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A model for the evolution of laminates in finite-strain elastoplasticity

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Abstract

We study the time evolution in elastoplasticity within the rate-independent framework of generalized standard materials. Our particular interest is the formation and the evolution of microstructure. Providing models where existence proofs are possible is a challenging task since the presence of microstructure comes along with a lack of convexity and, hence, compactness arguments cannot be applied to prove the existence of solutions. In order to overcome this problem, we will incorporate information on the microstructure into the internal variable, which is still compatible with generalized standard materials. More precisely, we shall allow for such microstructure that is given by simple or sequential laminates. We will consider a model for the evolution of these laminates and we will prove a theorem on the existence of solutions to any finite sequence of time-incremental minimization problems. In order to illustrate the mechanical consequences of the theory developed some numerical results, especially dealing with the rotation of laminates, are presented.

1 Introduction

We are going to show one way of incorporating microstructure into a model of elastoplasticity. Therefore, we start with a model in finite-strain elastoplasticity with kinematic hardening and show how it can be transformed into a model that describes the evolution of microstructure. All the necessary assumptions are collected in Section 2.

1.1 Finite-Strain Theory

We study a model in elastoplasticity where we are interested in the time evolution of the deformation ϕ and the plastic strain P. In terms of standard generalized materials, P serves as the internal variable. Let $\Omega \subseteq \mathbb{R}^m$, $m \in \{2,3\}$, be the reference configuration of an elastoplastic body. The variables ϕ and P are fields and can both be considered as functions of a spatial variable $x \in \Omega$. The following three conditions (1.1), (1.2) and (1.3) underline the non-linear character of finite-strain theory. They are supposed to hold for almost every $x \in \Omega$.

Physical requirements on the deformation lead to a condition for the deformation gradient $\nabla\phi$:

$$\nabla\phi(x) \in \mathrm{GL}^+(m) \overset{\text{def}}{=} \{F \in \mathbb{R}^{m\times m} \mid \det(F) > 0\}. \tag{1.1}$$

On the one hand, (1.1) guarantees that ϕ locally preserves the orientation of the ambient space $\mathbb{R}^m$ and, on the other hand, the self-penetration of matter is excluded, at least locally. The plastic strain is supposed to be volume preserving and, hence, an element of the special linear group $\mathrm{SL}(m)$:

$$P(x) \in \mathrm{SL}(m) \overset{\text{def}}{=} \{P \in \mathbb{R}^{m\times m} \mid \det(P) = 1\}. \tag{1.2}$$

Moreover, we assume a multiplicative split of the deformation gradient $\nabla\phi$ into the elastic strain F_{el} and the plastic strain P, such that

$$\nabla\phi(x) = F_{\mathrm{el}}(x)P(x). \tag{1.3}$$

In particular, this implies that $F_{\mathrm{el}}(x) \in \mathrm{GL}^+(m)$. Instead of the term $F_{\mathrm{el}}(x)$, we will often write $\nabla\phi(x)P^{-1}(x)$.

1.2 A Classical Model

Let $[0,T] \subseteq \mathbb{R}$ be a given time interval. Following [Mie03b, Mie03a, CT05], we associate an energy $\mathcal{E}$ to every time $t \in [0,T]$, every deformation ϕ and plastic strain P via the formula

$$\mathcal{E}(t,\phi,P) \stackrel{\text{def}}{=} \int_\Omega \big[W(\nabla\phi P^{-1}) + H(P)\big]\mathrm{d}x - \mathcal{F}(t,\phi). \tag{1.4}$$

The energy consists of three parts. The first part measures the stored elastic energy and is given by the density function $W\colon \mathbb{R}^{m\times m} \to \mathbb{R}\cup\{\infty\}$. In view of the condition (1.1), W has to satisfy $W(F) = \infty$ whenever $F \notin \mathrm{GL}^+(m)$. The second part is due to plastic kinematic hardening and given by $H\colon \mathrm{SL}(m) \to \mathbb{R}$. Finally, the term $\mathcal{F}(t,\phi)$ is supposed to be linear in ϕ and measures the work done with respect to a time-dependent external loading.

In the classical sense, a pair (ϕ,P) is said to be a solution of the time-continuous model if it meets a given initial condition $(\phi(0),P(0)) = (\phi_0,P_0)$ and if the following two equations are fulfilled for every $t \in [0,T]$: the equation of elastic equilibrium

$$\mathrm{D}_\phi\mathcal{E}(t,\phi(t),P(t)) = 0 \tag{1.5}$$

and the plastic flow rule (Biot's equation)

$$0 \in \partial_{\dot{P}}\mathcal{R}(P(t),\dot{P}(t)) + \mathrm{D}_P\mathcal{E}(t,\phi(t),P(t)), \tag{1.6}$$

where $\partial_{\dot{P}}\mathcal{R}$ denotes the subgradient with respect to the second variable. The dissipation potential $\mathcal{R}$ is given by a dissipation density R via

$$\mathcal{R}(P(t),\dot{P}(t)) \stackrel{\text{def}}{=} \int_\Omega R(P(t,x),\dot{P}(t,x))\,\mathrm{d}x. \tag{1.7}$$

We consider rate-independent materials, where R is positively homogeneous of degree 1 in the variable $\dot{P}$, that is $R(P,\lambda\dot{P}) = \lambda R(P,\dot{P})$ for all $\lambda > 0$. An example is given by $R(P,\dot{P}) = \sigma_{\mathrm{yield}}|\dot{P}P^{-1}|$ with yield stress $\sigma_{\mathrm{yield}} > 0$, see Section 3. The balance equations (1.5) and (1.6) are listed for the sake of completeness. The time-continuous model is not discussed further in this paper.

Now we replace the system (1.5) and (1.6) by the incremental minimization problem. As in [Mie03b, Mie02], we define the dissipation $D_{\mathrm{SL}}\colon \mathrm{SL}(m)\times\mathrm{SL}(m) \to [0,\infty]$ for plastic strains by

$$D_{\mathrm{SL}}(P_0,P_1) \stackrel{\text{def}}{=} \inf \int_0^1 R(P(s),\dot{P}(s))\,\mathrm{d}s \tag{1.8}$$

where the infimum is taken over the set of all paths $P \in \mathrm{C}^1([0,1],\mathrm{SL}(m))$ such that $P(0) = P_0$ and $P(1) = P_1$. We obtain a dissipation $\mathcal{D}$ between plastic-strain fields via

$$\mathcal{D}(P_0,P_1) \stackrel{\text{def}}{=} \|D_{\mathrm{SL}}(P_0,P_1)\|_{\mathrm{L}^1(\Omega)}.$$

Let $\mathcal{Q}$ be the state space of all admissible pairs (ϕ, P). For a partition $0 = t_0 < \cdots < t_N = T$ of the time interval and an initial state $(\phi_0, P_0) \in \mathcal{Q}$, the incremental minimization problem is to solve iteratively, for $l = 1, \ldots, N$,

$$(\phi_l, \Lambda_l) \in \operatorname{Argmin}\left[\mathcal{E}(t_l, \phi, P) + \mathcal{D}(P_{l-1}, P)\right]. \tag{IMP}$$

Analyzing this problem is the standard approach to find energetic solutions for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, see [MTL02, Mie05]. However, it is shown in [CHM02] that, in general, there are no solutions for (IMP) because of the formation of microstructure. Only in very special cases, the existence of solutions can be guaranteed, see [Mie04b]. Gradient plasticity is treated in [MM06, MM09] where the energy contains a regularizing term of the form $\int_\Omega \mathcal{G}(\nabla P)\mathrm{d}x$ that gives an internal length scale. Yet, neither approach allows for the modeling of microstructure.

1.3 Reformulation for Young Measures

In order to study systems that can develop microstructure, we are going to replace the plastic strain P by a new internal variable Λ that encodes both the micro-fluctuations around the deformation gradient $\nabla\phi$ as well as micro-fluctuations of the plastic strain. In fact, Λ lies in the set $\mathrm{YM}(\Omega, Z)$ of Young measures where $Z \overset{\text{def}}{=} \mathbb{R}^{m\times m} \times \mathrm{SL}(m)$. More details on Young measures are given in Section 2.4.

Following [Mie04a], we extend $\mathcal{E}$ in (1.4) to Young measures. Therefore, we define a new elastic strain $\widetilde{F_{\mathrm{el}}}$ as a function of the deformation gradient $\nabla\phi$ and $(A, P) \in Z$ via

$$\widetilde{F_{\mathrm{el}}}(\nabla\phi, A, P) \overset{\text{def}}{=} \nabla\phi(I{+}A)P^{-1}.$$

Here A is the placeholder for the micro-fluctuation around $\nabla\phi$ and P for the plastic strain, $I \in \mathbb{R}^{m\times m}$ denotes the identity matrix. Then, very similar to (1.4), the energy for Young measures takes the form

$$\widetilde{\mathcal{E}}(t, \phi, \Lambda) \overset{\text{def}}{=} \int_\Omega \langle W(\widetilde{F_{\mathrm{el}}}(\nabla\phi, A, P)) + H(P), \Lambda\rangle \,\mathrm{d}x - \mathcal{F}(t, \phi), \tag{1.9}$$

where $\langle ., . \rangle$ denotes the duality product between a continuous function and a measure, see (2.5). The energy (1.9) reduces to (1.4) if we set $\Lambda = \delta_{0,P}$ where $\delta_{0,P(x)}$ denotes the Dirac measure concentrated in $(0, P(x)) \in Z$.

Since $\nabla\phi$ is a gradient, its micro-fluctuations should also meet the geometric requirements of a gradient. This is related to the notion of gradient Young measures. In addition to that, modeling aspects as well as requirements of numerical implications make it necessary to work on a much smaller subset of $\mathrm{YM}(\Omega, Z)$, which will be done as follows. We fix $\mathcal{L}$, a given subset of probability measures over Z and reduce the analysis to Young measures that lie in the set

$$\mathrm{YM}_{\mathcal{L}}(\Omega, Z) \overset{\text{def}}{=} \{\Lambda \in \mathrm{YM}(\Omega, Z) \mid \Lambda(x) \in \mathcal{L} \text{ a.e. in } \Omega\}.$$

Let $\widetilde{\mathcal{Q}}$ be the extended state space of all admissible pairs (ϕ, Λ). Then the dissipation between Young measures $\Lambda_0, \Lambda_1 \in \mathrm{YM}_{\mathcal{L}}(\Omega, Z)$ is given via integration

$$\widetilde{\mathcal{D}}(\Lambda_0, \Lambda_1) \overset{\text{def}}{=} \int_\Omega D_{\mathcal{L}}(\Lambda_0(x), \Lambda_1(x))\mathrm{d}x. \tag{1.10}$$

Here the function $D_{\mathcal{L}} \colon \mathcal{L}\times\mathcal{L} \to [0, \infty]$ has to be modeled such that it measures the dissipation for elements in $\mathcal{L}$. As a first example for $D_{\mathcal{L}}$, we can think of the 1-Wasserstein distance, as was done in [MTL02, Mie04a, The02] and, in the context of damage, [FG06]. However, Young measures only take into account volume fractions, which is not enough to control rotating laminates (see Section 3). Hence, a better dissipation distance $D_{\mathcal{L}}$ has been introduced by [HK08], where $\mathcal{L}$ is a set of laminates, and will be studied in Section 3 in detail. This $D_{\mathcal{L}}$ distinguishes between the case where Λ_0 and Λ_1 are parallel and the case where Λ_1 is rotated against Λ_0. Thus $D_{\mathcal{L}}$ is not continuous and, hence, leads to new mathematical challenges in establishing an existence theory. Our analysis is the first that treats a model describing time-discontinuous rotations of laminates, which is a well-observed phenomenon in experiments.

We know that the incremental minimization problem (IMP) of the classical model that was discussed in the previous section, in general, does not admit solutions. Studying (IMP) for $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{E}}, \widetilde{\mathcal{D}})$, we still have to expect the formation of microstructures, since the subset $\mathrm{YM}_{\mathcal{L}}(\Omega, Z)$ may not be weakly closed in $\mathrm{YM}(\Omega, Z)$. Here we are going to circumvent this difficulty by the help of a regularization term

$$\mathcal{G}(\Lambda) = \int_\Omega \int_\Omega \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(y))^p}{|x-y|^{m+\theta p}} \,\mathrm{d}x\mathrm{d}y.$$

The energy $\widetilde{\mathcal{E}}(t, \phi, \Lambda)$ is then replaced by the regularized energy

$$\widetilde{\mathcal{E}}_{\mathrm{reg}}(t, \phi, \Lambda) \overset{\mathrm{def}}{=} \widetilde{\mathcal{E}}(t, \phi, \Lambda) + \mathcal{G}(\Lambda).$$

On the one hand, the novel form of $\mathcal{G}$ penalizes rapid changes of the microstructure in Λ and, hence, generates compactness, but, on the other hand, still allows for interfaces between pure states and laminates.

Under additional assumptions, which are given in Section 2, we will prove that the incremental minimization problem (IMP) for $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{E}}_{\mathrm{reg}}, \widetilde{\mathcal{D}})$ admits solutions. This is the main result of this paper. In order to prove this existence result, we will follow the direct method in the calculus of variations. All steps of the proof are contained in Section 4. As the major point, we have to show compactness of the sublevel sets of the energy $\widetilde{\mathcal{E}}_{\mathrm{reg}}$ in the appropriate function space, which involves tools from measure theory. Before we come to that, the necessary assumptions and, in particular, a metric structure on the set $\mathrm{YM}_{\mathcal{L}}$ of Young measures is given in Section 2. The example from [HK08], which was an important motivation for our analysis, is discussed in Section 3 in great detail.

2 Assumptions and Basic Properties

The reader exclusively interested in mechanics may skip this section and go directly to Section 3. In the following, we list all the necessary assumptions for the analysis. Moreover, we fix the space $\widetilde{\mathcal{Y}}$ of admissible deformation fields as well as the space $\widetilde{\mathcal{Z}}$ of admissible Young measures so that $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$ gives the extended state space. At the end of the section, we state our main result.

