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Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes

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Abstract. A general class of nonconforming meshes has been recently studied for stationary anisotropic heterogeneous diffusion problems, see [11]. Thanks to the basic ideas developed in [11] for stationary problems, we derive the new discretization scheme (4.16)–(4.17) in order to approximate the nonstationary heat problem (1.1)–(1.3). The unknowns of this scheme are the values at the centre of the control volumes, at some internal interfaces, and at the mesh points of the time discretization.
Although the numerical scheme stems from the finite volume methods, its formulation seems a discrete version for the weak formulation defined by (2.1) and (1.4) for the heat problem.
The main result is Theorem 4.1 which summarizes the obtained results of this work. We derive error estimates (4.33)–(4.35) in discrete norms $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$ and $\mathcal{W}^{1,\infty}(0,T;L^2(\Omega))$, and error estimate for an approximation for the gradient, in a general framework in which the discrete bilinear form involved in (4.16) and given by (4.29) is satisfying ellipticity (4.28). We prove in particular, see (4.36), when the discrete flux is calculated using a stabilized discrete gradient, the convergence order is $h_\mathcal{D}+k$, where $h_\mathcal{D}$ (resp. k) is the mesh size of the spatial (resp. time) discretization. This estimate is valid under the regularity assumption $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$ for the exact solution u.
These error estimates are useful because they allow us to get error estimates for the approximtions of the exact solution and its first derivatives.
Results of Theorem 4.1 have been obtained thanks to a comparison between the solution of scheme (4.33)–(4.35) and the auxiliary solution of (4.61) and to the use of the proof of [11, Theorem 4.8, Page 1033] with some special attention to determine the dependence of the constants which appear in the estimates on the exact solution.

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1. Aim of this paper and description of the main results

Let us consider the following heat problem:

$$u_t(x,t) - \Delta u(x,t) = f(x,t),\ (x,t) \in \Omega \times (0,T), \tag{1.1}$$

where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star = \mathbb{N} \setminus \{0\}$, $T > 0$, and f is a given function.
An initial condition is given by:

$$u(x,0) = u^0(x),\ x \in \Omega, \tag{1.2}$$

and, for the sake of simplicity, we consider homogeneous Dirichlet boundary conditions, that is

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

where, we denote by $\partial\Omega = \overline{\Omega} \setminus \Omega$ the boundary of Ω.
Heat equation (1.1) is typically used in different applications, such as fluid mechanics, heat and mass transfer, etc, and it is the prototypical parabolic partial differential equation which in turn arises, for instance, in many different models like Navier–Stokes and reaction–diffusions systems. It describes the distribution of heat (or variation in temperature) in a given region over time. Therefore parabolic equations are important from the mathematical viewpoint as well as in practice. For this reason, many works have been devoted to the numerical approximation of parabolic equations, see for instance [15, Chapter IV, Pages 837–868], [18], [16, Pages 331–341], [3, 4, 2, 1], the recently works [7, 8] which are devoted to *finite volume element methods*, and references therein.
The present paper is a continuation for our previous contributions [3, 4] which have been devoted

to error estimates for parablic equations on the so called admissible meshes given in [15], and it is an extended work of our recent notes [2, 1] in which we stated some particular cases of the present paper.

The first aim of the present work is to derive a discretization scheme approximating the nonstationary heat problem (1.1)–(1.3) using the new general class of spatial meshes which is introduced recently in [11] to approximate stationary problems. The second aim is to provide and to prove error estimates of our discretization scheme in possible different norms.

The general class of nonconforming multidimensional meshes introduced recently in [11] has the following advantages:

- The scheme can be applied on any type of grid: conforming or non conforming, 2D and 3D, or more, made with control volumes which are only assumed to be polyhedral (the boundary of each control volume is a finite union of subsets of hyperplanes).
- When the family of the discrete fluxes are satisfying some suitable conditions, the matrices of the generated linear systems are sparse, symmetric, positive and definite.
- A discrete gradient for the exact solution is formulated and converges to the gradient of the exact solution.

Thanks to the basic ideas of the finite volume scheme developed in [11] to approximate stationay problems, we first shall derive the new finite volume scheme (4.16)–(4.17) in order to approximate problem (1.1)–(1.3), see Section 4. The first equation of the finite volume scheme, i.e. (4.16), is a discrete version for the weak formulation (2.1) of the heat equation (1.1) (with, of course, the boundary condition (1.3)). Whereas, the discrete initial condition (4.17) of scheme (4.16)–(4.17) is a discrete version of the weak formulation for the orthogonal projection

$$a(u(0), v) = -\left(\Delta u^0, v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in H_0^1(\Omega), \tag{1.4}$$

where

$$a(w, v) = \int_\Omega \nabla w(x) \cdot \nabla v(x) dx. \tag{1.5}$$

This choice is useful as explained in Remark 6.

Although, the scheme (4.16)–(4.17) stems from the finite volume ideas developed these last years (Would say, integration over the control volumes and then we approximate the fluxes arising after integration by parts by some suitable numerical ones.), its formulation seems a discrete version for the weak formulation (2.1) and (1.4)–(1.5). From this point of view, the scheme (4.16)–(4.17) presented in this work looks like a *nonconforming finite element scheme* for the heat problem (1.1)–(1.3).

Thanks to the properties satisfied by the scheme presented in [11], the scheme we present, that is (4.16)–(4.17), also has the following advantages

- The scheme can be applied on any type of spatial grid: conforming or non conforming, 2D and 3D, or more, made with control volumes which are only assumed to be polyhedral (the boundary of each control volume is a finite union of subsets of hyperplanes).
- For each time level n, the scheme results in a linear system (4.16) with a number of unknowns being equal to card$(\mathcal{M})$ + card$(\mathcal{H})$, the sum of the number of control volumes and the cardinality of a certain subset of the set of edges of the mesh equations. So, the present scheme (4.16)–(4.17) has less unknowns than that presented in [1]

- When the discrete fluxes are satisfying some suitable conditions, the matrices generated by the scheme (4.16)–(4.17) are sparse, symmetric, positive and definite.
- For each level $n \in [\![0, N+1]\!]$, the finite volume solution of (4.16)–(4.17) converges to $u(\cdot, t_n)$ in the $\mathbb{L}^2(\Omega)$–norm, see first and fourth item of Remark 5.
- Using the discrete gradient provided in [11] for the stationary case, suitable discrete derivatives of the finite volume solution of (4.16)–(4.17) can be formulated in order to approximate spatial first derivatives of the exact solution of problem (1.1)–(1.3), see second and fourth item of Remark 5.
- A discrete time derivative is formulated in order to approximate the time derivative of the exact solution of (1.1)–(1.3), see third and fourth item of Remark 5.

The convergence analysis of the finite volume scheme (4.16)–(4.17), see Theorem 4.1, is provided in several discrete norms, namely in those which allow us to get error estimates for the approximation of the exact solution of (1.1)–(1.3) and its first derivatives. We derive error estimates (4.33)–(4.35) in discrete norms $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$ and $\mathcal{W}^{1,\infty}(0,T;L^2(\Omega))$, and error estimate for an approximation for the gradient, in a general framework in which the discrete bilinear form involved in the first equation (4.16) of the discretization scheme (4.16)–(4.17) and given by (4.29) is satisfying ellipticity condition (4.28). We prove in particular, see (4.36), when the discrete flux is given by (4.24)–(4.27), that the conve! rgence order is $h_{\mathcal{D}} + k$, where $h_{\mathcal{D}}$ (resp. k) is the mesh size of the spatial (resp. time) discretization. This estimate is valid under the regularity assumption $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$ for the exact solution u.
The proof of Theorem 4.1 is based on the comparison between the solution of scheme (4.16)–(4.17) and the *new auxiliary solution* defined by (4.61). As a first principal part of the proof of Theorem 4.1, we prove Lemma 4.5 and as a second principal part, we prove Lemma 4.6. The technical Lemma 4.7 will help us to conclude the proof of Theorem 4.1. Lemmata 4.1–4.4 ara some preliminary technical tools which are used in the proof of Lemmata 4.5 and 4.6 and Theorem 4.1. Lemma 4.5 provides us with some estimates on the error between the solution of (4.61) and the exact solution of (1.1)–(1.3), and its proof is based on the proof of [11, Theorem 4.8, Page 1033] with some special attention to determine the dependence of the constants, which appear in the estimates, on the exact solution. Lemma 4.6 provides us with some estimates on the error between the auxiliary solution of (4.61) and the finite volume solution of (4.16)–(4.17). So, the proof of Theorem 4.1 can be done by gathering results of Lemmata 4.5, 4.6, 4.7, and the triangle inequality.

The organization of this paper is as follows: in the second section, we state the weak formulation of the continuous problem and we recall some functional spaces which will be used throughout this paper. Third section is devoted to recall the definition of general nonconforming meshes as well as some discrete spaces given in [11]. In the fourth section, we derive and present the finite volume (4.16)–(4.17) and the main result of our paper, namely Theorem 4.1. The proof of Theorem 4.1 is performed thanks to Lemmata 4.5, 4.6, and 4.7. Among the tools used to prove Lemmata 4.5 and 4.6, we used some Lemmata and results from [11]. In fact, Lemma 4.1 (resp. 4.2) is the subject of [11, (4.6), Page 1026] (resp. [11, Lemma 4.2, Page 1026]), and we recall them here for the sake of completness. Whereas, Lemmata 4.3 and 4.4 are the subject of [11, Lemma 4.4, Page 1029] and [11, (4.20), Page 1031] in which the constants in estimates [11, (4.13), Lemma 4.4, Page 1029] and [11, (4.20), Page 1031] are depending on the function under consideration φ, whereas the constants which appear in estimate (4.40) of Lemma 4.3 and in estimate (4.52) of Lemma 4.4 are independent of the function under consideration φ. Writing Lemmata 4.3 and 4.4 in which the constants are independent of the function under consideration φ has at least two roles:

- The application of Lemmata 4.3 and 4.4 serves as to get constants independent of the exact solution in the error estimates, whereas a straightforward application of [11, Lemma 4.4, Page 1029] and [11, (4.20), Page 1031] leads to constants, which appear in error estimates, depending on $u(.,t_n)$ and consequently we obtain constants depending on the parameters of the time discretization.
- The required regularity in Lemmata 4.3 and 4.4 is $\varphi \in \mathcal{C}^2(\overline{\Omega})$. This regularity assumption together with the regularity assumptions in Lemmata 4.5, 4.6, and 4.7 yields the regularity assumption $u \in \mathcal{C}^2\left([0,T];\mathcal{C}(\overline{\Omega})\right)$ in Theorem 4.1 on the exact solution u of problem (1.1)–(1.3). So, we expect that this regularity may be weakened to $\mathcal{W}^{2,\infty}(0,T;H^2(\Omega))$, see Remark 2.

So, some efforts have been devoted in order to determine the dependence of the constants which appear in the estimates of [11, Theorem 4.8, Page 1033] on the exact solution.
Finally, fifth section is devoted to provide interesting tasks not resolved in this work and to work on in the future.

2. Weak problem and preliminaries

The following Theorem, provided in [10], gives a sense for a weak solution for problem (1.1)–(1.3) (recall that $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$), see also [6, Theorem X. 1, Page 205], [6, Theorem X. 1, Page 207], and [6, Theorem X. 9, Page 218] for more information:

Theorem 2.1. *(cf. [10, Theorems 3 and 4, Pages 356–358]) Let $f \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$ and $u_0 \in \mathbb{L}^2(\Omega)$. Then, there exists a unique weak solution for (1.1)–(1.3) in the following sense: there exists a function $u \in \mathbb{L}^2(0,T;H_0^1(\Omega))$ such that $u_t \in \mathbb{L}^2(0,T;H^{-1}(\Omega))$ and:*

(i) *For a.e. $0 \le t \le T$*

$$\langle u_t, v\rangle + \int_\Omega \nabla u(x,t)\cdot\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx, \text{ for } \forall v \in H_0^1(\Omega) \tag{2.1}$$

(ii)

$$u(0) = u_0. \tag{2.2}$$

The convergence of the finite volume scheme we want to present is analyzed using the space $\mathcal{C}^m\left([0,T];\ \mathcal{C}^l(\overline{\Omega})\right)$, where m and l are integers, of m–times continuously differentiable mappings of the interval $[0,T]$ with values in $\mathcal{C}^l(\overline{\Omega})$, see [16, Pages 47–48]. The space $\mathcal{C}^m\left([0,T];\ \mathcal{C}^l(\overline{\Omega})\right)$ is equipped with the norm

$$\|u\|_{\mathcal{C}^m\left([0,T];\ \mathcal{C}^l(\overline{\Omega})\right)} = \max_{j\in[\![1,m]\!]}\left\{\sup_{t\in[0,T]}\|\frac{d^j u}{dt^j}(t)\|_{\mathcal{C}^l(\overline{\Omega})}\right\}, \tag{2.3}$$

where $\|\cdot\|_{\mathcal{C}^l(\overline{\Omega})}$ denotes the usual norm of $\mathcal{C}^l(\overline{\Omega})$.

3. Meshes and discrete spaces

This paper deals with a finite volume scheme approximating (1.1)–(1.3) on a general class of nonconforming meshes which include the admissible mesh of [15, Definition 9.1, Page 762]. This general class of meshes is introduced in [11]. An example of two neighboring control volumes K and L is depicted in Figure 1. For the sake of completness, we recall the general finite volumes mesh given in [11].

