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Strong solutions for the interaction of a rigid body and a viscoelastic fluid

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Abstract. We study a coupled system of equations describing the movement of a rigid body which is immersed in a viscoelastic fluid. It is shown that under natural assumptions on the data and for general geometries of the rigid body, excluding contact scenarios, a unique local-in-time strong solution exists.

In this paper, we study the problem of interaction of rigid bodies and viscoelastic fluids in a strong regularity setting. To describe the fluid, we use the Johnson-Segalman model, so that the viscous stress tensor $\bar{\tau}$ satisfies the constitutive equation

$$\bar{\tau} + \lambda_1 \frac{D_a \bar{\tau}}{Dt} = 2\mu(D(v) + \lambda_2 \frac{D_a D(v)}{Dt}),$$

where $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the deformation tensor, v is the velocity field, $\lambda_1 > 0$ and $\lambda_2 > 0$ are relaxation and retardation times respectively and μ is a viscosity parameter. The derivative

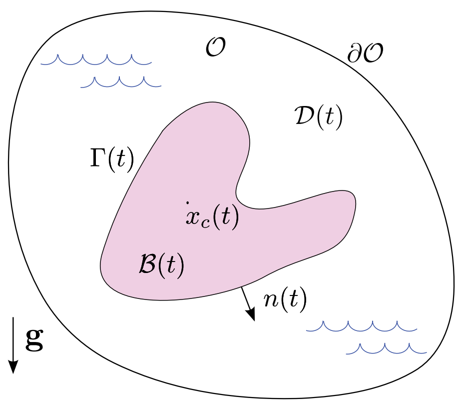
$$\frac{D_a \bar{\tau}}{Dt} := \partial_t \bar{\tau} + (v \cdot \nabla)\bar{\tau} + g_a(\bar{\tau}, \nabla v) \tag{0.1}$$

includes the material derivative and the objective function

$$g_a(\bar{\tau}, \nabla v) := \bar{\tau} W(v) - W(v)\bar{\tau} - a(D(v)\bar{\tau} + \bar{\tau} D(v)),$$

which guarantees that $\bar{\tau}$ is frame-invariant. Here, $W(v) := \frac{1}{2}(\nabla v - (\nabla v)^T)$ denotes the vorticity tensor and the parameter $a \in [-1, 1]$ is specific to the fluid. In particular, for $a = 1$, we recover the Oldroyd-B model [27, 25]. For a mathematical exposition of Johnson-Segalman fluid flow and many results concerning viscoelastic fluids in general, we refer to [28]. In particular, the existence of regular solutions for finite time or close to equilibrium was shown for several classes of data, cf. [21, 13] and [3] in critical spaces.

In order to include the rigid body in the flow problem, we assume that a container given as the bounded domain $\mathcal{O}$ holds the body in a bounded domain $\mathcal{B}(t)$ with outer normal $n(t)$ and that it is otherwise filled by the fluid, in the domain $\mathcal{D}(t) = \mathcal{O} \backslash \overline{\mathcal{B}(t)}$.



The outer boundary of $\mathcal{O}$ is denoted by $\partial\mathcal{O}$ with outer normal ν and the interface between body and fluid is denoted by $\Gamma(t)$. We write

$$\mathcal{Q}_{\mathcal{D}} := \{(t, x) \in \mathbb{R}^4 : t \in \mathbb{R}_+, x \in \mathcal{D}(t)\}$$

and similarly, $\mathcal{Q}_\Gamma$.

The balance of momentum and mass for incompressible fluids with constant density 1 is given by

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \operatorname{div} \mathbf{T} + f, & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \operatorname{div} v = 0, & \text{in } \mathcal{Q}_{\mathcal{D}}, \end{cases} \tag{0.2}$$

where we write $\mathbf{T}(\bar{\tau}, q) = \bar{\tau} - q\mathrm{Id}_{\mathbb{R}^3}$ for the full stress tensor, q for the pressure and f for some exterior force. If we split

$$\bar{\tau} =: 2\mu\frac{\lambda_2}{\lambda_1}D(v) + \tau, \qquad \text{and set } \alpha := \frac{\lambda_2}{\lambda_1},$$

as well as

$$D_{\mu,\alpha}(v) = 2\mu\alpha D(v),$$

and include initial conditions, then (0.1) and (0.2) are equivalent to the system

$$\left\{\begin{array}{rcll} \partial_t v + (v\cdot\nabla)v - \mu\alpha\Delta v + \nabla q &=& \operatorname{div}\tau + f, & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \operatorname{div} v &=& 0, & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ v(0) &=& v_0, & \text{in } \mathcal{D}(0), \\ \lambda_1(\partial_t\tau + (v\cdot\nabla)\tau + g_a(\tau,\nabla v)) + \tau &=& D_{\mu,1-\alpha}(v), & \text{in } \mathcal{Q}_{\mathcal{D}}, \\ \tau(0) &=& \tau_0, & \text{in } \mathcal{D}(0). \end{array}\right. \tag{0.3}$$

We say that no-slip conditions hold on the fluid boundaries, so that

$$v = 0 \quad \text{on } \partial\mathcal{O} \tag{0.4}$$

on the outer boundary of the container and

$$v(t,x) = \eta(t) + \theta(t)\times(x - x_c(t)), \quad \text{on } \mathcal{Q}_\Gamma, \tag{0.5}$$

on the interface, where the right-hand side denotes the velocity of the rigid body. It is given by a translational velocity vector η and an angular velocity vector θ, calculated with respect to the position of the center of mass x_c. They satisfy the equations of conservation of momentum and of angular momentum of the rigid body,

$$\left\{\begin{array}{rcll} \mathrm{m}\eta'(t) + \int_{\Gamma(t)}\mathbf{T}(\bar{\tau},q)(t,x)n(t,x)\,\mathrm{d}\Gamma &=& f_1(t), & t\in\mathbb{R}_+, \\ (J\theta)'(t) + \int_{\Gamma(t)}(x - x_c(t))\times\mathbf{T}(\bar{\tau},q)(t,x)n(t,x)\,\mathrm{d}\Gamma &=& f_2(t), & t\in\mathbb{R}_+, \\ \eta(0) &=& \eta_0, & \\ \theta(0) &=& \theta_0, & \end{array}\right. \tag{0.6}$$

which contain the drag force and the torque exerted by the fluid onto the body. The constant $\mathrm{m} > 0$ is the body's mass and J is its inertia tensor, whereas the functions f_1 and f_2 denote external forces and torques. For example, in order to model a free fall of the body under the influence of gravitation, we can set $f_0 = \mathbf{g}$, $f_1 = \mathrm{m}\mathbf{g}$ and $f_2 = 0$ for a constant vector $\mathbf{g}$.

For the *Newtonian* coupled problem given by (0.3) and (0.6) under the assumption that $\tau = 0$, an extensive exposition of known results and open problems is given in [15]. The existence of global-in-time weak solutions was shown for the 2D- and 3D-problems in [26, 32, 23, 4, 8]. It was shown moreover in the case of several moving bodies, when contact may occur, that weak solutions still exist [30, 10, 11]. Assuming regularity of solutions however, the modeling of possible contacts still seems unclear, cf. e.g. the result in [22]. Excluding contact, the existence of regular solutions on a finite time interval was shown first in [16] and [35, 5] in a Hilbert space setting. Recently, it was proved that the operator corresponding to a suitable linearization of the problem generates an analytic semigroup in $L^{6/5}\cap L^2(\mathbb{R}^3)$ and existence of local strong solutions was derived in the case that $\mathcal{B}$ is a ball [37]. In [19], it is shown that the linear problem satisfies maximal L^p-regularity estimates, corresponding to the estimates on the Stokes problem, and that local-in-time existence of strong solutions follows for general geometries of $\mathcal{B}$.

Looking at non-Newtonian fluids, existence of strong solutions can be carried over to the case of generalized Newtonian fluids of shear-thickening or -thinning type, [19]. In [12], it was moreover shown that for strongly shear-thickening power-law fluids, global-in-time solutions exist also in the case of several moving bodies and that contact will not occur.

Concerning the interaction of rigid structures and viscoelastic fluids, no results seem to be known in the instationary case. Using a second-order-fluid model, the existence of stationary solutions in the presence of gravity (the *steady fall* problem) was shown together with stability results for symmetric bodies [15, 18]. Corresponding to experiments, they suggest that for some positive Weissenberg number, normal stresses cause a rigid body to stabilize in the orientation opposite to the Newtonian stable steady fall [17]. These effects give a strong motivation to study, also in a regular setting where uniqueness can be expected, the interaction of rigid bodies and viscoelastic fluids. Considering this problem may thus be interesting for a number of applications such as industrial processes involving particle sedimentation or flow control using particles, as well as for the modeling and analysis of non-Newtonian fluids containing small and large particles.

From a mathematical point of view, the key points in our analysis are a general technique of how to deal with the geometric character of the interaction of rigid body movements and fluid flow (Subsection 4.1) and the construction of a suitable fixed point argument which combines the parabolic, hyperbolic and ODE-parts of the system (Subsection 4.2).

1. Main result and strategy of the proof

We combine equations (0.3), (0.4), (0.5) and (0.6) in one coupled system,

$$
(1.1)\qquad \left\{\begin{aligned}
\partial_t v + (v\cdot\nabla)v - \mu\alpha\Delta v + \nabla q &= \operatorname{div}\tau + f_0, && \text{in } \mathcal{Q}_{\mathcal{D}},\\
\operatorname{div} v &= 0, && \text{in } \mathcal{Q}_{\mathcal{D}},\\
v(0) &= v_0, && \text{in } \mathcal{D}(0),\\
v|_{\mathcal{Q}_\Gamma} &= \eta + \theta\times(x-x_c), && \text{on } \mathcal{Q}_\Gamma,\\
v|_{\partial\mathcal{O}} &= 0, && \text{on } \partial\mathcal{O},\\
\mathrm{m}\eta'(t) + \int_{\Gamma(t)} \mathbf{T}(\bar\tau,q)(t,x)n(t,x)\,\mathrm{d}\Gamma &= f_1(t), && t\in\mathbb{R}_+,\\
(J\theta)'(t) + \int_{\Gamma(t)}(x-x_c(t))\times \mathbf{T}(\bar\tau,q)(t,x)n(t,x)\,\mathrm{d}\Gamma &= f_2(t), && t\in\mathbb{R}_+,\\
\eta(0) &= \eta_0, &&\\
\theta(0) &= \theta_0, &&\\
\lambda_1(\partial_t\tau + (v\cdot\nabla)\tau + g_a(\tau,\nabla v)) + \tau &= D_{\mu,1-\alpha}(v), && \text{in } \mathcal{Q}_{\mathcal{D}},\\
\tau(0) &= \tau_0, && \text{in } \mathcal{D}(0).
\end{aligned}\right.
$$

in the unknowns v, q, η, θ and τ, and, implicitly, $\mathcal{D}(t)$. The main result of this paper is the existence of a unique strong solution to (1.1), under suitable assumptions on the data, requiring in particular that the body keeps some distance to the wall $\partial\mathcal{O}$. To make this notion more precise, we define the following spaces of functions.

Definition 1.1. Let $1 \le s, r \le \infty$ and $T > 0$, then

$$L^s(0,T;L^r(\mathcal{D}(\cdot))) := \{f \in L^1_{\mathrm{loc}}((0,T)\times\mathcal{D}(\cdot)) : \|f\|_{L^s(0,T;L^r(\mathcal{D}(\cdot)))} < \infty\},$$

where

$$\|f\|_{L^s(0,T;L^r(\mathcal{D}(\cdot)))} := \left(\int_0^T \|f(t)\|^s_{L^r(\mathcal{D}(t))}\,\mathrm{d}t\right)^{1/s}.$$

As an abbreviation, we write $\mathcal{D} := \mathcal{D}(0)$ and $J_T := (0,T) \subset \mathbb{R}$ for $T > 0$. Moreover, we shortly introduce the following function spaces on fixed domains and shortly repeat some well-known properties. For a domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$ and $1 \le r \le \infty$, the Sobolev spaces of order $m \in \mathbb{N}$ are denoted by $W^{m,r}(D)$. The positive scale of Bessel potential spaces is denoted by $H^{\beta,r}(D)$, $\beta \in \mathbb{R}_+$. On a domain of class C^m and for $0 < \beta \le m$, they can be given by complex interpolation of Sobolev spaces,

$$H^{\beta,r}(D) := [L^r(D), W^{m,r}(D)]_{\frac{\beta}{m}},$$

and they are compatible with the Sobolev spaces of integer order in this case. For every $0 < \beta < m, 1 \le r < \infty, 1 \le s \le \infty$, we define Besov spaces by real interpolation of Sobolev spaces,

$$B^{\beta}_{r,s}(D) := (L^r(D), W^{m,r}(D))_{\beta/m,s},$$

and we denote the Sobolev-Slobodeckiĭ spaces $W^{\beta,r}(D)$ by

$$W^{\beta,r}(D) := \begin{cases} W^{\beta,r}(D) & \text{if } \beta\in\mathbb{N},\\ B^{\beta}_{r,r}(D) & \text{if } \beta\notin\mathbb{N}. \end{cases}$$

They coincide with the Bessel potential spaces if $r = 2$ and we will also use their intrinsic characterization. For more information on these function spaces, we refer to [36] and we often cite [33] for interpolation and compactness results in vector-valued function spaces. In general, we do not distinguish in the notation for these spaces, whether function are $\mathbb{R}$- or $\mathbb{R}^d$-valued. In specifying the norm, we often use the short notation $L^s(L^r)$ for the space $L^s(J_T; L^r(\mathcal{D}))$ and a similar convention for other $\mathbb{R}^d$- or vector-valued function spaces.

We can now state the main theorem of the paper.

Theorem 1.2. *Let $1 < s < \infty$, $3 < r < \infty$ and $\mathcal{O}$ and $\mathcal{B} \subset \mathcal{O}$ be bounded domains with boundary of class $C^{2,1}$. Let η_0, θ_0 be in $\mathbb{R}^3$ and $v_0 \in B^{2-2/s}_{r,s}(\mathcal{D})$ satisfying the compatibility conditions given in Remark 1.3 below and let $\tau_0 \in W^{1,r}(\mathcal{D})$. Let $f_0 \in W^{\beta,s}(\mathbb{R}_+; L^r(\mathcal{O}))$ for some $\beta > 0$ and $f_1, f_2 \in L^s(\mathbb{R}_+; \mathbb{R}^3)$. If*

$$\operatorname{dist}(\mathcal{B}(0), \partial\mathcal{O}) > d \qquad \textit{for some } d > 0,$$

there exists a maximal interval $[0, T_*)$, such that problem (1.1) admits a unique strong solution

$$\begin{aligned} v &\in L^s(J_{T_*}; W^{2,r}(\mathcal{D}(\cdot))) \cap W^{1,s}(0, T_*; L^r(\mathcal{D}(\cdot))), \\ q &\in L^s(J_{T_*}; W^{1,r}(\mathcal{D}(\cdot))), \\ \eta &\in W^{1,s}(J_{T_*}; \mathbb{R}^3), \\ \theta &\in W^{1,s}(J_{T_*}; \mathbb{R}^3), \\ \tau &\in C(J_{T_*}; W^{1,r}(\mathcal{D}(\cdot))) \cap W^{1,s}(J_{T_*}; L^r(\mathcal{D}(\cdot))). \end{aligned}$$

Remark 1.3. The compatibility conditions on v_0, η_0 and θ_0 are $\operatorname{div} v_0 = 0$ and

$$\begin{aligned} v_0|_\Gamma(x) &= \theta_0 \times x + \eta_0, \\ v_0|_{\partial\mathcal{O}} &= 0, \end{aligned}$$

if $\frac{1}{2r} + \frac{1}{s} < 1$ and

$$\begin{aligned} v_0|_\Gamma(x) \cdot n(0,x) &= (\theta_0 \times x + \eta_0) \cdot n(0,x), \\ v_0|_{\partial\mathcal{O}} \cdot \nu(x) &= 0, \end{aligned}$$

if $\frac{1}{2r} + \frac{1}{s} > 1$. They are a consequence of the method of construction of the solution used below and of the characterization of the time-trace space

$$D_{s,r} := (L^r_\sigma, W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma)_{1-1/s,s} \tag{1.2}$$

of the Stokes operator, given in [1, Theorem 3.4].