In this paper, we work over finite-dimensional real vector spaces like $\mathbb{R}$, $\mathbb{R}^m$, $\mathbb{R}^{m\times m}$ and others. Every such space is associated with its Euclidean topology, unless stated otherwise. There will be no confusion if we denote the corresponding Euclidean norm always with the same symbol $|.|$. The quantity $\overline{\mathbb{R}}$ denotes the set $\mathbb{R} \cup \{\infty\}$. As usual, we assume that the set $\Omega \subseteq \mathbb{R}^m$, which defines the reference configuration, is sufficiently regular, namely Ω is a non-empty, open, connected, and bounded set with Lipschitz boundary $\partial\Omega$. We will consider different spaces that are defined over Ω, like Lebesgue and Sobolev spaces as well as spaces of Young measures. In such a space, every two elements are identified whenever they coincide almost everywhere (a.e.) in Ω, that means, outside a subset of Ω with Lebesgue measure 0.

2.1 Space of Deformations

We begin with assumptions for the functions W and H, which occur in the definition of the energy, see (1.4) and (1.9). A crucial assumption for our analysis is the following condition of polyconvexity on the energy density $W\colon \mathrm{GL}^+(m) \to \overline{\mathbb{R}}$. There exists a convex and lower semicontinuous function $\mathbb{W}\colon \mathbb{R}^\tau \to \overline{\mathbb{R}}$ such that

$$W(F_{\mathrm{el}}) = \mathbb{W}(\mathbb{M}(F_{\mathrm{el}})) \quad \text{for all } F_{\mathrm{el}} \in \mathrm{GL}^+(m). \tag{2.1a}$$

The quantity $\mathbb{M}(F_{\mathrm{el}}) \in \mathbb{R}^\tau$ with $\tau = \frac{(2m)!}{(m!)^2}$ denotes the vector of all minors of the matrix F_{el} in a fixed order. By (2.1a), the function W becomes polyconvex, see [Bal77] for the relevance of polyconvexity to the existence of solutions in elasticity theory. In addition, we assume W to be coercive: there exist $q_F > m$, $q_D > 0$ and $w_1, w_2 > 0$ such that

$$W(F_{\mathrm{el}}) \geq w_1(|F_{\mathrm{el}}|^{q_F} + \det(F_{\mathrm{el}})^{-q_D}) - w_2 \quad \text{for all } F_{\mathrm{el}} \in \mathrm{GL}^+(m). \tag{2.1b}$$

The hardening function $H\colon \mathrm{SL}(m) \to \mathbb{R}$ is assumed to be lower semicontinuous. In addition, we require that there exist $q_P > m$ and $h_1, h_2 > 0$ such that

$$H(P) \geq h_1|P|^{q_P} - h_2 \quad \text{for all } P \in \mathrm{SL}(m). \tag{2.1c}$$

As in [Mie04b, MM09], the exponents in (2.1b) and (2.1c) are real numbers that fulfill $q_F, q_P > m$ as well as

$$\frac{1}{q_Y} \overset{\text{def}}{=} \frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{m}. \tag{2.1d}$$

Later on we will use the following condition in order to bound the micro-fluctuation around the deformation gradient $\nabla\phi$:

$$\frac{1}{q_A} \overset{\text{def}}{=} \frac{1}{q_Y} + \frac{1}{mq_D} < 1. \tag{2.1e}$$

We shall see in Lemma 4.2 that the assumptions (2.1b), (2.1c) and (2.1d) restrict deformations ϕ with finite energy to the case $\phi \in \mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$. Here $\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$ denotes the Sobolev space of weakly differentiable functions over Ω with values in $\mathbb{R}^m$ such that function and derivative have both finite L^{q_Y}-norm. We denote the L^{q_Y}-norm by $\|.\|_{\mathrm{L}^{q_Y}}$ and the norm of the space $\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$ by $\|.\|_{\mathrm{W}^{1,q_Y}}$. The space $\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$ continuously embeds into the space $\mathrm{C}(\overline{\Omega}, \mathbb{R}^m)$ of continuous functions due to (2.1d).

We impose homogeneous Dirichlet boundary conditions on a measurable subset $\Gamma \subseteq \partial\Omega$ of positive surface measure. For simplicity, the Dirichlet datum ϕ_{Dir} is independent of time. Then the set of all admissible deformations reduces to

$$\widetilde{\mathcal{Y}} \overset{\text{def}}{=} \{\phi \in \mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m) \mid \phi = \phi_{\mathrm{Dir}} \text{ on } \Gamma\}. \tag{2.2}$$

The set $\widetilde{\mathcal{Y}}$ together with the norm $\|.\|_{\mathrm{W}^{1,q_Y}}$ is a closed affine subspace of $\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$ and, hence, weakly closed. Time-dependent Dirichlet boundary conditions could be treated as well, see, for example, [FM06, MM09].

For the term $\mathcal{F}$, which is connected to the external loading, we assume $\mathcal{F}(t, .)$ to be linear on $\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m)$, such that

$$\mathcal{F} \in \mathrm{C}^1([0, T], (\mathrm{W}^{1,q_Y}(\Omega, \mathbb{R}^m))^*). \tag{2.3}$$

2.2 Space for Internal Variables

Our aim is to define a metric structure on a subset of Young measures. We start on the level of the underlying space $Z = \mathbb{R}^{m\times m}\times \mathrm{SL}(m)$, which does not have a linear structure in the second component. We denote by E_Z the element $(0, I) \in Z$ where $I \in \mathrm{SL}(m)$ is the identity matrix. >From now on, we assume that $d_{\mathrm{SL}}\colon \mathrm{SL}(m)\times\mathrm{SL}(m) \to \mathbb{R}$ is a given distance on $\mathrm{SL}(m)$ that is related to the Euclidean norm in the following way. First, we assume that d_{SL} induces the Euclidean topology. This is equivalent to the fact that for every plastic tensors $P, P_1, P_2, \ldots$ in $\mathrm{SL}(m)$ we have

$$\lim_{k\to\infty} |P_k - P| = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} d_{\mathrm{SL}}(P_k, P) = 0. \tag{2.4a}$$

Second, we assume that there exist constants $\tau_0, \tau_1 > 0$ and a monotonously increasing function $\tau_2\colon \mathbb{R} \to \mathbb{R}$ such that for every $P \in Z$ we have

$$d_{\mathrm{SL}}(P, I) \le \tau_0 + \tau_1|P| \quad \text{and} \quad |P| \le \tau_2(d_{\mathrm{SL}}(P, I)). \tag{2.4b}$$

In particular, this guarantees that the notion of bounded sets is the same for d_{SL} and for the Euclidean norm. Finally, we define the distance $d_Z\colon Z\times Z \to \mathbb{R}$ on Z via

$$d_Z((A_0, P_0), (A_1, P_1)) \overset{\text{def}}{=} |A_0 - A_1| + d_{\mathrm{SL}}(P_0, P_1). \tag{2.4c}$$

The above conditions imply that (Z, d_Z) is separable and complete.

2.3 Probability Measures

Before we come to probability measures, we recall the pairing $\langle .,.\rangle$, which is defined between continuous functions and measures. Consider the set $\mathrm{C}_0(Z)$ of all continuous functions $g\colon Z \to \mathbb{R}$ that vanish at infinity, meaning, $g(A, P) \to 0$ as $d_Z((A, P), E_Z)$ tends to ∞. The supremum-norm turns $\mathrm{C}_0(Z)$ into a separable Banach space. The space $\mathcal{M}(Z)$ of all signed Radon measures ν over Z can be seen as the dual space of $\mathrm{C}_0(Z)$ with the pairing $\langle .,.\rangle$ given by

$$\langle g, \nu\rangle \overset{\text{def}}{=} \int_Z g(A, P)\,\mathrm{d}\nu(A, P). \tag{2.5}$$

Let $\mathcal{P}(Z) \subseteq \mathcal{M}(Z)$ be the subset of all probability measures over Z and define the subset where the first moment is finite

$$\mathcal{P}^1(Z) \overset{\text{def}}{=} \{\Lambda \in \mathcal{P}(Z) \mid \langle d_Z(E_Z, .), \Lambda\rangle < \infty\}.$$

For two probability measures $\mu_0, \mu_1 \in \mathcal{P}^1(Z)$, the 1-Wasserstein distance is given by

$$d_{\mathrm{W}}(\mu_0, \mu_1) \overset{\text{def}}{=} \sup\{|\langle g, \mu_0\rangle - \langle g, \mu_1\rangle| \mid g\colon Z \to \mathbb{R},\ \mathrm{Lip}_{d_Z}(g) \le 1\}. \tag{2.6}$$

Here the supremum is taken over all Lipschitz continuous functions g where the Lipschitz constant is formed using the distance d_Z and denotes the smallest constant $\mathrm{Lip}_{d_Z}(g)$ such that, for every $(A_0, P_0), (A_1, P_1) \in Z$, we have

$$|g(A_0, P_0) - g(A_1, P_1)| \le \mathrm{Lip}_{d_Z}(g) d_Z((A_0, P_0), (A_1, P_1)).$$

The 1-Wasserstein distance is usually defined the equivalent dual way, see (A.2) in the appendix for more. In what follows, we will often omit the exponent 1 and just write Wasserstein distance.

>From now on, we assume that $\mathcal{L}$ is a subset of $\mathcal{P}^1(Z)$ that is closed with respect to d_{W}. As a consequence, we get the following.

$$\text{The metric space } (\mathcal{L}, d_{\mathrm{W}}) \text{ is separable and complete.} \tag{2.7a}$$

In fact, since the metric space (Z, d_Z) is separable and complete, so is $(\mathcal{P}^1(Z), d_{\mathrm{W}})$ as shown in [AGS05, Proposition 7.1.5]. Moreover, we assume that the gradient part of each element in $\mathcal{L}$ is a homogenous gradient Young measure, see [KP91], and has mean value $0 \in \mathbb{R}^{m\times m}$, which means

$$\int_Z A \, \mathrm{d}\Lambda(A, P) = 0 \quad \text{for all } \Lambda \in \mathcal{L}. \tag{2.7b}$$

The integral is well-defined by definition of d_Z, see (2.4).

The following lemma gives a sufficient condition for a function to be lower semicontinuous with respect to the Wasserstein distance. Note that, by (2.4), the Euclidean norm and the distance d_Z create the same notion of lower semicontinuity for functions over Z.

Lemma 2.1. *Let $g\colon Z \to \overline{\mathbb{R}}$ be a lower semicontinuous function on (Z, d_Z) that is bounded from below. Moreover, let $\Lambda, \Lambda_1, \Lambda_2, \ldots \in \mathcal{P}^1(Z)$ be probability measures such that $d_{\mathrm{W}}(\Lambda_k, \Lambda) \to 0$ holds. Then we have*

$$\liminf_{k\to\infty} \langle g, \Lambda_k\rangle \geq \langle g, \Lambda\rangle.$$

Proof. We define functions g_l using the inf-convolution from convex analysis, also called Moreau-Yoshida regularization. For every integer $l > 0$, it is given by the function $g_l\colon Z \to \mathbb{R}$ with

$$g_l(\Theta) \overset{\text{def}}{=} \inf_{\hat{\Theta}\in Z} \left[g(\hat{\Theta}) + l\, d_Z(\Theta, \hat{\Theta}) \right]. \tag{2.8}$$

Since g is lower semicontinuous and bounded from below, g_l is well-defined and Lipschitz continuous with $\mathrm{Lip}_{d_Z}(g) \leq l$ for every $l > 0$. In addition, the functions $g_1, g_2, \ldots$ form a monotonously increasing sequence. The lower semicontinuity of g implies that $g_l \to g$ pointwise. Beppo Levi's monotone-convergence theorem for Λ-measurable functions gives

$$\lim_{l\to\infty} \langle g_l, \Lambda\rangle = \langle g, \Lambda\rangle. \tag{2.9}$$

Moreover, by the definition (2.6), we have $|\langle g_l, \Lambda_k\rangle - \langle g_l, \Lambda\rangle| \leq \mathrm{Lip}_{d_Z}(g_l) d_{\mathrm{W}}(\Lambda_k, \Lambda)$. Hence, the convergence $d_{\mathrm{W}}(\Lambda_k, \Lambda) \to 0$ implies that $\langle g_l/l, \Lambda_k\rangle \to \langle g_l/l, \Lambda\rangle$ holds for every $l > 0$. We get the estimate

$$\liminf_{k\to\infty} \langle g, \Lambda_k\rangle \geq \liminf_{k\to\infty} \langle g_l, \Lambda_k\rangle = \langle g_l, \Lambda\rangle \quad \text{for all } l > 0.$$

Together with (2.9), this finishes the proof. □

2.4 Young Measures

We begin with a short introduction to Young measures. More details of the following construction together with references can be found in [Bal89, p. 211].

Since $\mathrm{C}_0(Z)$ is a separable Banach space, so is $\mathrm{L}^1(\Omega, \mathrm{C}_0(Z))$. Consider the space $\mathrm{L}^\infty_{\mathrm{w}*}(\Omega, \mathcal{M}(Z))$ of all norm-bounded and weak* measurable functions $\mu\colon \Omega \to \mathcal{M}(Z)$, which contains the space of Young measures. Here an element μ is called weak* measurable if $\langle \mu(x), g\rangle$ is a measurable function in x for every g in $\mathrm{C}_0(Z)$. The space $\mathrm{L}^\infty_{\mathrm{w}*}(\Omega, \mathcal{M}(Z))$ can be identified with the dual space of $\mathrm{L}^1(\Omega, \mathrm{C}_0(Z))$ via the pairing

$$\langle\langle f, \mu\rangle\rangle = \int_\Omega \langle f(x), \mu(x)\rangle \,\mathrm{d}x. \tag{2.10}$$

The space of all Young measures is a subspace of $\mathrm{L}^\infty_{\mathrm{w}*}(\Omega, \mathcal{M}(Z))$ and given by

$$\mathrm{YM}(\Omega, Z) \overset{\text{def}}{=} \{\Lambda \in \mathrm{L}^\infty_{\mathrm{w}*}(\Omega, \mathcal{M}(Z)) \mid \Lambda(x) \in \mathcal{P}(Z) \text{ a.e. in } \Omega\}.$$

Weak* convergence is defined for Young measures using the pairing in (2.10). This is sometimes just called Young measure convergence. However, the set $\mathrm{YM}(\Omega, Z)$ is not closed under weak* convergence.

