Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

An asymptotic analysis for a nonstandard Cahn-Hilliard system with viscosity

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submitted: October 11, 2011

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No. 1652
Berlin 2011

2010 *Mathematics Subject Classification.* 74A15, 35K55, 35A05, 35B40.

Key words and phrases. Viscous Cahn-Hilliard system, phase field model, asymptotic limit, existence of solutions.

Acknowledgments. The authors gratefully acknowledge the financial support of the MIUR-PRIN Grant 2008ZKHAHN *"Phase transitions, hysteresis and multiscaling"*, the EU Marie Curie Research Training Network MULTIMAT *"Multi-scale Modeling and Characterization for Phase Transformations in Advanced Materials"*, the DFG Research Center MATHEON in Berlin, and the IMATI of CNR in Pavia.

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Abstract

This paper is concerned with a diffusion model of phase-field type, consisting of a parabolic system of two partial differential equations, interpreted as balances of microforces and microenergy, for two unknowns: the problem's order parameter ρ and the chemical potential μ; each equation includes a viscosity term – respectively, $\varepsilon\,\partial_t\mu$ and $\delta\,\partial_t\rho$ – with ε and δ two positive parameters; the field equations are complemented by Neumann homogeneous boundary conditions and suitable initial conditions. In a recent paper [5], we proved that this problem is well-posed and investigated the long-time behavior of its $(\varepsilon,\delta)-$solutions. Here we discuss the asymptotic limit of the system as ε tends to 0. We prove convergence of $(\varepsilon,\delta)-$solutions to the corresponding solutions for the case $\varepsilon = 0$, whose long-time behavior we characterize; in the proofs, we employ compactness and monotonicity arguments.

1 Introduction

The system we study was proposed for mathematical investigation in [5]; as to modeling issues, its most directly relevant antecedents are two papers by Fried & Gurtin [8] and Gurtin [9], and a paper by one of us [11].

A nonstandard phase-field evolution problem. The initial/boundary-value problem we dealt with in [5] consists in finding two *phase fields*, the *chemical potential* μ and the *order parameter* ρ, such that

$$\varepsilon\,\partial_t\mu + 2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0 \qquad \text{in } \Omega\times(0,+\infty), \tag{1.1}$$

$$\delta\,\partial_t\rho - \Delta\rho + f'(\rho) = \mu \qquad \text{in } \Omega\times(0,+\infty), \tag{1.2}$$

$$\partial_\nu\mu = \partial_\nu\rho = 0 \qquad \text{on } \Gamma\times(0,+\infty), \tag{1.3}$$

$$\mu(\,\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,,0) = \rho_0 \qquad \text{in } \Omega, \tag{1.4}$$

where Ω denotes a bounded domain of $\mathbb{R}^3$ with (sufficiently) smooth boundary Γ, and f' stands for the derivative of a double-well potential f. This nonstandard phase-field model can be regarded as a variant of the classic Cahn-Hilliard system for diffusion-driven phase segregation by atom rearrangement:

$$\partial_t\rho - \kappa\Delta\mu = 0\,, \qquad \mu = -\Delta\rho + f'(\rho). \tag{1.5}$$

Note, in (1.1), the unpleasant nonlinear terms involving time derivatives, and the fact that we have taken the *mobility* coefficient $\kappa > 0$ equal to 1. Moreover, recall that equations (1.5) are customarily combined so as to obtain the well-known *Cahn-Hilliard equation*:

$$\partial_t\rho = \kappa\Delta(-\Delta\rho + f'(\rho)). \tag{1.6}$$

Fried & Gurtin's generalization of Cahn-Hilliard equation. In [8, 9] a broad generalization of (1.6) was arrived at, with three measures:

(i) by regarding the second of (1.5) as a *balance of microforces:*

$$\operatorname{div}\boldsymbol{\xi}+\pi+\gamma=0, \tag{1.7}$$

where the distance microforce per unit volume is split into an internal part π and an external part γ, and the contact microforce per unit area of a surface oriented by its normal $\boldsymbol{n}$ is measured by $\boldsymbol{\xi}\cdot\boldsymbol{n}$ in terms of the *microstress* vector $\boldsymbol{\xi}$;[1]

(ii) by interpreting the first equation of (1.5) as a *balance law for the order parameter*:

$$\partial_t\rho=-\operatorname{div}\boldsymbol{h}+\sigma, \tag{1.8}$$

where the pair $(\boldsymbol{h},\sigma)$ is the *inflow* of ρ;

(iii) by requiring that the constitutive choices for $\pi,\boldsymbol{\xi},\boldsymbol{h}$, and the *free energy density* ψ, be consistent in the sense of Coleman and Noll [2] with a postulated "dissipation inequality that accomodates diffusion":

$$\partial_t\psi+(\pi-\mu)\partial_t\rho-\boldsymbol{\xi}\cdot\nabla(\partial_t\rho)+\boldsymbol{h}\cdot\nabla\mu\leq 0 \tag{1.9}$$

(cf. equation (3.6) in [9]).

In [9], the following set of constitutive prescriptions was shown to be consistent with (iii):

$$\left\{\begin{array}{c}\psi=\widehat{\psi}(\rho,\nabla\rho),\\ \widehat{\pi}(\rho,\nabla\rho,\mu)=\mu-\partial_\rho\widehat{\psi}(\rho,\nabla\rho),\\ \widehat{\boldsymbol{\xi}}(\rho,\nabla\rho)=\partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho)\end{array}\right\} \tag{1.10}$$

together with

$$\boldsymbol{h}=-\boldsymbol{M}\nabla\mu,\quad\text{with }\ \boldsymbol{M}=\widehat{\boldsymbol{M}}(\rho,\nabla\rho,\mu,\nabla\mu), \tag{1.11}$$

provided the tensor-valued *mobility mapping* $\widehat{\boldsymbol{M}}$ satisfies the *residual dissipation inequality*

$$\nabla\mu\cdot\widehat{\boldsymbol{M}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu\geq 0.$$

With the use of (1.7), (1.8), (1.10), and $(1.11)_1$, a general equation for diffusive phase segregation processes is arrived at:

$$\partial_t\rho=\operatorname{div}\Big(\boldsymbol{M}\nabla\left(\partial_\rho\widehat{\psi}(\rho,\nabla\rho)-\operatorname{div}\left(\partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho)\right)-\gamma\right)\Big)+\sigma;$$

in particular, the Cahn-Hilliard equation (1.6) is obtained by taking the external distance microforce γ and the order-parameter source term σ identically null, and by choosing

$$\widehat{\psi}(\rho,\nabla\rho)=f(\rho)+\frac{1}{2}|\nabla\rho|^2,\qquad\boldsymbol{M}=\kappa\boldsymbol{1}. \tag{1.12}$$

[1] In [7], the microforce balance is stated under form of a principle of virtual powers for microscopic motions.