Remark 1.4. The reason for the assumption $f_0 \in W^{\beta,s}(\mathbb{R}_+; L^r(\mathcal{O}))$ (instead of $f_0 \in L^s(\mathbb{R}_+; L^r(\mathcal{O}))$) will show in the proof of Lemma 4.3. Roughly speaking, the coupling of fluid pressure q and the rigid body velocities is so strong that q must be included as a variable in the Schauder fixed point argument employed in Section 4 for the existence proof. The potential part of f_0 on $\mathcal{D}(\cdot)$, which is not known a priori, must therefore yield compactness. Alternatively, we may ask that $f_0 \in L^s(\mathbb{R}_+; L^r_\sigma(\mathcal{D}(\cdot)))$, which seems unnatural.

Remark 1.5. The maximal time T_* of existence of the solution can be characterized as follows. Either T_* can be arbitrarily large or one of the functions

$$t \mapsto \|v(t)\|_{B^{2-2/s}_{r,s}(\mathcal{D}(t))}, \quad t \mapsto |\eta(t)|, \quad t \mapsto |\theta(t)|, \quad t \mapsto \|\tau(t)\|_{W^{1,r}(\mathcal{D}(t))}$$

is unbounded on $[0, T_*)$, because otherwise, the solution could be extended. The second limiting condition on T_* is that the rigid body may not move too close to the boundary, such that $\operatorname{dist}(\mathcal{B}(T_*), \partial\mathcal{O}) < \frac{d}{2}$. This distance can be estimated through the solution velocities η, θ.

The remainder of the paper is organized as follows. As a first step, in the next section we change coordinates in (1.1) so that we can consider the system on the fixed cylindrical domain $J_T \times \mathcal{D}$. Following the approach of Inoue and Wakimoto in [24], this transform is defined in a way as to preserve the solenoidal condition on the fluid velocity and and not to change the regularity of the solutions, cf. also [35], [29] and [9]. It is important to note that in our situation, the transform is an unknown part of the solution, so that we can only give a method of construction which should make sense for all possible body velocities in the solution space $W^{1,s}(J_T; \mathbb{R}^3)$. Section 3 contains a collection of preliminary results and estimates. In Section 4, we construct a Schauder fixed point argument to show the existence of at least one strong solution v, q, η, θ, τ as in Theorem 1.2. In the last section, it is shown that the solution is unique in its regularity class.

2. Change of coordinates

We rewrite (1.1) as a system of partial differential equations on the fixed domain $\mathbb{R}_+ \times \mathcal{D}$. For convenience, we assume that $x_c(0) = 0$. If $\mathcal{O}$ were the whole space $\mathbb{R}^3$, we could choose a frame of reference attached to the rigid body, centered at $x_c(0)$. This change of coordinates would be given by diffeomorphisms X_0, Y_0 defined in the following way, cf. [16]. For every $y \in \mathbb{R}^4$, we have $X_0(t,y) = Q(t)y + x_c(t)$ with some matrix $Q(t)$ in $SO(3)$, satisfying the set of differential equations

$$\left\{ \begin{array}{rcll} \partial_t X_0(t,y) &=& m(t)(X_0(t,y) - x_c(t)) + \eta(t), & J_T \times \mathbb{R}^3, \\ X_0(0,y) &=& y, & y \in \mathbb{R}^3, \end{array} \right. \tag{2.1}$$

where $m(t)$ denotes the skew-symmetric matrix satisfying $m(t)x = \theta(t) \times x$. Note that from this equation, we can see that $Q \in W^{2,s}(J_T; \mathbb{R}^{3\times 3})$, if $\eta, \theta \in W^{1,s}(J_T)$. The corresponding inverse $Y_0(t)$ of $X_0(t)$ is given by

$$Y_0(t,x) = Q^T(t)(x - x_c(t))$$

or the differential equation

$$\left\{ \begin{array}{rcll} \partial_t Y_0(t,x) &=& -M(t)Y_0(t,x) - \bar{\xi}(t), & J_T \times \mathbb{R}^3, \\ Y_0(0,x) &=& x, & x \in \mathbb{R}^3, \end{array} \right. \tag{2.2}$$

where

$$M(t) := Q^T(t)m(t)Q(t), \quad \bar{\xi}(t) := Q^T(t)\eta(t). \tag{2.3}$$

We modify X_0 and Y_0 such that they rotate and shift space only on a suitable open neighborhood of the rotating and translating body. The main reason for this modification is that we want to treat the case of bounded smooth fluid domains $\mathcal{D}$. Thus, we must not rotate or translate the outer boundary $\partial\mathcal{O}$. Let $\chi \in C^\infty(\mathbb{R}^3; [0,1])$ be a cut-off function,

$$\chi(x) := \left\{ \begin{array}{ll} 1, & \text{if } \operatorname{dist}(x, \partial\mathcal{O}) \geq d, \\ 0, & \text{if } \operatorname{dist}(x, \partial\mathcal{O}) \leq \frac{d}{2}, \end{array} \right. \tag{2.4}$$

and let $b : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field

$$b(t,x) := \chi(x - x_c(t))[m(t)(x - x_c(t)) + \eta(t)] - B_K(\nabla_x \chi(\cdot - x_c(t))m(t)(\cdot - x_c(t)))(x). \tag{2.5}$$

Here, $B_K : C_c^\infty(K;\mathbb{R}) \to C_c^\infty(K;\mathbb{R}^3)$ indicates the Bogovskiĭ operator corresponding to an open set K containing $\{x \in \mathbb{R}^n : d \leq \operatorname{dist}(x, \partial\mathcal{O}) \leq \frac{d}{2}\}$. It is a bounded operator yielding $\operatorname{div} B_K g = g$ if $\int_K g = 0$, cf. [2]. The function $b(t)$ belongs to $C^\infty(\mathbb{R}^3)$ and it is bounded. Due to

$$\int_{\overline{B_2}\setminus B_1} (\nabla \chi(y - x_c(t)))m(t)(y - x_c(t))\,\mathrm{d}y = \int_{\overline{B_2}\setminus B_1} \chi(y - x_c(t)) \operatorname{tr} m(t)\,\mathrm{d}y = 0,$$

the correction by the Bogovskiĭ term yields $\operatorname{div} b(t) = 0$ for all $t \in [0,T]$, so $b \in W^{1,s}(J_T; C^\infty_{c,\sigma}(\mathbb{R}^3))$, if b is extended by the rigid motion to $\mathcal{B}$ and by 0 to $\mathbb{R}^3 \setminus \mathcal{O}$.

We now consider the ordinary differential equation

$$\left\{ \begin{array}{rcll} \partial_t X(t,y) &=& b(t, X(t,y)), & J_T \times \mathbb{R}^3, \\ X(0,y) &=& y, & y \in \mathbb{R}^3. \end{array} \right. \tag{2.6}$$

Given $\eta, \theta \in W^{1,s}(J_T; \mathbb{R}^3)$, it yields a unique solution $X \in C^1(J_T; C^\infty(\mathbb{R}^3))$, by the Picard-Lindelöf theorem. The solution has continuous mixed partial derivatives $\frac{\partial^{|\beta|+1} X}{\partial t (\partial y_j)^\beta}$, $\frac{\partial^{|\beta|} X}{(\partial y_j)^\beta}$, where $\beta \in \mathbb{N}_0^3$ denotes a multi-index. By uniqueness, the function $X(t,\cdot)$ is bijective and we denote its inverse by $Y(t,\cdot)$. Since $\operatorname{div} b = 0$, Liouville's Theorem implies that X and Y are volume-preserving, i.e.

$$J_X(t,y) J_Y(t, X(t,y)) = \mathrm{Id} \text{ and } \det J_X(t,y) = \det J_Y(t,x) = 1,$$

for the Jacobians $(J_X)_{ij}(t.y) = \partial_j X_i(t,y)$ and $(J_Y)_{ij}(t,x) = \partial_j Y_i(t,x)$. Given X, the inverse transform Y satisfies the differential equation

$$\left\{ \begin{array}{rcll} \partial_t Y(t,x) &=& b^{(Y)}(t, Y(t,x)), & J_T \times \mathbb{R}^3, \\ Y(0,x) &=& x, & x \in \mathbb{R}^3, \end{array} \right. \tag{2.7}$$

where

$$b^{(Y)}(t,y) := -J_X^{-1}(t,y) b(t, X(t,y)). \tag{2.8}$$

Note that by this definition, $b^{(Y)}$ and Y obtain the same space and time regularity as b and X. Within the ball B_1, X, Y coincide with X_0, Y_0; whereas in the complement of $K \cup \overline{B_1}$, $\partial_t X(t,y) = \partial_t Y(t,x) = 0$.

For $(t,y) \in [0,T) \times \mathbb{R}^3$, let

$$
\begin{aligned}
\bar{u}(t,y) &:= J_Y(t,X(t,y))v(t,X(t,y)),\\
\bar{p}(t,y) &:= q(t,X(t,y)),\\
\bar{\omega}(t) &:= Q^T(t)\theta(t),\\
\bar{\xi}(t) &:= Q^T(t)\eta(t),\\
\mathcal{F}_0(t,y) &:= J_Y(t,X(t,y))f_0(t,y),\\
\mathcal{F}_i(t) &:= Q^T(t)f_i(t), \qquad i \in \{1,2\},\\
\sigma(t,y) &:= J_Y(t,X(t,y))\tau(t,X(t,y))J_Y^T(t,X(t,y)),\\
N(t,y) &:= Q^T(t)n(t,X(t,y)) \text{ on } \Gamma := \Gamma(0).
\end{aligned} \tag{2.9}
$$

It follows from (2.3) that

$$M(t)x = Q^T(t)m(t)Q(t)x = Q^T[\theta(t) \times Qx] = (Q^T(t)\theta(t)) \times x = \bar{\omega}(t) \times x,$$

and by definition, $N(t,y) = N(y)$ is the outer normal vector of Γ at $y \in \Gamma$. The transformed inertia tensor $I := Q^T(t)J(t)Q(t)$ no longer depends on time since for all $a,b \in B_1$,

$$a \cdot I \cdot b = \frac{|\mathcal{B}|}{m}\int_{\mathcal{B}(0)} (a \times y)\cdot(b \times y). \tag{2.10}$$

To get the transformed rigid body equations we use that

$$\int_{\Gamma(t)} \mathbf{T}(\bar{\tau},q)n(t)\,\mathrm{d}\Gamma = Q\int_{\Gamma} \mathbf{T}(D_{\mu,\alpha}(\bar{u})+\sigma,\bar{p})N\,\mathrm{d}\Gamma$$

and

$$\int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(\bar{\tau},q)n(t)\,\mathrm{d}\Gamma = Q\int_{\Gamma} y \times \mathbf{T}(D_{\mu,\alpha}(\bar{u})+\sigma,\bar{p})N\,\mathrm{d}\Gamma.$$

The first part of the system (1.1) transforms into

$$
\text{(NP)}\quad
\left\{
\begin{array}{rcll}
\partial_t\bar{u} + \mathcal{M}\bar{u} - 2\mu\alpha\mathcal{L}\bar{u} - \operatorname{div}\sigma + \mathcal{N}(\bar{u}) + \mathcal{G}\bar{p} &=& \mathcal{F}_0, & \text{in } J_T \times \mathcal{D},\\
\operatorname{div}\bar{u} &=& 0, & \text{in } J_T \times \mathcal{D},\\
\bar{u}(t,y) - \bar{\omega}(t) \times y - \bar{\xi}(t) &=& 0, & \text{on } J_T \times \Gamma,\\
\bar{u}|_{\partial\mathcal{O}} &=& 0, & \text{on } J_T \times \partial\mathcal{O},\\
\bar{u}(0) &=& v_0 & \text{in } \mathcal{D},\\
\mathrm{m}\bar{\xi}' + \mathrm{m}(\bar{\omega} \times \bar{\xi}) + \int_\Gamma \mathbf{T}(D_{\mu,\alpha}(\bar{u})+\sigma,\bar{p})N\,\mathrm{d}\Gamma &=& \mathcal{F}_1, & \text{in } J_T,\\
I\bar{\omega}' + \bar{\omega} \times (I\bar{\omega}) + \int_\Gamma y \times \mathbf{T}(D_{\mu,\alpha}(\bar{u})+\sigma,\bar{p})N\,\mathrm{d}\Gamma &=& \mathcal{F}_2, & \text{in } J_T,\\
\bar{\xi}(0) = \eta_0 & \text{and} & \bar{\omega}(0) = \theta_0 &
\end{array}
\right.
$$

on the cylindrical domain $J_T \times \mathcal{D}$. In this system, the operator $\mathcal{L}$ denotes the transformed Laplace operator and it is given by

$$
\begin{aligned}
(\mathcal{L}\bar{u})_i \;:=\;& \Sigma_{j,k=1}^3 \partial_j(g^{jk}\partial_k\bar{u}_i) + 2\Sigma_{j,k,l=1}^3 g^{kl}\Gamma_{jk}^i\partial_l\bar{u}_j\\
&+\Sigma_{j,k,l=1}^3\left(\partial_k(g^{kl}\Gamma_{jl}^i) + \Sigma_{m=1}^3 g^{kl}\Gamma_{jl}^m\Gamma_{km}^i\right)\bar{u}_j.
\end{aligned}
$$

The convection term becomes

$$(\mathcal{N}(\bar{u}))_i \;:=\; \Sigma_{j=1}^3\bar{u}_j\partial_j\bar{u}_i + \Sigma_{j,k=1}^3\Gamma_{jk}^i\bar{u}_j\bar{u}_k.$$

The transformed time derivative and the transformed gradient are given by

$$(\mathcal{M}\bar{u})_i \;:=\; \Sigma_{j=1}^3\dot{Y}_j\partial_j\bar{u}_i + \Sigma_{j,k=1}^3\left(\Gamma_{jk}^i\dot{Y}_k + (\partial_kY_i)(\partial_j\dot{X}_k)\right)\bar{u}_j$$

and

$$(\mathcal{G}\bar{p})_i \;:=\; \Sigma_{j=1}^3 g^{ij}\partial_j\bar{p},$$

respectively. The coefficients are given by the metric contravariant tensor

$$g^{ij} = \Sigma_{k=1}^3(\partial_kY_i)(\partial_kY_j), \tag{2.11}$$

the metric covariant tensor

$$g_{ij} = \Sigma_{k=1}^3(\partial_iX_k)(\partial_jX_k) \tag{2.12}$$

and the Christoffel symbol

$$\Gamma^i_{jk} = \Sigma^3_{l=1} g^{jk}(\partial_i g_{jl} + \partial_j g_{ik} - \partial_k g_{ij}) = \Sigma^3_{l=1}(\partial_j \partial_k X_l)\partial_l Y_i. \tag{2.13}$$