In a special case, there is a closedness property for Young measures. For every exponent $q \geq 1$ consider the set of Young measures given by

$$\mathrm{YM}^q(\Omega, Z) \overset{\text{def}}{=} \{\Lambda \in \mathrm{YM}(\Omega, Z) \mid \langle\langle |(A,P)|^q, \Lambda\rangle\rangle < \infty\}.$$

These are the Young measures with finite averaged qth moment. Note that here the qth moment is computed with respect to the Euclidean norm. We will need the following result later on.

Lemma 2.2. *Let $q \geq 1$ be an exponent and $\Lambda_1, \Lambda_2, \ldots$ a sequence in $\mathrm{YM}^q(\Omega, Z)$. If there exists a constant $C > 0$ such that for every $k > 0$ we have the estimate $\langle\langle |(A,P)|^q, \Lambda_k\rangle\rangle \leq C$, then there exists a subsequence (not relabeled) and a Young measure $\Lambda \in \mathrm{YM}^q(\Omega, Z)$ such that $\Lambda_k \to \Lambda$ weakly* and $\langle\langle |(A,P)|^q, \Lambda\rangle\rangle \leq C$.*

Proof. The Banach-Alaoglu theorem implies that there exists $\Lambda \in \mathrm{L}^\infty_{\mathrm{w}*}(\Omega, \mathcal{M}(Z))$ such that (up to a subsequence) $\Lambda_k \to \Lambda$ weakly*. As a direct consequence, we get the estimate $\langle\langle |(A,P)|^q, \Lambda\rangle\rangle \leq C$. It remains to prove that $\Lambda \in \mathcal{P}(Z)$ a.e. in Ω. This is shown by Ball [Bal89, Thm.(iii)] if for all $k > 0$ the Young measure takes the form $\Lambda_k = \delta_{0,P_k}$ where $P_k \in \mathrm{L}^q(\Omega, Z)$. Ball's idea can also be applied to the general case. $\square$

Let $p > 1$ be a given exponent that we will keep fixed for the rest of the paper. With the help of the Wasserstein distance d_{W}, we define a metric structure on a subspace of $\mathrm{YM}(\Omega, Z)$. Therefore, we restrict the set of admissible Young measures to

$$\widetilde{\mathcal{Z}} \overset{\text{def}}{=} \{\Lambda \in \mathrm{YM}_{\mathcal{L}}(\Omega, Z) \mid \|d_{\mathrm{W}}(\Lambda, \delta_{E_Z})\|_{\mathrm{L}^p} < \infty\}. \tag{2.11a}$$

The set $\widetilde{\mathcal{Z}}$ forms a metric space together with the distance $\mathrm{dist}_p\colon \widetilde{\mathcal{Z}} \times \widetilde{\mathcal{Z}} \to \mathbb{R}$ that is given by

$$\mathrm{dist}_p(\Lambda_0, \Lambda_1) \overset{\text{def}}{=} \|d_{\mathrm{W}}(\Lambda_0, \Lambda_1)\|_{\mathrm{L}^p}. \tag{2.11b}$$

This metric was used already in [Mie99].

Lemma 2.3. *Let $\Lambda_1, \Lambda_2, \ldots \in \widetilde{\mathcal{Z}}$ be a Cauchy sequence with respect to dist_p. Then there exists a subsequence (not relabeled) and a Young measure $\Lambda \in \widetilde{\mathcal{Z}}$ such that* *(i)* $d_{\mathrm{W}}(\Lambda_k(x), \Lambda(x)) \to 0$ *for almost every* $x \in \Omega$

(ii) $\mathrm{dist}_p(\Lambda_k, \Lambda) \to 0$.

In particular, $(\widetilde{\mathcal{Z}}, \mathrm{dist}_p)$ *is a complete metric space.*

Proof. We take a subsequence (not relabeled) such that the estimate $\mathrm{dist}_p(\Lambda_k, \Lambda_{k+1}) < 2^{-k}$ holds for every $k > 0$. This implies that we have

$$\sum_{k=1}^{\infty} d_{\mathrm{W}}(\Lambda_k(x), \Lambda_{k+1}(x)) < \infty \quad \text{for almost every } x \in \Omega.$$

Hence, $\Lambda_1(x), \Lambda_2(x), \ldots$ forms a Cauchy sequence in $(\mathcal{L}, d_{\mathrm{W}})$. Since $(\mathcal{L}, d_{\mathrm{W}})$ is a complete metric space, there exists a Young measure $\Lambda \in \widetilde{\mathcal{Z}}$ such that (i) is fulfilled. Moreover, we know that $\mathrm{dist}_p(\Lambda_k, \Lambda_l) < 2^{-k+1}$ for every $l > k > 0$. By Fatou's lemma, we conclude that $\mathrm{dist}_p(\Lambda_k, \Lambda) < 2^{-k+1}$ for every $k > 0$, which implies (ii). $\square$

2.5 Regularization and Main Result

We regularize the energy $\widetilde{\mathcal{E}}$ and set $\widetilde{\mathcal{E}}_{\mathrm{reg}} = \widetilde{\mathcal{E}} + \mathcal{G}$. Therefore, let $0 < \theta < m/p$ be a fixed number. We consider the function $\mathcal{G}\colon \widetilde{\mathcal{Z}} \to [0, \infty]$ given by

$$\mathcal{G}(\Lambda) \stackrel{\text{def}}{=} \int_\Omega \int_\Omega \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(y))^p}{|x-y|^{m+\theta p}} \,\mathrm{d}x\mathrm{d}y \quad \text{for } \Lambda \in \widetilde{\mathcal{Z}}. \tag{2.12}$$

Note that the term $\mathcal{G}$ remains finite even for certain Young measures that are discontinuous over Ω since $\theta < m/p$ holds. The corresponding term in Banach spaces would be the Sobolev-Slobodetsky norm leading to $\mathrm{W}^{\theta,p}(\Omega)$, which embeds into $\mathrm{C}^0(\overline{\Omega})$ if and only if $\theta > m/p$. In Section 4, we will see that $\mathcal{G}$ is still strong enough to imply compactness.

Our main result is as follows (equivalent to Theorem 4.7).

Theorem. *Let the functions* $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{D}}$ *and* $\mathcal{G}$ *as well as the spaces* $\widetilde{\mathcal{Y}}$ *and* $\widetilde{\mathcal{Z}}$ *be defined as in* (1.9), (1.10), (2.12), (2.2) *and* (2.11)*, respectively, such that the conditions* (2.1), (2.3), (2.4) *and* (2.7) *are fulfilled. Set* $\widetilde{\mathcal{E}}_{\mathrm{reg}} = \widetilde{\mathcal{E}} + \mathcal{G}$*. Moreover, assume that* $\widetilde{\mathcal{D}}\colon \widetilde{\mathcal{Z}} \times \widetilde{\mathcal{Z}} \to [0, \infty]$ *is lower semicontinuous.*
Let $0 = t_0 < t_1 < \ldots < t_N = T$ *be a finite partition of* $[0, T]$ *and* $(\phi_0, \Lambda_0) \in \widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$ *an initial state. Then the incremental minimization problem for* $(\widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}, \widetilde{\mathcal{E}}_{\mathrm{reg}}, \widetilde{\mathcal{D}})$*, which is to solve iteratively*

$$(\phi_l, \Lambda_l) \in \mathrm{Argmin}\left[\widetilde{\mathcal{E}}_{\mathrm{reg}}(t_l, \phi, \Lambda) + \widetilde{\mathcal{D}}(\Lambda_{l-1}, \Lambda)\right] \quad \textit{for } l = 1, 2, \ldots, N,$$

admits solutions in $\widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$.

Note that the lower semicontinuity of $\widetilde{\mathcal{D}}$ is ensured, for example, by the lower semicontinuity of $D_{\mathcal{L}}$ with respect to d_{W}. We will see, that the example in Section 3 fulfills this condition.

3 Example for $\mathcal{L}$ and $D_{\mathcal{L}}$

In this section, we give a mechanically relevant example of a set $\mathcal{L}$ and a dissipation $D_{\mathcal{L}}$ for which our main result (Theorem 4.7) is applicable. The example is due to [HK08] and for more detailed information see also [KH10b] and [KH11]. Various application, especially with respect to cyclic loading can be found in [KH10a] and [HK11].

3.1 Dissipation Distance on Z

In order to construct a suitable dissipation distance d_Z, we consider a specific dissipation density R by setting $R(P,\dot{P}) \overset{\text{def}}{=} \sigma_{\text{yield}}|\dot{P}P^{-1}|$ with yield stress $\sigma_{\text{yield}} > 0$. Having in mind the definition of D_{SL} in (1.8), we set

$$D_{\text{SL}}(P_0,P_1) = \inf_{P\in\mathcal{S}(P_0,P_1)} \int_0^1 \sigma_{\text{yield}}|\dot{P}(s)P(s)^{-1}|\mathrm{d}s \quad \text{for all } P_0,P_1 \in \mathrm{SL}(m) \tag{3.1}$$

where the infimum is taken over the set

$$\mathcal{S}(P_0,P_1) \overset{\text{def}}{=} \{P \in \mathrm{C}^1([0,1],\mathrm{SL}(m)) \mid P(0) = P_0 \text{ and } P(1) = P_1\}.$$

Then the property (2.4) is fulfilled if we set $d_{\text{SL}} = D_{\text{SL}}$, see Appendix B for the proof. Such kind of dissipations were introduced in [Mie02, Mie03b] and further studied in [HMM03, GMMM06]. Here the multiplicative structure of R (called plastic invariance in [Mie03b]) implies $D_{\text{SL}}(P_0,P_1) = D_{\text{SL}}(I,P_0P_1^{-1})$, which clearly shows that it is useless to treat $\mathrm{SL}(m)$ as a subset of the linear space $\mathbb{R}^{m\times m}$. In particular, we have the estimate $D_{\text{SL}}(I,\mathrm{e}^{\xi}) \leq \sigma_{\text{yield}}|\xi|$ which contradicts any coercivity of the type $D_{\text{SL}}(I,P) \geq c|P|^{\delta} - C$ for $\delta > 0$.

3.2 Dissipation for Simple-Laminate Fields

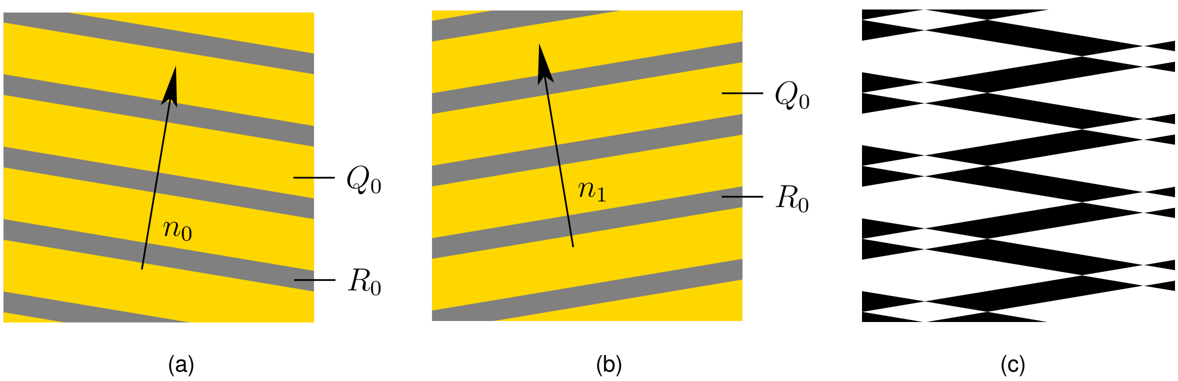

Figure 1: A pure rotation of a simple laminate: (a) before and (b) after the rotation. As a consequence, Q_0 and R_0 interchange roles on a subset which is depicted in (c), its volume ratio being $2\alpha_0(1-\alpha_0)$. The dissipation $D^{(2)}_{\text{lam}}$ amounts to $2\alpha_0(1-\alpha_0)D_{\text{SL}}(Q_0,R_0)$.

Let $\Lambda \in \mathcal{P}^1(Z)$ be a convex combination of two Dirac measures

$$\Lambda = \alpha\delta_{((1-\alpha)a\otimes n,Q)} + (1-\alpha)\delta_{(-\alpha a\otimes n,R)} \tag{3.2}$$

where $Q, R \in \mathrm{SL}(m)$ are plastic strains, $\alpha \in [0,1]$ a real number and $a, n \in \mathbb{R}^m$ vectors such that $|n| = 1$. The quantity $a \otimes n$ denotes the tensor product. In order to shorten notation, we write $A = (1-\alpha)a \otimes n$ and $B = -\alpha a \otimes n$. By definition, the matrices A and B are rank-one connected since the difference $A - B = a \otimes n$ has at most rank 1. Hence, Λ is a (homogenous) gradient Young measure. We define the set $\mathcal{L}_\mathrm{s} \subseteq \mathcal{P}^1(Z)$ of simple laminates as the collection of all those Λ that are of the form (3.2).

When working with simple laminates, we have to take care of the fact that the representation given by (3.2) is not unique. Note also that $(\mathcal{L}_\mathrm{s}, d_\mathrm{W})$ forms a separable and complete metric space, see Appendix A for a proof. In addition, the mean value of the micro-fluctuations is the zero matrix in $\mathbb{R}^{m\times m}$. Hence, the assumptions (2.7) listed in Section 2 are fulfilled.