Definition 3.1. *(Definition of a large class of finite volume grids, cf.* [11, Definition 2.1, Page 1012]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. A discretisation of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

1. *$\mathcal{M}$ is a finite family of non empty connected open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K; let $\mathrm{m}\,(K) > 0$ denote the measure of K and h_K denote the diameter of K.*
2. *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non empty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$–dimensional measure is strictly positive. We also assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{K;\ \sigma \in \mathcal{E}_K\}$. We then assume that, for any $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\,\Omega$ (the set of these interfaces, called boundary interfaces, denoted by $\mathcal{E}_{\mathrm{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, denoted by $\mathcal{E}_{\mathrm{int}}$). For all $\sigma \in \mathcal{E}$, we denote by x_σ the barycentre of σ. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by! $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K.*
3. *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K \in \mathcal{M}}$, such that for all $K \in \mathcal{M}$, $x_K \in K$ and K is assumed to be x_K–star-shaped, which means that for all $x \in K$, the property $[x_K, x] \subset K$ holds. Denoting by $d_{K,\sigma}$ the Euclidean distance between x_K and the hyperplane including σ, one assumes that $d_{K,\sigma} > 0$. We then denote by $\mathcal{D}_{K,\sigma}$ the cone with vertex x_K and basis σ.*

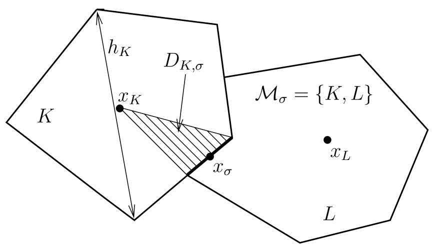

Figure 1. Notations for two neighboring control volumes in the case $d = 2$

Remark 1. *(Some properties of the mesh) It is useful to mention the difference between the admissible mesh considered in* [15, Definition 9.1, Page 762] *and the mesh considered in Definition 3.1. The class of meshes considered in 3.1 is larger than that considered in* [15, Definition 9.1, Page 762] *for the following reasons:*

- *the control volumes of the class of meshes in Definition 3.1 are not necessarily convex subsets of Ω, whereas the control volumes of the class of meshes in* [15, Definition 9.1, Page 762] *are convex polyogonal subsets of Ω.*
- *the class of meshes in Definition 3.1 does not satisfy the orthogonality property (iv) satisfied by the meshes considered in* [15, Definition 9.1, Page 762].

The discretization of Ω is then performed using the mesh $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ described in Definition 3.1, whereas the time discretization is performed with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in [\![0, N+1]\!]$.
For our need, we use the discrete spaces and their norms of the following Definition:

Definition 3.2. *(Discrete spaces and norms, cf.* [11]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$ and $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Throughout this paper we use the following spaces and norms:*

- *the space $\mathcal{X}_\mathcal{D}$*

$$\mathcal{X}_\mathcal{D} = \left\{ v = \left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}} \right);\ v_K \in \mathbb{R}, v_\sigma \in \mathbb{R} \right\}. \tag{3.1}$$

The space $\mathcal{X}_\mathcal{D}$ is equipped with the following semi–norm:

$$|\, v|_{\mathcal{X}}^2 = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}} (v_\sigma - v_K)^2. \tag{3.2}$$

- *the space $\mathcal{X}_{\mathcal{D},0}$*

$$\mathcal{X}_{\mathcal{D},0} = \left\{ v = \left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}} \right) \in \mathcal{X}_\mathcal{D};\ v_\sigma = 0,\, \forall \sigma \in \mathcal{E}_{\mathrm{ext}} \right\}. \tag{3.3}$$

The semi–norm $|\cdot|_\mathcal{X}$ given by (3.2) is a norm on the subspace $\mathcal{X}_{\mathcal{D},0}$ of $\mathcal{X}_\mathcal{D}$.

- *for a given family of real numbers $\{\beta_\sigma^K;\ K \in \mathcal{M}, \sigma \in \mathcal{E}_{\mathrm{int}}\}$, with $\beta_\sigma^K \neq 0$ only for some control volumes which are "close" to σ, and such that*

$$1 = \sum_{K\in\mathcal{M}} \beta_\sigma^K \quad \text{and} \quad x_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K\, x_K, \tag{3.4}$$

we define a space with dimension smaller than that of $\mathcal{X}_{\mathcal{D},0}$. This can be achieved by expressing u_σ, for all $\sigma \in \mathcal{B}$, where $\mathcal{B} \subset \mathcal{E}_{\mathrm{int}}$ as a consistent barycentric combination of the values u_K:

$$u_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K\, u_K. \tag{3.5}$$

We decompose then the set $\mathcal{E}_{\mathrm{int}}$ of interfaces into two non intersecting subsets, that is: $\mathcal{E}_{\mathrm{int}} = \mathcal{B} \cup \mathcal{H}$ and $\mathcal{H} = \mathcal{E}_{\mathrm{int}} \backslash \mathcal{B}$. The interface unknowns associated with $\mathcal{B}$ will be computed by using the barycentric formula (3.5). The unknowns of the scheme (see (4.16)–(4.17)) will be *then the quantities u_K for $K \in \mathcal{M}$ and u_σ for $\sigma \in \mathcal{H}$. Consider then the space $\mathcal{X}_{\mathcal{D},\mathcal{B}} \subset \mathcal{X}_{\mathcal{D},0}$ given by*

$$\mathcal{X}_{\mathcal{D},\mathcal{B}} = \{v \in \mathcal{X}_{\mathcal{D},0} \text{ such that } v_\sigma \text{ satisfying (3.5)}, \ \forall \sigma \in \mathcal{B}\}\,. \tag{3.6}$$

The semi–norm $|\cdot|_{\mathcal{X}}$ given by (3.2) is a norm on the subspace $\mathcal{X}_{\mathcal{D},\mathcal{B}}$ of $\mathcal{X}_{\mathcal{D},0}$.

- *the subspace $H_{\mathcal{M}}(\Omega)$ of $\mathbb{L}^2(\Omega)$ defined by the function which are constant on each control volume $K \in \mathcal{M}$. We then denote, for all $v \in H_{\mathcal{M}}(\Omega)$ and for all $\sigma \in \mathcal{E}_{\text{int}}$ with $\mathcal{M}_\sigma = \{K, L\}$, $D_\sigma\, v = |\, v_K - v_L|$ and $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, and for all $\sigma \in \mathcal{E}_{\text{ext}}$ with $\mathcal{M}_\sigma = \{K\}$, we denote $D_\sigma\, v = |\, v_K|$ and $d_\sigma = d_{K,\sigma}$. We then define the following norm:*

$$\forall\, v \in H_{\mathcal{M}}(\Omega), \ \|\, v\|^2_{1,2,\mathcal{M}} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_{K,\sigma} \left(\frac{D_\sigma\, v}{d_\sigma}\right)^2 = \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma) \frac{(\, D_\sigma\, v)^2}{d_\sigma}. \tag{3.7}$$

We also need the following interpolation operators:

Definition 3.3. *(Interpolation operators, cf.* [11]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$ and $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Throughout this paper we use the following interpolation operators:*

- *For all $v \in \mathcal{X}_{\mathcal{D}}$, we denote by $\Pi_{\mathcal{M}}\, v \in H_{\mathcal{M}}(\Omega)$ the piecewise constant function from Ω to $\mathbb{R}$ defined by $\Pi_{\mathcal{M}}\, v(x) = v_K$, for a.e. $x \in K$, for all $K \in \mathcal{M}$.*
- *For all $\varphi \in \mathcal{C}(\Omega)$, we denote by $\mathcal{P}_{\mathcal{D}}\varphi \in \mathcal{X}_{\mathcal{D}}$ the element defined by $\big((\varphi(x_K))_{K\in\mathcal{M}}\,, (\varphi(x_\sigma))_{\sigma\in\mathcal{E}}\big)$.*
- *For all $\varphi \in \mathcal{C}(\Omega)$, we denote by $\mathcal{P}_{\mathcal{D},\mathcal{B}}\varphi \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ the element $v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ such that*

$$v_K = \varphi(x_K), \ \forall K \in \mathcal{M}, \tag{3.8}$$

$$v_\sigma = 0, \ \forall \sigma \in \mathcal{E}_{\text{ext}}, \tag{3.9}$$

$$v_\sigma = \sum_{K\in\mathcal{M}} \beta^K_\sigma \varphi(x_K), \ \forall \sigma \in \mathcal{B}, \tag{3.10}$$

and

$$v_\sigma = \varphi(x_\sigma), \ \forall \sigma \in \mathcal{H}. \tag{3.11}$$

- *For all $\varphi \in \mathcal{C}(\Omega)$, we denote by $\mathcal{P}_{\mathcal{M}}\varphi \in H_{\mathcal{M}}(\Omega)$ the element defined by $\mathcal{P}_{\mathcal{M}}\varphi(x) = \varphi(x_K)$, for a.e. $x \in K$, for all $K \in \mathcal{M}$.*

In order to analyze the convergence, we need to consider the size of discretization $\mathcal{D}$, see [11, (4.1), Page 1025]

$$h_{\mathcal{D}} = \sup\{\mathrm{diam(K)};\ K \in \mathcal{M}\}, \tag{3.12}$$

and the regularity of the mesh is given by, see see [11, (4.2), Page 1025]

(3.13) $$\theta_{\mathcal{D}} = \max\left(\max_{\sigma\in\mathcal{E}_{\rm int}, K,L\in\mathcal{M}} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} \right).$$

For a given set $\mathcal{B} \subset \mathcal{E}_{\rm int}$ and for a given family $\left(\beta_\sigma^K\right)_{K\in\mathcal{M},\sigma\in\mathcal{E}_{\rm int}}$ satisfying property (3.4), we introduce some measure of the resulting regularity with

(3.14) $$\theta_{\mathcal{D},\mathcal{B}} = \max\left(\theta_{\mathcal{D}}, \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K\cap\mathcal{B}} \frac{\sum_{L\in\mathcal{M}} |\beta_\sigma^L|\,|x_\sigma - x_L|^2}{h_K^2} \right).$$

4. The discretization scheme and statement of the main result

The scheme we want to consider is to find an approximation for (1.1)–(1.3) by setting up systems of equations for a family of values $\left(\,(u_K^n)_{K\in\mathcal{M}}\,,(u_\sigma^n)_{\sigma\in\mathcal{E}}\right)$ in the control volumes and on the interfaces. Following the idea of finite volume method, we first integrate equation (1.1) over each control volume K and on each interval (t_n, t_{n+1}), and then we use an integration by parts to get (recall that $\mathbf{n}_{K,\sigma}$ is the unit vector normal to σ outward to K)

(4.1) $$\int_{t_n}^{t_{n+1}}\int_K u_t(x,t)dx\,dt - \sum_{\sigma\in\mathcal{E}_K}\int_{t_n}^{t_{n+1}}\int_\sigma \nabla\, u(x,t)\cdot\mathbf{n}_{K,\sigma}(x)d\gamma(x)\,dt = \int_{t_n}^{t_{n+1}}\int_K f(x,t)d\,x\,dt,$$

which gives

(4.2) $$\begin{aligned}\int_K (\,u(x,t_{n+1}) - u(x,t_n))\;dx \quad &- \quad \sum_{\sigma\in\mathcal{E}_K}\int_{t_n}^{t_{n+1}}\int_\sigma \nabla\, u(x,t)\cdot\mathbf{n}_{K,\sigma}(x)d\gamma(x)\,dt \\ &= \quad \int_{t_n}^{t_{n+1}}\int_K f(x,t)d\,x\,dt.\end{aligned}$$

The left hand side of the previous equation is the sum of two terms. We will then approximate these two terms.

- The first term $\int_K (\,u(x,t_{n+1}) - u(x,t_n))\;dx$ can be approximated using a zero order quadrature by
$$\mathrm{m}(K)\frac{u(x_K,t_{n+1}) - u(x_K,t_n)}{k}.$$
- For each $n \in [\![0, N+1]\!]$, the flux $-\displaystyle\int_{t_n}^{t_{n+1}}\int_\sigma \nabla\, u(x,t)\cdot\mathbf{n}_{K,\sigma}(x)d\gamma(x)\,dt$ is approximated by a function $kF_{K,\sigma}\left(u^{n+1}\right)$ of the values $\left(\,\left(u_K^{n+1}\right)_{K\in\mathcal{M}}\,,\left(u_\sigma^{n+1}\right)_{\sigma\in\mathcal{E}}\right)$ at the "centers" and the interfaces of the control volumes (in all particular cases, $F_{K,\sigma}\left(u^{n+1}\right)$ only depends on u_K^{n+1} and $\left(u_{\sigma'}^{n+1}\right)_{\sigma'\in\mathcal{E}_K}$), thus the the proposed scheme is implicit in time. The numerical flux $F_{K,\sigma}\left(u^{n+1}\right)$ satisfies the following conservativity:

(4.3) $$F_{K,\sigma}\left(u^{n+1}\right) + F_{L,\sigma}\left(u^{n+1}\right) = 0,\ \forall\,\sigma\in\mathcal{E}_{\rm int}\ \text{ such that }\ \mathcal{M}_\sigma = \{K,L\}.$$

Therefore, a discrete equation corresponding to (4.2) can be written as

$$\mathrm{m}(K)\partial^1 u_K^{n+1} + \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\left(u^{n+1}\right) = \mathrm{m}(K) f_K^n, \tag{4.4}$$

where $\partial^1 v^n$ denotes the value

$$\partial^1 v^n = \frac{v^n - v^{n-1}}{k}. \tag{4.5}$$

and

$$f_K^n = \frac{1}{k\mathrm{m}(K)} \int_{t_n}^{t_{n+1}} \int_K f(x,t) d\, x\, dt. \tag{4.6}$$

A discrete problem for (1.1) is then defined by

$$\mathrm{m}(K)\partial^1 u_K^{n+1} + \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\left(u^{n+1}\right) = \mathrm{m}(K) f_K^n v_K \quad \forall K \in \mathcal{M}. \tag{4.7}$$

The discretization of initial condition (1.2) is perfomed as:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\left(u^0\right) = -\int_K \Delta u^0(x)dx,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \tag{4.8}$$

The Dirichlet boundary condition (1.3) can be approximated as

$$u_\sigma^n = 0,\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}}. \tag{4.9}$$

Equation (4.7) could be written in some weak formulation; multiplying, for any $v \in \mathcal{X}_{\mathcal{D},0}$, both sides of (4.7) by the value v_K of v on the control volume, and summing over $K \in \mathcal{M}$ to get

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)\partial^1 u_K^{n+1} v_K + \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\left(u^{n+1}\right) v_K = \sum_{K\in\mathcal{M}} \mathrm{m}(K) f_K^n v_K. \tag{4.10}$$

Using (4.3), (4.7) yields the following discrete weak formulation: for any $n \in [\![0, N]\!]$, find $u^n \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)\partial^1 u_K^{n+1} v_K + \langle\, u^{n+1}, v\rangle_F = \sum_{K\in\mathcal{M}} \mathrm{m}(K) f_K^n v_K,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{4.11}$$

where

$$\langle\, w, v\rangle_F = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\left(w\right)\left(v_K - v_\sigma\right). \tag{4.12}$$

By the same way, (4.8) can be written in the following discrete weak form:

$$\langle\, u^0, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u^0(x)dx,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \tag{4.13}$$

It is useful to mention that (4.11) is equivalent to ((4.3),(4.7)); indeed, set $v \in \mathcal{X}_{\mathcal{D},0}$ in (4.11) such that $v_K = 1$ and $v_L = 0$, for all $L \neq K$, and $v_\sigma = 0$, for all $\sigma \in \mathcal{E}$, we get (4.7). Similarily, choosing $v \in \mathcal{X}_{\mathcal{D},0}$ such that $v_K = 0$, for all $K \in \mathcal{M}$, and $v_\sigma = 1$ and $v_\tau = 0$ for any et $\tau \in \mathcal{E}_{\mathrm{int}}$, such that $\mathcal{M}_\sigma = \{K, L\}$, leads to (4.3).
By the same way, we can justify that ((4.3),(4.8)) is equivalent to (4.13). This means that under

the conservativity property (4.3), problem (4.7)–(4.9) is equivalent to problem (4.11)–(4.13).