Similarly, the transport equation in the second part of (1.1) becomes

$$\text{(TE)} \qquad \begin{cases} \lambda_1(\partial_t \sigma + ((\dot{Y} + \bar{u}) \cdot \nabla)\sigma + \mathcal{C}(\bar{u}, \bar{\xi}, \bar{\omega}) : \sigma) + \sigma &= \mathcal{E}\bar{u} \quad \text{in } J_T \times \mathcal{D}, \\ \sigma(0) &= \tau_0 \quad \text{in } \mathcal{D}, \end{cases}$$

where

$$(\mathcal{E}\bar{u})_{ij} := \mu(1-\alpha)\Sigma^3_{k,l=1} g^{ik}\partial_k \bar{u}_j + g^{jk}\partial_k \bar{u}_i + (g^{ik}\Gamma^j_{kl} + g^{jk}\Gamma^i_{kl})\bar{u}_l,$$

and where

$$\begin{aligned} &\Sigma^3_{k,l=1}\mathcal{C}(\bar{u}, \bar{\xi}, \bar{\omega})_{ijkl}\sigma_{kl} \\ := \;& \Sigma^3_{k,l,m,n,\pi=1}\Big\{\delta_{jl}(\partial_m Y_i)(\partial_k \dot{X}_m) + \delta_{ik}(\partial_m Y_j)(\partial_l \dot{X}_m) + (\delta_{jl}\Gamma^i_{km} + \delta_{ik}\Gamma^j_{lm})\dot{Y}_m \\ + \;& \big[\frac{1}{2}(1-a)\partial_n\partial_m X_\pi(\delta_{jl}g^{in}\partial_k X_\pi + \delta_{ik}g^{jn}\partial_l X_\pi) - \frac{1}{2}(a-1)(\delta_{jl}\Gamma^i_{km} + \delta_{ik}\Gamma^j_{lm})\big]\bar{u}_m \\ + \;& \big[\frac{1}{2}(1-a)(\delta_{lj}g^{in}g_{km} + \delta_{ik}g^{jn}g_{lm}) - \frac{1}{2}(1+a)(\delta_{jl}\delta_{im}\delta_{kn} + \delta_{ik}\delta_{ln}\delta_{jm})\big]\partial_n \bar{u}_m\Big\}\sigma_{kl} \end{aligned}$$

Note that we often shortly write Y or $\dot{Y}$ for the functions given by $(t,y) \mapsto (Y \circ X)(t,y))$ and $(t,y) \mapsto ((\partial_t Y) \circ X)(t,y)$.

In the following, we use that a solution $(\bar{u}, \bar{p}, \bar{\xi}, \bar{\omega}, \sigma)$ to the coupled problem (NP) and (TE) yields a solution $(v, q, \eta, \theta, \tau)$ to the original problem by the definitions in (2.9). A posteriori, we can claim that the transforms X, Y are sufficiently regular for making the two systems of equations equivalent in the strong sense, cf. [24, Proposition 2.1, Theorem 2.5] and [29].

3. Preliminary Results

In this section, we state some preliminary results regarding the Stokes equation, the linearized transport equation corresponding to τ and the change of coordinates given by X, Y, which provide the estimates necessary for constructing a solution via a compactness argument in the following section.

For $1 < s, r < \infty$, the Stokes operator A_r with Dirichlet boundary conditions in $L^r_\sigma(\mathcal{D})$ is defined by

$$\begin{cases} A_r u &:= \mathbb{P}_{r,\mathcal{D}}\Delta u, \\ D(A_r) &:= W^{2,r}(\mathcal{D}) \cap W^{1,r}_0(\mathcal{D}) \cap L^r_\sigma(\mathcal{D})), \end{cases}$$

where $\mathbb{P}_{r,\mathcal{D}}$ denotes the Helmholtz projection on $L^r(\mathcal{D})$. In the following, we denote the corresponding space of maximal regularity by

$$X^T_{s,r,\sigma} := W^{1,s}(J_T; L^r_\sigma(\mathcal{D})) \cap L^s(J_T; D(A_r))$$

and the space for the associated pressure is

$$Y^T_{s,r,0} := \{p \in L^s(J_T; W^{1,r}(\mathcal{D})) : \int_{\mathcal{D}} p = 0\}.$$

The following proposition is a classical result due to Solonnikov [34] and Giga and Sohr [20] for the case $r \neq s$.

Proposition 3.1. *Let $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded or exterior domain of class C^2 and $1 < s, r < \infty$, $0 < T < T_0$, $f \in L^s(J_T; L^r(\Omega))$ and $u_0 \in D_{s,r}$ as defined in* (1.2). *Then there exists a unique solution $u \in X^T_{s,r,\sigma}$, $p \in Y^T_{s,r,0}$ to the Stokes problem*

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f, \quad \text{in } J_T \times \mathcal{D}, \\ \operatorname{div} u &= 0, \quad \text{in } J_T \times \mathcal{D}, \\ u|_{\partial\mathcal{D}} &= 0, \quad \text{on } J_T \times \Gamma, \\ u(0) &= u_0, \end{cases} \tag{3.1}$$

and there exists a constant $K^{Stokes}_{s,r} > 0$ independent of T, u_0 and f, such that

$$\|u\|_{X^T_{s,r,\sigma}} + \|p\|_{Y^T_{s,r,0}} \leq K^{Stokes}_{s,r}(\|f\|_{L^s(L^r)} + \|u_0\|_{D_{s,r}}).$$

This motivates the definition of the solution operators

$$\mathcal{U}(f,u_0) := u \in X^T_{s,r,\sigma}, \qquad \mathcal{P}(f,u_0) := p \in Y^T_{s,r,0} \tag{3.2}$$

for problem (3.1), which give isomorphisms from $L^s(L^r)\times D_{s,r}$ to $X^T_{s,r,\sigma}\times Y^T_{s,r,0}$. Note that the solutions satisfy the relation

$$\nabla\mathcal{P}(f,u_0) = (\mathrm{Id}_{L^r} - \mathbb{P}_{r,\mathcal{D}})(\Delta\mathcal{U}(f,u_0)+f). \tag{3.3}$$

In the following, we also deal with solutions u,p for the Stokes problem with in homogenous boundary data. We thus define the spaces

$$\begin{aligned} X^T_{s,r} &:= W^{1,s}(J_T;L^r(\mathcal{D}))\cap L^s(J_T;W^{2,r}(\mathcal{D})) \\ Y^T_{s,r} &:= L^s(J_T;W^{1,r}(\mathcal{D})) \\ W^T_s &:= W^{1,s}(J_T;\mathbb{R}^3). \end{aligned}$$

The following proposition yields embeddings properties of $X^T_{s,r}$ which will be needed later on.

Proposition 3.2. *Let $D\subset\mathbb{R}^n$ be a $C^{1,1}$-domain with compact boundary, let $s,r\in(1,\infty)$, $\beta\in(0,1)$ and $T_0>0$. Then*

$$X^{T_0}_{s,r} \hookrightarrow H^{\beta,s}(J_{T_0};H^{2(1-\beta),r}(D)). \tag{3.4}$$

In particular, if $\bar r,\bar s\in(1,\infty)\cup\{\infty\}$, $l\in\{0,1\}$ and

$$\frac{2-l}{2}+\frac{n}{2\bar r}-\frac{n}{2r} \geq \frac{1}{s}-\frac{1}{\bar s},$$

then for all $T\in J_{T_0}$

$$X^T_{s,r}\hookrightarrow L^{\bar s}(J_T;W^{l,\bar r}(D)).$$

Moreover, there exists a constant $C(T_0)>0$, independent of $T\in J_{T_0}$, such that the estimate

$$\|u\|_{L^{\bar s}(J_T;W^{l,\bar r}(D))} \leq C(T_0)\,\|u\|_{X^T_{s,r}}$$

holds true for all $u\in X^T_{s,r,0} := \{w\in X^T_{s,r} : w|_{t=0}=0\}$.

For a proof of this proposition by the mixed derivatives theorem (cf. [6]), we refer to [9, Lem. 4.2] and [7]. At this point we also note the more elementary embedding constants

$$\|f\|_{L^s} \leq T^{1/s-1/\bar s}\,\|f\|_{L^{\bar s}} \qquad \text{for all } f\in L^{\bar s}(J_T), \qquad \bar s\geq s$$

and

$$\|f\|_{L^\infty} \leq T^{1/s'}\,\|f\|_{W^T_s} \qquad \text{for all } f\in W^T_s,\quad f(0)=0,\quad \frac{1}{s}+\frac{1}{s'}=1,$$

which will be used frequently.

The estimates we need on the diffeomorphisms X and Y and on the coefficients $g_{ij},g^{ij},\Gamma^i_{jk}$, which characterize the change of coordinates are summarized in the following Lemmas. The proofs are elementary but tedious, so they are quoted from [19, Lemma 6.3, Lemma 6.5], where they were used to study the free movement of a rigid body in a Newtonian or generalized Newtonian fluid.

Lemma 3.3. *Let $T_0>0$ and $L>0$. Then for all $\xi,\omega\in C(J_{T_0};\mathbb{R}^3)$ such that $\|\xi\|_{C(J_{T_0})}+\|\omega\|_{C(J_{T_0})}\leq L$, then the solutions $X,Y\in C^1(J_{T_0};C^\infty(\mathbb{R}^n))$ of (2.6), (2.7) satisfy*

$$\|\partial^\beta X\|_{C(J_T;C(\mathcal{D}))} + \|\partial^\beta Y\|_{C(J_T;C(\mathcal{D}))} \leq K_L,$$

for all multi-indices $1\leq|\beta|\leq 3$ and for some constant $K_L>0$, independently of $0<T\leq T_0$.

Lemma 3.4. *Let $T_0>0$ and $L>0$. Then for all $\xi,\omega\in C(J_{T_0})$ such that $\|\xi\|_\infty+\|\Omega\|_\infty\leq L$,*

$$\sup_{t\in J_T}\|\partial^\beta g^{ij}(t)\|_{C(\mathcal{D})} + \|\partial^\beta g_{ij}(t)\|_{C(\mathcal{D})} + \|\partial^\beta\Gamma^i_{jk}(t)\|_{C(\mathcal{D})} \leq K_L$$

for all $0<T\leq T_0$ and for all multi-indices $0\leq|\beta|\leq 1$ and a constant $K_L>0$. Moreover,

$$\begin{aligned} \sup_{i,j}\|\partial_i Y_j-\delta_{ij}\|_{C(J_T;C(\mathcal{D}))} &\leq K_L T \qquad \textit{and} \\ \sup_{i,j}\|g^{ij}-\delta_{ij}\|_{C(J_T;C(\mathcal{D}))} &\leq K_L T, \end{aligned}$$

where g^{ij}, g_{ij} and Γ^i_{jk} are given by ξ,ω through X,Y as in (2.11), (2.12) and (2.13).

The unknown part of the stress tensor σ solves (TE) and belongs to the space

$$Z_{s,r}^T := C(J_T; W^{1,r}(\mathcal{D})) \cap W^{1,s}(J_T; L^r(\mathcal{D})). \tag{3.5}$$

We have the following preliminary result on the linearization of (TE).

Lemma 3.5. *Let $1 < p < \infty$ and $q > 3$, $T > 0$. Given $\tau_0 \in W^{1,r}(\mathcal{D})^{3\times 3}$, $w \in L^s(J_T; D(A_r))^3$, $\mathcal{C} \in L^s(J_T; W^{1,r}(\mathcal{D}))^{3\times 3\times 3\times 3}$ and $\mathcal{E} \in L^s(J_T; W^{1,r}(\mathcal{D}))^{3\times 3}$, there exists a unique solution $\sigma \in Z_{s,r}^T$ to*

$$\left\{ \begin{array}{rcll} \lambda_1[\partial_t \sigma + (w\cdot\nabla)\sigma + \mathcal{C} : \sigma] + \sigma &=& \mathcal{E}, & \text{in } (0,T)\times\mathcal{D}, \\ \sigma(0) &=& \tau_0, & \text{in } \mathcal{D}, \end{array} \right. \tag{3.6}$$

satisfying

$$\|\sigma\|_{L^\infty(W^{1,r})} \le (\|\tau_0\|_{W^{1,r}} + \frac{1}{\lambda_1 K_r}) \exp(C\|\nabla w\|_{L^1(L^\infty)} + \|\mathcal{C}\|_{L^1(W^{1,r})} + \|\mathcal{E}\|_{L^1(W^{1,r})}) =: \Lambda,$$

and

$$\|\partial_t \sigma\|_{L^s(L^r)} \le K_r \Lambda(\|w\|_{L^s(W^{1,r})} + \|\mathcal{C}\|_{L^s(L^r)} + \|\mathcal{E}\|_{L^s(L^r)} + \frac{T^{1/p}}{\lambda_1 K_r})$$

for some constant $K_r = K(r) > 0$.