For $\mathcal{L} = \mathcal{L}_\mathrm{s}$ we now give the nontrivial distance D_lam that was introduced in [HK08]. Motivated by experimental observations, this dissipation is made very sensible to a change in the orientation n. In particular, a "rotation" of a simple laminate produces dissipation. For simple laminates $\Lambda_0, \Lambda_1 \in \mathcal{L}_\mathrm{s}$ the distance D_lam distinguishes between two cases. First, suppose there exist representations of the form (3.2) for Λ_0 and Λ_1, respectively, so that $n_0 = n_1$ or $n_0 = -n_1$ holds. Then we write $\Lambda_0 \parallel \Lambda_1$ and the dissipation $D^{(1)}_\mathrm{lam}$ is modeled by the Wasserstein distance with respect to D_SL. Using the form (A.2) instead of (2.6), we find

$$D^{(1)}_\mathrm{lam}(\Lambda_0,\Lambda_1) = \inf \begin{bmatrix} \lambda D_\mathrm{SL}(Q_0,Q_1) + (1-\alpha_0-\alpha_1+\lambda) D_\mathrm{SL}(R_0,R_1) \\ +(\alpha_0-\lambda) D_\mathrm{SL}(Q_0,R_1) + (\alpha_1-\lambda) D_\mathrm{SL}(R_0,Q_1) \end{bmatrix}$$

where the infimum is taken over all λ such that the numbers λ, $\alpha_0 - \lambda$, $\alpha_1 - \lambda$ and $1-\alpha_0-\alpha_1+\lambda$ lie in $[0,1]$. Second, suppose that there are no representations of the form (3.2) so that $n_0 = n_1$ or $n_0 = -n_1$ holds. Then we write $\Lambda_0 \nparallel \Lambda_1$ and the dissipation is modeled by

$$\begin{aligned} D^{(2)}_\mathrm{lam}(\Lambda_0,\Lambda_1) = {} & \alpha_0\alpha_1 D_\mathrm{SL}(Q_0,Q_1) + \alpha_0(1-\alpha_1) D_\mathrm{SL}(Q_0,R_1) \\ & + (1-\alpha_0)\alpha_1 D_\mathrm{SL}(R_0,Q_1) + (1-\alpha_0)(1-\alpha_1) D_\mathrm{SL}(R_0,R_1). \end{aligned}$$

See Figure 1 for an example. Putting both functions together, we end up with

$$D_\mathrm{lam}(\Lambda_0,\Lambda_1) = \begin{cases} D^{(1)}_\mathrm{lam}(\Lambda_0,\Lambda_1) & \text{for } \Lambda_0 \parallel \Lambda_1 \\ D^{(2)}_\mathrm{lam}(\Lambda_0,\Lambda_1) & \text{for } \Lambda_0 \nparallel \Lambda_1, \end{cases}$$

which defines a dissipation on the set $\mathcal{L}_\mathrm{s}$ of simple laminates.

The function D_lam is lower semicontinuous with respect to the Wasserstein distance d_W. In fact, if we have in mind (A.2) as well as Remark A.2, it is not hard to prove that the functions $D^{(1)}_\mathrm{lam}$ and $D^{(2)}_\mathrm{lam}$ are continuous with respect to d_W. Moreover, we see that $D^{(2)}_\mathrm{lam} \geq D^{(1)}_\mathrm{lam}$ holds and that the set $\{(\Lambda_0,\Lambda_1) \in \mathcal{L}_\mathrm{s}^2 \mid \Lambda_0 \parallel \Lambda_1\}$ is closed with respect to d_W. This directly implies that D_lam is lower semicontinuous with respect to d_W. In particular, the associated dissipation $\widetilde{\mathcal{D}}$ is lower semicontinuous with respect to the distance dist_p, see (2.11b). Hence, the set $\mathcal{L} = \mathcal{L}_\mathrm{s}$, the distance $d_\mathrm{SL} = D_\mathrm{SL}$ and the dissipation $D_\mathcal{L} = D_\mathrm{lam}$, as defined above, form an example to which our main result can be applied.

3.3 A Study of a Rotating Laminate

Let us consider a particular case of the model introduced in [HK08] by allowing the inelastic deformation to assume only two distinct values, hence $P, Q \in \{I \pm \gamma s \otimes n_s\}$ in (3.2), where $\gamma > 0$ denotes a constant material parameter, and s and n_s are orthonormal vectors. Meaning, that $\mathrm{SL}(m)$ is replaced by $\{I \pm \gamma s \otimes n_s\} \subseteq \mathrm{SL}(m)$. This may constitute a model for a shape-memory-alloy possessing two martensitic variants. In this case the dissipation distance (1.8) reduces to

$$D_{\mathrm{SL}}(P_0, P_1) = \begin{cases} r & \text{for } P_0 \neq P_1 \\ 0 & \text{for } P_0 = P_1, \end{cases} \tag{3.3}$$

where $r > 0$. For laminates this yields the simple expression

$$D_{\mathrm{lam}}(\Lambda_0, \Lambda_1) = \begin{cases} r\,|\alpha_1 - \alpha_0| & \text{for } \Lambda_0 \| \Lambda_1 \\ r\,(\alpha_0(1-\alpha_1) + \alpha_1(1-\alpha_0)) & \text{for } \Lambda_0 \not\| \Lambda_1. \end{cases}$$

As in [HK08] we assume an incompressible neo-Hookean energy of the form

$$W(F_{\mathrm{el}}) = \begin{cases} \mu \|F_{\mathrm{el}}\|^2 & \text{for } \det(F_{\mathrm{el}}) = 1 \\ \infty & \text{for } \det(F_{\mathrm{el}}) \neq 1, \end{cases} \tag{3.4}$$

with $\mu > 0$ denoting the shear-modulus. Moreover we assume $H(I \pm \gamma s \otimes n_s) = 0$. The energy in (1.9) can then be calculated as

$$\widetilde{\mathcal{E}}(t, \phi, \Lambda) \overset{\text{def}}{=} \int_\Omega \tilde{W}(\nabla\phi, \Lambda)\,\mathrm{d}x - \mathcal{F}(t, \phi), \tag{3.5}$$

with

$$\begin{aligned} \tilde{W}(\nabla\phi, \Lambda) = {} & \alpha\, W(\nabla\phi(I + (1-\alpha)\, a \otimes n)(I + \gamma\, s \otimes n_s)) \\ & + (1-\alpha)\, W(\nabla\phi(I - \alpha\, a \otimes n)(I - \gamma\, s \otimes n_s)), \end{aligned}$$

where $a \cdot n = 0$.

In the plane-strain case, we can specify the quantities above as $n = (\cos\varphi, \sin\varphi)$ and $a = a_0\,(-\sin\varphi, \cos\varphi)$, $n_s = (\cos\psi, \sin\psi)$, $s = (-\sin\psi, \cos\psi)$. In the absence of dissipation the energy will be minimized with respect to all possible laminates resulting in a relaxed energy of the form

$$W^{\mathrm{rel}}(\nabla\phi) = \inf_{\alpha\in[0,1],\, \varphi\in[0,\pi)} \tilde{W}(\nabla\phi, \Lambda). \tag{3.6}$$

In the presence of dissipation, however, the minimization in Theorem 4.7 gives the stationarity conditions

$$0 = \frac{\partial \tilde{W}}{\partial a_{0l}}, \quad 0 \in \frac{\partial \tilde{W}}{\partial \alpha_l} + r\,\mathrm{sign}\,(\alpha_l - \alpha_{l-1}), \tag{3.7}$$

where a_{0l} and α_l denote the values at the end of time-increment number l, together with the condition for rotation of the laminate

$$f(\nabla\phi_l, \alpha_{l-1}, \alpha_l, \varphi_l) \overset{\text{def}}{=} \tilde{W}(\nabla\phi, \Lambda_l) - \tilde{W}(\nabla\phi_l, \Lambda_{l-1}) + r\,(\alpha_{l-1}(1-\alpha_l) + \alpha_l(1-\alpha_{l-1})) < 0, \tag{3.8}$$

which gives the evolution of φ as

$$\varphi_l = \begin{cases} \underset{\varphi\in[0,\pi)}{\mathrm{Argmin}} f(\nabla\phi, \alpha_{l-1}, \alpha_l, \varphi) & \text{for } \inf_{\varphi\in[0,\pi)} f(\nabla\phi, \alpha_{l-1}, \alpha_l, \varphi_{l-1}) < 0 \\ \varphi_{l-1} & \text{else.} \end{cases} \tag{3.9}$$

Given $\nabla\varphi_l$, α_{l-1}, φ_{l-1}, the equations (3.7) and (3.9) can be solved for α_l, φ_l. This allows one to compute the evolution of α and φ in the time-incremental problem (IMP).

As an example we present a simple shear test of the form $\nabla\varphi = \begin{pmatrix} 1 & 0 \\ \xi(t) & 1 \end{pmatrix}$, where $\xi(t) = t$ for $t \in [0,2]$. The model parameters chosen are: $\mu = 75$, $r = 1$, $\gamma = 0.2$, $\psi = \pi/10$. Hence, the inelastic shearing deformation is misaligned with respect to the applied shear.

In Figure 2 the laminate rotation angle φ is displayed as a function of ξ, once as result of the minimization in (3.6), and once as result of the time-incremental procedure in (3.9). The same is done for the volume ratio α as function of ξ in Figure 3. In Figure 4 the difference in α of the results from (3.7) and (3.9) is shown.

It can be seen that φ starts to deviate from the solution found by minimization, until finally the inequality (3.8) is satisfied. Then the minimization result is retrieved in a sudden way. This process repeats itself in a stick-slip-type behavior. After every jump in φ, the variable α remains constant within a certain interval, until the differential inclusion in (3.7) becomes nontrivial again.

4 Proof of the Main Result

In order to prove our main result, we use the direct method in the calculus of variations. In doing so, we have to show coercivity and lower semicontinuity of the energy as well as weak sequential compactness of the sublevel sets of the energy.

4.1 Coercivity

Before we prove the coercivity of the energy $\widetilde{\mathcal{E}}$, we recall an inequality between determinant and norm of a matrix.

Remark 4.1. *Let $F \in \mathrm{GL}^+(m)$ be a given matrix. Then we have*

$$\frac{|F|^m}{\sqrt{m^m}} \geq \det(F). \tag{4.1}$$

Proof. Let $\sigma_1, \ldots, \sigma_m \geq 0$ be a representation of the singular values of F counted with multiplicity. The Euclidean norm of F and the determinant of F are given by

$$|F| = \sqrt{\sigma_1^2 + \cdots + \sigma_m^2} \qquad \text{and} \qquad \det(F) = \sigma_1 \cdot \cdots \cdot \sigma_m.$$

The well-known inequality of the quadratic mean and the geometric mean reads

$$\sqrt{\frac{\sigma_1^2 + \cdots + \sigma_m^2}{m}} \geq \sqrt[m]{\sigma_1 \cdot \cdots \cdot \sigma_m}.$$

This directly implies (4.1). □

The energy $\widetilde{\mathcal{E}}$ is coercive in the following sense.

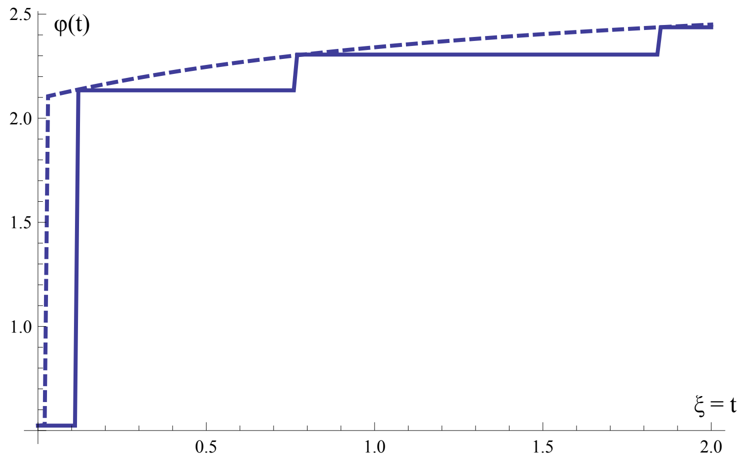

Figure 2: Evolution of φ as function of ξ. Dashed line: minimizer of W^{rel} in (3.6), solid line: solution via time-incremental evolution.

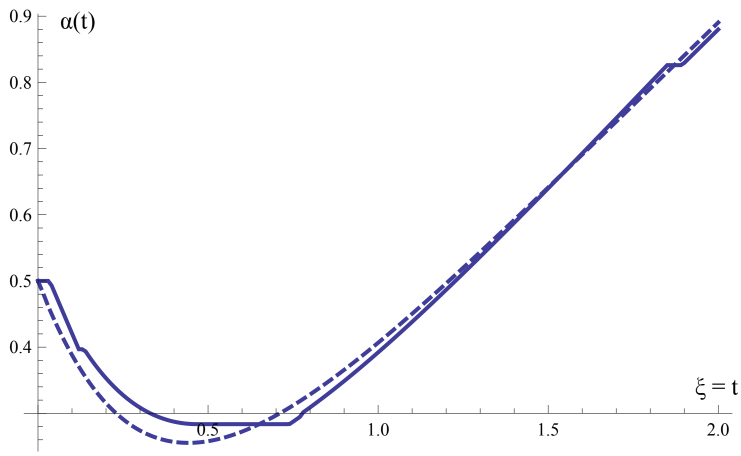

Figure 3: Evolution of α as function of ξ. Dashed line: minimizer of W^{rel} in (3.6), solid line: solution via time-incremental evolution.

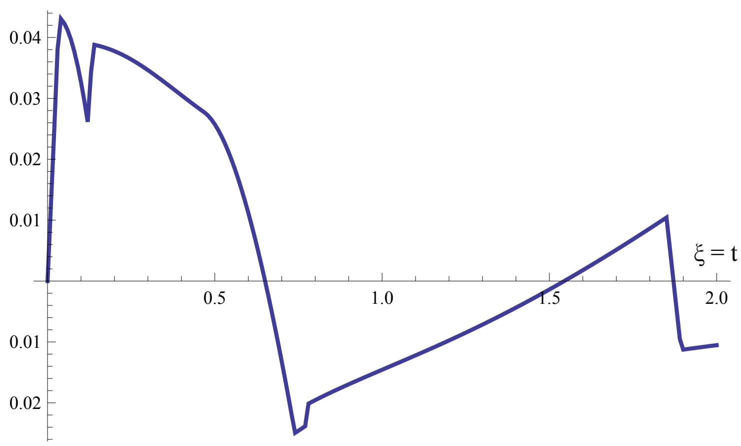

Figure 4: Difference of α as function of ξ between the minimizer of W^{rel} in (3.6) and the solution via time-incremental evolution.