An alternative generalization of Cahn-Hilliard equation. In [11], a major modification of Fried & Gurtin's approach to phase-segregation modeling was proposed. While the crucial step (i) was retained, both the order-parameter balance (1.8) and the dissipation inequality (1.9) were dropped and replaced, respectively, by the *microenergy balance*

$$\partial_t\varepsilon = e + w, \quad e := -\operatorname{div}\overline{\boldsymbol{h}} + \overline{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{1.13}$$

and the *microentropy imbalance*

$$\partial_t\eta \geq -\operatorname{div}\boldsymbol{h} + \sigma, \quad \boldsymbol{h} := \mu\overline{\boldsymbol{h}}, \quad \sigma := \mu\,\overline{\sigma}. \tag{1.14}$$

A further new feature was that the *microentropy inflow* $(\boldsymbol{h},\sigma)$ was deemed proportional to the *microenergy inflow* $(\overline{\boldsymbol{h}},\overline{\sigma})$ through the *chemical potential* μ, a positive field; consistently, the free energy was defined to be

$$\psi := \varepsilon - \mu^{-1}\eta, \tag{1.15}$$

with the chemical potential playing the same role as the *coldness* in the deduction of the heat equation.[2]

Combination of (1.13)-(1.15) gives:

$$\partial_t\psi \leq -\eta\partial_t(\mu^{-1}) + \mu^{-1}\,\overline{\boldsymbol{h}}\cdot\nabla\mu - \pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{1.16}$$

an inequality that replaces (1.9) in restricting *à la* Coleman-Noll the possible constitutive choices. On taking all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\overline{\boldsymbol{h}}$, dependent in principle on $\rho, \nabla\rho, \mu, \nabla\mu$, and on choosing

$$\psi = \widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu\,\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{1.17}$$

compatibility with (1.16) implies that we must have:

$$\left\{\begin{array}{c} \widehat{\pi}(\rho,\nabla\rho,\mu) = -\partial_\rho\widehat{\psi}(\rho,\nabla\rho,\mu) = \mu - f'(\rho), \\ \widehat{\boldsymbol{\xi}}(\rho,\nabla\rho,\mu) = \partial_{\nabla\rho}\widehat{\psi}(\rho,\nabla\rho,\mu) = \nabla\rho, \\ \widehat{\eta}(\rho,\nabla\rho,\mu) = \mu^2\partial_\mu\widehat{\psi}(\rho,\nabla\rho,\mu) = -\mu^2\rho \end{array}\right\} \tag{1.18}$$

together with

$$\widehat{\overline{\boldsymbol{h}}}(\rho,\nabla\rho,\mu,\nabla\mu) = \widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu, \quad \nabla\mu\cdot\widehat{\boldsymbol{H}}(\rho,\nabla\rho,\mu,\nabla\mu)\nabla\mu \geq 0.$$

If we now choose for $\widehat{\boldsymbol{H}}$ the simplest expression $\boldsymbol{H} = \kappa\boldsymbol{1}$, implying a constant and isotropic mobility, and if we once again assume that the external distance microforce γ and the source $\overline{\sigma}$ are null, then, with the use of (1.18) and (1.15), the microforce balance (1.7) and the energy balance (1.13) become, respectively,

$$\Delta\rho + \mu - f'(\rho) = 0 \tag{1.19}$$

[2] As much as absolute temperature is a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*; likewise, as argued in [11], the chemical potential can be seen as a macroscopic measure of microscopic *organization*.

and

$$2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \kappa\Delta\mu = 0, \tag{1.20}$$

a nonlinear system for the unknowns ρ and μ.

Two-parameter regularization. Compare now the systems (1.19)-(1.20) and (1.5): (1.19) and $(1.5)_2$ are one and the same 'static' relations between μ and ρ, whereas (1.20) is rather different from $(1.5)_1$, for more than one reason:

(R1) $(1.5)_1$ is linear, (1.20) is not;

(R2) the time derivatives of ρ and μ are both present in (1.20);

(R3) in front of both $\partial_t\mu$ and $\partial_t\rho$ there are *nonconstant factors* that *should remain nonnegative* during the evolution.

Thus, the system (1.19)-(1.20) deserves a careful analysis. We must confess that we boldly attacked this problem as is, prompted to optimism by the successful outcome of a previous joint research effort [3, 4] devoted to tackling the system of Allen-Cahn type one arrives at via the approach in [11] for no-diffusion phase-segregation processes. Unfortunately, the evolution problem ruled by (1.19)–(1.20) turned out to be too difficult for us. Therefore, we decided to study its regularized version (1.1)–(1.4), which we obtained by introducing the extra terms $\varepsilon\,\partial_t\mu$ in (1.20) and $\delta\,\partial_t\rho$ in (1.19), for small positive coefficients ε and δ. Motivations for the introduction of such terms are proposed and discussed in [5]; interestingly, while the second can be interpreted as a dissipative part of the distance microforce, so far we have not been able to find a convincing physical interpretation for the first. But, our present study demonstrates – so we believe – that it can legitimately be regarded as an efficient mathematical device.

Limit as the first parameter tends to 0. In [5], by assuming (as we did in [3, 4]) that f' is the sum of a strictly increasing C^1 function f_1' with domain $(0,1)$ that is singular at the endpoints, and of a smooth bounded perturbation f_2' (to allow for a double- or multi-well potential f), we first proved existence of a strong solution (μ,ρ) to (1.1)–(1.4) satisfying $\mu \geq 0$ and $0 < \rho < 1$ almost everywhere in $\Omega\times(0,+\infty)$ (of course, we stipulated that the initial data meet same requirements in Ω). Then, under some additional technical assumptions, we showed that the component μ is bounded, and so is $f'(\rho)$; as a consequence, ρ stays away from the threshold values 0 and 1. These boundedness properties are very useful in proving uniqueness of solutions.

In some sense, passing to the limit as the regularizing parameters tend to zero is the challenging final aim of our research project. For the moment being, we are able to deal with ε and to deduce, by a rather delicate asymptotic analysis, an existence theorem for the limit problem. Precisely, we let ε tend to zero and show that any weak or weak star limit of any subsequence of solutions $(\mu_\varepsilon,\rho_\varepsilon)$ to (1.1)–(1.4) yields a solution (μ,ρ) to the resulting limit problem, which is obtained by putting $\varepsilon = 0$ in (1.1)–(1.4) and rewriting the corresponding first equation in the form

$$2\partial_t(\mu\rho) - \Delta\mu = \mu\,\partial_t\rho \quad \text{rather than as} \quad 2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0. \tag{1.21}$$

This we do because it is not clear to us from the structure of the system whether a suitably regular representation for $\partial_t \mu$ could be recovered in the limit, while we are able to show that the time derivative $\partial_t(\mu\rho)$ actually exists, at least in some generalized sense.[3]

Here, as we did in [5], we also deal with the long-time behavior of the system. We prove that each element $(\mu_\omega, \rho_\omega)$ of the ω-limit set for a certain trajectory is a steady state solution to (1.1)–(1.4); therefore, in particular, μ_ω *is a constant* (cf. (1.21) and (1.3)).[4]

An outline of our paper is the following: in Section 2, we carefully state assumptions and results; Section 3 is devoted to the proof of the convergence theorem, so as to deduce the existence of solutions to the limit problem; finally, in Section 4, we develop our argument for the characterization of the ω-limit.

2 Assumptions and main results

First of all, we assume Ω to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ and set, for convenience,

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0\}. \tag{2.1}$$

We endow these spaces with their standard norms, for which we use self-explaining notation like $\|\cdot\|_V$. However, we write $\|\cdot\|_H$ for the norm in any power of H as well. The symbol $\langle \cdot\,, \cdot \rangle$ denotes the duality product between V^*, the dual space of V, and V itself. Since Ω is bounded and smooth, the embeddings $W \subset V \subset H$ are compact. Moreover, since V is dense in H, we can identify H with a subspace of V^* in the usual way, i.e., in order that $\langle u, v\rangle = (u, v)_H$, where $(\cdot\,, \cdot)_H$ denotes the inner product in H, holds for every $u \in H$ and $v \in V$. Then, also the embedding $H \subset V^*$ is compact.