Proof. The proof follows very closely the proof of [14, Lemma 10.3]. In particular, we obtain existence in the same way, by using the method of characteristics for smooth data and showing that the estimates allow for finding $\sigma \in Z_{s,r}^T$ for Sobolev data by passing to the limit. We repeat the arguments of how to obtain the above estimates, as they vary slightly and motivate the assumptions on the data. First, we formally multiply the first equation in (3.6) by $|\sigma|^{r-2}\sigma$, pointwise in time, and integrate over $\mathcal{D}$. This gives

$$\frac{\lambda_1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma\|_{L^r}^r + \|\sigma\|_{L^r}^r \quad \le \quad C\lambda_1\|\mathcal{C}\|_{L^\infty}\|\sigma\|_{L^r}^r + \|\mathcal{E}\|_{L^\infty}\|\sigma\|_{L^r}^{r-1},$$

as

$$\Sigma_{i,j,k=1}^3 (w_k(\partial_k\sigma_{ij}), \sigma_{ij}|\sigma|^{r-2})_{r,\mathcal{D}} \quad = \quad \Sigma_{k=1}^3 (w_k, \partial_k(|\sigma|^r))_{r,\mathcal{D}} = 0$$

by integration by parts. Secondly, we take the gradient of the first equation in (3.6) and formally multiply by $|\nabla\sigma|^{r-2}\nabla\sigma$ and then integrate to get

$$\begin{aligned} \frac{\lambda_1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sigma\|_{L^r}^r + \|\nabla\sigma\|_{L^r}^r \quad &\le \quad C\lambda_1\left(\|\nabla w\|_{L^\infty}\|\nabla\sigma\|_{L^r}^r + \|\mathcal{C}\|_{L^\infty}\|\nabla\sigma\|_{L^r}^r + \|\nabla\mathcal{C}\|_{L^r}\|\nabla\sigma\|_{L^r}^r\right) \\ &\quad + \|\nabla\mathcal{E}\|_{L^r}\|\nabla\sigma\|_{L^r}^{r-1} \end{aligned}$$

as

$$\Sigma_{i,j,k,l=1}^3 (w_l(\partial_k\partial_l\sigma_{ij}), \partial_k\sigma_{ij}|\nabla\sigma|^{r-2})_{r,\mathcal{D}} = 0$$

and

$$\begin{aligned} \Sigma_{i,j,k,l,m=1}^3 ((\partial_m\mathcal{C}_{klij})\sigma_{kl}, (\partial_m\sigma_{ij})|\nabla\sigma|^{r-2})_{r,\mathcal{D}} \quad &\le \quad C\|\nabla\mathcal{C}\|_{L^r}\|\sigma\|_{L^\infty}\left(\int_{\mathcal{D}} (|\nabla\sigma|^{r-1})^{r'}\right)^{1/r'} \\ &\le \quad C\|\nabla\mathcal{C}\|_{L^r}\|\sigma\|_{W^{1,r}}\|\nabla\sigma\|_{L^r}^{r-1} \end{aligned}$$

for $r' = \frac{r}{r-1}$ and using that $r > 3$. Together, this gives

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma\|_{W^{1,r}}^r + \|\sigma\|_{W^{1,r}}^r \le \|\mathcal{E}\|_{W^{1,r}}\|\sigma\|_{W^{1,r}}^{r-1} + \lambda_1 C\left(\|\nabla w\|_{L^\infty} + \|\mathcal{C}\|_{W^{1,r}}\right)\|\sigma\|_{W^{1,r}}^r.$$

We divide by $\lambda_1\|\sigma\|_{W^{1,r}}^{r-1}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sigma\|_{W^{1,r}} + \frac{1}{\lambda_1 C}\right) + \frac{1}{\lambda_1}\|\sigma\|_{W^{1,r}} \quad \le \quad C\left(\|\nabla w\|_{L^\infty} + \|\mathcal{C}\|_{W^{1,r}} + \|\mathcal{E}\|_{W^{1,r}}\right)\left(\|\sigma\|_{W^{1,r}} + \frac{1}{\lambda_1 C}\right).$$

It follows by Gronwall's Lemma that

$$\|\sigma\|_{L^\infty(W^{1,r})} \quad \le \quad \left(\|\tau_0\|_{W^{1,r}} + \frac{1}{\lambda_1 K_r}\right)\exp(C\|\nabla w\|_{L^1(L^\infty)} + \|\mathcal{C}\|_{L^1(W^{1,r})} + \|\mathcal{E}\|_{L^1(W^{1,r})}) = \Lambda.$$

Directly from (3.6),

$$\begin{aligned}\|\partial_t\sigma\|_{L^r} &\le \frac{1}{\lambda_1}(\|\mathcal{E}\|_{L^r}+\|\sigma\|_{L^r})+\tilde{C}(\|w\|_{W^{1,r}}+\|\mathcal{C}\|_{L^r})\|\sigma\|_{W^{1,r}}\\ &\le \tilde{C}(\|\sigma\|_{W^{1,,r}}+\frac{1}{\lambda_1\tilde{C}})(\|w\|_{W^{1,r}}+\|\mathcal{C}\|_{L^r}+\|\mathcal{E}\|_{L^r}+\frac{1}{\lambda_1\tilde{C}})\\ &\le K_r\Lambda(\|w\|_{W^{1,r}}+\|\mathcal{C}\|_{L^r}+\|\mathcal{E}\|_{L^r}+\frac{1}{\lambda_1K_r}),\end{aligned}$$

if we take $\tilde{C}=K_r$. This implies

$$\|\partial_t\sigma\|_{L^s(L^r)}\le K_r\Lambda(\|w\|_{L^s(W^{1,r})}+\|\mathcal{C}\|_{L^s(L^r)}+\|\mathcal{E}\|_{L^s(L^r)}+\frac{T^{1/s}}{\lambda_1K_r}).$$

□

Remark 3.6. The Lemma shows that in the full coupled problem, if τ_0 is chosen to be symmetric, then $\tau(t)$ remains symmetric for all $t\in J_{T_*}$. This is because for given u,ξ,ω, the corresponding coefficients $\mathcal{C},\mathcal{E}$ in equation (3.6) are symmetric, so that σ^T must satisfy the same equation as σ. By uniqueness, they must coincide. By the definition in (2.9), this property carries over to τ.

4. Proof of Existence

The existence proof is divided into three steps. In the first subsection, we define a suitable reformulation of the problem. It is necessary to reduce to homogeneous data in the parabolic part and to linearize the system in two steps. The first step is to use a modified Helmholtz projection on the fluid equations, which is appropriate for the transformed gradient of the pressure $\mathcal{G}p$. The second step consists in finding a linearization for the coupling of fluid and rigid body which yields strong estimates. After this, the fixed point map Φ is constructed in detail in Subsection 4.2. In the third subsection, it is shown that Schauder's fixed point theorem applies to Φ. This proves the existence claim in Theorem 1.2.

4.1. **Reformulation.** From [19, Theorem 4.1], we obtain that the linearized Newtonian coupled system

$$(4.1)\qquad \left\{\begin{aligned} \partial_t u^*-\mu\alpha\Delta u^*+\nabla p^* &= f_0, && \text{in } J_T\times\mathcal{D},\\ \operatorname{div}u^* &= 0, && \text{in } J_T\times\mathcal{D},\\ u^*(0) &= v_0, && \text{in } \mathcal{D},\\ u^*-\omega^*\times y-\xi^* &= 0, && \text{on } J_T\times\Gamma,\\ u^* &= 0, && \text{on } J_T\times\partial\mathcal{O},\\ \mathrm{m}(\xi^*)'+\int_\Gamma \mathbf{T}(D_{\mu,\alpha}(u^*),p^*)N\,\mathrm{d}\Gamma &= f_1, && \text{in } J_T,\\ I(\omega^*)'+\int_\Gamma y\times\mathbf{T}(D_{\mu,\alpha}(u^*),p^*)N\,\mathrm{d}\Gamma &= f_2, && \text{in } J_T,\\ \xi^*(0) &= \eta_0,\\ \omega^*(0) &= \theta_0.\end{aligned}\right.$$

has a unique strong solution $u^*,p^*,\xi^*\omega^*$ satisfying the estimate

$$\begin{aligned}&\|u^*\|_{X^T_{s,r}}+\|p^*\|_{Y^T_{s,r}}+\|\xi^*\|_{W^{1,s}(J_T)}+\|\omega^*\|_{W^{1,s}(J_T)}\\ \le\;& C\left(\|v_0\|_{B^{2-2/s}_{r,s}(\mathcal{D})}+|\eta_0|+|\theta_0|+\|f_0\|_{L^s(L^r)}+\|f_1\|_{L^s}+\|f_2\|_{L^s}\right)\\ (4.2)\qquad =:\;& K_*.\end{aligned}$$

We set

$$\begin{aligned}\hat{u} &:= \bar{u}-u^*,\\ \xi &:= \bar{\xi}-\xi^*,\\ \omega &:= \bar{\omega}-\omega^*.\end{aligned}$$

The Helmholtz projection applied to $\mathcal{G}\bar{p}(t)$ yields $\hat{p}\in Y^T_{s,r,0}$ such that

$$(4.3)\qquad \mathcal{G}\bar{p}=(\mathrm{Id}-\mathbb{P}_{r,\mathcal{D}})\mathcal{G}\bar{p}+\mathbb{P}_{r,\mathcal{D}}\mathcal{G}\bar{p}=:\nabla\hat{p}+\mathcal{H}(\bar{p}),$$

where $\mathcal{H}(\bar{p})(t) \in L^r_\sigma(\mathcal{D})$ and $\hat{p}$ satisfies the weak Neumann problem associated to

$$\left\{\begin{array}{rcll} \Delta\hat{p} &=& \operatorname{div}(\mathcal{G}\bar{p}), & \text{in } \mathcal{D}, \\ \frac{\partial\hat{p}}{\partial N} &=& \frac{\partial(\mathcal{G}\bar{p})}{\partial N} = \frac{\partial\bar{p}}{\partial N}, & \text{on } \Gamma, \\ \frac{\partial\hat{p}}{\partial\nu} &=& \frac{\partial(\mathcal{G}\bar{p})}{\partial\nu} = \frac{\partial\bar{p}}{\partial\nu}, & \text{on } \partial\mathcal{O}. \end{array}\right. \tag{4.4}$$

We write the system (NP) in terms of $\hat{u}, \hat{p}, \xi, \omega$ in a way that only linear terms appear on the left hand side:

$$\left\{\begin{array}{rcll} \partial_t\hat{u} - \mu\alpha\Delta\hat{u} + \nabla(\hat{p} - p^*) &=& \mathbf{F}_0 + \operatorname{div}\sigma, & \text{in } J_T \times \mathcal{D}, \\ \operatorname{div}\hat{u} &=& 0, & \text{in } J_T \times \mathcal{D}, \\ \hat{u}(0) &=& 0, & \text{in } \mathcal{D}, \\ \hat{u} &=& \omega \times y + \xi, & \text{on } J_T \times \Gamma, \\ \hat{u} &=& 0, & \text{on } J_T \times \partial\mathcal{O}, \\ \mathrm{m}\xi' + \int_\Gamma \mathbf{T}(D_{\mu,\alpha}(\hat{u}), \bar{p} - p^*)N\,\mathrm{d}\Gamma &=& \mathbf{F}_1 - \int_\Gamma \sigma N\,\mathrm{d}\Gamma, & \text{in } J_T, \\ I(\omega)' + \int_\Gamma y \times \mathbf{T}(D_{\mu,\alpha}(\hat{u}), \bar{p} - p^*)N\,\mathrm{d}\Gamma &=& \mathbf{F}_2 - \int_\Gamma y \times \sigma N\,\mathrm{d}\Gamma, & \text{in } J_T, \\ \xi(0) &=& 0, & \\ \omega(0) &=& 0, & \end{array}\right. \tag{4.5}$$

where

$$\begin{array}{rcl} \mathbf{F}_0 &:=& \mathcal{F}_0 - f_0 + \mu\alpha(\mathcal{L} - \Delta)\bar{u} - \mathcal{H}(\bar{p}) - \mathcal{M}(\bar{u}) - \mathcal{N}(\bar{u}), \\ \mathbf{F}_1 &:=& \mathcal{F}_1 - f_1 + \displaystyle\int_\Gamma (\hat{p} - \bar{p})N\,\mathrm{d}\Gamma\mathrm{m} - \mathrm{m}\bar{\omega} \times \bar{\xi}, \\ \mathbf{F}_2 &:=& \mathcal{F}_2 - f_2 + \displaystyle\int_\Gamma y \times [(\hat{p} - \bar{p})\mathrm{Id}N]\,\mathrm{d}\Gamma - \bar{\omega} \times I\bar{\omega}. \end{array} \tag{4.6}$$

Ignoring the $\mathbf{F}_i$ would linearize the system and reduce the effect of the rigid body on the fluid flow to the Dirichlet boundary condition on Γ. However, this condition does not satisfy $\hat{u}|_\Gamma \cdot N = 0$. In the following, we construct a potential field which corrects this condition and redefine $\hat{u}$ and $\hat{p}$ correspondingly. Then we show how this affects the rigid body equations, following [15].

Let $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ and let $a^{(i)}, \alpha^{(i)}$ be solutions of the Neumann problems

$$\left\{\begin{array}{rcll} \Delta a^{(i)} &=& 0, & \text{in } \mathcal{D}, \\ \frac{\partial a^{(i)}}{\partial N}|_\Gamma &=& N \cdot e_i, & \text{on } \Gamma, \\ \frac{\partial a^{(i)}}{\partial\nu}|_{\partial\mathcal{O}} &=& 0, & \text{on } \partial\mathcal{O}, \end{array}\right.$$

$$\left\{\begin{array}{rcll} \Delta\alpha^{(i)} &=& 0, & \text{in } \mathcal{D}, \\ \frac{\partial\alpha^{(i)}}{\partial N}|_\Gamma &=& N \cdot (e_i \times y), & \text{on } \Gamma, \\ \frac{\partial\alpha^{(i)}}{\partial\nu}|_{\partial\mathcal{O}} &=& 0, & \text{on } \partial\mathcal{O}. \end{array}\right.$$

As the domain is smooth, we can find special

$$a^{(i)}, \alpha^{(i)} \in W^{3,r}(\mathcal{D}) \tag{4.7}$$

solving these equations. For any given $\xi, \omega \in W^{1,s}_0(J_T)$, let

$$A_{\xi,\omega}(t) := \Sigma_i\omega_i(t)\alpha^{(i)} + \xi_i(t)a^{(i)} \qquad \text{for all } t \in J_T, \tag{4.8}$$

which implies that

$$\left\{\begin{array}{rcll} \Delta A_{\xi,\omega}(t) &=& 0, & \text{in } \mathcal{D}, \\ \frac{\partial A_{\xi,\omega}(t)}{\partial N}|_\Gamma &=& (\omega(t) \times y + \xi(t)) \cdot N, & \text{on } \Gamma, \\ \frac{\partial A_{\xi,\omega}(t)}{\partial\nu}|_{\partial\mathcal{O}} &=& 0, & \text{on } \partial\mathcal{O}. \end{array}\right.$$

Or simply, $\nabla A_{\xi,\omega}(t) = (\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})(\xi(t) + \omega(t) \times \cdot)$. Moreover, $\nabla A_{\xi,\omega} \in X^T_{s,r}$ and

$$\|\nabla A_{\xi,\omega}\|_{W^{1,s}(W^{2,r})} + \|\partial_t A_{\xi,\omega}\|_{L^s(W^{1,r})} \leq C(\|\xi\|_{W^{1,s}} + \|\omega\|_{W^{1,s}}).$$

We define

$$\begin{array}{rcl} u &:=& \hat{u} - \nabla A_{\xi,\omega}, \\ p &:=& \hat{p} - p^* + \partial_t A_{\xi,\omega}, \end{array}$$

to obtain the system of equations

$$
(4.9)\quad \begin{cases}
\partial_t u - \mu\alpha\Delta u + \nabla p &= \mathbf{F}_0 + \operatorname{div}\sigma, & \text{in } J_T\times\mathcal{D},\\
\operatorname{div} u &= 0, & \text{in } J_T\times\mathcal{D},\\
u(0) &= 0, & \text{in } \mathcal{D},\\
u|_\Gamma - \omega\times y - \xi &= -\nabla A_{\xi,\omega}|_\Gamma, & \text{on } J_T\times\Gamma,\\
u|_{\partial\mathcal{O}} &= -\nabla A_{\xi,\omega}|_{\partial\mathcal{O}}, & \text{on } J_T\times\partial\mathcal{O},\\
\mathrm{m}\xi' + \int_\Gamma \mathbf{T}(D_{\mu,\alpha}(u+\nabla A_{\xi,\omega}), p+\partial_t A_{\xi,\omega})N\,\mathrm{d}\Gamma &= \mathbf{F}_1 - \int_\Gamma \sigma N\,\mathrm{d}\Gamma, & \text{in } J_T,\\
I\omega' + \int_\Gamma y\times \mathbf{T}(D_{\mu,\alpha}(u+\nabla A_{\xi,\omega}), p+\partial_t A_{\xi,\omega})N\,\mathrm{d}\Gamma &= \mathbf{F}_2, - \int_\Gamma y\times\sigma N\,\mathrm{d}\Gamma, & \text{in } J_T,\\
\xi(0) &= 0,\\
\omega(0) &= 0,
\end{cases}
$$

equivalently to (4.5). Note that now u satisfies $u\cdot N = 0$ on Γ and $u\cdot\nu = 0$ on $\partial\mathcal{O}$. For all $0<\varepsilon<1-\frac{1}{r}$, we introduce the operator

$$
\mathcal{J} : W^{\varepsilon+1/r,r}(\mathcal{D};\mathbb{R}^{3\times 3}) \to \mathbb{R}^6,\ M\mapsto \begin{pmatrix} \int_\Gamma MN\,\mathrm{d}\Gamma \\ \int_\Gamma y\times MN\,\mathrm{d}\Gamma \end{pmatrix}.
$$