Lemma 4.2. *Let the functional* $\widetilde{\mathcal{E}}$ *as well as the spaces* $\widetilde{\mathcal{Y}}$ *and* $\widetilde{\mathcal{Z}}$ *be defined as in* (1.9)*,* (2.2) *and* (2.11)*, respectively, such that the conditions* (2.1)*,* (2.3) *and* (2.7) *are fulfilled. Moreover, let* $(t_1, \phi_1, \Lambda_1), (t_2, \phi_2, \Lambda_2), \dots$ *be a sequence in* $[0,T]\times\widetilde{\mathcal{Y}}\times\widetilde{\mathcal{Z}}$ *such that*

$$\sup_{k>0} \widetilde{\mathcal{E}}(t_k, \phi_k, \Lambda_k) < \infty.$$

Then both of the following conditions hold

(i) $\sup_{k>0} \|\phi_k\|_{\mathrm{W}^{1,q_Y}} < \infty$

(ii) $\sup_{k>0} \langle\langle |(A,P)|^q, \Lambda_k\rangle\rangle < \infty$ *where* $q = \min\{q_A, q_P\}$.

Proof. Within the proof, all the quantities that depend on the times t_k can be estimated uniformly, see (2.3). Hence, we can omit t_k.

The linear loading term $\mathcal{F}$ has no influence on coercivity. We can assume that $\mathcal{F} = 0$ for simplicity. The boundedness of $\widetilde{\mathcal{E}}$ together with (2.1c) and (2.1b) implies that there exist constants $c_1, c_2, c_3 > 0$ such that for every $k > 0$ we have

$$\langle\langle |P|^{q_P}, \Lambda_k\rangle\rangle \le c_1, \tag{4.2}$$
$$\langle\langle |\nabla\phi_k(I+A)P^{-1}|^{q_F}, \Lambda_k\rangle\rangle \le c_2, \tag{4.3}$$
$$\langle\langle \det(\nabla\phi_k(I+A))^{-q_D}, \Lambda_k\rangle\rangle \le c_3. \tag{4.4}$$

Now fix $k > 0$. Since the exponents fulfill condition (2.1d), we can use Hölder's inequality together with (4.2) and (4.3) (cf. [MM06]) and conclude

$$\|\langle |\nabla\phi_k(I+A)P^{-1}|^{q_F}, \Lambda_k\rangle^{1/q_F} \cdot \langle |P|^{q_P}, \Lambda_k\rangle^{1/q_P}\|_{\mathrm{L}^{q_Y}} \le c_1^{1/q_P} c_2^{1/q_F}.$$

We apply Hölder's inequality once again and obtain that

$$\|\langle |\nabla\phi_k(I+A)|^{q_Y}, \Lambda_k\rangle^{1/q_Y}\|_{\mathrm{L}^{q_Y}} \le c_1^{1/q_P} c_2^{1/q_F}. \tag{4.5}$$

The mean value of the micro-fluctuations is 0, see (2.7b). In addition, $|.|^{q_Y}$ is convex. Hence, Jensen's inequality applied to (4.5) yields $\|\nabla\phi_k\|_{\mathrm{L}^{q_Y}} \le c_1^{1/q_P} c_2^{1/q_F}$ and we have (i) since Poincaré's inequality gives the boundedness in the W^{1,q_Y}-norm.

As an assumption of Section 2, the first component of $\Lambda_k(x)$ (the gradient part) is a homogenous gradient Young measure for almost every $x \in \Omega$. Hence, Jensen's inequality can be applied for every quasiconvex function, see [KP91], and, in particular, for $\det(.)^{-q_D}$. Then (4.4) implies that $\|\det(\nabla\phi_k)^{-1}\|_{\mathrm{L}^{q_D}} \le c_3^{1/q_D}$ holds. Following Remark 4.1, we get the estimate

$$\|1/|\nabla\phi_k|\|_{\mathrm{L}^{mq_D}} \le c_4, \tag{4.6}$$

where $c_4 > 0$ is a constant independent of k. In view of (2.1e), Hölder's inequality together with (4.5) and (4.6) bounds the micro-fluctuations

$$\sup_{k>0} \langle\langle |A|^{q_A}, \Lambda_k\rangle\rangle \le (c_1^{1/q_P} c_2^{1/q_F} c_4)^{q_A}.$$

This estimate together with (4.2) implies that

$$\sup_{k>0} \langle\langle |A|^q + |P|^q), \Lambda_k\rangle\rangle \le (c_1^{1/q_P} c_2^{1/q_F} c_4)^{q_A} + c_1.$$

In fact, this is again an application of Hölders inequality where we use that the set Z has measure 1 and $q = \min\{q_D, q_P\}$. Hence, we have (ii), since simple computation shows that the inequality $2^{1-q/2}|A+P|^q \le |A|^q + |P|^q$ holds for every $(A,P) \in Z$. $\square$

4.2 Lower Semicontinuity

The lower-semicontinuity result makes use of polyconvexity defined in terms of the minors which are given by the quantity $\mathbb{M}(\nabla\phi(I{+}A)P^{-1})$. We will use an algebraic property of minors, which is provided in the next lemma for the sake of completeness.

Lemma 4.3. *There exists a bilinear function* $\mathbb{A}\colon\, \mathbb{R}^\tau{\times}\mathbb{R}^\tau \to \mathbb{R}^\tau$ *such that*

$$\mathbb{M}(UV) = \mathbb{A}(\mathbb{M}(U), \mathbb{M}(V)) \quad \textit{for every } U, V \in \mathbb{R}^{m\times m}.$$

Proof. The determinant has the property $\det(UV) = \det(U)\det(V)$ for all matrices $U, V \in \mathbb{R}^{m\times m}$. In a similar way, the Cauchy-Binet formula gives a multiplicative rule for every minor, see, for example, Gantmacher [Gan59, Binet-Cauchy formula]. Hence, there exists a bilinear function as desired. $\square$

Now we are in the position to prove the lower semicontinuity of the energy.

Lemma 4.4. *Let the functional* $\widetilde{\mathcal{E}}$ *as well as the spaces* $\widetilde{\mathcal{Y}}$ *and* $\widetilde{\mathcal{Z}}$ *be defined as in* (1.9)*,* (2.2) *and* (2.11)*, respectively, such that the conditions* (2.1)*,* (2.3) *and* (2.7) *are fulfilled. Moreover, let* $(t_1, \phi_1, \Lambda_1), (t_2, \phi_2, \Lambda_2), \ldots$ *be a sequence in* $[0, T]{\times}\widetilde{\mathcal{Y}}{\times}\widetilde{\mathcal{Z}}$ *such that* $t_k \to t$*,* $\phi_k \to \phi$ *weakly in* $\widetilde{\mathcal{Y}}$ *and* $\Lambda_k \to \Lambda$ *in* $\widetilde{\mathcal{Z}}$ *for some* $(t, \phi, \Lambda) \in [0, T]{\times}\widetilde{\mathcal{Y}}{\times}\widetilde{\mathcal{Z}}$*. Then we have the estimate*

$$\widetilde{\mathcal{E}}(t, \phi, \Lambda) \le \liminf_{k\to\infty} \widetilde{\mathcal{E}}(t_k, \phi_k, \Lambda_k).$$

Proof. Using Lemma 2.3, we can extract a subsequence (not relabeled) such that $\Lambda_k(x) \to \Lambda(x)$ holds with respect to d_{W} for almost every $x \in \Omega$.

The condition (2.3) implies that the loading term is weakly continuous in the second variable. We conclude that $\mathcal{F}(t_k, \phi_k) \to \mathcal{F}(t, \phi)$. Moreover, we can omit the time variable. The energy $\widetilde{\mathcal{E}}$ consists of two more parts. The hardening function H is lower semicontinuous and bounded from below, see (2.1c). As a consequence of Lemma 2.1, H is also lower semicontinuous with respect to d_{W} if H is seen as a linear function on $\mathcal{L}$. Applying Fatou's lemma, we end up with

$$\langle\langle H(P), \Lambda\rangle\rangle = \int_\Omega \langle H(P), \Lambda\rangle \,\mathrm{d}x \le \int_\Omega \liminf_{k\to\infty} \langle H(P), \Lambda_k\rangle \,\mathrm{d}x \underset{\text{Fatou}}{\le} \liminf_{k\to\infty} \langle\langle H(P), \Lambda_k\rangle\rangle.$$

It remains to prove lower semicontinuity of the stored elastic energy, which means

$$\langle\langle W(\nabla\phi(I + A)P^{-1}), \Lambda\rangle\rangle \le \liminf_{k\to\infty} \langle\langle W(\nabla\phi_k(I + A)P^{-1}), \Lambda_k\rangle\rangle. \tag{4.7}$$

Lemma 4.3 implies that there exists a bilinear function $\mathbb{A}\colon\, \mathbb{R}^\tau{\times}\mathbb{R}^\tau \to \mathbb{R}^\tau$ such that

$$\mathbb{M}(F(I + A)P^{-1}) = \mathbb{A}(\mathbb{M}(F), \mathbb{M}((I + A)P^{-1})) \quad \text{for } (F, A, P) \in \mathbb{R}^\tau{\times}Z.$$

With the help of $\mathbb{A}$, we define the function $f\colon\, \mathcal{L}{\times}\mathbb{R}^\tau \to \overline{\mathbb{R}}$ via

$$f(\Lambda, M) \overset{\text{def}}{=} \langle \mathbb{W}(\mathbb{A}(M, \mathbb{M}((I + A)P^{-1}))), \Lambda\rangle \quad \text{for } (\Lambda, M) \in \mathcal{L}{\times}\mathbb{R}^\tau.$$

The function f is designed in such a way that $f(\Lambda, \mathbb{M}(F)) = \langle W(F(I + A)P^{-1}), \Lambda\rangle$ holds for every $\Lambda \in \mathcal{L}$ and every $F \in \mathbb{R}^{m\times m}$, see (2.1a) for the relation between W and $\mathbb{W}$. By definition, f is convex and lower semicontinuous in the second variable. Moreover, f is lower semicontinuous in the first variable with respect to d_{W}. This follows by Lemma 2.1.

The rest of the proof is an application of a result by Eisen [Eis79]. In fact, Eisen assumes that f is a continuous function over a Euclidean space, which is not the case in our situation. Nevertheless, on the one hand, we can relax the continuity assumption by an approximation argument similar to that in the proof of Lemma 2.1. Note that the approximation of f via inf-convolution, see (2.8), preserves the convexity in the second variable. On the other hand, a careful look at his proof shows that his ideas also work if the first variable takes values in a complete metric space. Whenever $\mathbb{M}(\nabla\phi_k) \to \mathbb{M}(\nabla\phi)$ holds weakly in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$, we conclude that

$$\int_\Omega f(\Lambda, \mathbb{M}(\nabla\phi))\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f(\Lambda_k, \mathbb{M}(\nabla\phi_k))\mathrm{d}x.$$

This is equivalent to (4.7) and finishes the proof since, by the classical weak-continuity property of minors, we have that $\mathbb{M}(\nabla\phi_k) \to \mathbb{M}(\nabla\phi)$ weakly in $\mathrm{L}^{q_Y/m}(\Omega, \mathbb{R}^m)$, see for example [Dac89, §4, Thm. 2.6]. In view of (2.1d), we know that $q_Y/m > 1$. ☐

4.3 Compactness Result

We are going to use concepts from [AT04]. The set $\Omega \subseteq \mathbb{R}^m$ is a non-empty, open, connected, and bounded set with Lipschitz boundary. In this domain Ω, we consider the open balls

$$\mathcal{B}_{x,r} = \{y \in \Omega \mid |y - x| < r\} \quad \text{where } x \in \Omega \text{ and } r > 0. \tag{4.8}$$

Let $|\mathcal{B}_{x,r}|$ denote the Lebesgue measure of the ball $\mathcal{B}_{x,r}$ and $\mathrm{diam}(\Omega) \stackrel{\text{def}}{=} \sup\{|x - y| \mid x, y \in \Omega\}$ the diameter of Ω. Then the regularity of the boundary of Ω implies that there exists a positive constant $C_{\mathcal{B}} > 0$ such that

$$|\mathcal{B}_{x,r}| \geq C_{\mathcal{B}} \cdot \gamma(r)^m \quad \text{for all } x \in \Omega \text{ and } r > 0\text{, where } \gamma(r) = \min\{r, \mathrm{diam}(\Omega)\}. \tag{4.9}$$

In particular, we have the important doubling property of measures. This means that there exists a constant $c_{\mathrm{double}} > 0$ such that

$$|\mathcal{B}_{x,2r}| \leq c_{\mathrm{double}}|\mathcal{B}_{x,r}| \quad \text{for all } x \in \Omega \text{ and } r > 0.$$

Let $\Lambda \in \widetilde{\mathcal{Z}}$ be a Young measure for which $\mathcal{G}(\Lambda)$ is finite. Then we set

$$g_\Lambda(x) = \left[\int_\Omega \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(y))^p}{|x - y|^{m+\theta p}}\mathrm{d}y\right]^{1/p} \quad \text{for all } x \in \Omega. \tag{4.10}$$

The function g_Λ is non-negative and can be seen as the norm of a generalized gradient of Λ. As a consequence of Fubini's theorem, g_Λ lies in $\mathrm{L}^p(\Omega)$ as long as $\mathcal{G}(\Lambda) < \infty$, see (2.12).

Before we present our compactness result, we prove the following result. It generalizes the theory in [AT04], where the corresponding result for the classical Sobolev space $\mathrm{W}^{1,p}(\Omega, \mathcal{L})$ is established, i.e. the case $\theta = 1$.

