and (12), we have derived for $\tilde{\Pi}_1(\tau,\varepsilon)$ an inequality of type (21).
We define the function $\Pi_2(\tau,\varepsilon)$ as solution of the initial value problem

$$\begin{aligned} &\frac{d\Pi_2}{d\tau} = -h(0)\Big[\big(2+\varepsilon k(\tau,\varepsilon)\big)\tilde{\Pi}_0 + \varepsilon a(0)\Big]\Pi_2 + \pi_2(\tau,\varepsilon), \\ &\Pi_2(0,\varepsilon) = -\bar{y}_2(0), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\pi_2(\tau,\varepsilon) := \\ &-h(0)\Big(\tilde{\Pi}_1^2(\tau,\varepsilon) + 2\bar{y}_1(0)\tilde{\Pi}_1(\tau,\varepsilon) + 2\bar{y}_2(0)\tilde{\Pi}_0(\tau,\varepsilon) + 2\varepsilon\bar{y}_1'(0)\tau\tilde{\Pi}_0(\tau,\varepsilon)\Big) \\ &-h'(0)\tau\big(\tilde{\Pi}_0^2(\tau,\varepsilon) + 2\varepsilon\bar{y}_1(0)\tilde{\Pi}_0(\tau,\varepsilon)\big) \\ &+g_y(\varphi(0)+\tilde{\Pi}_0(\tau,\varepsilon),0)\big(\bar{y}_1(0)+\varphi'(0)\varepsilon\tau\big)\tilde{\Pi}_0(\tau,\varepsilon) \\ &+\bar{y}_1(0)\Pi_0 g + \varepsilon\, g_x(\varphi(0)+\tilde{\Pi}_0(\tau,\varepsilon),0)\tau\tilde{\Pi}_0 + \Pi_0 f_2. \end{aligned}$$

We note that the expression for $\pi_2(\tau,\varepsilon)$ contains some terms of order $O(\varepsilon\tau\tilde{\Pi}_0)$. The explanation to include these terms into the differential equation for Π_2 will be given in Remark 5.1.
For the function π_2 there holds the estimate

$$|\pi_2(\tau,\varepsilon)| \le c(1+\varepsilon\tau)\tilde{\Pi}_0(\tau,\varepsilon).$$

Therefore, the solution $\tilde{\Pi}_2(\tau,\varepsilon)$ of the initial value problem (14), which can be represented by means of an expression of type (24), satisfies the estimate

$$|\tilde{\Pi}_2(\tau,\varepsilon)| \le c(1+\tau+\varepsilon\tau^2)\tilde{\Pi}_0(\tau,\varepsilon).$$

Since we have $\tau\tilde{\Pi}_0(\tau,\varepsilon) \le \exp(-\varepsilon\kappa\tau)$ (this can be shown in the same way as we proved the estimate (26)), it holds finally

$$|\tilde{\Pi}_2(\tau,\varepsilon)| \le \exp(-\varepsilon\kappa\tau).$$

For the following we restrict ourselves to the terms $\bar{y}_i$ and Π_i with $i=0,1,2$.

5 Justification of the asymptotics in case $\bar{f}_1(x)\equiv 0$

We introduce the notation

$$Y_2(x,\varepsilon) := \sum_{i=0}^{2}\varepsilon^i\Big(\bar{y}_i(x)+\tilde{\Pi}_i(\tau,\varepsilon)\Big).$$

A lower solution to the initial value problem (1), (2) will be constructed in the form

$$\underline{Y}(x,\varepsilon) = Y_2(x,\varepsilon) - A\varepsilon^3, \tag{1}$$

where A is some positive number. We have

$$\underline{Y}(0,\varepsilon) = Y_2(0,\varepsilon) - A\varepsilon^3 = y^0 - A\varepsilon^3 < y^0,$$

that is, the condition 1^0 for a lower solution is fulfilled (see section 3).
Further, by using the differential equations for $\bar{y}_i$ and $\tilde{\Pi}_i$, it is not difficult to derive in analogy to (3) the following relation

$$L_\varepsilon\underline{Y} = \varepsilon^4\Big[-h(x)a(x)A(1+O(\varepsilon A))+O(1)\Big]$$
$$+\varepsilon^3\tilde{\Pi}_0(\tau,\varepsilon)\Big\{-2h(0)A(1+O(\varepsilon))+O(1)\Big\}. \tag{2}$$

As we have $h(x)>0, a(x)>0$ (see assumption (A_1) and (5)) and $\tilde{\Pi}_0(\tau,\varepsilon)>0$, we get from (2) for sufficiently large A and sufficiently small ε the inequality $L_\varepsilon\underline{Y}<0$, that is, the condition 2^0 for a lower solution is satisfied (see section 3). Hence, for sufficiently large A and sufficiently small ε, the function $\underline{Y}(x,\varepsilon)$ defined in (1) is a lower solution for the initial value problem (1), (2).