From the boundedness of the trace operator $\gamma : W^{\varepsilon/2+1/r,r}(\mathcal{D}) \to L^r(\partial\mathcal{D})$, it follows that

$$
(4.10)\qquad |\mathcal{J}(M)| \le C\|M\|_{W^{\varepsilon+1/r,r}(\mathcal{D})}, \quad \text{for all } M\in W^{\varepsilon+1/r,r}(\mathcal{D}).
$$

We introduce the matrix $\mathbb{I} := \begin{pmatrix} \mathrm{m}\mathrm{Id}_3 & 0 \\ 0 & I\end{pmatrix}$ and the *added mass matrix* $\mathbb{M}$ of $\mathcal{B}$, cf. [15, p. 685] in the following way. Let $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ be given by

$$
a_{ij} = \int_\Gamma a^{(i)} N_j\,\mathrm{d}\Gamma,\quad b_{ij} = \int_\Gamma \alpha^{(i)}(e_j\times y)\cdot N\,\mathrm{d}\Gamma,\quad c_{ij} = \int_\Gamma a^{(i)}(e_j\times y)\cdot N\,\mathrm{d}\Gamma,\quad d_{ij} = \int_\Gamma \alpha^{(i)} N_j\,\mathrm{d}\Gamma
$$

and

$$
\mathbb{M} := \begin{pmatrix}
a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13}\\
a_{21} & a_{22} & a_{23} & c_{21} & c_{22} & c_{23}\\
a_{31} & a_{32} & a_{33} & c_{31} & c_{32} & c_{33}\\
d_{11} & d_{12} & d_{13} & b_{11} & b_{12} & b_{13}\\
d_{21} & d_{22} & d_{23} & b_{21} & b_{22} & b_{23}\\
d_{31} & d_{32} & d_{33} & b_{31} & b_{32} & b_{33}
\end{pmatrix},
$$

where the point of this definition is that

$$
\mathcal{J}(\partial_t A_{\xi,\omega}) = \mathbb{M}\begin{pmatrix}\xi' \\ \omega'\end{pmatrix}.
$$

By a direct calculation, it can be shown that the matrix $\mathbb{M}$ is symmetric and semi positive-definite, cf. [19, Lemma 4.3], so that $(\mathbb{I}+\mathbb{M})$ is invertible. It remains to modify u in order to obtain homogeneous Dirichlet boundary conditions on Γ.

Consider, similarly to the definition in (2.5), the function

$$
b_A(t,y) := -\nabla A_{\xi,\omega}(t,y) + \chi(y)[\xi(t)+\omega(t)\times y] - B_K[(\nabla\chi)\xi(t)+\omega(t)\times\cdot](y).
$$

We have $b_A\in W^{1,s}(J_T; C^2(\mathcal{D})\cap L^r_\sigma(\mathcal{D}))$ and the estimate

$$
\|b_A\|_{X^T_{s,r}} \le C(\|\omega\|_{W^T_s} + \|\xi\|_{W^T_s}).
$$

The point of this definition is that

$$
\begin{aligned}
u &= \mathcal{U}(\xi,\omega) + b_A + \mathcal{U}(\mathbf{F}_0 + \operatorname{div}\sigma),\\
p &= \mathcal{P}(\xi,\omega) + \mathcal{P}(\mathbf{F}_0 + \operatorname{div}\sigma),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{U}(\xi,\omega) &= \mathcal{U}(-\partial_t b_A + \mu\alpha\Delta b_A)\\
\mathcal{P}(\xi,\omega) &= \mathcal{P}(-\partial_t b_A + \mu\alpha\Delta b_A).
\end{aligned}
$$

It follows that(NP) can be written equivalently as

$$
\begin{aligned}
\mathcal{R}(\xi,\omega) &= \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix} - \mathcal{J}(\sigma) - \mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}(\mathbf{F}_0+\operatorname{div}\sigma)), \mathcal{P}(\mathbf{F}_0+\operatorname{div}\sigma))),\\
\xi(0) &= 0,\\
\omega(0) &= 0,\\
u - b_A(\xi,\omega) - \mathcal{U}(\xi,\omega) &= \mathcal{U}(\mathbf{F}_0+\operatorname{div}\sigma),\\
p - \mathcal{P}(\xi,\omega) &= \mathcal{P}(\mathbf{F}_0+\operatorname{div}\sigma),\\
u(0) &= 0.
\end{aligned}
\tag{4.11}
$$

where we define the bounded linear operator $\mathcal{R} : W^T_s \times W^T_s \to L^s(J_T) \times L^s(J_T)$ by

$$\mathcal{R}(\xi,\omega) := (\mathbb{I} + \mathbb{M})\frac{\mathrm{d}}{\mathrm{d}t}(\xi,\omega) + \mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}(\xi,\omega)), \mathcal{P}(\xi,\omega))).$$

We used that

$$
\begin{aligned}
\mathcal{J}(D_{\mu,\alpha}(\hat{u})) &= \mathcal{J}(D_{\mu,\alpha}(\mathcal{U}(\xi,\omega) + b_A + \mathcal{U}(\mathbf{F}_0 + \mathcal{K}\sigma) + \nabla A_{\xi,\omega}))\\
&= \mathcal{J}(D_{\mu,\alpha}(\mathcal{U}(\xi,\omega) + \mathcal{U}(\mathbf{F}_0 + \mathcal{K}\sigma))),
\end{aligned}
$$

as near Γ, $D(b_A) = -D(\nabla A_{\xi,\omega})$. By the following lemma, the first equation in (4.11) can be solved for ξ,ω, given any right-hand side in $L^s(J_T;\mathbb{R}^6)$.

Lemma 4.1. *The operator $\mathcal{R}$ is bounded and invertible and*

$$\left\| \begin{pmatrix} \xi \\ \omega \end{pmatrix} \right\|_{W^T_s \times W^T_s} := \|\mathcal{R}^{-1}(f)\|_{W^T_s \times W^T_s} \le \|f\|_{L^s(J_T;\mathbb{R}^6)}$$

for all $f \in L^s(J_T;\mathbb{R}^6)$.

Proof. The lemma is shown in [19, Lemma 4.4]. The proof is based on the fact that the operator $\mathcal{R}_1 = \mathcal{J}(s(\mathcal{U}(\xi,\omega)))$ maps from $W^T_s \times W^T_s$ to $H^{\beta,s}(J_T) \overset{c}{\hookrightarrow} L^s(J_T)$ for some $\beta(s,r) > 0$ and that $\mathcal{J}(\mathrm{Id}_{\mathbb{R}^3}\mathcal{P}(\xi,\omega))$ can be controlled by $\mathcal{R}_1$. □

4.2. **Construction of the fixed point iteration.** To prove Theorem 1.2, we show the existence of a strong solution

$$
\begin{aligned}
u &\in X^{T_*}_{s,r},\\
p &\in Y^{T_*}_{s,r},\\
\xi &\in W^{T_*}_s,\\
\omega &\in W^{T_*}_s,\\
\sigma &\in Z^{T_*}_{s,r}.
\end{aligned}
$$

to the coupled transformed systems (4.11) and (TE) via a Schauder fixed point argument, similar to the method used in [13, 14] for studying the uncoupled viscoelastic flow. In the following, let

$$\mathbb{V} := \mathbb{V}^T_{R_1,R_2,R_3,R_4} := \{(U,P,\Xi,\Omega,\Sigma) \in X^{T,R_1}_{s,r,0} \times Y^{T,R_1}_{s,r} \times W^{T,R_2}_{s,0} \times W^{T,R_2}_{s,0} \times Z^{T,R_3,R_4}_{s,r}\},$$

where

$$
\begin{aligned}
X^{T,R_1}_{s,r,0} &:= \{U \in X^T_{s,r,0} : U|_{\partial\mathcal{O}} = 0, \|U\|_{X^T_{s,r}} \le R_1\},\\
Y^{T,R_1}_{s,r} &:= \{P \in Y^T_{s,r} : \|P\|_{Y^T_{s,r}} \le R_1\},\\
W^{T,R_2}_{s,0} &:= \{\Xi \in W^{1,s}(J_T;\mathbb{R}^3) : \Xi|_{t=0} = 0, \|\Xi\|_{W^{1,s}(J_T)} \le R_2\}\\
Z^{T,R_3,R_4}_{s,r} &:= \{\Sigma \in Z^T_{s,r} : \|\Sigma\|_{C(J_T);W^{1,r}(\mathcal{D})} \le R_3, \|\Sigma_t\|_{L^s(J_T;L^r(\mathcal{D}))} \le R_4\}.
\end{aligned}
$$

Let

$$\mathbb{W} := L^s(J_T;L^r_\sigma \cap W^{1,r}(\mathcal{D})) \times L^s(J_T;L^r(\mathcal{D})) \times C(J_T;\mathbb{R}^3) \times C(J_T;\mathbb{R}^3) \times C(J_T;L^r(\mathcal{D})).$$

For every $R_1, R_2, R_3, R_4, T > 0$ we construct a map

$$\begin{array}{rcl} \Phi : \mathbb{V} & \to & \mathbb{V}, \\ \begin{pmatrix} U \\ P \\ \Xi \\ \Omega \\ \Sigma \end{pmatrix} & \mapsto & \begin{pmatrix} \Phi_u \\ \Phi_p \\ \Phi_\xi \\ \Phi_\omega \\ \Phi_\sigma \end{pmatrix} \end{array}$$

such that a fixed point of Φ solves (4.11) and (TE). Then we show that $\Phi(\mathbb{V})$ is a convex, compact subset of $\mathbb{W}$ and that Φ is continuous in the topology of $\mathbb{W}$. By the Schauder fixed point theorem, the fixed point then exists. The map is defined in four steps.

(1) First, we calculate $\mathbf{F}_0, \mathbf{F}_1$ and $\mathbf{F}_2$ as defined in (4.6), as functions of (U, P, Ξ, Ω). For this, we need $\bar{U} = U + u^* + \nabla A_{\Xi,\Omega}$ and $\bar{\Xi} = \Xi + \xi^*$, $\bar{\Omega} = \Omega + \omega^*$. By definition, they satisfy the estimates

$$\|\bar{U}\|_{X^T_{s,r}} \le R_1 + K_* + CR_2$$

and

$$\begin{array}{rcl} \|\bar{\Xi}\|_{W^T_s} & \le & R_2 + K_*, \\ \|\bar{\Omega}\|_{W^T_s} & \le & R_2 + K_*, \end{array} \tag{4.12}$$

where K_* is the constant from (4.2). To obtain the pressure $\bar{P}$, we proceed in two steps. First, we get $\hat{P} = P + p^* - \partial_t A_{\Xi,\Omega}$. Then we solve the weak Neumann problem associated to

$$\left\{ \begin{array}{rcll} \operatorname{div}(\mathcal{G}\bar{P}) & = & \Delta \hat{P}, & \text{in } \mathcal{D}, \\ \frac{\partial \bar{P}}{\partial N} & = & \frac{\partial \hat{P}}{\partial N}, & \text{on } \Gamma, \\ \frac{\partial \bar{P}}{\partial \nu} & = & \frac{\partial \hat{P}}{\partial \nu}, & \text{on } \partial\mathcal{O}, \end{array} \right. \tag{4.13}$$

for $\bar{P} \in W^{1,r}(\mathcal{D})$. This is possible as the matrix g, for any $\Xi, \Omega \in W^{T,R_2}_s$, given by $g^{ij} = (J_Y^T J_Y)_{ij}$, is symmetric and has determinant 1, so that it is positive. Moreover, for $\lambda_i > 0$ the eigenvalues of g, we know that $\Pi_i \lambda_i = 1$ and $\Sigma_i \lambda_i = \Sigma_i g_{ii} = \operatorname{tr} g$, where $0 < \operatorname{tr} g \le CR_2$. It follows that $\inf \lambda_i \ge \frac{1}{(\max \lambda_i)^2} \ge \frac{1}{(\operatorname{tr} g)^2} > \frac{1}{C^2 R_2^2}$ and that uniformly in Ξ, Ω, the system remains elliptic (cf. also [24, p. 309]). Using classical estimates, e.g. [31, Theorem 3.1], and by Poincaré's inequality, it follows that

$$\|\nabla \bar{P}\|_{L^r} \le C \|\nabla \hat{P}\|_{L^r}.$$

We have that $\mathcal{G}\bar{P} - \nabla \hat{P} = \mathcal{H}(\bar{P})$ by construction, where $\nabla \hat{P} = (\mathrm{Id}_{L^r} - \mathbb{P}_{r,\mathcal{D}})\mathcal{G}\bar{P}$ and $\mathcal{H}(\bar{P}) = \mathbb{P}_{r,\mathcal{D}}(\mathcal{G}\bar{P})$, see (4.3). We have $\sup_{i,j} \|g^{ij} - \delta_{ij}\|_{\infty,\infty} \le K_{R_2+K_*} T$, where here and in the following, $K_{R_2+K_*}$ denotes some polynomial of the constant $K_{R_2+K_*}$ introduced in Lemma 3.4. It follows that

$$\begin{array}{rcl} \|\mathcal{H}(\bar{P})\|_{L^s(L^r)} & = & \mathbb{P}_{r,\mathcal{D}}(\mathcal{G}\bar{P} - \nabla \bar{P}) \\ & \le & K_{R_2+K_*} T \|\nabla \bar{P}\|_{L^s(L^r)} \\ & \le & K_{R_2+K_*} T (R_1 + K_* + R_2). \end{array}$$