We may also choose a space with dimension smaller than that of $\mathcal{X}_{\mathcal{D},0}$. This can be achieved by expressing u_σ, for all $\sigma \in \mathcal{E}_{\rm int}$, as the consistent barycentric combination (3.5) of the values u_K, where $\{\beta_\sigma^K;\ K \in \mathcal{M}, \sigma \in \mathcal{E}_{\rm int}\}$ is a family of real numbers, with $\beta_\sigma^K \neq 0$ only for some control volumes which are "close" to σ, and satisfies (3.4).
Hence the new scheme could be written as: for any $n \in [\![0, N]\!]$, find $u^n \in \mathcal{X}_{\mathcal{D},0}$ such that $u_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K u_K$, for all $\sigma \in \mathcal{E}_{\rm int}$

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)\partial^1 u_K^{n+1} v_K + \langle u^{n+1}, v\rangle_F$$
$$= \sum_{K\in\mathcal{M}} \mathrm{m}(K) f_K^n v_K,\ \forall v \in \mathcal{X}_{\mathcal{D},0} \text{ with } v_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K u_K,\ \forall \sigma \in \mathcal{E}_{\rm int}, \tag{4.14}$$

and find $u^0 \in \mathcal{X}_{\mathcal{D},0}$ such that $u_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K u_K$, for all $\sigma \in \mathcal{E}_{\rm int}$

$$\langle u^0, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u^0(x)dx,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \tag{4.15}$$

Let us decompose the set $\mathcal{E}_{\rm int}$ of interfaces into two non intersecting subsets, that is: $\mathcal{E}_{\rm int} = \mathcal{B} \cup \mathcal{H}$ and $\mathcal{H} = \mathcal{E}_{\rm int} \setminus \mathcal{B}$. The interface unknowns associated with $\mathcal{B}$ will be computed by using the barycentric formula (3.5).
In terms of the space $\mathcal{X}_{\mathcal{D},\mathcal{B}}$ given by (3.6), we suggest the following composite scheme, which is based on the ideas of the finite volume approximation of anisotropic diffusion equations considered in [11]: for any $n \in [\![0, N]\!]$, find $u_\mathcal{D}^n \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ such that

$$\left(\partial^1 \Pi_\mathcal{M} u_\mathcal{D}^{n+1}, \Pi_\mathcal{M} v\right)_{\mathbb{L}^2(\Omega)} + \langle u_\mathcal{D}^{n+1}, v\rangle_F = \sum_{K\in\mathcal{M}} \mathrm{m}(K) f_K^n v_K,\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}, \tag{4.16}$$

where f_K^n is given by (4.6), and find $u_\mathcal{D}^0 \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ such that

$$\langle u_\mathcal{D}^0, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u^0(x)dx,\ \forall\, v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}, \tag{4.17}$$

where $(\,\cdot,\cdot)_{\mathbb{L}^2(\Omega)}$ denotes the $\mathbb{L}^2$ inner product, and $\Pi_\mathcal{M} v$, for all $v \in \mathcal{X}_\mathcal{D}$, is the piecewise constant function from Ω to $\mathbb{R}$ defined by $\Pi_\mathcal{M} v(x) = v_K$, for a.e. $x \in K$, for all $K \in \mathcal{M}$, see Definition 3.3.

4.1. **Construction of the numerical flux using the discrete gradient.** We recall here an example of an explicit expression for a numerical flux $F_{K,\sigma}$ given in [11]. This numerical flux is derived using a discrete gradient and can be calculated as follows:

$$\langle u, v\rangle_F = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma) = \int_\Omega \nabla_\mathcal{D} u(x) \cdot \nabla_\mathcal{D} v(x) d\,x,\ \forall u \in \mathcal{X}_\mathcal{D},\ \forall v \in \mathcal{X}_\mathcal{D}. \tag{4.18}$$

Let us consider the discrete gradient given in [11]:

$$\nabla_\mathcal{D} u(x) = \nabla_{K,\sigma} u,\ \text{ a. e. }\ x \in \mathcal{D}_{K,\sigma}, \tag{4.19}$$

where $\mathcal{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u\, \mathbf{n}_{K,\sigma}, \tag{4.20}$$

$$\nabla_K u = \frac{1}{\mathrm{m}(K)} \sum_{\sigma \in \mathcal{E}_K} \mathrm{m}(\sigma)\,(\,u_\sigma - u_K)\,\mathbf{n}_{K,\sigma}, \tag{4.21}$$

and

$$R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}} \left(u_\sigma - u_K - \nabla u_K \cdot (x_\sigma - x_K)\right). \tag{4.22}$$

Let us set

$$\nabla_{K,\sigma} u = \sum_{\sigma' \in \mathcal{E}_K} (\,u_{\sigma'} - u_K)\; y^{\sigma\sigma'}, \tag{4.23}$$

where

$$y^{\sigma\sigma'} = \begin{cases} \frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)}\mathbf{n}_{K,\sigma} + \frac{\sqrt{d}}{d_{K,\sigma}} \left(1 - \frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)}\mathbf{n}_{K,\sigma} \cdot (\,x_\sigma - x_K)\right)\mathbf{n}_{K,\sigma},\ \sigma' = \sigma \\ \dfrac{\mathrm{m}(\sigma')}{\mathrm{m}(K)}\mathbf{n}_{K,\sigma'} - \dfrac{\sqrt{d}}{d_{K,\sigma}\mathrm{m}(K)}\mathrm{m}(\sigma')\mathbf{n}_{K,\sigma'} \cdot (\,x_\sigma - x_K)\,\mathbf{n}_{K,\sigma},\ \sigma' \neq \sigma. \end{cases} \tag{4.24}$$

Therefore, using (4.19) and (4.23)

$$\int_\Omega \nabla_{\mathcal{D}}\, u(x) \cdot \nabla_{\mathcal{D}}\, v(x) d\,x = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \sum_{\sigma' \in \mathcal{E}_K} A^{\sigma\sigma'}(u_\sigma - u_K)(v_{\sigma'} - v_K),\ \forall u \in \mathcal{X}_{\mathcal{D}},\ \forall v \in \mathcal{X}_{\mathcal{D}}, \tag{4.25}$$

where

$$A^{\sigma\sigma'} = \sum_{\sigma'' \in \mathcal{E}_K} \Lambda_{K,\sigma''}\, y^{\sigma''\sigma} \cdot y^{\sigma''\sigma'} \text{ and } \Lambda_{K,\sigma''} = \int_{\mathcal{D}_{K,\sigma''}} \mathcal{I} d\,x. \tag{4.26}$$

The identification, using (4.18) and (4.25), leads to

$$F_{K,\sigma}(u) = \sum_{\sigma' \in \mathcal{E}_K} A^{\sigma\sigma'}(u_K - u_{\sigma'}). \tag{4.27}$$

The convergence of the discretization scheme (4.16)–(4.17) is provided in the following theorem.

Theorem 4.1. *(Error estimates for the finite volume scheme (4.16)–(4.17)) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the weak solution of (1.1)–(1.3) in the sense of Theorem 2.1 satisfies $u \in \mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Let $\mathcal{B} \subset \mathcal{E}_{\mathrm{int}}$ be given and let $\{\beta_\sigma^K;\ \sigma \in \mathcal{B}, K \in \mathcal{M}\}$ be a subset of $\mathbb{R}$ satisfying (3.4). Assume that $\theta_{\mathcal{D},\mathcal{B}}$, given by (3.14), satisfies $\theta \geq \theta_{\mathcal{D},\mathcal{B}}$. Let $(F_{K,\sigma})_{K \in \mathcal{M}, \sigma \in \mathcal{E}}$ be a family of linear mappings from $\mathcal{X}_{\mathcal{D}}$ into $\mathbb{R}$ such that there exists a positive constant α with*

$$\alpha |\, v\,|_{\mathcal{X}}^2 \leq \langle v, v\rangle_F\,,\ \forall\, v \in \mathcal{X}_{\mathcal{D}}, \tag{4.28}$$

where $\langle \cdot, \cdot \rangle_F$ *is defined by (4.12), that is*

$$\langle u, v \rangle_F = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma), \ \forall u, v \in \mathcal{X}_{\mathcal{D}}. \tag{4.29}$$

Then there exists a unique solution $(u_{\mathcal{D}}^n)_{n=0}^{N+1}$ *for problem (4.16)–(4.17).*
For a function $u \in \mathcal{C}^1(\overline{\Omega})$*, we define the following expressions:*

$$\mathcal{R}_{K,\sigma}(u) = F_{K,\sigma}\left(\mathcal{P}_{\mathcal{D},\mathcal{B}}(u)\right) + \int_\sigma \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x), \tag{4.30}$$

and

$$\mathbb{E}_{\mathcal{D}}(u) = \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{\mathrm{m}(\sigma)} \left(\mathcal{R}_{K,\sigma}(u) \right)^2 \right)^{\frac{1}{2}}. \tag{4.31}$$

Let $(u_{\mathcal{D}}^n)_{n=0}^{N+1}$ *be the solution (4.16)–(4.17). For each* $n \in [\![0, N+1]\!]$*, let us define the error* $e_{\mathcal{M}}^n \in H_{\mathcal{M}}(\Omega)$ *by:*

$$e_{\mathcal{M}}^n = \mathcal{P}_{\mathcal{M}} u(\cdot, t_n) - \Pi_{\mathcal{M}} u_{\mathcal{D}}^n. \tag{4.32}$$

Then, the following error estimates hold, for positive constants C_1*,* C_2*, and* C_3 *only depending on* Ω*,* d*,* α*,* θ *and* T

- *discrete* $\mathbb{L}^\infty(0, T; H_0^1(\Omega))$*–estimate: for all* $n \in [\![0, N+1]\!]$

$$\| e_{\mathcal{M}}^n \|_{1,2,\mathcal{M}} \le C_1 \left(\max_{j \in [\![0,2]\!]} \max_{m \in [\![j, N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m) \right) + (h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\, \mathcal{C}^2(\overline{\Omega}))} \right). \tag{4.33}$$

- $\mathcal{W}^{1,\infty}(0, T; \mathbb{L}^2(\Omega))$*–estimate: for all* $n \in [\![1, N+1]\!]$

$$\| \partial^1 e_{\mathcal{M}}^n \|_{\mathbb{L}^2(\Omega)} \le C_2 \left(\max_{j \in [\![1,2]\!]} \max_{m \in [\![j, N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m) \right) + (h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\, \mathcal{C}^2(\overline{\Omega}))} \right). \tag{4.34}$$

- *error estimate in the gradient approximation: for all* $n \in [\![0, N+1]\!]$

$$\| \nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(\cdot, t_n) \|_{\mathbb{L}^2(\Omega)} \le C_3 \left(\max_{j \in [\![0,2]\!]} \max_{m \in [\![j, N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m) \right) + (h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\, \mathcal{C}^2(\overline{\Omega}))} \right). \tag{4.35}$$

Moreover, in the particular case where $(F_{K,\sigma})_{K \in \mathcal{M}, \sigma \in \mathcal{E}}$ *is defined by (4.24)–(4.27), there exists a constent* C_4 *only depending on* θ*,* Ω*, and* d *such that, for all* $j \in [\![0, 2]\!]$

$$\max_{m \in [\![j, N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m) \right) \le C_4\, h_{\mathcal{D}} \| u \|_{\mathcal{C}^2\left([0,T];\, \mathcal{C}^2(\overline{\Omega})\right)}. \tag{4.36}$$

Remark 2. *(Regularity required for the results of Theorem 4.1) It seems that the extension of Theorem 4.1 to* $u \in \mathcal{W}^{2,\infty}\left(0, T;\ H^2(\Omega)\right)$ *can be studied for the case* $d = 2$ *or* $d = 3$ *(see* [11, Remark 4.9, Page 1033]*). Indeed, e.g., in the case when the mesh is admissible in the sense of* [15, Definition 9.1, Pages 762–763] *and* $d = 2$ *or* $d = 3$*, it is maybe possible to show that results of Lemmata 4.5, 4.6 and 4.7 (and then the results of Theorem 4.1 thanks to the triangle inequality) hold when only*

$u \in \mathcal{W}^{2,\infty}\left(0, T;\ H^2(\Omega)\right)$, *and the estimates will be obtained under some condition on the mesh, see* [17, Theorem 3.2, Page 1942].