As in [5], we assume that

$$f = f_1 + f_2, \quad \text{where} \quad f_1, f_2 : (0,1) \to \mathbb{R} \text{ are functions satisfying:} \tag{2.2}$$
$$f_1 \text{ is } C^1 \text{ and convex}, \quad f_2 \text{ is } C^2, \quad \text{and} \quad f_2'' \text{ is bounded}, \tag{2.3}$$
$$\lim_{r\searrow 0} f_1'(r) = -\infty \quad \text{and} \quad \lim_{r\nearrow 0} f_1'(r) = +\infty. \tag{2.4}$$

As to initial data, we start with the assumptions

$$\mu_0 \in V \quad \text{and} \quad \mu_0 \geq 0 \quad \text{a.e. in } \Omega; \tag{2.5}$$
$$\rho_0 \in W, \quad 0 < \rho_0 < 1 \quad \text{in } \Omega; \quad \text{and} \quad f'(\rho_0) \in H. \tag{2.6}$$

The reader is referred to the forthcoming Remark 2.5 for weaker conditions.

[3] In fact, (1.1) has the equivalent formulation:

$$\partial_t(\varepsilon\mu + 2\mu\rho) - \Delta\mu = \mu\partial_t\rho,$$

which singles out the time derivative of the auxiliary variable $\varepsilon\mu + 2\mu\rho$ for $\varepsilon > 0$.

[4] Note that the steady state problem associated with both cases $\varepsilon > 0$ and $\varepsilon = 0$ is the same.

Since we aim to let ε tend to zero, we stress the dependence of the solution found in [5] on the parameter ε. In that paper, for any fixed $T > 0$, the following a priori regularity is required:

$$\mu_\varepsilon \in H^1(0,T;H) \cap L^2(0,T;W), \tag{2.7}$$
$$\rho_\varepsilon \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.8}$$
$$\mu_\varepsilon \geq 0 \quad \text{a.e. in } Q_T, \tag{2.9}$$
$$0 < \rho_\varepsilon < 1 \quad \text{a.e. in } Q_T \quad \text{and} \quad f'(\rho_\varepsilon) \in L^\infty(0,T;H), \tag{2.10}$$

where $Q_T := \Omega \times (0,T)$. We note that homogeneous Neumann boundary conditions follow from (2.7)–(2.8), in view of the definition of W (see (2.1)). Then, the ε-problem is:

$$\Big[(\varepsilon + 2\rho_\varepsilon)\partial_t\mu_\varepsilon + \mu_\varepsilon\,\partial_t\rho_\varepsilon - \Delta\mu_\varepsilon = 0 \quad \text{or}\Big]$$
$$\partial_t(\varepsilon\mu_\varepsilon + 2\mu_\varepsilon\rho_\varepsilon) - \Delta\mu_\varepsilon = \mu_\varepsilon\partial_t\rho_\varepsilon \qquad \text{a.e. in } Q_T, \tag{2.11}$$
$$\delta\partial_t\rho_\varepsilon - \Delta\rho_\varepsilon + f'(\rho_\varepsilon) = \mu_\varepsilon \qquad \text{a.e. in } Q_T, \tag{2.12}$$
$$\mu_\varepsilon(0) = \mu_0 \quad \text{and} \quad \rho_\varepsilon(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{2.13}$$

We recall the existence result of [5].

Theorem 2.1. *Let $T \in (0,+\infty)$, and assume that* (2.2)–(2.4) *and* (2.5)–(2.6) *are satisfied. Then, there exists a pair $(\mu_\varepsilon, \rho_\varepsilon)$ satisfying* (2.7)–(2.10) *and solving problem* (2.11)–(2.13).

As to uniqueness, the result in [5, Thm. 2.2] holds for solutions that, in addition to (2.7)–(2.10), have certain properties that, in turn, are guaranteed whenever the initial data fulfil the following conditions, additional to (2.5) and (2.6):

$$\mu_0 \in L^\infty(\Omega), \quad \inf\rho_0 > 0, \quad \text{and} \quad \sup\rho_0 < 1$$

(see [5, Thm 2.3]). Within such a framework, since $T > 0$ is arbitrary, the existence of a unique solution $(\mu_\varepsilon, \rho_\varepsilon)$ defined for every positive time was ensured, and its long-time behavior was studied.

Here, our first concern is to construct a global solution $(\mu_\varepsilon, \rho_\varepsilon)$ to problem (2.11)–(2.13) that satisfies (2.7)–(2.10) for every finite T, without assuming the just mentioned stronger conditions on the initial data. We cannot ensure uniqueness, of course. The corresponding result reads:

Proposition 2.2. *Assume that* (2.2)–(2.4) *and* (2.5)–(2.6)*are fulfilled. Then, there exists a pair $(\mu_\varepsilon, \rho_\varepsilon) : [0,+\infty) \to W \times W$ satisfying* (2.7)–(2.10) *and solving problem* (2.11)–(2.13) *for every $T \in (0,+\infty)$.*

Starting from any family $\{(\mu_\varepsilon, \rho_\varepsilon)\}_{\varepsilon>0}$ of solutions of this type, we then let ε tend to zero. To do this, we need to assume that

$$\inf\rho_0 > 0, \tag{2.14}$$

in addition to (2.5)–(2.6). Under this assumption, we show that ρ_ε is bounded away from zero and that $(\mu_\varepsilon, \rho_\varepsilon)$ tends to some pair (μ, ρ) as $\varepsilon \searrow 0$ in a suitable topology, at least for a subsequence. Moreover, we determine the limit problem solved by (μ, ρ). The a priori regularity we require for (μ, ρ) on every finite time interval is the following:

$$\mu \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.15}$$
$$\rho \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.16}$$
$$\mu \ge 0 \quad \text{a.e. in } Q_T \quad \text{and} \quad \inf \rho > 0, \tag{2.17}$$
$$\rho < 1 \quad \text{a.e. in } Q_T \quad \text{and} \quad f'(\rho) \in L^2(0,T;H), \tag{2.18}$$
$$\mu\rho \in W^{1,p}(0,T;V^*) \quad \text{for some } p \in (1,+\infty); \tag{2.19}$$

the corresponding limit problem is:

$$2\langle \partial_t(\mu\rho)(t), v\rangle + \int_\Omega \nabla\mu(t) \cdot \nabla v = \int_\Omega \mu(t)\,\partial_t\rho(t)\,v$$
$$\text{for every } v \in V \text{ and for a.a. } t \in (0,T), \tag{2.20}$$
$$\delta\partial_t\rho - \Delta\rho + f'(\rho) = \mu \quad \text{a.e. in } Q_T, \tag{2.21}$$
$$(\mu\rho)(0) = \mu_0\rho_0 \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{a.e. in } \Omega. \tag{2.22}$$

Remark 2.3. The last integral in (2.20) makes sense because $V \subset L^3(\Omega)$ and $\mu\partial_t\rho$ belongs at least to $L^1(0,T;L^{3/2}(\Omega))$, as a consequence of (2.15)–(2.16). Note that (2.20) incorporates the homogeneous Neumann boundary condition for μ in a generalized sense. Moreover, note that (2.19) implies that $\mu\rho$ is a continuous V^*-valued function, so that the first equality in (2.22) has a precise meaning. On the contrary, no continuity for μ is ensured at the moment.

Here is our convergence result.