(2) From Lemma 3.4, it follows that the coefficients in $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are bounded by $K_{R_2+K_*}$ and continuous, so that $\mathbf{F}_0, \mathbf{F}_1$ and $\mathbf{F}_2$ can be calculated from $\bar{U}, \bar{P}, \bar{\Xi}$ and $\bar{\Omega}$. We use that by Proposition 3.2,

$$X^T_{s,r,0} \hookrightarrow H^{1/2,s}(W^{1,r}) \hookrightarrow L^{\bar{s}}(W^{1,r})$$

with $\bar{s} = \frac{2s}{2-s}$, to get

$$\|U + \nabla A_{\Xi,\Omega}\|_{L^s(W^{1,r})} \le CT^{1/2} \|U + \nabla A_{\Xi,\Omega}\|_{L^{\bar{s}}(W^{1,r})} \le CT^{1/2}(R_1 + R_2)$$

for $C > 0$ a constant independent of $0 < T < T_0$. This allows us to get

$$\begin{aligned}
\|\mathbf{F}_0\|_{L^s(L^r)} &\leq \|\mathcal{F}_0 - f_0\|_{L^s(L^r)} + \mu\alpha\|(\mathcal{L}-\Delta)\bar{U}\|_{L^s(L^r)} \\
&\quad + \|\mathcal{M}(\bar{U})\|_{L^s(L^r)} + \|\mathcal{N}(\bar{U})\|_{L^s(L^r)} + \|\mathcal{H}(\bar{P})\|_{L^s(L^r)} \\
&\leq \sup_{i,j}\|\partial_i Y_j - \delta_{ij}\|_{L^\infty(L^\infty)}\|f_0\|_{L^s(L^r)} \\
&\quad + \alpha\mu\sup_{i,j}\|g^{ij} - \delta_{ij}\|_{L^\infty(L^\infty)}\|\Delta\bar{U}\|_{L^s(L^r)} + C_{R_2+K_*}\|\bar{U}\|_{L^s(W^{1,r})} \\
&\quad + \|\bar{U}\|_{L^\infty(L^r)}\|\nabla\bar{U}\|_{L^s(L^\infty)} + C_{R_2+K_*}(R_1+R_2+K_*) \\
&\leq K_{R_2+K_*}[TK_* + T(R_1+R_2) + T^\delta(R_1+R_2)] + R_1^2 \\
&=: K_{R_2+K_*}C^T_{R_1,R_2} \\
\|\mathbf{F}_1\|_{L^s} + \|\mathbf{F}_2\|_{L^s} &\leq \|\mathcal{F}_1 - f_1\|_{L^s} + \|\mathcal{F}_2 - f_2\|_{L^s} + \mathrm{m}\|\bar{\Omega}\times\bar{\Xi}\|_{L^s} + |I|\|\bar{\Omega}\|^2, \\
&\leq K_{R_2+K_*}T(K_*+R_2) + C(K_*+R_2)^2 \\
&=: K_{R_2+K_*}L^T_{R_1,R_2},
\end{aligned}$$

see also [19, Lemma 6.6]. We also have that

$$\|\mathrm{div}\,\Sigma\|_{L^s(L^r)} \leq T^{1/s}R_3.$$

(3) From the second step and Proposition 3.1, we get $\mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)), \mathcal{P}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)))$ with the estimate

$$\|\mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)), \mathcal{P}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)))\|_{L^s} \leq K^{Stokes}_{s,r}K_{R_2+K_*}(C^T_{R_1,R_2} + T^{1/s}R_3),$$

and

$$\|\mathcal{J}(\Sigma)\|_{L^s} \leq K_{R_2+K_*}T^{1/s}R_3.$$

By Lemma 4.1 on $\mathcal{R}$, this gives unique

$$\begin{pmatrix}\Phi_\xi \\ \Phi_\omega\end{pmatrix} := \mathcal{R}^{-1}\left(\begin{pmatrix}\mathbf{F}_1 \\ \mathbf{F}_2\end{pmatrix} - \mathcal{J}(\Sigma) - \mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}(\mathbf{F}_0+\mathrm{div}\,\Sigma)), \mathcal{P}(\mathbf{F}_0+\mathrm{div}\,\Sigma)))\right), \tag{4.14}$$

satisfying

$$\begin{aligned}
\|\Phi_\xi\|_{W^T_s} + \|\Phi_\omega\|_{W^T_s} &\leq K_{R_2+K_*}(K^{Stokes}_{s,r}C^T_{R_1,R_2} + L^T_{R_1,R_2} + T^{1/s}R_3) \\
&=: M^T_{R_1,R_2,R_3}.
\end{aligned}$$

We define Φ_u, Φ_p from the last three equations in (4.11), i.e.

$$\begin{aligned}
\Phi_u &:= b_A(\Phi_\xi, \Phi_\omega) + \mathcal{U}(\Phi_\xi, \Phi_\omega) + \mathcal{U}(\mathbf{F}_0 + \mathrm{div}\,\Sigma), \\
\Phi_p &:= \mathcal{P}(\Phi_\xi, \Phi_\omega) + \mathcal{P}(\mathbf{F}_0 + \mathrm{div}\,\Sigma),
\end{aligned}$$

with the estimate

$$\begin{aligned}
\|\Phi_u\|_{X^T_{s,r}} &\leq \|\mathcal{U}(\Phi_\xi,\Phi_\omega)\|_{X^T_{s,r}} + \|\mathcal{U}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)\|_{X^T_{s,r}} + \|b_A(\Phi_\xi,\Phi_\omega)\|_{X^T_{s,r}} \\
&\leq CK^{Stokes}_{s,r}M^T_{R_1,R_2,R_3}, \\
\|\Phi_p\|_{Y^T_{s,r}} &\leq \|\mathcal{P}(\Phi_\xi,\Phi_\omega)\|_{Y^T_{s,r}} + \|\mathcal{P}(\mathbf{F}_0 + \mathrm{div}\,\Sigma)\|_{Y^T_{s,r}} \leq CK^{Stokes}_{s,r}M^T_{R_1,R_2,R_3}.
\end{aligned}$$

(4) In the last step, by Lemma 3.5, we solve the following linearization of (TE),

$$\left\{\begin{aligned}
\lambda_1(\partial_t\Phi_\sigma + ((\dot{Y}+\bar{U})\cdot\nabla)\Phi_\sigma + \mathcal{C}(\bar{U},\bar{\Xi},\bar{\Omega}):\Phi_\sigma) + \Phi_\sigma &= \mathcal{E}(\bar{U}) && \text{in } J_T\times\mathcal{D}, \\
\Phi_\sigma(0) &= \tau_0 && \text{in } \mathcal{D}.
\end{aligned}\right. \tag{4.15}$$

From the definitions (2.6) and (2.7) of X and Y through $\bar{\Xi}, \bar{\Omega}$, we get that

$$\begin{aligned}
\|\mathcal{C}(\bar{U},\bar{\Xi},\bar{\Omega})\|_{L^s(W^{1,r})} &\leq K_{R_2+K_*}\left(\|\bar{U}\|_{L^s(W^{2,r})} + \|\dot{J}_X\|_{L^s(W^{1,r})} + \|\dot{Y}\|_{L^s(W^{1,r})}\right) \\
&\leq K_{R_2+K_*}(R_1 + TR_2 + K_*), \\
\|\mathcal{C}(\bar{U},\bar{\Xi},\bar{\Omega})\|_{L^s(L^r)} &\leq K_{R_2+K_*}\left(\|\bar{U}\|_{L^s(W^{1,r})} + \|\dot{J}_X\|_{L^s(L^r)} + \|\dot{Y}\|_{L^s(L^r)}\right) \\
&\leq K_{R_2+K_*}\left(T^{1/2}R_1 + TR_2 + K_*\right)
\end{aligned}$$

and we have

$$\begin{aligned}\|\mathcal{E}(\bar{U})\|_{L^s(W^{1,r})} &\leq K_{R_2+K_*}R_1,\\ \|\mathcal{E}(\bar{U})\|_{L^s(L^r)} &\leq K_{R_2+K_*}T^{1/2}R_1.\end{aligned}$$

It only remains to consider

$$w := \dot{Y} + U + u^* + \nabla A_{\Xi,\Omega}$$

and show that $w \in L^s(J_T; D(A_r))$. By construction, $w|_\Gamma = w|_{\partial\mathcal{O}} = 0$. We only need to check that $\operatorname{div}\dot{Y} = 0$. First note that $\det J_Y = 1$ by Liouville's Theorem, so

$$0 = \partial_m \det J_Y = \Sigma_{k,l=1}^3 (J_X)_{lk}\partial_l(J_Y)_{km}, \quad m \in \{1,2,3\}.$$

Directly from equation (2.7), it follows that

$$\begin{aligned}\Sigma_{k=1}^3 \partial_k[\dot{Y}(t,X(t,y)]_k &= -\Sigma_{k,m=1}^3 \partial_k[(J_X^{-1})_{km}(t,y)b_m(t,X(t,y))]\\ &= -\Sigma_{k,l,m=1}^3 \partial_l(J_Y)_{km}(J_X)_{lk}b_m - (J_X^{-1})_{km}(J_X)_{lk}\partial_l b_m\\ &= -\operatorname{div} b = 0.\end{aligned}$$

We thus obtain the estimate

$$\|w\|_{X_{s,r}^T} \leq CR_2 + R_1 + K_*.$$

It follows that (4.15) can be solved by Lemma 3.5 and

$$\begin{aligned}\|\Phi_\sigma\|_{L^\infty(W^{1,r})} &\leq (\|\tau_0\|_{W^{1,r}} + \frac{1}{\lambda_1 K_r})\exp(C\|\nabla w\|_{L^1(L^\infty)} + \|\mathcal{C}\|_{L^1(W^{1,r})} + \|\mathbf{F}_3\|_{L^1(W^{1,r})})\\ &\leq (\|\tau_0\|_{W^{1,r}} + \frac{1}{\lambda_1 K_r})\exp(C_1(1+K_{R_2+K_*})T^{1/s}[R_1+R_2+K_*])\\ &=: \Lambda_1 \qquad (4.16)\end{aligned}$$

for some $C_1 > 0$ and

$$\begin{aligned}\|\partial_t\Phi_\sigma\|_{L^s(L^r)} &\leq C_*\Lambda_1(\|w\|_{L^s(W^{1,r})} + \|\mathcal{C}\|_{L^s(L^r)} + \|\mathbf{F}_3\|_{L^s(L^r)} + \frac{T^{1/s}}{\lambda_1 C})\\ &\leq C_*\Lambda_1\left(C_2T^\delta(R_2+R_1+K_*) + \frac{T^{1/s}}{\lambda_1 C}\right) \qquad (4.17)\end{aligned}$$

for some $C_2 > 0$. Note that the constants may depend on T, but only in a way that $C_i \overset{T\to 0}{\to} C_* > 0$ for some $C_* > 0$.

4.3. **Proof of Theorem 1.2.** From the construction of Φ and the estimates in the previous subsection, we get the following.

Lemma 4.2. *The map Φ is well-defined, i.e. $\Phi(\mathbb{V}) \subset \mathbb{V}$, if $R_1, R_2, R_3, R_4 > 0$ and $T > 0$ are chosen suitably.*

Proof. Due to the presence of "quadratic" terms, we must choose $R_1, R_2, T < 1$ and T possibly even smaller, such that $K_* \leq 1$. Considering the estimate (4.16), we set

$$R_3 = \Lambda_1 = C_0\exp(3C_1(1+K_2))$$

in this case. It remains to show that given R_3, we can make R_1, R_2, T even smaller to obtain simultaneously that

$$\begin{aligned}\|\Phi_u\|_{X_{s,r}^T} + \|\Phi_p\|_{Y_{s,r,0}^T} &\leq CM_{R_1,R_2,R_3}^T\\ &=: C_1R + C_1(R_1^2 + R_2^2)\\ &\leq R_1,\end{aligned}$$

and

$$\begin{aligned}\|\Phi_\xi\|_{W_s^T} + \|\Phi_\omega\|_{W_s^T} &\leq CM_{R_1,R_2,R_3}^T\\ &=: C_2R + C_2(R_1^2 + R_2^2)\\ &\leq R_2,\end{aligned}$$

where

$$R = C[T(R_1 + R_2 + R_3 + K_*) + K_*(K_* + R_2)].$$

W.l.o.g., we take $R_2 \geq R_1$ and $C_1 = C_2$. Regardless of R_1, R_2, R_3, we can always choose T so small that $C_1 R < R_1$. Now choose R_2 such that $2C_1 R_2^2 \leq R_1 - C_1 R$, then $C_1(R_2^1 + R_2^2) \leq R_1 - C_1 R \leq R_2 - C_1 R$ and the claim is proved. Setting $R_4 = C_* R_3 \left(C_2 T^\delta (R_2 + R_1 + K_*) + \frac{T^{1/s}}{\lambda_1 C} \right)$ as in the estimate (4.17) proves the lemma. □

In the following, not only when denoting norms, we shortly write $L^s(L^r)$ for $L^s(J_T; L^r(\mathcal{D}))$ and use similar conventions for other vector-valued spaces.

Lemma 4.3. *The set* $\Phi(\mathbb{V})$ *is a convex, bounded and relatively compact subset of* $\mathbb{W}$.

Proof. By linearity of the equations and by Lemma 4.2, we have convexity and boundedness. From [33, Corollary 4], we get that the embeddings

$$\begin{aligned} X_{s,r}^T &\overset{c}{\hookrightarrow} L^s(L_\sigma^r \cap W^{1,r}), \\ Z_{s,r}^T &\overset{c}{\hookrightarrow} C(L^r), \quad \text{and} \\ W_s^T &\overset{c}{\hookrightarrow} C(J_T; \mathbb{R}^3), \end{aligned}$$

are compact. It only remains to show relative compactness for the set of pressures

$$\{\Phi_p\} := \{\Phi_p \in Y_{s,r}^{T,R_1} : (\Phi_u, \Phi_p, \Phi_\xi, \Phi_\omega, \Phi_\sigma) = \Phi(U, P, \Xi, \Omega, \Sigma) \text{ for some } (U, P, \Xi, \Omega, \Sigma) \in \mathbb{V}\}$$

which appear in the image of Φ. By [33, Theorem 3], we must show that $\{\Phi_p\}$ is bounded in $L^1_{loc}(J_T; X)$ for some space $X \overset{c}{\hookrightarrow} L^r(\mathcal{D})$ and that

$$\|d_\delta \Phi_p\|_{L^s(L^r)} \to 0$$

uniformly in $\{\Phi_p\}$ as $\delta \to 0$, where

$$d_\delta f(t) := f(t + \delta) - f(t).$$

By construction, the first property clearly holds and Φ_p satisfies $\Phi_p(t) \in L_0^r(\mathcal{D}) := \{f \in L^r(\mathcal{D}) : \int_\mathcal{D} f = 0\}$. The dual space of L_0^r is $L_0^{r'}$, so that for every $\psi \in L_0^{r'}$, there is some $\varphi \in W^{2,r}(\mathcal{D})$, such that

$$\begin{cases} \Delta\varphi = \psi & \text{in } \mathcal{D}, \\ \frac{\partial\varphi}{\partial N} = 0 & \text{on } \partial\mathcal{O} \cup \Gamma. \end{cases}$$

We split $\Phi_p = \mathcal{P}(\Phi_\xi, \Phi_\omega) + \mathcal{P}(\mathbf{F}_0 + \operatorname{div}\Sigma)$ First, we look at $\mathcal{P}(\Phi_\xi, \Phi_\omega)$. Since $\partial_t b_A(t) - \Delta b_A(t) \in L_\sigma^r$, we have, using the identity (3.3),

$$\begin{aligned} (\mathcal{P}(\Phi_\xi, \Phi_\omega)(t), \psi)_{r,\mathcal{D}} &= (\mathcal{P}(\Phi_\xi, \Phi_\omega)(t), \Delta\varphi)_{r,\mathcal{D}} = -(\nabla\mathcal{P}(\Phi_\xi, \Phi_\omega)(t), \nabla\varphi)_{r,\mathcal{D}} \\ &= ((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\Delta\mathcal{U}(\Phi_\xi, \Phi_\omega)(t), \nabla\varphi)_{r,\mathcal{D}} \\ &= -(\nabla\mathcal{U}(\Phi_\xi, \Phi_\omega)(t), D^2\varphi)_{r,\mathcal{D}} + (\nabla\mathcal{U}(\Phi_\xi, \Phi_\omega)(t)N, \nabla\varphi)_{r,\Gamma} \\ &\quad + (\nabla\mathcal{U}(\Phi_\xi, \Phi_\omega)(t)N, \nabla\varphi)_{r,\partial\mathcal{O}} \qquad (4.18) \\ &\leq \|\mathcal{U}(\Phi_\xi, \Phi_\omega)(t)\|_{W^{1+1/r+\varepsilon,r}} \|\psi\|_{L^{r'}}. \qquad (4.19) \end{aligned}$$