Remark 3. *(Sufficent conditions on the data to get the required regularity of Theorem 4.1) The required regularity assumption $u \in \mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))$ in Theorem 4.1 can be reached by assuming sufficient regularity for the data u_0, f, and Ω and some compatibility conditions, see for instance* [6, Theorem X. 10, Page 219] *and* [10, Theorem 5, Pages 360–361], *and* [10, Theorem 7, Page 367].

Remark 4. *(A semi–discretization scheme) The present work is devoted to the full discretization scheme (which is the more practical) (4.16)–(4.17), i.e. discretization in time and space, but the analysis presented here can be extended also to a semi–discretization scheme, i.e. discretization only in space.*

Remark 5. *(Some applications of Theorem 4.1) Results of Theorem 4.1 are useful since they allow us to get error estimates for approximations for the first derivatives of the exact solution, of order* $\max_{j \in [\![0,2]\!]} \max_{m \in [\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m)\right) + (h_{\mathcal{D}} + k)\| u\|_{\mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))}$; *indeed*

- *Estimate (4.33) implies that using* [11, (5.10), Lemma 5.4, Page 1038] *and the triangle inequality, for all $n \in [\![0, N+1]\!]$, $\Pi_{\mathcal{M}} u_{\mathcal{D}}^n$ approximates $u(\cdot, t_n)$ by order* $\max_{j \in [\![0,2]\!]} \max_{m \in [\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m)\right) + (h_{\mathcal{D}} + k)\| u\|_{\mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))}$, *in $\mathbb{L}^2(\Omega)$–norm.*
- *Estimate (4.35) implies that, for all $n \in [\![0, N+1]\!]$, the i–th component of the discrete gradient $\nabla_{\mathcal{D}} u_{\mathcal{D}}^n$, defined by (4.19)–(4.22) by replacing u with $u_{\mathcal{D}}^n$, approximates the i–th component of the gradient $\nabla u(\cdot, t_n)$ by order* $\max_{j \in [\![0,2]\!]} \max_{m \in [\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m)\right) + (h_{\mathcal{D}} + k)\| u\|_{\mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))}$, *in $\mathbb{L}^2(\Omega)$–norm.*
- *Estimate (4.34) implies that (using the triangle inequality), for all $n \in [\![0, N]\!]$, $\dfrac{\mathcal{P}_{\mathcal{M}}\, u_{\mathcal{D}}^{n+1} - \mathcal{P}_{\mathcal{M}}\, u_{\mathcal{D}}^n}{k}$ approximates $u_t(\cdot, t_n)$ by order* $\max_{j \in [\![1,2]\!]} \max_{m \in [\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot, t_m)\right) + (h_{\mathcal{D}} + k)\| u\|_{\mathcal{C}^2([0,T];\ \mathcal{C}^2(\overline{\Omega}))}$, *in $\mathbb{L}^2(\Omega)$–norm.*
- *In the particular case where $\left(F_{K,\sigma}\right)_{K \in \mathcal{M}, \sigma \in \mathcal{E}}$ is defined by (4.24)–(4.27), $\Pi_{\mathcal{M}} u_{\mathcal{D}}^n$, the i–th component of the discrete gradient $\nabla_{\mathcal{D}} u_{\mathcal{D}}^n$, and $\dfrac{\mathcal{P}_{\mathcal{M}}\, u_{\mathcal{D}}^{n+1} - \mathcal{P}_{\mathcal{M}}\, u_{\mathcal{D}}^n}{k}$ approximate respectively $u(\cdot, t_n)$, the i–th component of the gradient $\nabla u(\cdot, t_n)$, and $u_t(\cdot, t_n)$ by order $h_{\mathcal{D}} + k$ in $\mathbb{L}^2(\Omega)$–norm.*

Remark 6. *(Discretization (4.17) of the initial condition (1.2)) The choice of the discretization (4.17) of the initial condition (1.2) is useful in the proof of Lemma 4.6, on which the proof of Theorem 4.1 is based. Indeed, the choice (4.17) implies (4.98) below, see (4.96)–(4.98) below. The property (4.98) will allow to obtain (4.128) for the first time step. Error estimates for the finite volume scheme (4.16) with another choice of discretization for the initial condition (1.2) but different from then that of (4.17) could be studied, see Section 5.*

The proof of Theorem 4.1 is performed thanks to several technical lemmata. We will then quote these lemmata and then we prove Theorem 4.1. We begin with the following lemma which is concerned with some interpolatory relations and norm inequalities. Results of Lemma 4.1 are given in [11], and we recall them here for the sake of completness.

Lemma 4.1. *(Some interpolatory relations and norm inequalities, cf.* [11, (4.6), Page 1026]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$ and $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Let $\mathcal{B} \subset \mathcal{E}_{\text{int}}$ be given and let $\{\beta_\sigma^K;\ \sigma \in \mathcal{B}, K \in \mathcal{M}\}$ be a subset of $\mathbb{R}$ satisfying (3.4).*

(1) *Interpolatory relations: Let $\mathcal{P}_\mathcal{M}$, $\mathcal{P}_\mathcal{D}$, and $\mathcal{P}_{\mathcal{D},\mathcal{B}}$ be the interpolatory operators given in Definition 3.3, and $\varphi \in \mathcal{C}(\Omega)$. The following relation holds:*

$$\mathcal{P}_\mathcal{M}\varphi = \Pi_\mathcal{M}\mathcal{P}_\mathcal{D}\varphi = \Pi_\mathcal{M}\mathcal{P}_{\mathcal{D},\mathcal{B}}\varphi. \tag{4.37}$$

(2) *Norm inequalities: let $\|\cdot\|_{1,2,\mathcal{M}}$ and $|\cdot|_\mathcal{X}$ be the norm and the semi norm given in Definition 3.2. Then, the following inequality holds:*

$$\|\Pi_\mathcal{M}\, v\|_{1,2,\mathcal{M}} \le |\, v|_\mathcal{X},\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \tag{4.38}$$

The following lemma, which is the subject of [11, Lemma 4.2, Page 1026], provides us with the equivalence between the norm of the gradient, given in (4.19)–(4.22), and the norm $|\cdot|_\mathcal{X}$, given in (3.2). This lemma is useful since it allows us, for instancce, to get the uniqueness (and then the existence) of the solution $u_\mathcal{D}^n$ of (4.16)–(4.17) when $(F_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}}$ is defined by (4.24)–(4.27), and also to prove error estimate (4.35) of Theorem 4.1, see for example (4.80)–(4.81).

Lemma 4.2. *(Stability property for the discrete gradient, cf.* [11, Lemma 4.2, Page 1026]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$ and $\mathcal{D}$ be a discretisation of Ω in the sense of Definition 3.1, and let $\theta \ge \theta_\mathcal{D}$ be given (where $\theta_\mathcal{D}$ is defined by (3.13)). Then there exists $C_5 \ge 1$ only depending on θ and d such that:*

$$C_5^{-1}|v|_\mathcal{X} \le \|\nabla_\mathcal{D}\, v\|_{\mathbb{L}^2(\Omega)} \le C_5|v|_\mathcal{X},\ \forall v \in \mathcal{X}_\mathcal{D}, \tag{4.39}$$

where $\nabla_\mathcal{D}$ is the discrete gradient given in (4.19)–(4.22).

Lemmata 4.3 and 4.4, given below, provide us, respectively, with error estimate for the gradient approximation and a consistency result. Lemma 4.3 (resp. 4.4) is the subject of [11, Lemma 4.4, Page 1029] (resp. [11, (4.20), Page 1031]) with some slight modification on the r.h.s. (right hand side) of [11, (4.13), Lemma 4.4, 1029] (resp. [11, (4.20), Page 1031]) . Indeed, the constants which appear in [11, (4.13), Lemma 4.4, 1029] and [11, (4.20), Page 1031] are depending on the function φ, so when we apply [11, (4.13), Lemma 4.4, 1029] (resp. [11, (4.20), Page 1031]) directly, for instance, in (4.87)–(4.88) (resp. (4.83)–(4.84)), we get constants depending on $u(x, t_n)$ and then on n, whereas the application of Lemmata 4.3 and 4.4, given below, leads to constants indepndent of disretiz! ation parameters. In addition to this, the application of Lemmata 4.3 and 4.4 below helps us that to see clearly which regularity is required to get results of Theorem 4.1, see also Remark 2.

Lemma 4.3. *(Consistency result for the discrete gradient, see* [11, Lemma 4.4, Page 1029]*) Let $\mathcal{D}$ be a discretisation of Ω in the sense of Definition 3.1, and let $\theta \ge \theta_\mathcal{D}$ be given (where $\theta_\mathcal{D}$ is defined by (3.13)). Then, for any function $\varphi \in \mathcal{C}^2(\overline{\Omega})$, the following estimate holds:*

$$\|\nabla_\mathcal{D}\mathcal{P}_\mathcal{D}\varphi - \nabla\,\varphi\|_{(\mathbb{L}^\infty(\Omega))^d} \le C_6 h_\mathcal{D} \max_{|\alpha|=2} \|\, D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}, \tag{4.40}$$

where $\nabla_\mathcal{D}$ is the discrete gradient given in (4.19)–(4.22) and $C_6 = d^3\,\theta + d^{\frac{7}{2}}\theta^2 + d^{\frac{5}{2}}\theta + 1$.

Proof. Using the triangle inequality, and the definitions (4.20) and (4.22), we get

$$|\nabla_{K,\sigma}\mathcal{P}_{\mathcal{D}}\varphi - \nabla\,\varphi(x_K)| \le |\nabla_K\mathcal{P}_{\mathcal{D}}\varphi - \nabla\,\varphi(x_K)| + |R_{K,\sigma}\mathcal{P}_{\mathcal{D}}\varphi|. \tag{4.41}$$

We then estimate each term on the r.h.s. of the previous inequality; thanks to (4.21) and the Taylor expansion

$$\begin{aligned}\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi &= \frac{1}{\mathrm{m}(K)}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\left(\varphi(x_\sigma)-\varphi\left(x_K\right)\right)\mathbf{n}_{K,\sigma}\\ &\le \frac{1}{\mathrm{m}(K)}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\left(\left(x_\sigma - x_K\right)^T\nabla\varphi(x_K) + d^2h_K^2\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\right)\mathbf{n}_{K,\sigma},\end{aligned} \tag{4.42}$$

where $\left(x_\sigma - x_K\right)^T$ denotes the transpose of $x_\sigma - x_K \in \mathbb{R}^d$.
We use the following geometrical relation, it is the subject of [11, (2.17), Page 1017]:

$$\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\mathbf{n}_{K,\sigma}\left(x_\sigma - x_K\right)^T = \mathrm{m}(K)\mathcal{I},\ \forall K\in\mathcal{M}, \tag{4.43}$$

where $\mathcal{I}$ is the $d\times d$ identity matrix (Recall that $\left(x_\sigma - x_K\right)^T$ is a $1\times d$ matrix and $\mathbf{n}_{K,\sigma}$ is a $d\times 1$ matrix, therefore the product $\mathbf{n}_{K,\sigma}\left(x_\sigma - x_K\right)^t$ is meaningful, namely $\mathbf{n}_{K,\sigma}\left(x_\sigma - x_K\right)^T$ is a $d\times d$ matrix; consequently equality (4.43) makes sense.)
Therefore (4.42) with (4.43), and the definition (3.13) of $\theta_{\mathcal{D}}$, yields that

$$|\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi - \nabla\varphi(x_K)| \le \frac{d^2h_K\theta_{\mathcal{D}}}{\mathrm{m}(K)}\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)d_{K,\sigma}\,. \tag{4.44}$$

Thanks to the assumption that K is x_K-star-shaped, the following property holds, cf. [11, (4.3), Page 1025]

$$\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)d_{K,\sigma} = d\mathrm{m}(K). \tag{4.45}$$

Consequently, (4.44) with (4.45) and the fact that $\theta_{\mathcal{D}}\le\theta$, implies that

$$|\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi - \nabla\varphi(x_K)| \le \max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\,d^3\theta\,h_K. \tag{4.46}$$

Let us move to estimate the second term on the r.h.s. of (4.41); using definition (4.22) combined with (4.42) and (4.46), we get for some values $\rho_{K,\sigma}$ such that $|\rho_{K,\sigma}| \le d^2\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}$:

$$
\begin{aligned}
|\,R_{K,\sigma}\mathcal{P}_{\mathcal{D}}\,\varphi| &= |\frac{\sqrt{d}}{d_{K,\sigma}}\left(\varphi(x_\sigma)-\varphi(x_K)-\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi\cdot(\,x_\sigma-x_K)\right)|\\
&= |\frac{\sqrt{d}}{d_{K,\sigma}}\big((\,x_\sigma-x_K)\cdot\nabla\varphi(x_K)+h_K^2\,\rho_{K,\sigma}-\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi\cdot(x_\sigma-x_K)\big)\,|\\
&= |\frac{\sqrt{d}}{d_{K,\sigma}}\big((\,x_\sigma-x_K)\cdot(\nabla\varphi(x_K)-\nabla_K\mathcal{P}_{\mathcal{D}}\,\varphi)+h_K^2\,\rho_{K,\sigma}\big)\,|\\
&\leq \frac{\sqrt{d}}{d_{K,\sigma}}\left(\theta d_{K,\sigma}\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\,d^3\theta\,h_K+d^2h_K\theta d_{K,\sigma}\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\right)\\
&= \sqrt{d}d^2\theta\,\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\,(\theta d+1)\;h_K, \qquad (4.47)
\end{aligned}
$$

Combining then inequalities (4.41), (4.46), and (4.47), we get

$$
|\nabla_{K,\sigma}\mathcal{P}_{\mathcal{D}}\varphi-\nabla\,\varphi(x_K)|\leq\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\,\sqrt{d}d^2\theta\,\left(\sqrt{d}+\theta d+1\right)\,h_K. \qquad (4.48)
$$