Remark 5.1 *If we would not include terms of the order $O(\varepsilon\tau\tilde{\Pi}_0)$ into the expression of the function $\pi_2(\tau,\varepsilon)$, then these terms multiplied by ε^2 would arise in the expression for $L_\varepsilon\underline{Y}$ such that the last term in the curly braces on the right hand side of (2) would have the order $O(\tau)$ and not the order $O(1)$. Thus, for any chosen A, the expression in the curly brackets can be positive for sufficiently large τ.*

Analogously we can prove that for sufficiently large A and sufficiently small ε the function

$$\overline{Y}(x,\varepsilon) := Y_2(x,\varepsilon) + A\varepsilon^3$$

is an upper solution for the initial value problem (1), (2). Thus, we have the following result

Theorem 5.2 *Assume the hypotheses $(A_1), (A_2''), (A_3), (A_4)$ to be valid. Then for sufficiently small ε the initial value problem (1), (2) has a unique solution $y(x,\varepsilon)$ satisfying*

$$y(x,\varepsilon) = Y_2(x,\varepsilon) + O(\varepsilon^3) \quad \textit{for} \quad x \in I. \tag{3}$$

Corollary 5.3 *For $k = 0, 1$, we get from (3)*

$$y(x,\varepsilon) = Y_k(x,\varepsilon) + O(\varepsilon^{k+1}) \quad \textit{for} \quad x \in I,$$

where

$$Y_k(x,\varepsilon) := \sum_{i=0}^{k} \varepsilon^i \Big(\bar{y}_i(x) + \tilde{\Pi}_i(\tau,\varepsilon)\Big), \quad k = 0, 1.$$

Remark 5.4 *Outside the boundary layer, that is for $x \in [\delta, X]$, we have for any natural number k*

$$y(x,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i \bar{y}_i(x) + O(\varepsilon^{k+1}).$$

6 Example

Consider the initial value problem

$$\begin{aligned}
&\varepsilon^2 \frac{dy}{dx} = -(1+x)(y+2x)^2 + \varepsilon(y+2x) + \varepsilon^2 y^3, \quad x \in I := [0, X],\\
&y(0,\varepsilon) = 1.
\end{aligned} \tag{1}$$

We have the case $\bar{f}_1(x) \equiv 0$. Compared with (6) and (1), it holds

$$h(x) \equiv 1 + x, g(y,x) \equiv 1, f_2(y,x) \equiv y^3.$$

Using the algorithm for the construction of an asymptotic expansion described in section 4, we get

$$\bar{y}_0(x) = \varphi(x) \equiv -2x,$$

and the equation for the determination of $\bar{y}_1(x)$ reads

$$(1+x)\bar{y}_1^2 - \bar{y}_1 - 2 - 8x^3 = 0.$$

This equation has two different solutions. As $\bar{y}_1(x)$ we choose the larger solution

$$\bar{y}_1(x) = \frac{1 + \sqrt{1 + 8(1+x)(1+4x^3)}}{2(1+x)}$$

satisfying $\bar{y}_1(x) > 0$ for $x \in I$.
For the determination of the boundary layer function $\tilde{\Pi}_0(\tau, \varepsilon)$ we obtain the initial value problem

$$\frac{d\tilde{\Pi}_0}{d\tau} = -(\tilde{\Pi}_0^2 + 3\varepsilon\tilde{\Pi}_0), \quad \tilde{\Pi}_0(0, \varepsilon) = 1.$$

Its solution reads

$$\tilde{\Pi}_0(\tau, \varepsilon) = \frac{3\varepsilon \exp(-3\varepsilon\tau)}{3\varepsilon + \big(1 - \exp(-3\varepsilon\tau)\big)}.$$

For $\tilde{\Pi}_1(\tau, \varepsilon)$ we get the initial value problem

$$\frac{d\tilde{\Pi}_1}{d\tau} = -2(\tilde{\Pi}_0(\tau, \varepsilon) + 3\varepsilon)\tilde{\Pi}_1, \quad \tilde{\Pi}_1(0, \varepsilon) = -\bar{y}_1(0) = -2$$

whose solution $\tilde{\Pi}_1(\tau, \varepsilon)$ can be given explicitly

$$\tilde{\Pi}_1(\tau, \varepsilon) = -\frac{18\varepsilon^2 \exp(-3\varepsilon\tau)}{\Big[3\varepsilon + \big(1 - \exp(-3\varepsilon\tau)\big)\Big]^2}.$$

It is also not difficult to determine $\bar{y}_2(x)$ and $\tilde{\Pi}_2(\tau, \varepsilon)$. If we restrict ourselves on terms of zeroth and first order, then the solution $y(x, \varepsilon)$ of (1) can be represented for $x \in I$ in the form

$$y(x, \varepsilon) = -2x + \varepsilon \frac{1 + \sqrt{1 + 8(1+x)(1+4x^3)}}{2(1+x)} +$$

$$+ \frac{3\varepsilon \exp(\frac{-3x}{\varepsilon})}{3\varepsilon + \big(1 - \exp(\frac{-3x}{\varepsilon})\big)} \Big[1 - \frac{6\varepsilon^2}{3\varepsilon + \big(1 - \exp(\frac{-3x}{\varepsilon})\big)}\Big] + O(\varepsilon^2).$$

References

[1] A.B. Vasil'eva and V.F. Butuzov, *Asymptotic Methods in the Theory of Singular Perturbation*, (in Russian), Vysshaya Shkola, Moscow, 1990.

[2] A.B. Vasil'eva and V.S. Pilyugin, Singularly perturbed boundary value problems with a power-law boundary layer., *Diff. Equs.* **45**(3) (2009), 323-334.

[3] A.B. Vasil'eva , A two-point boundary value problem for singularly perturbed equations with a reduced equation having multiple roots. *Zh. Vychisl. Mat. Mat. Fiz.* **49**, (2009), 1067-1079.

[4] S.A. Chaplygin, *A new method for integrating differential equations*, (in Russian), GITTL, Moscow, 1950.