By linearity, we obtain $\|d_\delta \mathcal{P}(\Phi_\xi, \Phi_\omega)\|_{L^s(L^r)} \leq \|d_\delta \mathcal{U}(\Phi_\xi, \Phi_\omega)\|_{L^s(W^{1+1/r+\varepsilon,r})}$. From the embedding (3.4), we obtain that, up to some $\varepsilon > 0$, $\mathcal{U}(\Phi_\xi, \Phi_\omega) \in W^{\alpha,s}(W^{1+1/r+\varepsilon,r})$ for $\alpha = \frac{1}{2}(1 - \frac{1}{r} - \varepsilon)$ and

$$\int_0^T \int_0^T \frac{(\|\mathcal{U}(\Phi_\xi, \Phi_\omega)(t + \delta)\|_{W^{1+1/r+\varepsilon,r}} - \|\mathcal{U}(\Phi_\xi, \Phi_\omega)(t)\|_{W^{1+1/r+\varepsilon,r}})^s}{\delta^{\alpha s + 3}} \, \mathrm{d}\delta \, \mathrm{d}t \leq C R_1.$$

This implies

$$\|d_\delta \mathcal{P}(\Phi_\xi, \Phi_\omega)\|_{L^s(L^r)} \leq \|\mathcal{U}(\Phi_\xi, \Phi_\omega)\|_{L^s(W^{1+1/r+\varepsilon,r})} \leq \delta^{\alpha s + 3} C R_1 \to 0,$$

uniformly in $\{\mathcal{P}(\Phi_\xi, \Phi_\omega)\}$, cf. also [33, Lemma 5]. Similar arguments work for $\mathcal{P}(\mathbf{F}_0 + \operatorname{div}\Sigma)$. As in (4.19), we get

$$\begin{aligned} (\mathcal{P}(\mathbf{F}_0 + \operatorname{div}\Sigma)(t), \psi)_{r,\mathcal{D}} &= ((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})(\mu\alpha\mathcal{L}\bar{U} - \mathcal{M}\bar{U} - \mathcal{N}(\bar{U}) + \mathcal{F}_0 - f_0), \nabla\varphi)_{r,\mathcal{D}} \\ &\quad + ((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\operatorname{div}\Sigma, \nabla\varphi)_{r,\mathcal{D}}, \end{aligned}$$

where we use that $(\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\mathcal{H}(\bar{P}) = 0$. Setting

$$l_{ikl} = \Sigma_{j=1}^3 (\partial_j g^{jk}) \delta_{li} + 2 g^{jk} \Gamma_{lj}^i$$

and

$$L_{ij} = \Sigma_{k,l,m=1}^3 \partial_k (g^{kl}\Gamma_{jl}^i) + g^{kl}\Gamma_{jl}^m \Gamma_{km}^i,$$

it follows that,

$$\begin{aligned}
((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\mathcal{L}(\bar{U}), \nabla\varphi)_{r,\mathcal{D}} &= \Sigma_{i,j,k,l=1}^3 (g^{kj}(\partial_k\partial_j \bar{U}_i) + l_{ikl}(\partial_k \bar{U}_l) + L_{ij}\bar{U}_j, \partial_i \varphi)_{r,\mathcal{D}} \\
&= -\Sigma_{i,j,k=1}^3 ((\partial_j \bar{U}_i)(\partial_k g^{kj}), \partial_i\varphi)_{r,\mathcal{D}} - ((\partial_j \bar{U}_i) g^{kj}, (\partial_k \partial_i \varphi))_{r,\mathcal{D}} \\
&\quad + (\partial_k \bar{U}_i N_k, (\partial_i \varphi))_{r,\Gamma} + (\partial_k \bar{U}_i \nu_k, \partial_i \varphi)_{r,\partial\mathcal{O}} \\
&\leq K_{R_2+K_*} \|\bar{U}(t)\|_{W^{1+1/r+\varepsilon,r}} \|\psi\|_{L^{r'}},
\end{aligned}$$

and

$$\begin{aligned}
((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\mathcal{M}(\bar{U}), \nabla\varphi)_{r,\mathcal{D}} &= \Sigma_{i,j,k=1}^3 (\dot{Y}_j(\partial_j \bar{U}_i) + [\Gamma_{jk}^i \dot{Y}_k + (\partial_k Y_i)(\partial_j \dot{X}_k)]\bar{U}_j, \partial_i \varphi)_{r,\mathcal{D}} \\
&\leq K_{R_2+K_*} \|\bar{U}(t)\|_{W^{1,r}} \|\psi\|_{L^{r'}}
\end{aligned}$$

and moreover,

$$\begin{aligned}
((\mathrm{Id} - \mathbb{P}_{r,\mathcal{D}})\mathcal{N}(\bar{U}), \nabla\varphi)_{r,\mathcal{D}} &= \Sigma_{i,j,k=1}^3 (\bar{U}_j(\partial_j \bar{U}_i) + \Gamma_{jk}^i \bar{U}_j \bar{U}_k, \partial_i \varphi)_{r,\mathcal{D}} \\
&= -\Sigma_{i,j,k=1}^3 \frac{1}{2} (\bar{U}_j \bar{U}_i, \partial_j \partial_i \varphi)_{r,\mathcal{D}} + (\bar{U}_j N_j \bar{U}_i, \partial_i \varphi)_{r,\Gamma} + (\Gamma_{jk}^i \bar{U}_j \bar{U}_k, \partial_i \varphi)_{r,\mathcal{D}} \\
&\leq K_{R_2+K_*} \|\bar{U}(t)\|_{L^r} \|\bar{U}(t)\|_{L^\infty} \|\psi\|_{L^{r'}} + C_{\Gamma,r}(|\bar{\Xi}| + |\bar{\Omega}|)^2 \|\psi\|_{L^{r'}}.
\end{aligned}$$

Similarly,

$$(\operatorname{div}\Sigma, \nabla\varphi)_{r,\mathcal{D}} \quad \leq \quad \|\Sigma(t)\|_{W^{1/r+\varepsilon,r}} \|\psi\|_{L^{r'}}.$$

From this, it follows that

$$\begin{aligned}
\|d_\delta \Phi_p\|_{L^s(L^r)} &\leq C(\|d_\delta \bar{U}\|_{L^s(W^{1+1/r+\varepsilon,r})} + \|d_\delta(|\bar{\Xi}| + |\bar{\Omega}|)^2\|_{L^s} \\
&\quad + \|d_\delta \Sigma\|_{L^s(W^{1/r+\varepsilon,r})} + \|d_\delta(\mathcal{F}_0 - f_0)\|_{L^s(L^r)}) \\
&\leq C\delta^{\min(\alpha,\beta)} \qquad \text{for } \delta < 1 \\
&\to 0 \qquad \text{uniformly as } \delta \to 0,
\end{aligned}$$

as we have $\|\bar{U}\|_{W^{\alpha,s}(W^{1+1/r+\varepsilon,r})} \leq R_1 + K_* + CR_2$, $\|\bar{\Xi}\|_{W_s^T} + \|\bar{\Omega}\|_{W_s^T} \leq R_2 + K_*$ and

$$\|\Sigma\|_{W^{2\alpha,s}(W^{1/r+\varepsilon,r})} \leq R_3 + R_4$$

from the embedding $Z_{s,r}^T \hookrightarrow W^{2\alpha,s}(W^{1-2\alpha,r})$, $0 < 2\alpha < 1$, cf. ([33, Lemma 7]). Moreover, by assumption, $\|\mathcal{F}_0 - f_0\|_{W^{\beta,s}(L^r)} \leq T(R_2 + K_*)$. □

Lemma 4.4. *The set $\Phi(\mathbb{V})$ is closed in $\mathbb{W}$.*

Proof. The proof works similarly as in [14, Theorem 9.1], where the authors treat pure viscoelastic flow. We make adjustments for the new fixed point variables P, Ξ, Ω. Given a sequence $(\sigma_n)_n \subset Z_{s,r}^{T,R_3,R_4}$ which converges strongly to σ in $C(J_T; L^r(\mathcal{D}))$, we know that there exist a subsequence $(\sigma_k)_k$ of $(\sigma_n)_n$ and $\tilde{\sigma} \in Z_{s,r}^T$ such that

$$\begin{aligned}
\sigma_k &\to \tilde{\sigma}, && \text{weakly}^* \text{ in } L^\infty(W^{1,r}) \text{ and} \\
& && \text{strongly in } C(L^\infty), \\
\partial_t \sigma_k &\to \partial_t \tilde{\sigma}, && \text{weakly in } L^s(L^r) \text{ and} \\
\sigma_k(0) &= \tilde{\sigma}(0),
\end{aligned} \tag{4.20}$$

so

$$\begin{aligned}
\|\tilde{\sigma}\|_{L^\infty(W^{1,r})} &\leq \liminf \ \|\sigma_k\|_{L^\infty(W^{1,r})} \leq R_3, \\
\|\partial_t \tilde{\sigma}\|_{L^s(L^r)} &\leq \liminf \ \|\partial_t \sigma_k\|_{L^s(L^r)} \leq R_4.
\end{aligned}$$

It follows immediately that $\tilde{\sigma} \in Z_{s,r}^{T,R_3,R_4}$ and $\tilde{\sigma} = \sigma$. Similarly, a sequence $(u_n)_n \subset X_{s,r,0}^{T,R_1}$ which converges strongly to u in $L^s(L^r_\sigma \cap W^{1,r})$ has a subsequence $(u_k)_k$ such that

$$\begin{aligned}
u_k &\to \tilde{u}, && \text{weakly* in } L^\infty(L^r) \text{ and} \\
& && \text{weakly in } L^p(L^q_\sigma \cap W^{2,q}) \text{ and} \\
& && \text{strongly in } C(W^{-1,\infty}), \\
\partial_t u_k &\to \partial_t \tilde{u}, && \text{weakly in } L^s(L^r) \text{ and} \\
u_k(0) &= 0,
\end{aligned} \tag{4.21}$$

for some $\tilde{u} \in X^T_{s,r,0}$. It follows that

$$\|\tilde{u}\|_{X^T_{s,r,0}} \leq \liminf \|u_k\|_{X^T_{s,r,0}} \leq R_1,$$

so $\tilde{u} = u \in X^{T,R_1}_{s,r,0}$. Similarly, for $(p_n)_n \subset L^s(W^{1,r})$ strongly converging to $p \in L^s(L^r)$, we obtain

$$p_k \to \tilde{p}, \qquad \text{weakly in } L^s(W^{1,r}) \tag{4.22}$$

for a subsequence $(p_k)_k$ and some $\tilde{p} \in L^s(W^{1,r})$. From $\tilde{p} \leq \liminf \|p_k\|_{L^s(W^{1,r})} \leq R_1$ it follows that $p \in Y^{T,R_1}_{r,s}$. Moreover, for $(\xi_n)_n, (\omega_n)_n \subset W^T_s$, converging in $C(J_T)$ and bounded by R_2, we have subsequences and $\xi, \omega \in W^T_s$, such that

$$\begin{aligned} \xi_k, \omega_k \quad &\to \quad \xi, \omega, \quad \text{weakly in } W^T_s \text{ and} \\ & \qquad\qquad\quad \text{strongly in } C(J_T), \\ \xi_k, \omega_k(0) \quad &= \quad 0, \end{aligned} \tag{4.23}$$

and the bound and the initial value are preserved by the weak limit. □

Lemma 4.5. Φ *is continuous in the topology of* $\mathbb{W}$.

Proof. Let $(U_n, P_n, \Xi_n, \Omega_n, \Sigma_n)_n \subset \mathbb{V}$ be a sequence such that $(U_n, P_n, \Xi_n, \Omega_n, \Sigma_n) \to (U, P, \Xi, \Omega, \Sigma)$ in the topology of $\mathbb{W}$. If follows that there is a subsequence $(U_k, P_k, \Xi_k, \Omega_k, \Sigma_k)_k$ converging as in (4.20), (4.21), (4.22) and (4.23). We must show that

$$\begin{aligned} \Phi(U, P, \Xi, \Omega, \Sigma) \quad &= \quad \lim_{n\to\infty} \Phi(U_n, P_n, \Xi_n, \Omega_n, \Sigma_n) \\ &=: \quad \lim_{n\to\infty} ((\Phi_u)_n, (\Phi_p)_n, (\Phi_\xi)_n, (\Phi_\omega)_n, (\Phi_\sigma)_n) \\ &=: \quad (\Phi_u, \Phi_p, \Phi_\xi, \Phi_\omega, \Phi_\sigma). \end{aligned}$$

We split the proof into four steps which correspond to the steps in defining Φ above.

(1) We directly obtain

$$\bar{U}_k = U_k + u^* + \nabla A_k \to \bar{U} = U + u^* + \nabla A_{\Xi,\Omega}$$

as in (4.21) and

$$\hat{P}_k = P_k + p^* - \partial_t A_k \quad \to \quad \hat{P} = P + p^* - \partial_t A, \qquad \text{weakly in } L^s(W^{1,r}).$$

Here, we have $A_k := A_{\Xi_k,\Omega_k}$ and $(\Xi_k, \Omega_k) \to (\Xi, \Omega)$ weakly in $L^s(J_T)$, so $\partial_t A_k \to \partial_t A$ weakly in $L^s(W^{1,r})$ and $\nabla A_k \to \nabla A_{\Xi,\Omega}$ as in (4.21). We also have

$$\bar{\Xi}_k = \Xi_k + \xi^* \to \bar{\Xi} = \Xi + \xi^*$$

and

$$\bar{\Omega}_k = \Omega_k + \omega^* \to \bar{\Omega} = \Omega + \omega^*$$

as in (4.23). We find $\bar{P}_k$ and $\mathcal{H}(\bar{P}_k)$ from solving (4.13) as in step 1 of the construction of Φ. From $(g^{ij})_k \to g^{ij}$ strongly in $C(J_T; C(\mathcal{D}))$, it follows that

$$\bar{P}_k \to \bar{P}, \text{ weakly in } L^s(W^{1,r})$$

and

$$\mathcal{H}(\bar{P}_k) \to \mathcal{H}(\bar{P}), \text{ weakly in } L^s(L^r_\sigma).$$

(2) From the definition in (4.6), together with the first step and Lemma 3.4, it follows that

$$\begin{aligned} (\mathbf{F}_0(\bar{U}_k, \bar{P}_k, \bar{\Xi}_k, \bar{\Omega}_k)) \quad &\to \quad \mathbf{F}_0(\bar{U}, \bar{P}, \bar{\Xi}, \bar{\Omega}), \text{ weakly in } L^s(L^r), \\ (\mathbf{F}_1(\bar{U}_k, \bar{P}_k, \hat{P}_k, \bar{\Xi}_k, \bar{\Omega}_k)) \quad &\to \quad \mathbf{F}_1(\bar{U}, \bar{P}, \bar{\Xi}, \bar{\Omega}), \text{ weakly in } L^s, \\ (\mathbf{F}_2(\bar{U}_k, \bar{P}_k, \hat{P}_k, \bar{\Xi}_k, \bar{\Omega}_k)) \quad &\to \quad \mathbf{F}_2(\bar{U}, \bar{P}, \bar{\Xi}, \bar{\Omega}), \text{ weakly in } L^s, \end{aligned}$$

We have $\operatorname{div} \Sigma_k \to \operatorname{div} \Sigma$ weakly in $L^s(L^r)$, so that

$$\mathcal{U}_n := \mathcal{U}((\mathbf{F}_0)_n + \operatorname{div} \Sigma_n) \to \mathcal{U}(\mathbf{F}_0 + \operatorname{div} \Sigma)$$

as in (4.21) and

$$\mathcal{P}_n := \mathcal{P}((\mathbf{F}_0)_n + \operatorname{div} \Sigma_n) \to \mathcal{P}(\mathbf{F}_0 + \operatorname{div} \Sigma)$$

as in (4.22) by the uniqueness of solutions of the linear Stokes problem.