It is easily seen that, since $|\,x_K-x|\leq h_K$, for all $x\in\mathcal{D}_{K,\sigma}$, for all $\sigma\in\mathcal{E}_K$

$$
|\nabla\,\varphi(x_K)-\nabla\,\varphi(x)|_{(\,L^\infty(\mathcal{D}_{K,\sigma}))^d}\leq\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\,h_K. \qquad (4.49)
$$

Using the triangle inequality combined with (4.48)–(4.49), we get, for all $\sigma\in\mathcal{E}_K$, for all $K\in\mathcal{M}$

$$
\|\nabla_{K,\sigma}\mathcal{P}_{\mathcal{D}}\varphi-\nabla\,\varphi(x)\|_{(\,L^\infty(\mathcal{D}_{K,\sigma}))^d}\leq\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\left(d^3\,\theta+d^{\frac{7}{2}}\theta^2+d^{\frac{5}{2}}\theta+1\right)\,h_K. \qquad (4.50)
$$

Which implies, since $h_K\leq h_{\mathcal{D}}$, for all $K\in\mathcal{M}$

$$
\|\nabla_{K,\sigma}\mathcal{P}_{\mathcal{D}}\varphi-\nabla\,\varphi(x)\|_{(\,L^\infty(\Omega))^d}\leq\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\left(d^3\,\theta+d^{\frac{7}{2}}\theta^2+d^{\frac{5}{2}}\theta+1\right)\,h_{\mathcal{D}}. \qquad (4.51)
$$

This concludes the proof of the desired inequality (4.40). □

Lemma 4.4. *(See* [11, (4.20), Page 1031]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d\in\mathbb{N}^\star$ and $\mathcal{D}=(\mathcal{M},\mathcal{E},\mathcal{P})$ be a discretization in the sense of Definition 3.1. Let $\mathcal{B}\subset\mathcal{E}_{\rm int}$ be given and let $\{\beta_\sigma^K;\ \sigma\in\mathcal{B},K\in\mathcal{M}\}$ be a subset of $\mathbb{R}$ satisfying (3.4). Let φ be a function satisfying $\varphi\in\mathcal{C}^2(\overline{\Omega})$. Then, for the following estimate holds:*

$$
|\varphi(x_\sigma)-\varphi_\sigma|\leq d^2\max_{|\alpha|=2}\|\,D^\alpha\varphi\|_{\mathcal{C}(\overline{\Omega})}\theta_{\mathcal{D},\mathcal{B}}\,h_{\mathcal{D}}^2, \qquad (4.52)
$$

where $\theta_{\mathcal{D},\mathcal{B}}$ is given by (3.14) and

$$
\varphi_\sigma=\sum_{L\in\mathcal{M}}\beta_\sigma^L\varphi(x_L). \qquad (4.53)
$$

Proof. Thanks to a Taylor expansion, for $\varphi \in \mathcal{C}^2(\overline{\Omega})$, we have

$$\varphi(x_L) = \varphi(x_\sigma) + \nabla u(x_\sigma) \cdot (x_L - x_\sigma) + \int_0^1 H(\varphi)(tx_\sigma + (1-t)x_L)(x_L - x_\sigma) \cdot (x_L - x_\sigma) dt, \tag{4.54}$$

where $H(\varphi)(z)$ denotes the Hessian matrix of φ at the point z.
This implies, using (3.4)

$$\begin{aligned} \sum_{L\in\mathcal{M}} \beta_\sigma^L \varphi(x_L) &= \sum_{L\in\mathcal{M}} \beta_\sigma^L \varphi(x_\sigma) + \sum_{L\in\mathcal{M}} \beta_\sigma^L \nabla u(x_\sigma) \cdot (x_L - x_\sigma) + \mathcal{L}_\sigma \\ &= \varphi(x_\sigma) + \left(\sum_{L\in\mathcal{M}} \beta_\sigma^L\, x_L - \sum_{L\in\mathcal{M}} \beta_\sigma^L\, x_\sigma \right) \cdot \nabla u(x_\sigma) + \mathcal{L}_\sigma \\ &= \varphi(x_\sigma) + \left(\left(\sum_{L\in\mathcal{M}} \beta_\sigma^L\, x_L \right) - x_\sigma \right) \cdot \nabla u(x_\sigma) + \mathcal{L}_\sigma \\ &= \varphi(x_\sigma) + \mathcal{L}_\sigma \end{aligned} \tag{4.55}$$

where

$$\mathcal{L}_\sigma = \sum_{L\in\mathcal{M}} \beta_\sigma^L \int_0^1 H(\varphi)(tx_\sigma + (1-t)x_L)(x_L - x_\sigma) \cdot (x_L - x_\sigma) dt \tag{4.56}$$

It is easily seen that

$$|\mathcal{L}_\sigma| \le d^2 \max_{|\alpha|=2} \| D^\alpha \varphi\|_{\mathcal{C}(\overline{\Omega})} \sum_{L\in\mathcal{M}} |\beta_\sigma^L|\, x_L - x_\sigma|^2. \tag{4.57}$$

But, using (3.14)

$$\sum_{L\in\mathcal{M}} |\beta_\sigma^L|\, x_L - x_\sigma|^2 \le \theta_{\mathcal{D},\mathcal{B}}\, h_{\mathcal{D}}^2. \tag{4.58}$$

This with (4.57) implies that

$$|\mathcal{L}_\sigma| \le d^2 \max_{|\alpha|=2} \| D^\alpha \varphi\|_{\mathcal{C}(\overline{\Omega})} \theta_{\mathcal{D},\mathcal{B}}\, h_{\mathcal{D}}^2. \tag{4.59}$$

Which gives

$$|\varphi(x_\sigma) - \sum_{L\in\mathcal{M}} \beta_\sigma^L \varphi(x_L)| \le d^2 \max_{|\alpha|=2} \| D^\alpha \varphi\|_{\mathcal{C}(\overline{\Omega})} \theta_{\mathcal{D},\mathcal{B}}\, h_{\mathcal{D}}^2. \tag{4.60}$$

Which completes the proof of Lemma 4.4. □

To analyse the convergence of the finite volume scheme (4.16)–(4.17), we need to use the following auxiliary scheme: for any $n \in [\![0, N+1]\!]$, find $\bar{u}_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ such that

$$\langle\, \bar{u}_{\mathcal{D}}^n, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta\, u(x, t_n) dx, \ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}. \tag{4.61}$$

Note that, taking $n = 0$ in (4.61) with (1.2) leads to

$$(4.62) \qquad \langle \bar{u}^0_{\mathcal{D}}, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta\, u^0(x)dx,\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}},$$

which together with (4.17) implies, when the condition (4.28) is satisfied (and then the uniqueness of the solution of (4.62) holds)

$$(4.63) \qquad \bar{u}^0_{\mathcal{D}} = u^0_{\mathcal{D}},$$

where $u^0_{\mathcal{D}}$ is given by (4.17).
The following lemma concerns the convergence of the auxiliary scheme (4.61). The proof of Lemma 4.5 is based on the use of the proof of [11, Theorem 4.8, Page 1033] with special attention to the constants which appear in the error estimates in the isotropic case.

Lemma 4.5. *(Some error estimates for the auxiliary scheme (4.61), see* [11, Theorem 4.8, Page 1033]*) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega}\setminus\Omega$ its boundary. Assume that the weak solution of (1.1)–(1.3) in the sense of Theorem 2.1 satisfies $u \in \mathcal{C}([0,T];\mathcal{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Let $\mathcal{B} \subset \mathcal{E}_{\text{int}}$ be given and let $\{\beta^K_\sigma;\ \sigma \in \mathcal{B}, K \in \mathcal{M}\}$ be a subset of $\mathbb{R}$ satisfying (3.4). Assume that $\theta_{\mathcal{D},\mathcal{B}}$, given by (3.14), satisfies $\theta \geq \theta_{\mathcal{D},\mathcal{B}}$. Let $(F_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}}$ be a family of linear mappings from $\mathcal{X}_{\mathcal{D}}$ into $\mathbb{R}$ such that (4.28) holds for some positive constant α. For a function $u \in \mathcal{C}^1(\overline{\Omega})$, we define the expressi! ons $\mathcal{R}_{K,\sigma}(u)$ and $\mathbb{E}_{\mathcal{D}}(u)$ given respectively by (4.30) and (4.31) in Theorem 4.1.*
Then for each $n \in [\![0, N+1]\!]$, there exits a unique solution $\bar{u}^n_{\mathcal{D}}$ for the auxiliary scheme (4.61). In addition to this, the following error estimates hold

- *discrete $\mathbb{L}^\infty(0,T;H^1_0(\Omega))$–error estimate: for all $n \in [\![0, N+1]\!]$*

$$(4.64) \qquad \alpha\|\, \mathcal{P}_{\mathcal{M}}\, u(\cdot, t_n) - \Pi_{\mathcal{M}}\, \bar{u}^n_{\mathcal{D}}\|_{1,2,\mathcal{M}} \leq \max_{m\in[\![0,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\, u(\cdot, t_m)\right).$$

- *$\mathcal{W}^{j,\infty}(0,T;\mathbb{L}^2(\Omega))$–error estimate, for all $j \in [\![0, 2]\!]$: for all $n \in [\![j, N+1]\!]$*

$$(4.65) \qquad \alpha\|\, \partial^j\left(\mathcal{P}_{\mathcal{M}}\, u(\cdot, t_n) - \Pi_{\mathcal{M}}\, \bar{u}^n_{\mathcal{D}}\right)\|_{\mathbb{L}^2(\Omega)} \leq C_{\text{ p}} \max_{m\in[\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j\, u(\cdot, t_m)\right),$$

where we have denoted $\partial^0 v^n = v^n$, $\partial^1 v^n$ is given by (4.5), and $\partial^2 v^n = \dfrac{1}{k}\left(\partial^1 v^n - \partial^1 v^{n-1}\right)$, and $C_{\text{ p}}$ is the constant which appears in the Sobolev inequality [11, (5.10), Lemma 5.4, Page 1038]
- *error estimate in the gradient approximation: for a constent C_7 only depending on θ, d, Ω, and α such that, for all $n \in [\![0, N+1]\!]$*

$$(4.66) \qquad \|\, \nabla_{\mathcal{D}}\, \bar{u}^n_{\mathcal{D}} - \nabla\, u(\cdot, t_n)\|_{(\,\mathbb{L}^2(\Omega))^d} \leq C_7\left(\max_{m\in[\![0,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\, u(\cdot, t_m)\right) + h_{\mathcal{D}}\|\, u\|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)}\right).$$

Moreover, in the particular case where $(F_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}}$ is defined by (4.24)–(4.27) and $u \in \mathcal{C}^2\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)$, there exists a constant C_4 only depending on θ, Ω, and d such that, for all $j \in [\![0, 2]\!]$

(4.67) $$\max_{m\in[\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}\left(\partial^j u(\cdot,t_m)\right) \le C_4\, h_{\mathcal{D}} \|u\|_{\mathcal{C}^2\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)}.$$

Proof. Let us first remark that, thanks to the regularity assumption $u \in \mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)$, equation (4.61) is meaningfull.

1. Proof of existence and uniqueness : For each $n \in [\![0, N+1]\!]$, equation (4.61) is equivalent to a linear system of N unknowns, namely $\{(\bar{u}_K^n, \bar{u}_\sigma^n);\ K \in \mathcal{M}, \sigma \in \mathcal{H}\}$, and N equations, where $N = \mathrm{card}(\mathcal{M}) + \mathrm{card}(\mathcal{H})$ (recall that $\mathcal{H} = \mathcal{E}_{\mathrm{int}} \setminus \mathcal{B}$ and $\bar{u}_\sigma = \sum_{K\in\mathcal{M}} \beta_\sigma^K \bar{u}_K$, for all $\sigma \in \mathcal{B}$ where the set $\{\beta_\sigma^K;\ \sigma \in \mathcal{B}, K \in \mathcal{M}\}$ is satisfying (3.4)).

For a fixed $n \in [\![0, N+1]\!]$, assume that the r.h.s. of (4.61) equals to zero, taking $v_{\mathcal{D}} = \bar{u}_{\mathcal{D}}^n$, and using (4.28) yields that $\bar{u}_{\mathcal{D}}^n = 0$. This uniqueness implies the existence.

2. Proof of estimate (4.64) : Using an integration by parts yields that

(4.68) $$-\sum_{K\in\mathcal{M}} v_K \int_K \Delta\, u(x,t_n)dx = -\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} v_K \int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{K,\sigma}(x)d\,\gamma(x).$$

Since $v_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$ and $\int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{K,\sigma}(x)d\,\gamma(x) + \int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{L,\sigma}(x)d\,\gamma(x)$, for all $\sigma \in \mathcal{E}$ such that $\mathcal{M}_\sigma = \{K, L\}$ (it stems from the fact that $\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}$), we have

(4.69) $$-\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} v_\sigma \int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{K,\sigma}(x)d\,\gamma(x) = 0.$$

This with (4.68) leads to

(4.70) $$-\sum_{K\in\mathcal{M}} v_K \int_K \Delta\, u(x,t_n)dx = -\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} (v_K - v_\sigma) \int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{K,\sigma}(x)d\,\gamma(x).$$

Substituting this in (4.61) and multiplying both sides of the resulting equation by -1, we get, for all $n \in [\![0, N+1]\!]$

(4.71) $$-\langle\, \bar{u}_{\mathcal{D}}^n, v\rangle_F = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} (v_K - v_\sigma) \int_\sigma \nabla\, u(x,t_n)\cdot \mathbf{n}_{K,\sigma}(x)d\,\gamma(x),\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}.$$

Adding $\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n))(v_K - v_\sigma)$ to both sides of the previous equality, and using definition (4.29), we get, for all $v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$

(4.72) $$\langle \mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n) - \bar{u}_{\mathcal{D}}^n, v\rangle_F = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \mathcal{R}_{K,\sigma}\left(\, u(\cdot,t_n)\right)\left(\, v_K - v_\sigma\right),$$

where $\mathcal{R}_{K,\sigma}$ is given by (4.30).