Theorem 2.4. *Assume that* (2.2)–(2.4)*,* (2.5)–(2.6)*, and* (2.14) *are satisfied, and let* $\{(\mu_\varepsilon, \rho_\varepsilon)\}_{\varepsilon\in(0,1)}$ *be a family of solutions to problem* (2.11)–(2.13) *satisfying* (2.7)–(2.10)*. Then, there exists a pair* (μ, ρ)*, satisfying* (2.15)–(2.19) *and solving problem* (2.20)–(2.22) *for every* $T \in (0,+\infty)$*, such that* $(\mu_\varepsilon, \rho_\varepsilon)$ *converges to* (μ, ρ) *in a suitable topology,*[5] *at least for a subsequence* $\varepsilon_n \searrow 0$*.*

Remark 2.5. The assumptions (2.5)–(2.6) are strong. In fact, while they are needed for Theorem 2.1 and Proposition 2.2, some of them will not play any role in the following, as it can be seen by looking at our a priori estimates. For instance, the last condition in (2.6) will not be important, since just $f(\rho_0) \in L^1(\Omega)$ will be used. Accordingly, one can prove a result similar to Theorem 2.4, but involving ε-approximations of less regular initial data that, this notwithstanding, satisfy (2.5)–(2.6). Precisely, suppose we assume that

$$\mu_0 \in H, \quad \mu_0 \ge 0 \quad \text{a.e. in } \Omega, \quad \rho_0 \in V, \quad f(\rho_0) \in L^1(\Omega), \quad \text{and} \quad \inf \rho_0 > 0. \tag{2.23}$$

Then, it would be possible to construct ε-approximations $(\mu_{0\varepsilon}, \rho_{0\varepsilon})$ of such initial data (μ_0, ρ_0) that satisfy (2.5)–(2.6) and whose norms of type (2.23) remain bounded as $\varepsilon \searrow 0$: e.g., as to $\rho_{0\varepsilon}$, one could take the solution to the elliptic equation

$$\frac{\rho_{0\varepsilon} - \rho_0}{\varepsilon} - \Delta\rho_{0\varepsilon} + f_1'(\rho_{0\varepsilon}) = 0 \quad \text{a.e. in } \Omega,$$

supplemented by homogeneous Neumann boundary conditions.

[5] to be specified in the course of the proof given in the next section.

Remark 2.6. Theorem 2.4 and the previous Remark offer us the possibility of defining and obtaining a weaker solution to problem (2.11)–(2.13) (that is, also for the case $\varepsilon > 0$), if one writes equation (1.1) in the form (2.11). To see that this solution is weaker than the one provided by Theorem 2.1, it suffices to compare (2.15)–(2.16) with (2.7)–(2.8). On the other hand, we can just assume (2.23) and point out that in this approach one should consider (2.19), (2.20), (2.22) with $\mu\rho$ replaced by $(\varepsilon/2)\mu + \mu\rho$.

Our final aim is to study the long-time behavior of any solution constructed according to Theorem 2.4. To this end, we introduce the ω-limit of the trajectory in a proper topology, and prove that every element of it coincides with a steady state. We set:

$$\begin{aligned}\omega(\mu,\rho) \ &= \big\{(\mu_\omega,\rho_\omega) \in H \times V : \ \big(\mu(t_n),\rho(t_n)\big) \to (\mu_\omega,\rho_\omega) \\ &\qquad \text{weakly in } H \times V \text{ for some sequence } t_n \nearrow +\infty\big\}. \end{aligned} \tag{2.24}$$

The above definition has a precise meaning, because the pointwise values of the $(H \times V)$-valued function (μ,ρ) are well defined thanks to the continuity properties stated in our next result.

Theorem 2.7. *Assume that* (2.2)–(2.4)*,* (2.5)–(2.6) *and* (2.14) *are satisfied, and let* (μ,ρ) : $[0,+\infty) \to H \times V$ *be given by Theorem 2.4. Then,* (μ,ρ) *is bounded, and its components* μ *and* ρ *are weakly and strongly continuous, respectively. In particular, the* ω*-limit* (2.24) *is nonempty. Moreover, every element of* $\omega(\mu,\rho)$ *coincides with a pair* (μ_s,ρ_s) *such that*

$$\begin{aligned}&\mu_s \text{ is a nonnegative constant,}\\ &\rho_s \in W, \quad 0 < \rho_s < 1, \quad f'(\rho_s) \in H, \quad \text{and} \quad -\Delta\rho_s + f'(\rho_s) = \mu_s \quad \text{a.e. in } \Omega,\end{aligned} \tag{2.25}$$

i.e., it coincides with a steady state.

We stress that the above result does not necessarily hold for all possible solutions. Indeed, it only deals with solutions obtained as limits of solutions to the ε-problem as $\varepsilon \searrow 0$. We also observe that there is no reason for the function ρ_s mentioned in the statement to be a constant, since f is not required to be convex.

The rest of the paper is organized as follows: in the next section, we prove both Proposition 2.2 and Theorem 2.4; the proof of Theorem 2.7 will be given in the last section.

3 Global solutions

We first prove the existence of a global solution to the ε-problem. The major part of the section is devoted to the proof of Theorem 2.4 and the subsequent existence of a global solution to the limit problem.

Proof of Proposition 2.2. We imitate, with minor changes, the proof of Thm 2.1 in [5], where the final time T was fixed once and for all. Let ε be fixed and, for notational conciseness, let the dependence on ε be omitted. The main tool used in [5] was an approximation procedure using a time delay $\tau = T/N$, for N a positive integer. Approximating τ-problems were constructed and solved step by step in the time intervals $I_n := [0, n\tau]$, $n = 1, \ldots, N$. It turned out that the resulting unique solution (μ^τ, ρ^τ) coincided a posteriori with the one obtained by glueing together solutions on the time steps $[(n-1)\tau, n\tau]$, $n = 1, \ldots, N$.

The necessary slight modification is the following. Take, e.g., $\tau = 1/N$, and solve the same problems as before step by step, now for every $n \geq 1$. This provides a global solution (μ^τ, ρ^τ) to the approximating τ-problem. Then, for every fixed $T > 0$, the argument of [5] applies, and a solution for (2.11)–(2.13) on $[0, T]$ is constructed as the limit of the approximating solutions as $\tau \searrow 0$, at least for a subsequence. This holds, in particular, for $T = 1, 2, \ldots$. Therefore, there is a subsequence $\tau_{1,n} \searrow 0$ such that the restriction of (μ^τ, ρ^τ) to $[0, 1]$ converges to a solution to problem (2.11)–(2.13) with $T = 1$. We denote this solution by (μ_1, ρ_1). Now, take the restriction of (μ^τ, ρ^τ) to $[0, 2]$ with $\tau = \tau_{1,n}$. Then, for the same reason, there is a subsequence $\{\tau_{2,n}\}$ of $\{\tau_{1,n}\}$ such that the restriction we are considering converges to a solution (μ_2, ρ_2) to problem (2.11)–(2.13) with $T = 2$. However, as $\{\tau_{2,n}\}$ is a subsequence of $\{\tau_{1,n}\}$, the restriction of (μ_2, ρ_2) to $[0, 1]$ coincides with (μ_1, ρ_1). Proceeding inductively in this way, and then using a diagonal procedure, leads to a global solution to problem (2.11)–(2.13). $\square$