Lemma 4.5. *Let the functional $\mathcal{G}$ and the space $\widetilde{\mathcal{Z}}$ be defined as in* (2.12) *and* (2.11)*, respectively. There exists a positive constant $C_{d_{\mathrm{W}}} > 0$ that only depends on Ω and θ with the following property. Let $\Lambda \in \widetilde{\mathcal{Z}}$ be a given Young measure with $\mathcal{G}(\Lambda) < \infty$ and g_Λ as defined in* (4.10)*. Then we have*

$$d_{\mathrm{W}}(\Lambda(x), \Lambda(y)) \leq C_{d_{\mathrm{W}}}|x - y|^\theta (g_\Lambda(x) + g_\Lambda(y)) \quad \text{for a.a. } (x, y) \in \Omega \times \Omega. \tag{4.11}$$

Proof. It is sufficient to show (4.11) for every $(x, y) \in \Omega \times \Omega$ with $g_\Lambda(x), g_\Lambda(y) < \infty$ and $x \neq y$. Fix such a pair (x, y), set $r = |x - y|$, and consider the set given by

$$\omega_{x,y} \overset{\text{def}}{=} \mathcal{B}_{x,2r} \cap \mathcal{B}_{y,2r}.$$

The triangle inequality in $\mathbb{R}^m$ implies that $\mathcal{B}_{x,r} \subseteq \omega_{x,y}$. Using (4.9), we have

$$|\omega_{x,y}| \geq C_{\mathcal{B}} |x - y|^m. \tag{4.12}$$

In fact, here we know that $\gamma(r) = r$ since $r = |x - y| \leq \operatorname{diam}(\Omega)$. Now we apply the triangle inequality for d_{W} and obtain the following estimate after integration

$$d_{\mathrm{W}}(\Lambda(x), \Lambda(y)) \leq \frac{1}{|\omega_{x,y}|} \left[\int_{\omega_{x,y}} d_{\mathrm{W}}(\Lambda(x), \Lambda(z)) \mathrm{d}z + \int_{\omega_{x,y}} d_{\mathrm{W}}(\Lambda(z), \Lambda(y)) \mathrm{d}z \right]. \tag{4.13}$$

The integrals on the right-hand side of (4.13) can be bounded from above as follows. We use (4.12) and we add non-negative terms in order to find that

$$d_{\mathrm{W}}(\Lambda(x), \Lambda(y)) \leq \frac{r^{-m}}{C_{\mathcal{B}}} \left[\int_{\mathcal{B}_{x,2r}} d_{\mathrm{W}}(\Lambda(x), \Lambda(z)) \mathrm{d}z + \int_{\mathcal{B}_{y,2r}} d_{\mathrm{W}}(\Lambda(z), \Lambda(y)) \mathrm{d}z \right]. \tag{4.14}$$

We deal with the two integrals separately. We write the first integrand as a product

$$\int_{\mathcal{B}_{x,2r}} d_{\mathrm{W}}(\Lambda(x), \Lambda(z)) \mathrm{d}z = \int_{\mathcal{B}_{x,2r}} \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(z))}{|x - z|^{\theta + m/p}} \cdot |x - z|^{\theta + m/p} \mathrm{d}z$$

and apply Hölder's inequality. In doing so, we find that the integral is bounded by

$$\left[\int_{\mathcal{B}_{x,2r}} \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(z))^p}{|x - z|^{m + \theta p}} dz \right]^{\frac{1}{p}} \cdot \left[|\mathcal{B}_{x,2r}| (2r)^{(\theta + m/p)p'} \right]^{\frac{1}{p'}}.$$

Here p' denotes the conjugate exponent to p such that $1/p + 1/p' = 1$. The first factor is bounded by $g_\Lambda(x)$. Hence, using $\gamma(2r) \leq 2r$ together with (4.9) implies that

$$\int_{\mathcal{B}_{x,2r}} d_{\mathrm{W}}(\Lambda(x), \Lambda(z)) \mathrm{d}z \leq g_\Lambda(x) \left[c_1 \cdot (2r)^m \right]^{1/p'} (2r)^{\theta + m/p} \leq c_1 \cdot (2r)^{\theta + m} g_\Lambda(x)$$

for some positive constant $c_1 > 0$ that does not depend on Λ. The second integral in (4.14) can be bounded analogously by interchanging x and y. Finally, recalling $r = |x - y|$, we end up with the following estimate

$$d_{\mathrm{W}}(\Lambda(x), \Lambda(y)) \leq \frac{r^{-m}}{C_{\mathcal{B}}} \cdot c_1 \cdot (2r)^{\theta + m} (g_\Lambda(x) + g_\Lambda(y)) = C_{d_{\mathrm{W}}} |x - y|^\theta (g_\Lambda(x) + g_\Lambda(y)),$$

where we have set $C_{d_{\mathrm{W}}} = 2^{\theta + m} c_1 / C_{\mathcal{B}}$. This finishes the proof. □

Now we can prove the compactness result, as the main ingredient for the proof of the existence of solutions to the incremental minimization problem for $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{E}}_{\mathrm{reg}}, \widetilde{\mathcal{D}})$.

Lemma 4.6. *Let the functions $\widetilde{\mathcal{E}}$ and $\mathcal{G}$ as well as the spaces $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Z}}$ be defined as in* (1.9)*,* (2.12)*,* (2.2) *and* (2.11)*, respectively, such that the conditions* (2.1)*,* (2.3)*,* (2.4) *and* (2.7) *are fulfilled. Set $\widetilde{\mathcal{E}}_{\mathrm{reg}} = \widetilde{\mathcal{E}} + \mathcal{G}$.*
Let $t_* \in [0,T]$ be a given time and let $(\phi_1,\Lambda_1),(\phi_2,\Lambda_2),\ldots$ be a sequence in $\widetilde{\mathcal{Y}}\times\widetilde{\mathcal{Z}}$. If the regularized energy $\widetilde{\mathcal{E}}_{\mathrm{reg}}$ stays uniformly bounded, that means

$$\sup_{k>0} \widetilde{\mathcal{E}}_{\mathrm{reg}}(t_*,\phi_k,\Lambda_k) < \infty,$$

then there exists a subsequence (not relabeled), a deformation field $\phi_0 \in \widetilde{\mathcal{Y}}$ and a Young measure $\Lambda_0 \in \widetilde{\mathcal{Z}}$ such that all the following conditions are fulfilled

(i) $\phi_k \to \phi_0$ *weakly in* $\widetilde{\mathcal{Y}}$

(ii) $\Lambda_k \to \Lambda_0$ *in* $\widetilde{\mathcal{Z}}$

(iii) $\Lambda_k(x) \to \Lambda_0(x)$ *pointwise with respect to* d_{W} *for almost every* $x \in \Omega$

(iv) $\mathcal{G}(\Lambda_0) \le \liminf\limits_{k\to\infty} \mathcal{G}(\Lambda_k)$.

Proof. Within the proof, all the quantities that depend on time are evaluated at an arbitrary but fixed time t_* and, hence, we can omit t_*. We are going to extract subsequences of $(\phi_1,\Lambda_1),(\phi_2,\Lambda_2),\ldots$ in an iterative way. In order to shorten notation, we shall keep the original labeling.

We apply the sequential form of the Banach-Alaoglu theorem. Lemma 4.2 implies that the sequence $\phi_1,\phi_2,\ldots$ is uniformly bounded in $\widetilde{\mathcal{Y}}$. Hence, by the Banach-Alaoglu theorem, there exists a deformation $\phi_0 \in \widetilde{\mathcal{Y}}$ and a subsequence such that $\phi_k \to \phi_0$ weakly in $\widetilde{\mathcal{Y}}$. We have (i). In addition, Lemma 4.2 also implies that the Young measures $\Lambda_1,\Lambda_2,\ldots$ lie in the space $\mathrm{YM}^q(\Omega,Z)$ with uniformly bounded averaged qth moment. In particular, for some constant $C>0$ we have that

$$\sup_{k>0} \langle\langle |(A,P)|^q, \Lambda_k\rangle\rangle \le C. \tag{4.15}$$

By Lemma 2.2, there exists a Young measure $\Lambda_0 \in \mathrm{YM}^q(\Omega,Z)$ and a further subsequence such that $\Lambda_k \to \Lambda_0$ weakly* and $\langle\langle |(A,P)|^q,\Lambda_0\rangle\rangle \le C$.

As a first step towards showing (ii), we are going to analyze spatial averages of the Young measures $\Lambda_0,\Lambda_1,\ldots$. Therefore, fix a radius $r>0$, a point $x\in\Omega$, an integer $k\ge 0$ and define the average $\mu_k^r(x)$ given by

$$\langle g,\mu_k^r(x)\rangle = \frac{1}{|\mathcal{B}_{x,r}|}\int_{\mathcal{B}_{x,r}} \langle g,\Lambda_k(y)\rangle \mathrm{d}y \quad \text{for all } g\in \mathrm{C}_0(Z).$$

See (4.8) for the definition of $\mathcal{B}_{x,r}$. Then $\mu_k^r(x)$ lies in $\mathcal{P}(Z)$. Due to (4.9) and (4.15), we have the following estimate

$$\langle |(A,P)|,\mu_k^r(x)\rangle = \frac{1}{|\mathcal{B}_{x,r}|}\int_{\mathcal{B}_{x,r}} \langle |(A,P)|,\Lambda_k(y)\rangle \mathrm{d}y \le C C_{\mathcal{B}}^{-1}\gamma(r)^{-m}.$$

In view of (2.4), we conclude that

$$\langle d_Z((A,P),E_Z),\mu_k^r(x)\rangle \le \tau_0 + (\tau_1+1)CC_{\mathcal{B}}^{-1}\gamma(r)^{-m} \quad \text{for all integers } k\ge 0. \tag{4.16}$$

That means that $\mu_k^r(x)$ lies in $\mathcal{P}^1(\Omega)$ and its first moment is bounded uniformly with respect to k. Moreover, since we know that $\Lambda_k \to \Lambda_0$ weakly*, we get that $\mu_k^r(x) \to \mu_0^r(x)$ weakly*. This implies that $d_{\mathrm{W}}(\mu_k^r(x), \mu_0^r(x)) \to 0$ holds for every $x \in \Omega$ and $r > 0$, see [AGS05, Prop. 7.1.5].

The triangle inequality for d_{W} gives the estimate

$$d_{\mathrm{W}}(\mu_k^r(x), \mu_0^r(x)) \leq d_{\mathrm{W}}(\mu_k^r(x), \delta_{E_Z}) + d_{\mathrm{W}}(\mu_0^r(x), \delta_{E_Z}).$$

Both terms in the right-hand side can be computed explicitly via the equation

$$d_{\mathrm{W}}(\mu_k^r(x), \delta_{E_Z}) = \langle d_Z((A,P), E_Z), \mu_k^r(x)\rangle.$$

Then (4.16) bounds the quantity $d_{\mathrm{W}}(\mu_k^r(x), \mu_0^r(x))$ independently of $x \in \Omega$ and k. Together with the pointwise convergence, we obtain convergence in $\widetilde{\mathscr{Z}}$, meaning that

$$\operatorname{dist}_p(\mu_k^r, \mu_0^r) = \|d_{\mathrm{W}}(\mu_k^r(x), \mu_0^r(x))^p\|_{\mathrm{L}^p} \to 0 \quad \text{for all } r > 0. \tag{4.17}$$

Next, we are going to bound the quantity $\operatorname{dist}_p(\Lambda_k, \mu_k^r)$. Therefore, fix a positive integer $k > 0$ and a radius $\operatorname{diam}(\Omega) > r > 0$. The following holds for almost every $x \in \Omega$. By the definition of the Wasserstein distance, we know that

$$\begin{aligned} d_{\mathrm{W}}(\Lambda_k(x), \mu_k^r(x)) &= \sup_{\mathrm{Lip}(g)\leq 1} \tfrac{1}{|\mathcal{B}_{x,r}|} \int_{\mathcal{B}_{x,r}} [\langle g, \Lambda_k(x)\rangle - \langle g, \Lambda_k(y)\rangle] \mathrm{d}y \\ &\leq \tfrac{1}{|\mathcal{B}_{x,r}|} \int_{\mathcal{B}_{x,r}} d_{\mathrm{W}}(\Lambda_k(x), \Lambda_k(y)) \mathrm{d}y \\ &\leq \tfrac{1}{|\mathcal{B}_{x,r}|} \int_{\mathcal{B}_{x,r}} C_{d_{\mathrm{W}}} |x-y|^\theta (g_{\Lambda_k}(x) + g_{\Lambda_k}(y)) \mathrm{d}y. \end{aligned}$$

In the last line, we have used Lemma 4.5 where g_{Λ_k} is given by (4.10). Clearly, the term $|x - y|$ is bounded by $\operatorname{diam}(\Omega)$. With the help of (4.9), we conclude that

$$d_{\mathrm{W}}(\Lambda_k(x), \mu_k^r(x)) \leq \frac{C_{d_{\mathrm{W}}}\, r^\theta}{C_{\mathcal{B}}\, r^d} \int_{\mathcal{B}_{x,r}} (g_{\Lambda_k}(x) + g_{\Lambda_k}(y)) \mathrm{d}y.$$

If we extend the function g_{Λ_k} by 0, it can be seen as an element of $\mathrm{L}^p(\mathbb{R}^m)$. We set $z = y - x$ as well as $B_r \overset{\text{def}}{=} \{y \in \mathbb{R}^m \mid |y| < r\}$ and find that

$$d_{\mathrm{W}}(\Lambda_k(x), \mu_k^r(x)) \leq \frac{C_{d_{\mathrm{W}}}\, r^\theta}{C_{\mathcal{B}}\, r^d} \int_{B_r} (g_{\Lambda_k}(x) + g_{\Lambda_k}(x+z)) \mathrm{d}z.$$

The function $|.|^p$ is convex. Now if we integrate and apply Jensen's inequality to it, we get that

$$\operatorname{dist}_p(\Lambda_k, \mu_k^r) \leq \left(\frac{C_{d_{\mathrm{W}}}\, r^\theta}{C_{\mathcal{B}}\, r^m} \int_\Omega \int_{B_r} |g_{\Lambda_k}(x) + g_{\Lambda_k}(x+z)|^p \mathrm{d}z \mathrm{d}x \right)^{1/p}.$$

We can change the order of integration by Fubini's theorem. In order to get the integral uniformly bounded for all integers $k > 0$, we use that there exists a constant $C_{\mathcal{G}} > 0$ such that $\mathcal{G}(\Lambda_k) \leq C_{\mathcal{G}}$, which follows because the energy is uniformly bounded. Hence, $\|g_{\Lambda_k}\|_{\mathrm{L}^p} \leq C_{\mathcal{G}}^{1/p}$ holds for every integer $k > 0$ and we conclude that

$$\operatorname{dist}_p(\Lambda_k, \mu_k^r) \leq \left(\frac{C_{d_{\mathrm{W}}}\, r^\theta}{C_{\mathcal{B}}\, r^m} |B_r|\, C_{\mathcal{G}}\, 2^p \right)^{1/p} = c_1\, r^{\theta/p} \tag{4.18}$$

where $c_1 > 0$ is a constant that is independent of r and k.