Taking $\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n) = v + \bar{u}_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$ (threfore $v = \mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n) - \bar{u}_{\mathcal{D}}^n$) in the previous equality, we get, for all $n \in [\![0, N+1]\!]$

$$(4.73)\qquad \langle v,v\rangle_F=\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\mathcal{R}_{K,\sigma}\left(u(\cdot,t_n)\right)\left(v_K-v_\sigma\right).$$

The previous inequality with the coercivity (4.28), the Cauchy Schwarz inequality and the definitions (3.2) and (4.31) yields to

$$(4.74)\qquad \alpha|\,v|_{\mathcal{X}}\le \mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_n)\right).$$

The previous inequality implies, since $v=\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)-\bar{u}^n_{\mathcal{D}}$

$$(4.75)\qquad \alpha|\,\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)-\bar{u}^n_{\mathcal{D}}|_{\mathcal{X}}\le \mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_n)\right).$$

Using now (4.38), (4.75) implies

$$(4.76)\qquad \alpha\|\,\Pi_{\mathcal{M}}\mathcal{P}_{\mathcal{D},\mathcal{B}}\,u(\cdot,t_n)-\Pi_{\mathcal{M}}\,\bar{u}^n_{\mathcal{D}}\|_{1,2,\mathcal{M}}\le \mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_n)\right).$$

This with (4.37) of Lemma 4.1 yields

$$(4.77)\qquad \alpha\|\mathcal{P}_{\mathcal{M}}\,u(\cdot,t_n)-\Pi_{\mathcal{M}}\,\bar{u}^n_{\mathcal{D}}\|_{1,2,\mathcal{M}}\le \mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_n)\right),$$

which implies the required estimate (4.64).
3. Proof of estimate (4.65) : Estimate (4.77) with the Sobolev inequality [11, (5.10), Lemma 5.4, Page 1038] (by taking $p=2$ in [11, (5.10), Lemma 5.4, Page 1038]) implies, since $\mathcal{P}_{\mathcal{M}}\,u(\cdot,t_n)-\Pi_{\mathcal{M}}\,\bar{u}^n_{\mathcal{D}}\in H_{\mathcal{M}}(\Omega)$ (see Definition 3.3), for all $n\in[\![\,0,N+1]\!]$

$$(4.78)\qquad \alpha\|\,\mathcal{P}_{\mathcal{M}}\,u(\cdot,t_n)-\Pi_{\mathcal{M}}\,\bar{u}^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)}\le C_p\max_{m\in[\![\,0,N+1]\!]}\mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_m)\right),$$

which is the required estimate (4.65) when $j=0$.
Using the definition of ∂^j and (4.61) and the fact that $\langle\,\cdot,\cdot\rangle_F$ is a bilinear form, to deduce that for any $n\in[\![\,j,N+1]\!]$, $\partial^j\bar{u}^n_{\mathcal{D}}\in\mathcal{X}_{\mathcal{D},\mathcal{B}}$ is the solution of the following problem

$$(4.79)\qquad \langle\,\partial^j\bar{u}^n_{\mathcal{D}},v\rangle_F=-\sum_{K\in\mathcal{M}}v_K\int_K\Delta\,\partial^j\,u(x,t_n)dx,\ \forall v\in\mathcal{X}_{\mathcal{D},\mathcal{B}}.$$

Therefore, we can apply estimate (4.78) to get (4.65), for any $j\in\{1,2\}$.
4. Proof of estimate (4.66): Using the triangle inequality to get

$$(4.80)\qquad \begin{aligned}\|\,\nabla_{\mathcal{D}}\mathcal{P}_{\mathcal{D}}\,u(\cdot,t_n)-\nabla_{\mathcal{D}}\,\bar{u}^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)^d}\ &\le\ \|\,\nabla_{\mathcal{D}}\mathcal{P}_{\mathcal{D}}\,u(\cdot,t_n)-\nabla_{\mathcal{D}}\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)\|_{\mathbb{L}^2(\Omega)^d}\\ &+\ \|\,\nabla_{\mathcal{D}}\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)-\nabla_{\mathcal{D}}\,\bar{u}^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)^d}.\end{aligned}$$

The second term on the r.h.s. of the previous inequality can be written as $\|\,\nabla_{\mathcal{D}}\left(\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)-\,\bar{u}^n_{\mathcal{D}}\right)\|_{L^2(\Omega)^d}$; gathering (4.39) of Lemma 4.2 and (4.75) leads to

$$(4.81)\qquad \|\,\nabla_{\mathcal{D}}\mathcal{P}_{\mathcal{D},\mathcal{B}}u(\cdot,t_n)-\nabla_{\mathcal{D}}\,\bar{u}^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)^d}\le\frac{C_5}{\alpha}\,\mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_n)\right).$$

The first term on the r.h.s. of (4.80) could be written as $\| \nabla_{\mathcal{D}} (\mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n)) \|_{L^2(\Omega)^d}$; using then (4.39) of Lemma 4.2 to get

$$\| \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}} u - \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D},\mathcal{B}} u \|_{\mathbb{L}^2(\Omega)^d} \le C_5 \| \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n) \|_{\mathcal{X}}, \tag{4.82}$$

On the other hand, using definition (3.2) of the norm $| \cdot |_{\mathcal{X}}$, we get

$$| \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n) |^2_{\mathcal{X}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{B}} \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}} \left(u(x_\sigma, t_n) - u^n_\sigma \right)^2 . \tag{4.83}$$

Using the fact that $u^n_\sigma = \sum_{L \in \mathcal{M}} \beta^L_\sigma u(x_L, t_n)$ and estimate (4.52) of Lemma 4.4 yields, since $\theta_{\mathcal{D},\mathcal{B}} \le \theta$

$$| \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n) |^2_{\mathcal{X}} \le d^4 \| u \|^2_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \theta^2 h^2_{\mathcal{D}} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{B}} \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}} h^2_K. \tag{4.84}$$

Using (3.13) and (4.45), the previous inequality implies that

$$| \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n) |^2_{\mathcal{X}} \le \| u \|^2_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \theta^4 d^5 \mathrm{m}(\Omega) h^2_{\mathcal{D}}. \tag{4.85}$$

This with (4.82) implies that

$$\| \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D},\mathcal{B}} u(\cdot, t_n) \|_{\mathbb{L}^2(\Omega)^d} \le C_5 \| u \|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \theta^2 d^{\frac{5}{2}} \sqrt{\mathrm{m}(\Omega)} h_{\mathcal{D}}. \tag{4.86}$$

Gathering now (4.80), (4.81), and (4.86) yields that

$$\begin{aligned} \| \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}} u(\cdot, t_n) - \nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}} \|_{\mathbb{L}^2(\Omega)^d} \quad &\le \quad C_5 \| u \|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \theta^2 d^{\frac{5}{2}} \sqrt{\mathrm{m}(\Omega)} h_{\mathcal{D}} \\ &+ \quad \frac{C_5}{\alpha} \mathbb{E}_{\mathcal{D}} \left(u(\cdot, t_n) \right). \end{aligned} \tag{4.87}$$

This with (4.40) of Lemma 4.3 and the triangle inequality implies

$$\begin{aligned} \| \nabla u(\cdot, t_n) - \nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}} \|_{\mathbb{L}^2(\Omega)^d} \quad &\le \quad C_5 \| u \|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \theta^2 d^{\frac{5}{2}} \sqrt{\mathrm{m}(\Omega)} h_{\mathcal{D}} + \frac{C_5}{\alpha} \mathbb{E}_{\mathcal{D}} \left(u(\cdot, t_n) \right) \\ &+ \quad C_6 h_{\mathcal{D}} \| u \|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)} \sqrt{\mathrm{m}(\Omega)}, \end{aligned} \tag{4.88}$$

which leads to (4.66).

5. Proof of estimate (4.67) : Estimate (4.67), when $j = 0$, is given in [11, (4.27), Theorem 4.8, Page 1033] but with a constant depending on u. Thanks to the proof of [11, (4.27), Theorem 4.8, Page 1033] and the previous techniques, we can prove that there exists a constent C_4 only depending on θ, Ω, and d such that, for all $j \in [\![0, 2]\!]$

$$\mathbb{E}_{\mathcal{D}} \left(\partial^j u(\cdot, t_n) \right) \le C_4 \, h_{\mathcal{D}} \| \partial^j u(\cdot, t_n) \|_{\mathcal{C}^2(\overline{\Omega})}. \tag{4.89}$$

On the other hand

$$\|\partial^j u(\cdot,t_n)\|_{\mathcal{C}^2(\overline{\Omega})} = \max_{|\alpha|\le 2}\sup_{x\in\overline{\Omega}} |D^\alpha \partial^j u(x,t_n)|$$

$$(4.90) \qquad = \max_{|\alpha|\le 2}\sup_{x\in\overline{\Omega}} |\partial^j \left(D^\alpha u(x,t_n)\right)|.$$

For $j=0$, the previous inequality leads to

$$(4.91) \qquad \|\partial^j u(\cdot,t_n)\|_{\mathcal{C}^2(\overline{\Omega})} \le \| u\|_{\mathcal{C}\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)}.$$

For $j=1$, we remark that

$$(4.92) \qquad \partial^1 \left(D^\alpha u(x,t_n)\right) = \frac{1}{k}\int_{t_{n-1}}^{t_n} \left(D^\alpha u\right)_t (x,t)dt,$$

which impies that

$$|\partial^1 \left(D^\alpha u(x,t_n)\right)| \le \frac{1}{k}\int_{t_{n-1}}^{t_n} \sup_{x\in\overline{\Omega}} |\left(D^\alpha u\right)_t (x,t)|\, dt$$

$$\le \max_{t\in[0,T]}\sup_{x\in\overline{\Omega}} |\left(D^\alpha u\right)_t (x,t)|$$

$$(4.93) \qquad = \| u\|_{\mathcal{C}^1\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)},\ \forall \alpha\in\mathbb{N}^d \text{ satisfying } |\alpha|\le 2.$$

For $j=2$, we remark that

$$(4.94) \qquad \partial^2 \left(D^\alpha u(x,t_n)\right) = \frac{1}{k^2}\int_{t_{n-1}}^{t_n}\int_{t-h}^{t} \left(D^\alpha u\right)_{tt} (x,t)dsdt,$$

which yield, thanks to the technique used to prove (4.93), that

$$(4.95) \qquad |\partial^2 \left(D^\alpha u(x,t_n)\right)| \le \| u\|_{\mathcal{C}^2\left([0,T];\mathcal{C}^2(\overline{\Omega})\right)},\ \forall \alpha\in\mathbb{N}^d \text{ satisfying } |\alpha|\le 2.$$

Gathering now (4.89)–(4.95) to get the desired estimate (4.67). □

The previous lemma gives error estimates for the auxiliary finite volume approximation $(\bar{u}_{\mathcal{D}}^n)_{n\in[\![0,N+1]\!]}$, given by (4.61). We move now to compare the approximation $(\bar{u}_{\mathcal{D}}^n)_{n\in[\![0,N+1]\!]}$, with the solution $(u_{\mathcal{D}}^n)_{n\in[\![0,N+1]\!]}$ of our finite volume scheme (4.16)–(4.17). For this reason, we set, for all $n\in[\![0,N+1]\!]$

$$(4.96) \qquad \eta_{\mathcal{D}}^n = \bar{u}_{\mathcal{D}}^n - u_{\mathcal{D}}^n.$$

Equality (4.96) means that

$$(4.97) \qquad \eta_K^n = \bar{u}_K^n - u_K^n,\ \forall K\in\mathcal{M} \text{ and } \eta_\sigma^n = \bar{u}_\sigma^n - u_\sigma^n,\ \forall \sigma\in\mathcal{E},$$

where we have denoted $u_{\mathcal{D}}^n = \left((u_K^n)_{K\in\mathcal{M}}, (u_\sigma^n)_{\sigma\in\mathcal{E}}\right)$ and $\bar{u}_{\mathcal{D}}^n = \left((\bar{u}_K^n)_{K\in\mathcal{M}}, (\bar{u}_\sigma^n)_{\sigma\in\mathcal{E}}\right)$.
When (4.28) is satisfied, (4.63) implies that

$$(4.98) \qquad \eta_{\mathcal{D}}^0 = 0.$$

The following lemma provides us with some estimates concerning $\eta_{\mathcal{D}}^n$ given by (4.96). These estimates together with that of the previous lemma will help us to get some estimates for the solution $u_{\mathcal{D}}^n = \big((u_K^n)_{K\in\mathcal{M}}, (u_\sigma^n)_{\sigma\in\mathcal{E}} \big)$ of the scheme (4.16)–(4.17).