Preliminaries to the proof of Theorem 2.4. We begin by listing some of the tools we shall use. First of all, the well-known continuous embedding, with the related Sobolev inequality, holds in our 3-dimensional case:

$$W^{1,p}(\Omega) \subset L^q(\Omega) \quad \text{and} \quad \|v\|_{L^q(\Omega)} \leq C_p \|v\|_{W^{1,p}(\Omega)} \quad \text{for every } v \in W^{1,p}(\Omega), \tag{3.1}$$

$$\text{provided that} \quad 1 \leq p < 3 \quad \text{and} \quad 1 \leq q \leq p^* := \frac{3p}{3-p}\,, \tag{3.2}$$

with the constant C_p in (3.1) depending only on Ω and p; moreover,

$$\text{the embedding} \quad W^{1,p}(\Omega) \subset L^q(\Omega) \quad \text{is compact if} \quad 1 \leq q < p^*. \tag{3.3}$$

In particular, $V \subset L^q(\Omega)$ for $1 \leq q \leq 6$, and

$$\|v\|_{L^q(\Omega)} \leq C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \leq q \leq 6, \tag{3.4}$$

where C depends only on Ω and the embedding $V \subset L^q(\Omega)$ is compact if $q < 6$. Furthermore (see, e.g., [6, formula (3.2), p. 8]), we have the continuous embedding

$$L^\infty(0, T; H) \cap L^2(0, T; V) \subset L^{10/3}(Q_T)$$

and the related inequality

$$\begin{aligned} &\|v\|_{L^{10/3}(Q_T)} \leq C_T \|v\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \\ &\quad \text{for every } v \in L^\infty(0, T; H) \cap L^2(0, T; V), \end{aligned} \tag{3.5}$$

where C_T depends on Ω and T. In our proof, we shall make use also of the well-known Hölder inequality, mainly in the form

$$\|v_1 \cdots v_n\|_{L^p(0,T;L^q(\Omega))} \leq \prod_{i=1}^n \|v_i\|_{L^{p_i}(0,T;L^{q_i}(\Omega))} \quad \text{for} \quad v_i \in L^{p_i}(0, T; L^{q_i}(\Omega)),\ i = 1, \ldots, n,$$

$$\text{provided that} \quad p, q, p_i, q_i \in [1, +\infty], \quad \frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i} \quad \text{and} \quad \frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i}\,.$$

Remark 3.1. To avoid a cumbersome notation, the lowercase letter c stands for different constants, each of which may depend on one or another of the data involved in our current statement and on the coefficient δ, but never depends either on ε or on the final time T; consequently, the relative estimates continue to hold when we discuss both the system's asymptotic limit as ε tends to 0 and its long-time behavior. Moreover, a notation like c_σ signals that that constant has an additional dependence on the parameter σ. Hence, the meaning of c and c_σ may change from line to line, and even in the same chain of inequalities. On the contrary, we use the uppercase letter C for precise constants we are going to refer to after their introduction, such as C_p in (3.1) or C_T in (3.4). Finally, in order to lighten our notation, we do not write the subscript ε in performing our a priori estimates until each estimate is completely proved; the same we do for the auxiliary function

$$u_\varepsilon := \varepsilon\mu_\varepsilon + 2\mu_\varepsilon\rho_\varepsilon. \tag{3.6}$$

Next, we prove that ρ_ε is bounded away from zero uniformly with respect to ε. Such a result is essentially known from the proof of [5, Thm 2.3], among other properties there established for a fixed ε. Nevertheless, we prefer to repeat the proof here, in order to make sure that the constructed lower bound is in fact independent of ε, and that just the additional assumption (2.14) is used.

Lemma 3.2. *There exists some $r_* \in (0,1)$ such that $\rho_\varepsilon \geq r_*$ a.e. for every $\varepsilon \in (0,1)$.*

Proof. We set for convenience $\rho_* := \inf \rho_0$ and $M := \sup_{r\in(0,1)} |f_2'(r)|$ and recall that $\rho_* > 0$ by (2.14). Thus, owing to (2.4), we can choose $r_* \in (0, \rho_*]$ such that $f_1'(r_*) \leq -M$. Then, we test (2.21) by $-(\rho_\varepsilon - r_*)^-$ and integrate over $\Omega \times (0,t)$ where $t \in (0,T)$ is arbitrary. By omitting the subscript ε for simplicity, we have

$$\begin{aligned}
&\frac{\delta}{2}\int_\Omega |(\rho - r_*)^-(t)|^2 + \int_0^t\!\!\int_\Omega |\nabla(\rho - r_*)^-|^2 - \int_0^t\!\!\int_\Omega \bigl(f_1'(\rho) - f_1'(r_*)\bigr)(\rho - r_*)^- \\
&= \frac{\delta}{2}\int_\Omega |(\rho - r_*)^-(0)|^2 - \int_0^t\!\!\int_\Omega \mu(\rho - r_*)^- + \int_0^t\!\!\int_\Omega \bigl(f_1'(r_*) - f_2'(\rho)\bigr)(\rho - r_*)^-.
\end{aligned}$$

Every term on the left-hand side is nonnegative; in the right-hand side, the first term vanishes, because $\rho_0 \geq r_*$, and the other two are nonpositive, because $\mu \geq 0$ and $f_1'(r_*) - f_2'(\rho) \leq f_1'(r_*) + M \leq 0$. Hence, $(\rho - r_*)^- = 0$, and the assertion is proved. $\square$

Proof of Theorem 2.4. Our proof will proceed as follows. For a fixed finite final time T, we shall perform a number of a priori estimates and use well-known compactness results to prove that, as ε tends to 0, the solution $(\mu_\varepsilon, \rho_\varepsilon)$ to the ε-problem (2.11)–(2.13) we are considering converges to a solution (μ, ρ) to problem (2.20)–(2.22), at least for a subsequence $\varepsilon_n \searrow 0$; in particular, this holds for $T = 1, 2, \ldots$. Having established this result, we shall be able to argue as in the proof of Proposition 2.2. Indeed, a diagonal procedure provides a subsequence $\varepsilon_n \searrow 0$ such that $(\mu_\varepsilon, \rho_\varepsilon)$ with $\varepsilon = \varepsilon_n$ converges to a global solution (μ, ρ) to problem (2.20)–(2.22) defined in the whole of $[0, +\infty)$. Therefore, just the case of a fixed final time T has to be considered.

First a priori estimate. We test (2.11) (e.g., the equation within square brackets) by μ_ε, and integrate over $\Omega \times (0,t)$, for an arbitrary $t \in (0,T)$. We obtain

$$\int_0^t \int_\Omega \partial_t \Big(\frac{\varepsilon}{2} \mu^2 + \rho\mu^2 \Big) + \int_0^t \int_\Omega |\nabla \mu|^2 = 0,$$

whence

$$\frac{\varepsilon}{2} \int_\Omega |\mu(t)|^2 + \int_\Omega (\rho\mu^2)(t) + \int_0^t \int_\Omega |\nabla\mu|^2 = \frac{\varepsilon}{2} \|\mu_0\|_H^2 + \|\rho_0 \mu_0^2\|_{L^1(\Omega)} \le c.$$

Since $\rho\mu^2 \ge r_* \mu^2$ thanks to Lemma 3.2, we immediately deduce that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\mu_\varepsilon\|_{L^2(0,T;H)} \le c. \tag{3.7}$$