We show that $\mathrm{dist}_p(\mu_0^r, \Lambda_0) \to 0$ as r tends to 0. Fix a function $g \in \mathrm{C}_0(Z)$. For every radius $r > 0$ and every point $x \in \Omega$, we know that

$$\langle g, \mu_0^r(x)\rangle = \frac{1}{|\mathcal{B}_{x,r}|} \int_{\mathcal{B}_{x,r}} \langle g, \Lambda_0(y)\rangle \mathrm{d}y.$$

The Lebesgue differentiation theorem implies that we obtain $\langle g, \mu_0^r(x)\rangle \to \langle g, \Lambda_0(x)\rangle$ as r tends to 0. Hence, we find that $\mu_0^r(x) \to \Lambda_0(x)$ weakly* for almost every $x \in \Omega$. Fix two radii $r_1, r_2 > 0$. For every integer $k > 0$, the triangle inequality for dist_p implies that

$$\mathrm{dist}_p(\mu_0^{r_1}, \mu_0^{r_2}) \leq \mathrm{dist}_p(\mu_0^{r_1}, \mu_k^{r_1}) + \mathrm{dist}_p(\mu_k^{r_1}, \Lambda_k) + \mathrm{dist}_p(\Lambda_k, \mu_k^{r_2}) + \mathrm{dist}_p(\mu_k^{r_2}, \mu_0^{r_2}).$$

Applying the estimates (4.17) and (4.18) and taking the limit $k \to \infty$ gives

$$\mathrm{dist}_p(\mu_0^{r_1}, \mu_0^{r_2}) \leq c_1(r_1^{\theta/p} + r_2^{\theta/p}).$$

Thus, μ_0^r forms a Cauchy sequence in $\widetilde{\mathcal{Z}}$ as r tends to 0. The convergence with respect to d_{W} implies weak* convergence, see [AGS05, Prop. 7.1.5]. Since we already know that $\mu_0^r(x) \to \Lambda_0(x)$ weakly* for almost every $x \in \Omega$, we conclude that Λ_0 is the limit of the Cauchy sequence. Moreover, we get the estimate

$$\mathrm{dist}_p(\mu_0^r, \Lambda_0) \leq c_1 r^{\theta/p}. \tag{4.19}$$

Now we are in the position to show (ii). For every radius $r > 0$ and every integer $k > 0$, the triangle inequality for dist_p implies that

$$\mathrm{dist}_p(\Lambda_k, \Lambda_0) \leq \mathrm{dist}_p(\Lambda_k, \mu_k^r) + \mathrm{dist}_p(\mu_k^r, \mu_0^r) + \mathrm{dist}_p(\mu_0^r, \Lambda_0).$$

As a consequence of (4.17), (4.18) and (4.19), we obtain $\mathrm{dist}_p(\Lambda_k, \Lambda_0) \to 0$, which is (ii). Eventually extracting a further subsequence, we get (iii) by Lemma 2.3. Finally, (iv) follows by (iii) if we apply Fatou's lemma. ☐

4.4 Existence of Solutions

Now we are in the position to prove the existence of solutions to the time incremental minimization problem with regularized energy. We follow the strategy of the direct methods in the calculus of variations.

Theorem 4.7. *Let the functions $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{D}}$ and $\mathcal{G}$ as well as the spaces $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Z}}$ be defined as in* (1.9)*,* (1.10)*,* (2.12)*,* (2.2) *and* (2.11)*, respectively, such that the conditions* (2.1)*,* (2.3)*,* (2.4) *and* (2.7) *are fulfilled. Set $\widetilde{\mathcal{E}}_{\mathrm{reg}} = \widetilde{\mathcal{E}} + \mathcal{G}$. Moreover, assume that $\widetilde{\mathcal{D}} \colon \widetilde{\mathcal{Z}} \times \widetilde{\mathcal{Z}} \to [0, \infty]$ is lower semicontinuous.*
Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a finite partition of $[0, T]$ and $(\phi_0, \Lambda_0) \in \widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$ an initial state. Then the regularized incremental minimization problem for $(\widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}, \widetilde{\mathcal{E}}_{\mathrm{reg}}, \widetilde{\mathcal{D}})$, which is to solve iteratively, for every $l = 1, 2, \ldots, N$,

$$(\phi_l, \Lambda_l) \in \mathrm{Argmin}\left[\widetilde{\mathcal{E}}_{\mathrm{reg}}(t_l, \phi, \Lambda) + \widetilde{\mathcal{D}}(\Lambda_{l-1}, \Lambda)\right],$$

admits solutions in $\widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$.

Proof. Fix a positive integer $l \in \{1, 2, \ldots, N\}$ and assume that the pairs (ϕ_0, Λ_0), ..., $(\phi_{l-1}, \Lambda_{l-1})$ form a solution for the times $t_0, \ldots, t_{l-1}$. Let $(\phi_{l,1}, \Lambda_{l,1})$, $(\phi_{l,2}, \Lambda_{l,2})$, $\ldots \in \widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$ be an infimizing sequence for the sum $\widetilde{\mathcal{E}}_{\mathrm{reg}}(t_l, \phi, \Lambda) + \widetilde{\mathcal{D}}(\Lambda_{l-1}, \Lambda)$. As a consequence of Lemma 4.4 and Lemma 4.6(iv), the energy $\widetilde{\mathcal{E}}_{\mathrm{reg}}$ is lower semicontinuous. Moreover, $\widetilde{\mathcal{D}}$ is assumed to be lower semicontinuous, too. Together with the compactness result of Lemma 4.6, this implies that there exists a pair $(\phi_l, \Lambda_l) \in \widetilde{\mathcal{Y}} \times \widetilde{\mathcal{Z}}$ such that

$$\widetilde{\mathcal{E}}_{\mathrm{reg}}(t_l, \phi_l, \Lambda_l) + \widetilde{\mathcal{D}}(\Lambda_{l-1}, \Lambda_l) = \liminf_{k \to \infty} \left[\widetilde{\mathcal{E}}_{\mathrm{reg}}(t_l, \phi_{l,k}, \Lambda_{l,k}) + \widetilde{\mathcal{D}}(\Lambda_{l-1}, \Lambda_{l,k}) \right].$$

Hence, the pairs $(\phi_0, \Lambda_0), \ldots, (\phi_l, \Lambda_l)$ form a solution of the incremental minimization problem for the times $t_0, \ldots, t_l$. The rest of the proof follows by induction over l. $\square$

A Wasserstein Distance in $\mathcal{L}_{\mathrm{s}}$

Let $\mu_0, \mu_1 \in \mathcal{P}^1(Z)$ two probability measures with finite first moment. Then the 1-Wasserstein distance is given by

$$d_{\mathrm{W}}(\mu_0, \mu_1) \overset{\text{def}}{=} \min_{\mu \in \Gamma(\mu_0, \mu_1)} \int_{Z \times Z} d_Z(\Theta_0, \Theta_1) \mathrm{d}\mu(\Theta_0, \Theta_1) \tag{A.1}$$

where $\Gamma(\mu_0, \mu_1) \subseteq \mathcal{P}^1(Z \times Z)$ denotes the set of transport plans between μ_0 and μ_1, see, for example, [AGS05, §5.2, §7.1]. The equivalence between (2.6) and (A.1) follows by the duality theorem of Kantorovich and Rubinstein, see, for example, [Rac91, §5.3].

In the special situation of simple laminates $\Lambda_0, \Lambda_1 \in \mathcal{L}_{\mathrm{s}}$, every element in $\Gamma(\Lambda_0, \Lambda_1)$ is given by a doubly stochastic 2×2-matrix. Consequently, the set $\Gamma(\Lambda_0, \Lambda_1)$ can be parameterized by one real variable λ such that the Wasserstein distance d_{W} defined by (A.1) reduces to

$$d_{\mathrm{W}}(\Lambda_0, \Lambda_1) = \inf \left[\begin{array}{l} \lambda d_Z((A_0, Q_0), (A_1, Q_1)) + (1 - \alpha_0 - \alpha_1 + \lambda) d_Z((B_0, R_0), (B_1, R_1)) \\ + (\alpha_0 - \lambda) d_Z((A_0, Q_0), (B_1, R_1)) + (\alpha_1 - \lambda) d_Z((B_0, R_0), (A_1, Q_1)) \end{array} \right] \tag{A.2}$$

where the infimum is taken over all λ such that the numbers λ, $\alpha_0 - \lambda$, $\alpha_1 - \lambda$ and $1 - \alpha_0 - \alpha_1 + \lambda$ lie in $[0, 1]$. Since (2.6) and (A.1) are equivalent, the value of $d_{\mathrm{W}}(\Lambda_0, \Lambda_1)$ in (A.2) does not depend on the representations of Λ_0 and Λ_1 that we have chosen via (3.2).

A.1 Separability and Completeness of $(\mathcal{L}_{\mathrm{s}}, d_{\mathrm{W}})$

For every integer $i_0 \geq 2$ we consider the set $\mathcal{P}[i_0](Z))$ of all probability measures $\mu \in \mathcal{P}^1(Z))$ that are supported on at most i_0 points. Clearly, $\mathcal{L}_{\mathrm{s}}$ is a subset of $\mathcal{P}[2](Z))$. Note also that $\mathcal{P}[1](Z))$ can be identified with Z.

Lemma A.1. *If (Z, d_Z) is a complete and separable metric space, so is, for every integer $i_0 \geq 2$, the space $(\mathcal{P}[i_0](Z), d_{\mathrm{W}})$.*

Proof. Using [AGS05, Proposition 7.1.5], the set $\mathcal{P}^1(Z)$ of probability measures is separable and complete with respect to the Wasserstein distance d_{W}. Clearly, the subset $\mathcal{P}[i_0](Z)$ is separable.

It remains to show that $\mathcal{P}[i_0](Z)$ is a closed subset with respect to d_{W}. Assume that this is not the case. Then there exist probability measures $\mu_1, \mu_2, \ldots \in \mathcal{P}[i_0](Z)$ and $\mu \in \mathcal{P}^1(Z)$ such that $\mu_k \to \mu$ with respect to d_{W} and μ fulfills the following property: There exist positive real numbers $\rho, \epsilon > 0$ and points $\Theta_1, \ldots, \Theta_{i_0+1} \in Z$ such that the balls $\mathcal{B}_{\Theta_i,\rho} = \{\Xi \in \mathbb{R}^{m\times m} \mid d_Z(\Xi, \Theta_i) < \rho\}$, for $i = 1, \ldots, i_0+1$, are pairwise disjoint and

$$\langle \chi_{\mathcal{B}_{\Theta_i,\rho/2}}, \mu\rangle > \epsilon \quad \text{for all } i = 1, \ldots, i_0 + 1 \tag{A.3}$$

where $\chi_{\mathcal{B}_{\Theta_i,\rho/2}}$ denotes the characteristic function of the ball $\mathcal{B}_{\Theta_i,\rho/2}$. Condition (A.3) characterizes the case where the support of μ consists of (at least) $i_0 + 1$ points.

Consider the functions $g_1, \ldots, g_{i_0+1} \colon Z \to \mathbb{R}$ given by

$$g_i(\Xi) = \begin{cases} \rho/2 & \text{for } d_Z(\Xi, \Theta_i) \leq \rho/2 \\ \rho - d_Z(\Xi, \Theta_i) & \text{for } \rho/2 < d_Z(\Xi, \Theta_i) \leq \rho \\ 0 & \text{for } \rho < d_Z(\Xi, \Theta_i). \end{cases}$$

The functions $g_1, \ldots, g_{i_0}$ are non-negative and Lipschitz continuous with Lipschitz constant 1. Moreover, by construction, $g_1, \ldots, g_{i_0}$ have pairwise disjoint support and (A.3) implies that $\langle g_i, \mu\rangle > \epsilon\rho/2$ for every $i = 1, \ldots, i_0 + 1$. We know that $\mu_k \to \mu$ holds with respect to d_{W}. Hence, there exists an index k_0 such that $\langle g_i, \mu_{k_0}\rangle > \epsilon\rho/4$ holds for $i = 1, \ldots, i_0 + 1$, simultaneously. This forms a contradiction since μ_{k_0} is supported on at most i_0 points only. ☐

Using the same tools as in the proof of the previous lemma, we can show the following.

Remark A.2. *Let $i_0 \geq 2$ be a fixed integer and $\mu_0, \mu_1, \mu_2, \ldots \in \mathcal{P}[i_0](Z)$ probability measures such that $\mu_0 \notin \mathcal{P}[i_0-1](Z)$ and $\mu_k \to \mu_0$ with respect to the Wasserstein distance d_{W}. Then, for every integer $k \geq 0$, there exist a representation*

$$\mu_k = \sum_{i=1}^{i_0} \alpha_k^i \delta_{\Theta_k^i}$$

where $\alpha_k^1, \ldots, \alpha_k^{i_0}$ lie in $[0, 1]$ and $\Theta_k^1, \ldots, \Theta_k^{i_0}$ lie in Z such that $\alpha_k^i \to \alpha_0^i$ holds with respect to the Euclidean norm in $\mathbb{R}$ and $\Theta_k^i \to \Theta_0^i$ holds with respect to the distance d_Z on Z for every $i = 1, \ldots, i_0$.