(3) By [33, Lemma 7], $(\mathcal{U}_n)_n \subset X^{T,R_1}_{s,r,0} \overset{c}{\hookrightarrow} L^s(W^{1+1/r+\varepsilon,r})$ for all $0 < \varepsilon < 1 - 1/r$, so that we have a subsequence $(\mathcal{U}_k)_k$ which converges strongly in $L^s(W^{1+1/r+\varepsilon,r})$. From the boundedness of $\mathcal{J}$, see (4.10), we obtain $\mathcal{J}(D_{\mu,\alpha}(\mathcal{U}_k)) \to \mathcal{J}(D_{\mu,\alpha}(\mathcal{U}))$ strongly in $L^s(J_T)$. Since we have a subsequence $(\mathcal{P}_k)_k$ of $(\mathcal{P}_n)_n$ converging strongly in $L^s(L^r)$ and weakly in $L^s(W^{1,r})$, we get strong convergence of $\mathcal{J}(\mathcal{P}_k)$ to $\mathcal{J}(\mathcal{P})$ in $L^s(J_T)$. Similarly, from $\Sigma_n \to \Sigma$ weakly in $L^s(W^{1,r})$, we obtain $\mathcal{J}(\Sigma_k) \to \mathcal{J}(\Sigma)$ strongly in L^s for some subsequence. We can thus take limits in the definition (4.14) of Φ_ξ, Φ_ω to obtain

$$\begin{pmatrix} (\Phi_\xi)_n \\ (\Phi_\omega)_n \end{pmatrix} = \mathcal{R}^{-1}\left[\begin{pmatrix} (\mathbf{F}_1)_n \\ (\mathbf{F}_2)_n \end{pmatrix} - \mathcal{J}(\Sigma_n) - \mathcal{J}(\mathbf{T}(D_{\mu,\alpha}(\mathcal{U}_n), \mathcal{P}_n))\right] \to \begin{pmatrix} \Phi_\xi \\ \Phi_\omega \end{pmatrix}$$

strongly in $W^{1,s}(J_T;\mathbb{R}^6)$. We see that therefore also there are subsequences

$$(\Phi_u)_k = \mathcal{U}_k + b_h((\Phi_\xi)_k, (\Phi_\omega)_k) + \mathcal{U}((\Phi_\xi)_k, (\Phi_\omega)_k) \to \Phi_u$$

and

$$(\Phi_p)_k = \mathcal{P}_k + \mathcal{P}((\Phi_\xi)_k, (\Phi_\omega)_k) \to \Phi_p$$

as in (4.20) and (4.22), so that continuity of the components Φ_u, Φ_p, Φ_ξ and Φ_ω of Φ follows.

(4) In the transport equation, we can also take the weak limit in $L^s(L^r)$ to get that $\lim_{n\to\infty}(\Phi_\sigma))_n$ as well as Φ_σ satisfy

$$\begin{cases} \lambda_1(\partial_t \Phi_\sigma + ((\dot{Y} + \bar{U}) \cdot \nabla)\Phi_\sigma + \mathcal{C}(\bar{U}, \bar{\Xi}, \bar{\Omega}) : \Phi_\sigma) + \Phi_\sigma = \mathcal{E}(\bar{U}) & \text{in } J_T \times \mathcal{D}, \\ \Phi_\sigma(0) = \tau_0 & \text{in } \mathcal{D}, \end{cases}$$

where $\bar{U}, \bar{\Xi}, \bar{\Omega}$ are given as in step 1 and $\dot{Y}, \mathcal{C}, \mathcal{E}$ are constructed from $\bar{\Xi}, \bar{\Omega}$. By uniqueness, they thus coincide.

□

We have shown that $\mathbb{V}$ is a convex, compact subset of $\mathbb{W}$ and that Φ is continuos in the topology of $\mathbb{W}$. By Schauder's fixed point theorem, it follows that Φ has a fixed point $u, p, \xi, \omega, \sigma \in \mathbb{V}$ solving (4.1) and (TE) and that we can find $\bar{u}, \bar{p}, \bar{\xi}, \bar{\omega}$ correspondingly, so that (NP) and (TE) are solved. By construction, the backward change of coordinates yields a solution v, q, η, θ, τ of (1.1), proving the existence claim in Theorem 1.2.

5. Proof of Uniqueness

In order to show uniqueness of the strong solution, we apply, in the usual way, Gronwall's Lemma to an energy estimate for the difference of two solutions. As the fluid domain is unknown a priori, we work with the quasilinear equations (NP) and (TE) to obtain this estimate.

Given $v_0, \theta_0, \eta_0, \tau_0, f_0, f_1, f_2$ as in Theorem 1.2, let $v^i, q^i, \eta^i, \theta^i, \tau^i$, $i \in \{1,2\}$ be two solutions of (1.1). We choose a fixed χ as a cut-off function in (2.4) for both solutions and then define $u^i, p^i, \xi^i, \omega^i, \sigma^i$, $i \in \{1,2\}$ as in (2.9). This gives two strong solutions of (NP) and (TE), such that their difference satisfies

$$\begin{aligned} u &:= u^1 - u^2 && \in X^T_{s,r}, \\ p &:= p^1 - p^2 && \in Y^T_{s,r}, \\ \xi &:= \xi^1 - \xi^2 && \in W^T_s, \\ \omega &:= \omega^1 - \omega^2 && \in W^T_s, \\ \sigma &:= \sigma^1 - \sigma^2 && \in Z^T_{s,r}, \end{aligned}$$

and the equations

$$(5.1)\quad \begin{cases} \partial_t u + \mathcal{M}^1 u + \mathcal{M} u^2 - \mu\alpha(\mathcal{L}^1 u + \mathcal{L} u^2) - \operatorname{div}\sigma + \mathcal{N}^1(u) + \mathcal{N}(u^2) + \mathcal{G}^1 p + \mathcal{G} p^2 = \mathcal{F}^1_0 - \mathcal{F}^2_0, \\ \operatorname{div} u = 0, \\ u|_\Gamma - \omega \times \cdot - \xi = 0, \\ u|_{\partial\mathcal{O}} = 0, \\ u(0) = 0, \\ \mathrm{m}\xi' + \mathrm{m}(\omega^1 \times \xi + \omega \times \xi^2) + \int_\Gamma \mathbf{T}(D_{\mu,\alpha}(u) + \sigma, p) N \,\mathrm{d}\Gamma = \mathcal{F}^1_1 - \mathcal{F}^2_1, \\ I\omega' + \omega^1 \times I\omega + \omega \times I\omega^2 + \int_\Gamma y \times \mathbf{T}(D_{\mu,\alpha}(u) + \sigma, p) N \,\mathrm{d}\Gamma = \mathcal{F}^1_2 - \mathcal{F}^2_2, \\ \xi(0) = 0, \\ \omega(0) = 0, \\ \lambda_1(\partial_t \sigma + ((\dot{Y}^1 + u^1) \cdot \nabla)\sigma + ((\dot{Y} + u) \cdot \nabla)\sigma^2 + \mathcal{C}^1 : \sigma + \mathcal{C} : \sigma^2) + \sigma = \mathcal{E}^1 u + \mathcal{E} u^2, \\ \sigma(0) = 0, \end{cases}$$

on their respective domains. The operators $\mathcal{M}^2, \mathcal{L}^2, \ldots$ are given by ξ^2, ω^2 and $\mathcal{M} = \mathcal{M}^1 - \mathcal{M}^2, \mathcal{L} = \mathcal{L}^1 - \mathcal{L}^2, \ldots$ Multiplying the i-th component of the first line by $g^1_{ij}u_j$ and integrating over the sum gives

$$\sum_{i,j,k,l=1}^{3} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathcal{D}} u_i g^1_{ij} u_j\Big) + \mu\alpha \int_{\mathcal{D}} (g^1)^{lk} g^1_{ij} (\partial_k u_i)(\partial_l u_j)$$

$$(5.2) \qquad = \sum_{i,j,k,l=1}^{3} \int_{\mathcal{D}} l^1_{ijk}(\partial_i u_k)u_j + \int_{\Gamma} u_i \mathbf{T}_{ij}(D_{\mu,\alpha}(u) + \sigma, p)N_j - \int_{\mathcal{D}} g^1_{ij}\sigma_{ik}\partial_k u_j + \int_{\mathcal{D}} r,$$

where

$$l^1_{ijk} = -\mu\alpha[(g^1)^{il}(\partial_l g^1_{kj} - 2(\Gamma^1)^l_{ki} g^1_{lj})],$$

and where r satisfies, a.e. in J_T,

$$\|r\|_{L^1(\mathcal{D})} \leq C(|\xi^\pi|, |\omega^\pi|)\big[1 + \|u^\pi\|_{W^{2,2}} + \|u^\pi\|_{L^2}\|u^\pi\|_{W^{1,3}} + \|\nabla p^2\|_{L^2} + \|f_0\|_{L^2}\big] \cdot (\|u\|^2_{L^2} + |\xi|^2 + |\omega|^2 + \|\sigma\|^2_{L^2}), \qquad \pi \in \{1, 2\},$$

where we have used Lemmas 3.3 and 3.4 and chosen $C(|\xi^\pi|, |\omega^\pi|) > 0$ such that it bounds a polynomial of order 8 in $|\xi^\pi|, |\omega^\pi|$. Note that the matrices $(g^1)^{lk}$ and (g^1_{ij}) are bounded and symmetric positive definite uniformly in $J_T \times \mathcal{D}$ (cf. the argument in step 1 in the construction of Φ), so that there are constants $c_1, C_1 > 0$, such that

$$c_1\|\nabla u\|_{L^2} \leq \mu\alpha \int_{\mathcal{D}} (g^1)^{lk} g^1_{ij}(\partial_k u_i)(\partial_l u_j) \leq C_1\|\nabla u\|_{L^2}$$

and thus the first term on the right-hand-side of (5.2) can be absorbed. Multiplying the sixth and seventh line of (5.1) by ξ and ω, respectively, gives

$$-\Sigma^3_{i,j=1}\int_{\Gamma} u_i \mathbf{T}_{ij}(D_{\mu,\alpha}(u) + \sigma, p)N_j = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{m}|\xi|^2 + \omega^T I\omega) + \mathrm{m}\xi^T(\omega \times \xi^2) + \omega^T(\omega^1 \times I\omega)$$

$$(5.3) \qquad + \xi^T Q^T f_1 + \omega^T Q^T f_2.$$

Moreover, multiplying the i, j-component of the tenth line of (5.1) by $\bar{\mu}(g^1_{ik})(g^1_{jl})\sigma_{kl}$, where $\bar{\mu} := \frac{1}{2\mu(1-\alpha)}$, gives

$$\int_{\mathcal{D}} g^1_{ij}\sigma_{ik}\partial_k u_j = \bar{\mu}\Big(\int_{\mathcal{D}} (\mathcal{E}^1 u)_{ij}(g^1_{ij})(g^1_{kl})\sigma_{kl} - \int_{\mathcal{D}} g^1_{jl}(\Gamma^1)^j_{kn} u_n \sigma_{kl}\Big)$$

$$(5.4) \qquad = \lambda_1\bar{\mu}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathcal{D}} \sigma_{ij}\sigma_{kl}(g^1_{ik})(g^1_{jl})\Big) + \int_{\mathcal{D}} \bar{r},$$

where $\bar{r}$ is a function such that a.e. in J_T,

$$\|\bar{r}\|_{L^1(\mathcal{D})} \leq \frac{c_1}{2}\|\nabla u\|^2_{L^2} + C(|\xi^\pi|, |\omega^\pi|)\Sigma^2_{\pi=1}(1 + \|u^\pi\|_{W^{2,3}} + \|\sigma^2\|_{W^{1,3}} + \|\sigma^2\|^2_{W^{1,3}} + \|\sigma^2\|_{L^\infty}\|u^\pi\|_{W^{1,2}}) \cdot(\|u\|^2_{L^2} + |\xi|^2 + |\omega|^2 + \|\sigma\|^2_{L^2}).$$

Again, the constant $C(|\xi^\pi|, |\omega^\pi|) > 0$ bounds a polynomial in $|\xi^\pi|, |\omega^\pi|$ of order 16. To obtain this estimate, we have used that, for example,

$$\int_{\mathcal{D}} (\dot{Y} + u)_m(\partial_m \sigma^2_{ij})(g^1_{ik})(g^1_{jl})\sigma_{kl} \leq C_{\mathcal{D}}\|g^1\|^2_{L^\infty}\|\nabla\sigma^2\|_{L^3}\|\sigma\|_{L^2}(\|(b^{(Y)})^1 - (b^{(Y)})^2\|_{W^{1,2}} + \|u\|_{W^{1,2}})$$

$$\leq \frac{c_1}{4}\|\nabla u\|_{L^2} + C_{\mathcal{D}}\frac{4}{c_1}\|g^1\|^4_{L^\infty}\|\sigma^2\|^2_{W^{1,3}}\|\sigma\|^2_{L^2} + C_{\mathcal{D}}(|\xi^\pi|, |\omega^\pi|)\|\sigma^2\|_{W^{1,3}}(\|u\|^2_{L^2} + |\xi|^2 + |\omega|^2 + \|\sigma\|^2_{L^2}),$$

where we note that the elementary estimate on $b^{(Y)}$ is shown in detail in [19, Lemmas 6.2-6.4]. Combining (5.2), (5.3) and (5.4), we obtain

$$\sum_{i,j,k,l=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}} u_i g^1_{ij} u_j + \mathrm{m}|\xi|^2 + \omega^T I \omega + \lambda_1 \bar{\mu} \int_{\mathcal{D}} \sigma_{ij} g^1_{ik} g^2_{jl} \sigma_{kl}\right) + \frac{c_1}{4}\|\nabla u\|^2_{L^2}$$
$$\leq C(\|u\|^2_{L^2} + |\xi|^2 + |\omega|^2 + \|\sigma\|^2_{L^2})$$
$$\leq \frac{C}{c_1} \sum_{i,j,k,l=1}^{3} \left(\int_{\mathcal{D}} u_i g^1_{ij} u_j + \mathrm{m}|\xi|^2 + \omega^T I \omega + \lambda_1 \bar{\mu} \int_{\mathcal{D}} \sigma_{ij} g^1_{ik} g^2_{jl} \sigma_{kl}\right).$$

As $u(0) \equiv 0$, $\xi(0) = \omega(0) = 0$ and $\sigma(0) \equiv 0$ by definition, Gronwall's Lemma implies that $u^1 \equiv u^2$, $\xi^1 \equiv \xi^2$, $\omega^1 \equiv \omega^2$ and $\sigma^1 \equiv \sigma^2$ on J_T and moreover that $p^1 \equiv p^2$ in the space $Y^T_{s,r,0}$. This shows that the corresponding backward changes of coordinates coincide, so that also the solution to the original problem (1.1) is unique.

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