Second a priori estimate. We test (2.12) by $\partial_t \rho_\varepsilon$, and use the second of (2.11) in order to compute the right-hand side we get; we also recall (3.6). For $t \in (0,T)$, we obtain:

$$\begin{aligned}
&\delta \int_0^t \int_\Omega |\partial_t \rho|^2 + \frac{1}{2} \int_\Omega |\nabla\rho(t)|^2 - \frac{1}{2} \int_\Omega |\nabla\rho_0|^2 + \int_\Omega f(\rho(t)) - \int_\Omega f(\rho_0) \\
&= \int_0^t \int_\Omega \mu \partial_t \rho = \int_0^t \int_\Omega \partial_t \big(\varepsilon\mu + 2\mu\rho \big) - \int_0^t \int_\Omega \Delta\mu \\
&= \int_0^t \int_\Omega \partial_t u = \int_\Omega u(t) - \int_\Omega \big(\varepsilon\mu_0 + 2\rho_0\mu_0 \big) \le 3 \int_\Omega \mu(t) + c.
\end{aligned}$$

Since (3.7) holds and f is bounded from below, we easily infer that

$$\|\partial_t \rho_\varepsilon\|_{L^2(0,T;H)} + \|\nabla\rho_\varepsilon\|_{L^\infty(0,T;H)} + \|f(\rho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{3.8}$$

Moreover, because $0 < \rho_\varepsilon < 1$ a.e. in Q_T for every $\varepsilon \in (0,1)$, we conclude that

$$\|\rho_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \le c_T. \tag{3.9}$$

Third a priori estimate. Taking into account (3.7)–(3.9) and the boundedness of f_2', we see that (2.12) yields

$$\|-\Delta\rho_\varepsilon + f_1'(\rho_\varepsilon)\|_{L^2(0,T;H)} = \|\mu_\varepsilon - \partial_t \rho_\varepsilon - f_2'(\rho_\varepsilon)\|_{L^2(0,T;H)} \le c_T \,.$$

By a standard argument (test formally by $f_1'(\rho_\varepsilon)$, for instance) and elliptic regularity, we conclude that

$$\|f_1'(\rho_\varepsilon)\|_{L^2(0,T;H)} + \|\rho_\varepsilon\|_{L^2(0,T;W)} \le c_T \,. \tag{3.10}$$

First conclusions. The above estimates allow us to use standard weak and weak star compactness results. Thus, a triplet (μ, ρ, φ) exists such that

$$\mu_\varepsilon \to \mu \quad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V) \tag{3.11}$$

$$\rho_\varepsilon \to \rho \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.12}$$

$$f_1'(\rho_\varepsilon) \to \varphi \quad \text{weakly in } L^2(0,T;H), \tag{3.13}$$

at least for a subsequence $\varepsilon_n \searrow 0$.[6] We note that $\mu \geq 0$ and that $\rho \geq r_*$ a.e. in Q_T (the former inequality holds because $\mu_\varepsilon \geq 0$ for every ε, the latter by Lemma 3.2). Moreover, by (3.12) and the compact embedding $V \subset L^p(\Omega)$ for $p < 6$, we infer that

$$\rho_\varepsilon \to \rho \quad \text{strongly in } C^0([0,T]; L^p(\Omega)) \quad \text{for every } p < 6, \tag{3.14}$$

thanks to [12, Sect. 8, Cor. 4]. Hence, $f_2'(\rho_\varepsilon)$ converges to $f_2'(\rho)$ in a suitable topology, for instance, strongly in $L^2(0,T;H)$, since f_2' is Lipschitz continuous. In particular, we deduce that

$$\delta \partial_t \rho - \Delta \rho + \varphi + f_2'(\rho) = \mu \quad \text{a.e. in } Q_T.$$

Furthermore, invoking both (3.14) and (3.13), and using a standard monotonicity technique (see, e.g., [1, Lemma 1.3, p. 42]), we conclude that

$$0 < \rho < 1 \quad \text{and} \quad \varphi = f_1'(\rho) \quad \text{a.e. in } Q_T.$$

Finally, (3.14) implies that $\rho_\varepsilon(0)$ converges to $\rho(0)$ strongly in H, whence $\rho(0) = \rho_0$.

In summary, so far we have proved (2.15)–(2.18), (2.21), and the second condition in (2.22). It remains for us to show (2.19), (2.20), and the first condition in (2.22). For this purpose, further arguments are needed.

Fourth a priori estimate. We recall (3.6), and we notice that the second (2.11) reads:

$$\partial_t u_\varepsilon = \mu_\varepsilon \partial_t \rho_\varepsilon + \Delta \mu_\varepsilon \, . \tag{3.15}$$

Moreover, μ_ε satisfies homogeneous Neumann boundary conditions, since it is W-valued. Therefore, we have

$$\int_0^T \int_\Omega \partial_t u_\varepsilon \, v = \int_0^T \int_\Omega \mu_\varepsilon \partial_t \rho_\varepsilon \, v - \int_0^T \int_\Omega \nabla \mu_\varepsilon \cdot \nabla v \quad \text{for every } v \in L^2(0,T;V). \tag{3.16}$$

From (3.16), we derive an estimate for $\partial_t u_\varepsilon$ as a V^*-valued function, in the framework of the Hilbert triplet (V, H, V^*). We treat the integrals on the right-hand side individually (for a while, we omit the subscript ε in order to simplify the notation).

Assume that $v \in L^5(0,T;V)$. Then the Hölder inequality and the Sobolev inequality (3.4) with $q = 5$ yield:

$$\begin{aligned} \Big| \int_0^T \int_\Omega \mu \partial_t \rho \, v \Big| &\leq \|\mu\|_{L^{10/3}(Q_T)} \, \|\partial_t \rho\|_{L^2(Q_T)} \, \|v\|_{L^5(Q_T)} \\ &\leq c \|\mu\|_{L^{10/3}(Q_T)} \, \|\partial_t \rho\|_{L^2(Q_T)} \, \|v\|_{L^5(0,T;V)} \, . \end{aligned}$$

On the other hand, inequality (3.5) holds. Therefore, on taking into account (3.7) and (3.9), we conclude that

$$\Big| \int_0^T \int_\Omega \mu \partial_t \rho \, v \Big| \leq c_T \|v\|_{L^5(0,T;V)} \quad \text{for every } v \in L^5(0,T;V). \tag{3.17}$$

[6] Incidentally, we anticipate that the convergence results stated below will hold only for suitable subsequences. Nevertheless, we will not mention such a detail.

Next, we consider the second term on the right-hand side of (3.16). By assuming again that $v \in L^5(0,T;V)$, and invoking (3.7) once more, we immediately find that

$$\Big|\int_0^T\int_\Omega \nabla\mu\cdot\nabla v\Big| \le \|\mu\|_{L^2(0,T;V)}\,\|v\|_{L^2(0,T;V)} \le c_T\|v\|_{L^5(0,T;V)}\,.$$

Combining this estimate with (3.17) and (3.16), we obtain that

$$|\langle\partial_t u, v\rangle| \le c_T\|v\|_{L^5(0,T;V)} \quad \text{for every } v\in L^5(0,T;V).$$

In other words, we have that

$$\|\partial_t u_\varepsilon\|_{L^{5/4}(0,T;V^*)} \le c_T\,. \tag{3.18}$$

Consequence. From (i) the strong convergence (3.14) with $p=4$, (ii) the weak convergence $\mu_\varepsilon \to \mu$ in $L^2(0,T;L^4(\Omega))$ implied by (3.11), and (iii) the Sobolev inequality (3.4) with $q=4$, we infer that

$$\mu_\varepsilon\rho_\varepsilon \to \mu\rho \quad \text{weakly in } L^2(0,T;H), \text{ whence } u_\varepsilon \to 2\mu\rho \text{ weakly in } L^2(0,T;H), \tag{3.19}$$

since $\varepsilon\mu_\varepsilon \to 0$ strongly in $L^2(0,T;V)$, by (3.7). Hence, accounting for (3.18), we conclude that

$$\partial_t u_\varepsilon \to 2\partial_t(\mu\rho) \text{ weakly in } L^{5/4}(0,T;V^*), \text{ whence } u_\varepsilon \to 2\mu\rho \text{ weakly in } W^{1,5/4}(0,T;V^*), \tag{3.20}$$

so that (2.19) holds with $p = 5/4$. Moreover, (3.20) also implies that u_ε converges to $2\mu\rho$ weakly in $C^0([0,T];V^*)$; in particular, $u_\varepsilon(0) \to (2\mu\rho)(0)$ weakly in V^*. On the other hand, $u_\varepsilon(0) = \varepsilon\mu_0 + 2\mu_0\rho_0$ converges to $2\mu_0\rho_0$, e.g., strongly in H. Thus, the Cauchy condition for $\mu\rho$ in (2.22) follows.