Lemma A.3. *If (Z, d_Z) is a complete and separable metric space, so is $(\mathcal{L}_{\mathrm{s}}, d_{\mathrm{W}})$.*

Proof. In view of Lemma A.1, it remains to show that $\mathcal{L}_{\mathrm{s}}$ is a closed subset of $\mathcal{P}[2](Z)$ with respect to d_{W}. Let $\Lambda_1, \Lambda_2 \ldots \in \mathcal{L}_{\mathrm{s}}$ be simple laminates and $\mu \in \mathcal{P}[2](Z)$ a measure such that $\Lambda_k \to \mu$ with respect to d_{W}.

First, consider the function $G\colon Z \to \mathbb{R}^{m\times m}$ given by $G(F, P) = F$. This function is Lipschitz continuous with respect to d_{W}. The definition of the Wasserstein distance (2.6) implies that $\langle G, \Lambda_k\rangle \to \langle G, \mu\rangle$. The quantity $\langle G, \Lambda_k\rangle$ is equal to the mean value of the first part of Λ_k. We conclude that the mean value of the first part of μ is 0 since it is 0 for all $\Lambda_1, \Lambda_2, \ldots$.

Second, we show that μ can be represented in the form

$$\mu = \alpha\delta_{(A,Q)} + (1 - \alpha)\delta_{(B,R)} \tag{A.4}$$

where the matrices A and B are rank-one connected. Suppose that μ lies in $\mathcal{P}[1](Z)$. Having in mind the above result, we can write $\mu = 1 \cdot \delta_{(0,Q)} + 0 \cdot \delta_{(0,Q)}$ for some $Q \in \mathrm{SL}(m)$. Now suppose that μ does not lie in $\mathcal{P}[1](Z)$. Then we apply Remark A.2. Since the rank is a lower semicontinuous function with respect to the Euclidean norm, we conclude that there exists a representation of the form (A.4) with A and B rank-one connected.

We put both results together. The mean value of the first part of μ is 0. Hence, we know that

$$\alpha A + (1-\alpha)B = 0. \tag{A.5}$$

Since A and B are rank-one connected, there exist vectors $a, n \in \mathbb{R}^m$ such that $|n| = 1$ and

$$A - B = a \otimes n. \tag{A.6}$$

The unique common solution of the equations (A.5) and (A.6) is $A = (1-\alpha)a\otimes n$ and $B = -\alpha a \otimes n$. As a consequence, we have shown that μ lies in $\mathcal{L}_{\mathrm{s}}$. ☐

B Topology on Z

We are going to show that the distance D_{SL} given by (3.1) is a possible choice for d_{SL}. In particular, the properties (2.4a) and (2.4b) are fulfilled for $d_{\mathrm{SL}} = D_{\mathrm{SL}}$. Results in this direction can also be found in [Mie02, HMM03].Within the proofs, we will assume that $\sigma_{\mathrm{yield}} = 1$ in order to simplify the formulas. The assertions are true for every $\sigma_{\mathrm{yield}} > 0$ if they are true for $\sigma_{\mathrm{yield}} = 1$.

Lemma B.1. *The distance D_{SL} given by* (3.1) *fulfills the condition* (2.4a).

Proof. We have to show that

$$\lim_{k\to\infty} |P_k - I| = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} D_{\mathrm{SL}}(P_k, I) = 0$$

for every sequence $P_1, P_2, \ldots$ in $\mathrm{SL}(m)$. In fact, it is sufficient to investigate the situation around the identity matrix I because the following two facts are true for every matrices $P, P_1, P_2, \ldots$ in $\mathrm{SL}(m)$: On the one hand, we know that $D_{\mathrm{SL}}(P_k, P) = D_{\mathrm{SL}}(P_k P^{-1}, I)$ holds by definition. On the other hand, we have the estimate

$$|P_k P^{-1} - I| \frac{1}{|P^{-1}|} \leq |P_k - P| \leq |P_k P^{-1} - I|\,|P|$$

since the Euclidean norm in $\mathbb{R}^{m\times m}$ is sub-multiplicative, that is, the estimate $|AB| \leq |A|\,|B|$ holds for every matrices $A, B \in \mathbb{R}^{m\times m}$. Hence, $|P_k - P|$ tends to 0 if and only if $|P_k P^{-1} - I|$ does.

First, we show the direction "$\Leftarrow$". Fix an integer $k > 0$. We apply again the fact that the Euclidean norm is sub-multiplicative and we get that

$$D_{\mathrm{SL}}(P_k, I) = \inf_{P\in\mathcal{S}(P_k,I)} \int_0^1 |\dot{P}(s)P(s)^{-1}|\mathrm{d}s \geq \inf_{P\in\mathcal{S}(P_k,I)} \int_0^1 |\dot{P}(s)|\frac{1}{|P(s)|}\mathrm{d}s. \tag{B.1}$$

For every path $P \in \mathcal{S}(P_k, I)$ we consider the function $p\colon [0,1] \to \mathbb{R}$ given by $p(s) = |P(s)|$. The triangle inequality for the Euclidean norm implies that $|\dot{P}| \geq \dot{p}$ holds. We conclude that

$$D_{\mathrm{SL}}(P_k, I) \geq \inf_{P\in\mathcal{S}(P_0,P_1)} \int_0^1 \dot{p}(s)p(s)^{-1}\mathrm{d}s = \ln(|P_k|) - \ln(|I|). \tag{B.2}$$

Let $P \in \mathcal{S}(P_k, I)$ be any path such that

$$\int_0^1 |\dot{P}(s)P(s)^{-1}|\mathrm{d}s \leq 2D_{\mathrm{SL}}(P_k, I).$$

Then, for every real number $s \in [0,1]$, the estimates (B.2) implies that $|P(s)| \leq C$ holds where $C = \mathrm{e}^{2D_{\mathrm{SL}}(P_k,I)+\ln(|I|)}$ is constant. In view of (B.1), we end up with

$$D_{\mathrm{SL}}(P_k, I) \geq C^{-1} \inf_{P\in\mathcal{S}(P_k,I)} \int_0^1 |\dot{P}(s)|\mathrm{d}s \underset{\text{Jensen}}{\geq} C^{-1}|P_k - I| \tag{B.3}$$

where we have also applied Jensen's inequality. Hence, if $D_{\mathrm{SL}}(P_k, I)$ tends to 0, so does $|P_k - I|$.

Second, we show the direction "$\Rightarrow$". For every integer $k > 0$ we consider the polar decomposition $P_k = U_k S_k$. Here $S_k \in \mathbb{R}^{m\times m}$ is a symmetric positive-definite matrix and U_k lies in the special orthogonal group

$$\mathrm{SO}(m) = \{U \in \mathbb{R}^{m\times m} \mid U^T U = I \text{ and } \det(U) = 1\}.$$

Both the Euclidean norm and the distance D_{SL} are invariant under the multiplication with an element of $\mathrm{SO}(m)$. Indeed, it is not hard to show that

$$D_{\mathrm{SL}}(P_0, P_1) = D_{\mathrm{SL}}(UP_0\tilde{U}, UP_1\tilde{U}) \quad \text{and} \quad |P_0 - P_1| = |UP_0\tilde{U} - UP_1\tilde{U}| \tag{B.4}$$

holds for every $P_0, P_1 \in \mathrm{SL}(m)$ and $U, \tilde{U} \in \mathrm{SO}(m)$. In particular, we get the inequality

$$D_{\mathrm{SL}}(P_k, I) = D_{\mathrm{SL}}(S_k, U_k^{-1}) \leq D_{\mathrm{SL}}(S_k, I) + D_{\mathrm{SL}}(U_k, I) \quad \text{for all } k > 0. \tag{B.5}$$

If $|P_k - I|$ tends to 0, both quantities $|S_k - I|$ and $|U_k - I|$ tend to 0, too. Otherwise, there would be a subsequence (not relabeled), a matrix $\tilde{U} \in \mathrm{SO}(m) \setminus \{I\}$ and a symmetric positive-semidefinite matrix $\tilde{S} \in \mathbb{R}^{m\times m} \setminus \{I\}$ such that $|U_k - \tilde{U}| \to 0$ and $|S_k - \tilde{S}| \to 0$. We get that $\tilde{U}\tilde{S} = I$. Yet, this is impossible since the unique polar decomposition of the identity matrix is $I = I \cdot I$.

Let $\mathrm{J}(U_k)$ and $\mathrm{J}(S_k)$ be the Jordan normal forms of U_k and S_k, respectively, for every integer $k > 0$. Note that the Jordan normal form is not unique. However, the argument works with any fixed order of the Jordan blocks. In view of (B.4) and (B.5), it remains to show that both the following conditions hold

(i) $\lim_{k\to\infty} |\mathrm{J}(U_k) - I| = 0 \quad \Rightarrow \quad \lim_{k\to\infty} D_{\mathrm{SL}}(\mathrm{J}(U_k), I) = 0,$

(ii) $\lim_{k\to\infty} |\mathrm{J}(S_k) - I| = 0 \quad \Rightarrow \quad \lim_{k\to\infty} D_{\mathrm{SL}}(\mathrm{J}(S_k), I) = 0.$

Fix an integer $k > 0$. The matrices U_k and S_k are both normal. In particular, the spectral theorem can be applied to them. Hence, the Jordan normal form $\mathrm{J}(U_k)$ of U_k consists of Jordan blocks

$$Z(\beta_i) \overset{\text{def}}{=} \begin{pmatrix} \cos(\beta_i) & -\sin(\beta_i) \\ \sin(\beta_i) & \cos(\beta_i) \end{pmatrix}$$

in a fixed order where $\beta_i \in [-\pi, \pi)$ is some angle for every $1 \leq i \leq m/2$. If the dimension m is odd, there is an additional Jordan block (1), a 1×1-block. The Jordan normal form $\mathrm{J}(S_k)$ of S_k is even simpler, it is a diagonal matrix: $\mathrm{J}(S_k) = \mathrm{diag}(\sigma_1, \dots, \sigma_m)$ where $\sigma_1, \dots, \sigma_m > 0$ are the singular values of P_k.

We define a path from $\mathrm{J}(U_k)$ to U_k in a block-wise fashion. We set $s \mapsto Z((1-s)\beta_i)$ for every $1 \leq i \leq m/2$. The 1×1-block (1) remains unaltered. We get a corresponding path $P \in \mathcal{S}(\mathrm{J}(U_k), I)$ such that

$$|\dot{P}(s)P(s)^{-1}| = |\dot{P}(s)| \leq m^{1/2} \cdot \max_{1 \leq i \leq m/2} \{|\beta_i|\} \tag{B.6}$$

holds for every $s \in [0,1]$. As $\mathrm{J}(U_k)$ tends to I, the angle β_i converges to 0 for every $1 \leq i \leq m/2$. Hence, we have (i). Now consider the path in $\mathcal{S}(\mathrm{J}(S_k), I)$ given by $P(s) = \mathrm{diag}(\sigma_1^{1-s}, \dots, \sigma_m^{1-s})$ for $s \in [0,1]$. Then we end up with the estimate

$$|\dot{P}(s)P(s)^{-1}| = |\mathrm{diag}(\ln(\sigma_1), \dots, \ln(\sigma_m))| \leq m^{1/2} \max\{|\ln(\sigma_1)|, \dots, |\ln(\sigma_m)|\}. \tag{B.7}$$

This holds for every $s \in [0,1]$. As $\mathrm{J}(S_k)$ tends to I, the entry σ_i converges to 1 and, hence, $|\ln(\sigma_i)|$ converges to 0 for every $i = 1, \dots, m$. This shows (ii) and finishes the proof. $\square$

Lemma B.2. *The distance D_{SL} given by* (3.1) *fulfills the condition* (2.4b).

Proof. We are going to use arguments of the previous lemma. It is sufficient to show that there exist constants $\widetilde{\tau}_0, \widetilde{\tau}_1 > 0$ and a monotonously increasing function $\widetilde{\tau}_2 \colon \mathbb{R} \to \mathbb{R}$ such that for every matrix $P \in \mathrm{SL}(m)$ both the following conditions are fulfilled

(i) $D_{\mathrm{SL}}(P, I) \leq \widetilde{\tau}_0 + \widetilde{\tau}_1 |P - I|$,

(ii) $|P - I| \leq \widetilde{\tau}_2(D_{\mathrm{SL}}(P, I))$.

Clearly, (ii) is a direct consequence of (B.3). It remains to show (i). Fix a matrix $P \in \mathrm{SL}(m)$ and let $P = US$ be its polar decomposition where U lies in $\mathrm{SO}(m)$ and $S \in \mathbb{R}^{m \times m}$ is a symmetric positive-definite matrix. Like in the previous lemma, we consider their Jordan normal forms. In particular, we get that $\mathrm{J}(S) = \mathrm{diag}(\sigma_1, \dots, \sigma_d)$ where $\sigma_1, \dots, \sigma_d > 0$ are the singular values of P.

With the help of (B.5) together with the estimates (B.6) and (B.7), we see that there exists some constant $C > 0$ (we can actually set $C = m^{1/2}\pi$) such that

$$D_{\mathrm{SL}}(P, I) \leq C + m^{1/2} \max\{|\ln(\sigma_1)|, \dots, |\ln(\sigma_m)|\}. \tag{B.8}$$

We know that $\det(P) = \det(S) = 1$. This immediately implies that $\ln(\sigma_1) + \dots + \ln(\sigma_m) = 0$ holds. Moreover, for the Euclidean norm we have that $|P| \geq \max\{\sigma_1, \dots, \sigma_m\}$. We end up with the estimate

$$\max\{|\ln(\sigma_1)|, \dots, |\ln(\sigma_m)|\} \leq m \ln(\max\{\sigma_1, \dots, \sigma_m\}) \leq m \ln(|P|).$$

Together with (B.8), this implies that $D_{\mathrm{SL}}(P, I) \leq C + m^{3/2} \ln(|P - I| + |I|)$ and, hence, we have (i). $\square$

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