Lemma 4.6. *(Some error estimates for $\eta_{\mathcal{D}}^n$ given by (4.96)) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the weak solution of (1.1)–(1.3) in the sense of Theorem 2.1 satisfies $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1. Let $\mathcal{B} \subset \mathcal{E}_{\rm int}$ be given and let $\{\beta_\sigma^K;\ \sigma \in \mathcal{B}, K \in \mathcal{M}\}$ be a subset of $\mathbb{R}$ satisfying (3.4). Assume that $\theta_{\mathcal{D},\mathcal{B}}$, given by (3.14), satisfies $\theta \geq \theta_{\mathcal{D},\mathcal{B}}$. Let $(F_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}}$ be a family of linear mappings from $\mathcal{X}_{\mathcal{D}}$ into $\mathbb{R}$ satisfying (4.28), for some positive constant α, where $\langle \cdot, \cdot \rangle_F$ is define! d by (4.29).*
Then, for each $n \in [\![0, N+1]\!]$, there exists a unique solution $\bar{u}_{\mathcal{D}}^n$, for the auxiliary scheme (4.61), and there exists a unique solution $(u_{\mathcal{D}}^n)_{n\in[\![0,N+1]\!]}$ for the finite volume scheme (4.16)–(4.17). Let $\eta_{\mathcal{D}}^n$, $n \in [\![0, N+1]\!]$, be given by (4.96).
For a function $u \in \mathcal{C}^1(\overline{\Omega})$, we define the expressions $\mathcal{R}_{K,\sigma}(u)$ and $\mathbb{E}_{\mathcal{D}}(u)$ given respectively by (4.30) and (4.31) in Theorem 4.1 and we define the following new expressions, for $j \in \{0,1,2\}$

$$\mathbb{S}_j = \max_{m\in[\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}} \left(\partial^j u(\cdot, t_m) \right), \tag{4.99}$$

where we denote by $\partial^0 v_n = v_n$, $\partial^1 v_n$ is given by (4.5), and $\partial^2 v_n = \dfrac{1}{k}\left(\partial^1 v_n - \partial^1 v_{n-1} \right)$.
Let us consider the following expressions, for all $n \in [\![2, N+1]\!]$

$$\begin{aligned} \mathrm{m}(K)\mathbb{T}_K^n &= \int_K \partial^2 u(x,t_n)\,dx - \frac{1}{k}\int_K \partial^1 \left(\int_{t_n}^{t_n} \Delta\, u(x,t)dt \right) dx \\ &- \mathrm{m}(K)\partial^2 u(x_K,t_n) + \int_K \Delta\partial^1 u(x,t_n)dx, \end{aligned} \tag{4.100}$$

with

$$\mathbb{T}_K^n = 0,\ \forall\, n \in \{0,1\}, \tag{4.101}$$

and, for all $n \in [\![1, N+1]\!]$

$$\begin{aligned} \mathrm{m}(K)\mathbb{K}_K^n &= \int_K \partial^1 u(x,t_n)\,dx - \frac{1}{k}\int_K \int_{t_n}^{t_n} \Delta\, u(x,t)dt\,dx \\ &- \mathrm{m}(K)\partial^1 u(x_K,t_n) + \int_K \Delta u(x,t_n)dx, \end{aligned} \tag{4.102}$$

with

$$\mathbb{K}_K^0 = 0. \tag{4.103}$$

where $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$ is the solution of (1.1)–(1.3).
Set

$$\mathbb{T} = \max_{n\in[\![0,N+1]\!]} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K) \left(\mathbb{T}_K^n\right)^2 \right)^{\frac{1}{2}}, \tag{4.104}$$

and

$$\mathbb{K} = \max_{n\in[\![0,N+1]\!]} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)\, (\mathbb{K}_K^n)^2 \right)^{\frac{1}{2}}. \tag{4.105}$$

Then, the following error estimates hold

- *discrete* $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$*–estimate: for all* $n \in [\![0,N+1]\!]$

$$\alpha\|\Pi_{\mathcal{M}}\eta_{\mathcal{D}}^n\|_{1,2,\mathcal{M}} \leq 2\frac{C_{\mathrm{p}}^2}{\alpha}\mathbb{S}_1 + T\,\frac{C_{\mathrm{p}}^2}{\alpha}\mathbb{S}_2 + 2C_{\mathrm{p}}\mathbb{K} + TC_{\mathrm{p}}\mathbb{T}, \tag{4.106}$$

  where C_{p} *(the letter "* p*" for Poincaré) is the constant which appears in* [11, (5.10), Lemma 5.4, Page 1038] *when* $p = 2$.
- $\mathcal{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$*–estimate: for all* $n \in [\![1,N+1]\!]$

$$\|\,\partial^1\Pi_{\mathcal{M}}\,\eta_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \leq \frac{C_{\mathrm{p}}}{\alpha}\mathbb{S}_1 + T\,\frac{C_{\mathrm{p}}}{\alpha}\mathbb{S}_2 + T\mathbb{T} + \mathbb{K}. \tag{4.107}$$

- *error estimate in the gradient approximation: for all* $n \in [\![0,N+1]\!]$

$$\alpha\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \leq 2C_5\frac{C_{\mathrm{p}}^2}{\alpha}\mathbb{S}_1 + T\,C_5\frac{C_{\mathrm{p}}^2}{\alpha}\mathbb{S}_2 + TC_5C_{\mathrm{p}}\mathbb{T} + 2C_5C_{\mathrm{p}}\mathbb{K}, \tag{4.108}$$

  where C_5 *is the constant which appears in (4.39) of Lemma 4.2.*

Proof.
1. Proof of existence and uniqueness results: The existene and uniqueness of the solution $\bar{u}_{\mathcal{D}}^n$, for each $n \in [\![0,N+1]\!]$, for the auxiliary scheme (4.61) is provided in Lemma 4.5.
To prove the existence and uniqueness of the solution $(u_{\mathcal{D}}^n)_{n\in[\![0,N+1]\!]}$ for the composite scheme (4.16)–(4.17), we set $f_K^n = 0$ and $u_K^n = 0$, and taking $v = u_{\mathcal{D}}^{n+1}$ in (4.16) yields, thanks to (4.28), $u_{\mathcal{D}}^{n+1} = 0$. This yields the uniqueness of the solution $u_{\mathcal{D}}^{n+1}$ for (4.16) for given $\{f_K^n;\ K \in \mathcal{M}\}$ and $u_{\mathcal{D}}^n$. The existence of $u_{\mathcal{D}}^{n+1}$ follows immediately, since (4.16) is a finite dimensional linear system with respect to the unknowns $\{(u_K^n, u_\sigma^n);\ K \in \mathcal{M}, \sigma \in \mathcal{H}\}$ (with as many unknowns as many equations). This with the existence and uniqueness (thanks to Lemma 4.5) of $u_{\mathcal{D}}^0$, implies, successively on n, the existence and uniqueness of $u_{\mathcal{D}}^n$ for all $n \in [\![0,N+1]\!]$.
We first prove (4.107) and then we prove (4.106) and (4.108) respectively.
2. Proof of estimate(4.107): Let us write equation (4.61) in the step n, for all $n \in [\![0,N]\!]$

$$\langle\, \bar{u}_{\mathcal{D}}^{n+1}, v\rangle_F = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta\, u(x,t_{n+1})dx,\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}. \tag{4.109}$$

Subtracting (4.16) from (4.109) and using (4.96)–(4.97), we get, for all $n \in [\![0,N]\!]$

$$\begin{aligned}\left(\partial^1\Pi_{\mathcal{M}}\,\eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{\mathbb{L}^2(\Omega)} + \langle\,\eta_{\mathcal{D}}^{n+1}, v\rangle_F \quad &= \quad -\sum_{K\in\mathcal{M}} \left(\mathrm{m}(K)f_K^n + \int_K \Delta\, u(x,t_{n+1})dx\right) v_K \\ &+ \quad \left(\partial^1\,\Pi_{\mathcal{M}}\,\bar{u}_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{\mathbb{L}^2(\Omega)},\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}.\end{aligned} \tag{4.110}$$

Acting the discrete operator ∂^1 on the both sides of the previous equality, we get, for all $n \in [\![1, N]\!]$

$$\begin{aligned}\left(\partial^2\Pi_{\mathcal{M}}\,\eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{L^2(\Omega)} + \langle \partial^1\eta_{\mathcal{D}}^{n+1}, v\rangle_F &= -\sum_{K\in\mathcal{M}}\left(\mathrm{m}(K)\partial^1 f_K^n + \int_K \Delta\,\partial^1 u(x,t_{n+1})dx\right) v_K\\ &+ \left(\partial^2\,\Pi_{\mathcal{M}}\,\bar{u}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{L^2(\Omega)},\ \forall v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}.\end{aligned} \tag{4.111}$$

Substituting f by $u_t - \Delta u$ (subject of equation (1.1)), and recall that f_K^n is given by (4.6)

$$\begin{aligned}\mathrm{m}(K)\partial^1 f_K^n &= \frac{1}{k}\int_K \partial^1\left(\int_{t_n}^{t_{n+1}} f(x,t)dt\right)dx\\ &= \int_K \partial^2 u(x,t_{n+1})\,dx - \frac{1}{k}\int_K \partial^1\left(\int_{t_n}^{t_{n+1}} \Delta\,u(x,t)dt\right)dx.\end{aligned} \tag{4.112}$$

From (4.112) and (4.100), we write

$$\mathrm{m}(K)\partial^1 f_K^n + \int_K \Delta\partial^1 u(x,t_{n+1})dx = \mathrm{m}(K)\mathbb{T}_K^{n+1} + \mathrm{m}(K)\partial^2 u(x_K,t_{n+1}). \tag{4.113}$$

Inserting this in (4.111) yields that, for all $n \in [\![1, N]\!]$

$$\begin{aligned}\left(\partial^2\,\Pi_{\mathcal{M}}\,\eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{\mathbb{L}^2(\Omega)} &+ \langle \partial^1\eta_{\mathcal{D}}^{n+1}, v\rangle_F = -\left(\partial^2\,\Pi_{\mathcal{M}}\,\xi_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\,v\right)_{\mathbb{L}^2(\Omega)}\\ &- \sum_{K\in\mathcal{M}} \mathrm{m}(K)\mathbb{T}_K^{n+1} v_K,\end{aligned} \tag{4.114}$$

where $\xi_{\mathcal{D}}^n$ is given by, for all $n \in [\![0, N+1]\!]$

$$\xi_{\mathcal{D}}^n = \mathcal{P}_{\mathcal{D}}\,u(\cdot,t_n) - \bar{u}_{\mathcal{D}}^n. \tag{4.115}$$

Taking $v = \partial^1\eta_{\mathcal{D}}^{n+1}$ (this is possible since $\eta_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$, $n \in [\![0, N+1]\!]$, see (4.96)) in (4.114), using (4.28), and the Cauchy Schwarz inequality leads to (recall that $\partial^2\eta_{\mathcal{D}}^{n+1} = \dfrac{\partial^1\eta_{\mathcal{D}}^{n+1} - \partial^1\eta_{\mathcal{D}}^n}{k}$), for all $n \in [\![1, N]\!]$

$$\|\,\partial^1\Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} \le \|\,\partial^1\Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)} + k\|\,\partial^2\,\Pi_{\mathcal{M}}\,\xi_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} + k\mathbb{T}. \tag{4.116}$$

One remarks that

$$\Pi_{\mathcal{M}}\mathcal{P}_{\mathcal{D}}\,\varphi = \mathcal{P}_{\mathcal{M}}\varphi,\ \forall\,\varphi \in \mathcal{C}(\overline{\Omega}), \tag{4.117}$$

one deduces that, using (4.115)

$$\Pi_{\mathcal{M}}\xi_{\mathcal{D}}^n = \mathcal{P}_{\mathcal{M}}\,u(\cdot,t_n) - \Pi_{\mathcal{M}}\bar{u}_{\mathcal{D}}^n, \tag{4.118}$$

and therefore, using (4.65) and (4.99), for all $n \in [\![1, N]\!]$

$$\|\partial^2\Pi_{\mathcal{M}}\xi_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} \le \frac{C_{\mathrm{p}}}{\alpha}\,\mathbb{S}_2, \tag{4.119}$$

where C_{p} is the constant which appears in the Sobolev inequality [11, (5.10), Lemma 5.4, Page 1038] when $p = 2$.
This with (4.116) implies that, for all $n \in [\![1, N]\!]$

$$\|\,\partial^1\Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} - \|\,\partial^1\Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)} \le k\,\frac{C_{\mathrm{p}}}{\alpha}\,\mathbb{S}_2 + k\mathbb{T}. \tag{4.120}$$

One remarks that, for all $n \in [\![1, N]\!]$

$$\| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1} \|_{\mathbb{L}^2(\Omega)} - \| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{1} \|_{\mathbb{L}^2(\Omega)} = \sum_{j=1}^{n} \Big(\| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{j+1} \|_{\mathbb{L}^2(\Omega)} - \| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{j} \|_{\mathbb{L}^2(\Omega)} \Big), \tag{4.121}$$

one deduces using (4.120)

$$\| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1} \|_{\mathbb{L}^2(\Omega)} \leq \| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{1} \|_{\mathbb{L}^2(\Omega)} + kn \frac{C_{\rm P}}{\alpha} \mathbb{S}_2 + kn\mathbb{T}, \tag{4.122}$$

which gives, since $nk \leq T$

$$\| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1} \|_{\mathbb{L}^2(\Omega)} \leq \| \partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{1} \|_{\mathbb{L}^2(\Omega)} + T \frac{C_{\rm P}}{\alpha} \mathbb{S}_2 + T\mathbb{T}. \tag{4.123}$$

Let us estimate the first term on the first term on the r.h.s. of the previous inequality; set $n = 0$ in (4.110) to get, for all $v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$

$$\begin{aligned} \big(\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^1, \Pi_{\mathcal{M}}\, v \big)_{\mathbb{L}^2(\Omega)} + \langle \eta_{\mathcal{D}}^1, v \rangle_F &= - \sum_{K \in \mathcal{M}} \left(\mathrm{m}(K) f_K^0 + \int_K \Delta\, u(x, t_1) dx \right) v_K \\ &+ \big(\partial^1\, \Pi_{\mathcal{M}}\, \bar{u}_{\mathcal{D}}^1, \Pi_{\mathcal{M}}\, v \big)_{\mathbb{L}^2(\Omega)}. \end{aligned} \tag{4.124}$$

Using once again the fact that $f = u_t - \Delta u$ (subject of equation (1.1)), and recall that f_K^n, for all $n \in [\![0, N]\!]$, is given by (4.6))

$$\begin{aligned} \mathrm{m}(K)\, f_K^n &= \frac{1}{k} \int_K \left(\int_{t_n}^{t_{n+1}} f(x, t) dt \right) dx \\ &= \int_K \partial^1\, u(x, t_{n+1})\, dx - \frac{1}{k} \int_K \int_{t_n}^{t_{n+1}} \Delta\, u(x, t) dt dx. \end{aligned} \tag{4.125}$$