In order to prove (2.20), one can try to let $\varepsilon \searrow 0$ in (3.16) first, then to get rid of time integration. But, a trouble arises in dealing with the first term on the right-hand side, since, for the moment being, both μ_ε and $\partial_t\rho_\varepsilon$ are just weakly convergent. Hence, we have to prepare a new tool.

Fifth a priori estimate. We want to find a bound for ∇u_ε, i.e., for the partial derivatives $D_i u_\varepsilon$, $i=1,2,3$. As usual, we omit the subscript ε for a while. We have:

$$|D_i u| = |\varepsilon D_i\mu + \rho D_i\mu + \mu D_i\rho| \le 2|D_i\mu| + \mu|D_i\rho|.$$

Now, on taking (3.7) into account, we see that $D_i\mu$ is bounded in $L^2(0,T;H)$, while μ is bounded in $L^2(0,T;L^6(\Omega))$ thanks to the Sobolev inequality (3.4). On the other hand, (3.9) provides a bound for $D_i\rho$ in $L^\infty(0,T;H)$. Hence, using Hölder inequality, we see that the product $\mu D_i\rho$ is bounded in $L^2(0,T;L^{3/2}(\Omega))$. Therefore, we conclude that

$$\|u_\varepsilon\|_{L^2(0,T;W^{1,3/2}(\Omega))} \le c_T\,. \tag{3.21}$$

Consequence. As (3.19) holds, from (3.21) we infer that

$$u_\varepsilon \to 2\mu\rho \quad \text{weakly in } L^2(0,T;W^{1,3/2}(\Omega)). \tag{3.22}$$

Now, we observe that the embedding $W^{1,3/2}(\Omega) \subset L^q(\Omega)$ is compact for every $q < 3$, by (3.3). On the other hand, (3.20) holds. By using the Aubin-Lions lemma (see, e.g., [10, Thm. 5.1, p. 58]), we deduce the strong convergence

$$u_\varepsilon \to 2\mu\rho \quad \text{strongly in } L^2(0,T;L^q(\Omega)) \quad \text{for every } q < 3. \tag{3.23}$$

We stress that, in particular, $u_\varepsilon \to 2\mu\rho$ strongly in $L^2(0,T;H)$.

Lemma 3.3. *The strong convergence $\mu_\varepsilon \to \mu$ holds in $L^2(0,T;H)$.*

Proof. We set $u := 2\mu\rho$ and argue a.e. in Q_T for a while. Thanks to Lemma 3.2, we have

$$|\mu_\varepsilon - \mu| = \Big|\frac{u_\varepsilon}{\varepsilon + 2\rho_\varepsilon} - \frac{u}{2\rho}\Big| = \Big|\frac{2\rho u_\varepsilon - \varepsilon u - 2\rho_\varepsilon u}{2\rho(\varepsilon + 2\rho_\varepsilon)}\Big| \le \frac{\varepsilon|u| + 2|\rho u_\varepsilon - \rho_\varepsilon u|}{4r_*^2}\,.$$

On the other hand, we have

$$|\rho u_\varepsilon - \rho_\varepsilon u| \le |\rho|\,|u_\varepsilon - u| + |u|\,|\rho - \rho_\varepsilon| \le |u_\varepsilon - u| + 2\mu\,|\rho - \rho_\varepsilon|.$$

By combining these inequalities, we deduce that

$$\begin{aligned}&\|\mu_\varepsilon - \mu\|_{L^2(0,T;H)} \\ &\le c\left(\varepsilon\|u\|_{L^2(0,T;H)} + \|u_\varepsilon - u\|_{L^2(0,T;H)} + \|\mu\|_{L^2(0,T;L^4(\Omega))}\|\rho - \rho_\varepsilon\|_{C^0([0,T];L^4(\Omega))}\right).\end{aligned} \tag{3.24}$$

The first two terms on the right-hand side tend to zero as $\varepsilon \searrow 0$, by (3.23); as to the last term, it suffices to recall (2.15), (3.14), and (3.4) for $q = 4$. $\square$

Conclusion. The strong convergence guaranteed by Lemma 3.3, together with the weak convergence $\partial_t\rho_\varepsilon \to \partial_t\rho$ in $L^2(0,T;H)$ given by (3.12), imply that

$$\mu_\varepsilon\,\partial_t\rho_\varepsilon \to \mu\,\partial_t\rho \quad \text{weakly in } L^1(Q_T).$$

On the other hand, (3.20) and (3.11) hold. Hence, by letting $\varepsilon \searrow 0$ in (3.16), we easily obtain that

$$2\int_0^T \langle\partial_t(\mu\rho)(t), z(t)\rangle\,dt = \int_0^T\!\!\int_\Omega \mu\partial_t\rho\,z - \int_0^T\!\!\int_\Omega \nabla\mu\cdot\nabla z \tag{3.25}$$

for every $z \in L^5(0,T;V) \cap L^\infty(Q_T)$. Now, take any $v \in V \cap L^\infty(\Omega)$ and any $\zeta \in L^\infty(0,T)$. Then the function $z : t \mapsto \zeta(t)v$ is admissible in (3.25), and a standard argument yields

$$2\langle\partial_t(\mu\rho)(t), v\rangle + \int_\Omega \nabla\mu(t)\cdot\nabla v = \int_\Omega \mu(t)\,\partial_t\rho(t)\,v,$$

for every $v \in V \cap L^\infty(\Omega)$ and for a.a. $t \in (0,T)$. Now, we note that, for a.a. $t \in (0,T)$, each term in the above equation defines an element of V^*. This is clear as far the left-hand side is concerned, since $\partial_t(\mu\rho)$ is V^*-valued and μ is V-valued. For the remaining term, we recall Remark 2.3. On the other hand, $V \cap L^\infty(\Omega)$ is dense in V. Therefore, (2.20) follows, and the proof is complete. $\square$

4 Long-time behavior

This section is devoted to proving Theorem 2.7. We first derive some a priori estimates, then we prove the continuity property announced in the statement; finally, we characterize the ω-limit.