From (4.125) and (4.102), we write

$$\mathrm{m}(K) f_K^n + \int_K \Delta\, u(x, t_{n+1}) dx = \mathrm{m}(K) \mathbb{K}_K^{n+1} + \mathrm{m}(K) \partial^1 u(x_K, t_{n+1}). \tag{4.126}$$

Set $n = 0$ in the previous expansion and inserting the result in (4.124) yields that

$$\begin{aligned} \big(\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^1, \Pi_{\mathcal{M}}\, v \big)_{\mathbb{L}^2(\Omega)} + \langle \eta_{\mathcal{D}}^1, v \rangle_F &= - \big(\partial^1\, \Pi_{\mathcal{M}}\, \xi_{\mathcal{D}}^1, \Pi_{\mathcal{M}}\, v \big)_{\mathbb{L}^2(\Omega)} \\ &- \sum_{K \in \mathcal{M}} \mathrm{m}(K) \mathbb{K}_K^1\, v_K. \end{aligned} \tag{4.127}$$

Taking $v = \partial^1 \eta_{\mathcal{D}}^1$ in (4.127), using (4.28), and the Cauchy Schwarz inequality lead to (recall that $\partial^1 \eta_{\mathcal{D}}^1 = \dfrac{\eta_{\mathcal{D}}^1 - \eta_{\mathcal{D}}^0}{k} = \dfrac{\eta_{\mathcal{D}}^1}{k}$ since $\eta_{\mathcal{D}}^0 = 0$, thanks to (4.98)), we get

$$\| \partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^1 \|_{\mathbb{L}^2(\Omega)} \leq \| \partial^1 \Pi_{\mathcal{M}}\, \xi_{\mathcal{D}}^1 \|_{\mathbb{L}^2(\Omega)} + \mathbb{K}. \tag{4.128}$$

Thanks to (4.118) and (4.65), the previous inequality implies

$$\| \partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^1 \|_{\mathbb{L}^2(\Omega)} \leq \frac{C_{\rm P}}{\alpha} \mathbb{S}_1 + \mathbb{K}. \tag{4.129}$$

This with (4.123) implies that, for all $n \in [\![1, N]\!]$

$$\|\partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} \leq T\frac{C_{\rm p}}{\alpha}\mathbb{S}_2 + \frac{C_{\rm p}}{\alpha}\mathbb{S}_1 + T\mathbb{T} + \mathbb{K}. \tag{4.130}$$

Gathering the previous two inequalities yields that, for all $n \in [\![0, N+1]\!]$

$$\|\partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)} \leq T\frac{C_{\rm p}}{\alpha}\mathbb{S}_2 + \frac{C_{\rm p}}{\alpha}\mathbb{S}_1 + T\mathbb{T} + \mathbb{K}. \tag{4.131}$$

Which is the required estimate (4.107).

3. Proof of estimate (4.106) : Let us turn to (4.110); inserting (4.126) in (4.110) leads to, for all $v \in \mathcal{X}_{\mathcal{D},\mathcal{B}}$

$$\begin{aligned}\left(\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} + \langle\, \eta_{\mathcal{D}}^{n+1}, v\rangle_F \quad &= \quad -\left(\partial^1\, \Pi_{\mathcal{M}}\, \xi_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} \\ &- \sum_{K\in\mathcal{M}} \mathrm{m}(K)\mathbb{K}_K^{n+1}\, v_K.\end{aligned} \tag{4.132}$$

Taking $v = \eta_{\mathcal{D}}^{n+1}$ in the previous inequality, and using the Cauchy Schwarz inequality yields that, for all $n \in [\![0, N]\!]$

$$\langle\, \eta_{\mathcal{D}}^{n+1}, \eta_{\mathcal{D}}^{n+1}\rangle_F \leq \mathbb{M}_{\mathcal{D}}^{n+1}\|\, \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)}, \tag{4.133}$$

where

$$\mathbb{M}_{\mathcal{D}}^{n+1} = \|\,\partial^1 \Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} + \|\,\partial^1\, \Pi_{\mathcal{M}}\, \xi_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)} + \mathbb{K}. \tag{4.134}$$

Inequality (4.133) with estimate [11, (5.10), Lemma 5.4, Page 1038] when $p = 2$, (4.28), and (4.38) (recall that $\Pi_{\mathcal{M}}\ \eta_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},\mathcal{B}} \subset \mathcal{X}_{\mathcal{D},0}$ since (3.6)) yields that

$$\alpha\|\eta_{\mathcal{D}}^{n+1}\|_{\mathcal{X}} \leq C_{\rm p}\mathbb{M}_{\mathcal{D}}^{n+1}. \tag{4.135}$$

Estimate (4.135) with the expression (4.134), (4.131), (4.65), and the fact that $\eta_{\mathcal{D}}^0 = 0$ (see (4.98)), implies that, for all $n \in [\![0, N+1]\!]$

$$\alpha\|\eta_{\mathcal{D}}^{n}\|_{\mathcal{X}} \leq T\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_2 + TC_{\rm p}\mathbb{T} + 2C_{\rm p}\mathbb{K} + 2\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_1\,. \tag{4.136}$$

This with (4.38) yields that, for all $n \in [\![0, N+1]\!]$

$$\alpha\|\Pi_{\mathcal{M}}\eta_{\mathcal{D}}^{n}\|_{1,2,\mathcal{M}} \leq T\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_2 + TC_{\rm p}\mathbb{T} + 2\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_1 + 2C_{\rm p}\mathbb{K}, \tag{4.137}$$

which is (4.106).

Proof of estimate (4.108) : Thanks to (4.39) of Lemma 4.2, (4.136) implies that, for all $n \in [\![0, N+1]\!]$

$$\alpha\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)} \leq TC_5\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_2 + TC_5C_{\rm p}\mathbb{T} + 2C_5C_{\rm p}\mathbb{K} + 2C_5\frac{C_{\rm p}^2}{\alpha}\mathbb{S}_1, \tag{4.138}$$

which concludes the proof of (4.108), and then the proof of the Lemma is completed. □

The following Lemma is devoted to estimate $\mathbb{T}^n$ and $\mathbb{K}^n$ defined respectively by (4.104).

Lemma 4.7. *(A technical lemma) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the weak solution of (1.1)–(1.3) in the sense of Theorem 2.1 satisfies $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 3.1.*
Let $\{\mathbb{T}_K^n;\ n \in [\![0, N+1]\!],\ K \in \mathcal{M}\}$ (resp. $\{\mathbb{K}_K^n;\ n \in [\![0, N+1]\!],\ K \in \mathcal{M}\}$) be the set of expressions given by (4.100)–(4.101) (resp. (4.102)–(4.103)).
Then the following estimates hold:

$$\mathbb{T} \leq (h_{\mathcal{D}} + k)\sqrt{\mathrm{m}(\Omega)} d \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))},\ \forall n \in [\![0, N+1]\!], \tag{4.139}$$

and

$$\mathbb{K} \leq (h_{\mathcal{D}} + k)\sqrt{\mathrm{m}(\Omega)} d \| u \|_{\mathcal{C}^1([0,T];\mathcal{C}^2(\overline{\Omega}))},\ \forall n \in [\![0, N+1]\!], \tag{4.140}$$

where $\mathbb{T}$ and $\mathbb{K}$ are respectively given by (4.104) and (4.105).

Proof. We first remark that, for all $n \in [\![1, N+1]\!]$

$$\left| \int_K \partial^2 u(x,t_n)\, dx - \mathrm{m}(K)\partial^2 u(x_K,t_n) \right| \leq h_{\mathcal{D}} \mathrm{m}(K) d \| \partial^2 u(\cdot,t_n) \|_{\mathcal{C}^1(\overline{\Omega})}. \tag{4.141}$$

This with the representation (4.94) implies that

$$\left| \int_K \partial^2 u(x,t_n)\, dx - \mathrm{m}(K)\partial^2 u(x_K,t_n) \right| \leq h_{\mathcal{D}} \mathrm{m}(K) d \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^1(\overline{\Omega}))}. \tag{4.142}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{k}\int_K \partial^1 \left(\int_{t_{n-1}}^{t_n} \Delta\, u(x,t) dt \right) dx - \int_K \Delta \partial^1 u(x,t_n) dx \\ &= \frac{1}{k^2} \int_K \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \int_s^{s+k} \Delta \frac{d^2 u}{dl^2}(x,l) dl ds dt dx, \end{aligned} \tag{4.143}$$

which implies that

$$\left| \frac{1}{k}\int_K \partial^1 \left(\int_{t_{n-1}}^{t_n} \Delta\, u(x,t) dt \right) dx - \int_K \Delta \partial^1 u(x,t_n) dx \right| \leq k \mathrm{m}(K) d \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{4.144}$$

Gathering (4.100)–(4.101), estimates (4.142) and (4.144), and the triangle inequality, leads to, for all $n \in [\![0, N+1]\!]$, for all $K \in \mathcal{M}$

$$|\mathbb{T}_K^n| \leq (h_{\mathcal{D}} + k) d \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{4.145}$$

This with (4.104) implies (4.139).
A similar reasoning to that presented in (4.141)–(4.145) implies (4.140). □

Proof of Theorem 4.1 The results of Theorem 4.1 can be justified easily using Lemmata 4.5, 4.6, and 4.7 together with the triangle inequality. ■

5. Conclusion

We considered the nonstationary heat equation with homogeneous Dirichlet boundary conditions posed on polygonal domain at any space dimension. The scheme we presented, that is (4.16)–(4.17), can be applied on any type of spatial grid: conforming or non conforming, 2D and 3D, or more, made with control volumes which are only assumed to be polyhedral (the boundary of each control volume is a finite union of subsets of hyperplanes). The estimates obtained, i.e. (4.33)–(4.35) allow to get error estimates for approximations for the exact solution u of (1.1)–(1.3) and its first derivatives, see Remark 5.

The first equation of the finite volume scheme, i.e. (4.16), is a discretization of the weak formulation (2.1) of the heat equation (1.1) (with, of course, the boundary condition (1.3)). Whereas, the discrete initial condition (4.17) is a discrete version of an orthogonal projection (1.4). From this point of view the discretization scheme (4.33)–(4.35) can be viewed as a nonconforming finite element method although the scheme stems from finite volume ideas. The choice of the discretization (4.17) for the initial condition (1.2) is useful as explained in Remark 6.

It is worth to discuss the case when the discretization of initial condition (1.2) is performed using the following *obvious* choice (recall that $u^0_{\mathcal{D}} = \Big(\big(u^0_K\big)_{K\in\mathcal{M}}, \big(u^0_\sigma\big)_{\sigma\in\mathcal{E}}\Big)$ is an element of $\mathcal{X}_{\mathcal{D},0}$):

$$u^0_K = u^0(x_K),\ \forall K\in\mathcal{M}, \tag{5.1}$$

and

$$u^0_\sigma = u^0(x_\sigma),\ \forall \sigma\in\mathcal{E}. \tag{5.2}$$

Concerning the finite volume scheme (4.16) with (5.1)–(5.2), we could only prove that, for some positive constants C only depending on T, Ω, d, and θ, see the case of admissible mesh [4, Section 3, Pages 239–240]:

- discrete $\mathbb{L}^2(0,T;H^1_0(\Omega))$–estimate

$$\begin{aligned}&\sum_{n=0}^{N} k\| \mathcal{P}_{\mathcal{M}}u(\cdot,t_{n+1}) - \Pi_{\mathcal{M}}\, u^{n+1}_{\mathcal{D}}\|^2_{1,2,\mathcal{M}}\\ &\quad\le\ C\left(\sum_{n=0}^{N} k\left(\mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_{n+1})\right)\right)^2 + (h_{\mathcal{D}}+k)^2\| u\|^2_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}\right).\end{aligned} \tag{5.3}$$

- discrete semi–norm $H^1(0,T;\mathbb{L}^2(\Omega))$–estimate:

$$\begin{aligned}&\sum_{n=0}^{N}\sum_{K\in\mathcal{M}} \mathrm{m}(K)k\left(\frac{e^{n+1}_K - e^n_K}{k}\right)^2\\ &\quad\le\ C\left(\sum_{n=0}^{N}\left(\mathbb{E}_{\mathcal{D}}\left(u(\cdot,t_{n+1})\right)\right)^2 + \frac{(h_{\mathcal{D}}+k)^2}{k}\| u\|^2_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}\right).\end{aligned} \tag{5.4}$$

So, in the case when $\left(F_{K,\sigma}\right)_{K\in\mathcal{M},\sigma\in\mathcal{E}}$ is defined by (4.24)–(4.27), estimates (5.3)–(5.4) become as, thanks to (4.36)

- discrete $\mathbb{L}^2(0,T;H_0^1(\Omega))$–estimate

$$\left(\sum_{n=0}^{N} k\| \mathcal{P}_{\mathcal{M}}u(\cdot,t_{n+1}) - \Pi_{\mathcal{M}}\, u_{\mathcal{D}}^{n+1}\|_{1,2,\mathcal{M}}^2\right)^{\frac{1}{2}} \leq C(h_{\mathcal{D}}+k)\| u\|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{5.5}$$

- discrete semi–norm $H^1(0,T;\mathbb{L}^2(\Omega))$–estimate:

$$\left(\sum_{n=0}^{N}\sum_{K\in\mathcal{M}} \mathrm{m}(K)k\left(\frac{e_K^{n+1}-e_K^n}{k}\right)^2\right)^{\frac{1}{2}} \leq C\frac{h_{\mathcal{D}}+k}{\sqrt{k}}\| u\|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{5.6}$$

Therefore, there are many investigations to take care of in the future, and among them we quote:

(1) Although, the efficiency of the finite volume schemes arising from the new class of general meshes was proved numerically in the stationary case in [11], it is a worth to justify numerically Theorem 4.1 and estimates (5.5)–(5.6).

(2) It is worth to care of the task if it is possible to weakened the regularity assumption $u \in \mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))$ of Theorem 4.1.

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