A priori estimates. We recall Lemma 3.2 and the a priori estimates (3.7) and (3.8), which involve constants that do not depend on the final time. Hence, we immediately obtain that

$$\rho(t) \geq r_* \quad \text{and} \quad \|\mu(t)\|_H + \|\nabla\rho(t)\|_H \leq c \quad \text{for a.a. } t > 0. \tag{4.1}$$

Recalling that $0 < \rho < 1$, we see, in particular, that (μ, ρ) is a bounded $(H \times V)$-valued function, as stated. For the same reason, we also deduce that

$$\|(\mu\rho)(t)\|_H \leq c \quad \text{for a.a. } t > 0. \tag{4.2}$$

Moreover, the estimates (3.7) and (3.8) also yield the bounds

$$\|\nabla\mu\|_{L^2(0,T;H)} + \|\partial_t\rho\|_{L^2(0,T;H)} \leq c \quad \text{for every } T > 0,$$

and we conclude that

$$\int_0^\infty \int_\Omega |\nabla\mu|^2 < +\infty \quad \text{and} \quad \int_0^\infty \int_\Omega |\partial_t\rho|^2 < +\infty. \tag{4.3}$$

Strong and weak continuity. As far as ρ is concerned, we have $\rho \in H^1(0,T;H) \cap L^2(0,T;W)$ for every $T < +\infty$ by (3.12). Since the embedding

$$H^1(0,T;H) \cap L^2(0,T;W) \subset C^0([0,T];V)$$

holds, we immediately deduce that ρ is a strongly continuous V-valued function. The weak continuity of μ is less obvious: we prove it by using the following well-known tool, whose proof is left as an exercise to the reader.

Proposition 4.1. *Let $\mathcal{Z}$ be a Hausdorff topological space, and let Z be a reflexive Banach space such that $Z \subset \mathcal{Z}$, where the embedding is continuous with respect to the weak topology of Z. Assume that $z : [0,T] \to \mathcal{Z}$ is continuous and that $z(t) \in Z$ and $\|z(t)\|_Z \leq M$ for a.a. $t \in (0,T)$ for some constant M. Then z is Z-valued, i.e., $z(t)$ belongs to Z for every $t \in [0,T]$, and is continuous with respect to the weak topology of Z. Moreover, $\|z(t)\|_Z \leq M$ for every $t \in [0,T]$.*

In our case, we argue on any fixed finite time interval $[0,T]$ and apply Proposition 4.1 twice, first with $\mathcal{Z} = V^*$, with either the weak or the strong topology, then with $\mathcal{Z} = L^1(\Omega)$, endowed with the weak topology. We set:

$$u := 2\mu\rho, \tag{4.4}$$

in order to agree with (3.6), and we recall that $u \in W^{1,5/4}(0,T;V^*)$, by (3.20); in particular, $u \in C^0([0,T];V^*)$. On the other hand, we have proved (4.2). We conclude that $u(t) \in H$ for every $t \in [0,T]$ and that u is continuous with respect to the weak topology of H.

Besides, ρ is strongly continuous as an H-valued function, and the first condition in (4.1) holds. Hence, the same is true for $1/\rho$, and we infer that $\mu = u/(2\rho)$ is a weakly continuous $L^1(\Omega)$-valued function. Now, we recall the estimate of μ given by (4.1) and conclude that μ is weakly continuous as an H-valued function.

Conclusion. It remains for us to show that every element of the ω-limit is a steady state. To this end, we pick any $(\mu_\omega, \rho_\omega) \in \omega(\mu, \rho)$ and consider a corresponding sequence $t_n \nearrow +\infty$, as given by definition (2.24). We set:

$$\mu_n(t) := \mu(t_n + t), \quad \rho_n(t) := \rho(t_n + t), \quad \text{and} \quad u_n(t) := u(t + t_n), \quad \text{for } t \geq 0, \tag{4.5}$$

and study the sequence $\{(\mu_n, \rho_n)\}$ on a fixed finite time interval $[0, T]$ by using u_n as well. From (4.1), (4.3), and weak star compactness, we immediately deduce that

$$\begin{aligned} &\mu_n \to \mu_\infty \quad \text{weakly star in } L^\infty(0, T; H), \quad \rho_n \to \rho_\infty \quad \text{weakly star in } L^\infty(0, T; V), \\ &|\nabla \mu_n| \to 0 \quad \text{and} \quad \partial_t \rho_n \to 0 \quad \text{strongly in } L^2(0, T; H), \end{aligned}$$

at least for a subsequence. It follows that μ_∞ is space- and ρ_∞ time-independent. Thus, we can write $\rho_\infty(t) = \rho_s$ for a.a. $t \in (0, T)$ for some $\rho_s \in V$. On the other hand, we can reproduce the estimates (3.10), (3.18), and (3.21), on the time interval $[t_n, t_n + T]$ instead of $[0, T]$. We obtain:

$$\|f_1'(\rho)\|_{L^2(t_n, t_n+T; H)} + \|\rho\|_{L^2(t_n, t_n+T; W)} + \|u_n\|_{W^{1,5/4}(t_n, t_n+T; V^*) \cap L^2(t_n, t_n+T; W^{1,3/2}(\Omega))} \leq c_T,$$

where c_T does not depend on n. This means that

$$\|f_1'(\rho_n)\|_{L^2(0,T;H)} + \|\rho_n\|_{L^2(0,T;W)} + \|u_n\|_{W^{1,5/4}(0,T;V^*) \cap L^2(0,T;W^{1,3/2}(\Omega))} \leq c_T\,. \tag{4.6}$$

Thus, $\rho_\infty \in L^2(0, T; W)$, i.e., $\rho_s \in W$. Moreover, to derive a strong convergence for ρ_n in $C^0([0, T]; H)$, we can argue as in the previous section. This allows us to ensure that $f_2'(\rho_n)$ converges to $f_2'(\rho_\infty)$ strongly in $L^2(0, T; H)$ and that the weak limit of $f_1'(\rho_n)$ in $L^2(0, T; H)$, given by weak compactness, is $f_1'(\rho_\infty)$. All this yields that

$$0 < \rho_s < 1 \quad \text{and} \quad -\Delta \rho_s + f'(\rho_s) = \mu_\infty \quad \text{a.e. in } Q_T,$$

and we deduce that μ_∞ is even time-independent. Thus, $\mu_\infty(x, t) = \mu_s$ for a.a. $(x, t) \in Q_T$ for some constant μ_s. Furthermore, μ_s is nonnegative, since $\mu_n \geq 0$ for every n. This concludes the proof that (μ_s, ρ_s) is a steady state.

Lastly, we show that (μ_s, ρ_s) coincides with $(\mu_\omega, \rho_\omega)$. Because $\rho_n \to \rho$ strongly in $C^0([0, T]; H)$, we see that $\rho_n(0)$ converges to $\rho_\infty(0) = \rho_s$ strongly in H. On the other hand, by assumption $\rho_n(0) = \rho(t_n)$ converges to ρ_ω weakly in V. Hence, $\rho_s = \rho_\omega$. The corresponding argument for μ_s and μ_ω is a bit more involved. We remind that the embedding $W^{1,3/2}(\Omega) \subset H$ is compact. Hence, from (4.6) and the Aubin-Lions lemma, we conclude that there is some u_∞ such that

$$u_n \to u_\infty \quad \text{weakly in } W^{1,5/4}(0, T; V^*) \cap L^2(0, T; W^{1,3/2}(\Omega)), \tag{4.7}$$

$$\text{whence} \quad u_n \to u_\infty \quad \text{strongly in } L^2(0, T; H). \tag{4.8}$$

On the other hand, the strong convergence $\rho_n \to \rho_\infty$ in $C^0([0,T];H)$ and the uniform inequality $\rho_n \geq r_*$ imply the strong convergence $1/\rho_n \to 1/\rho_\infty$ in $C^0([0,T];H)$. We infer that

$$\mu_n = \frac{u_n}{2\rho_n} \to \frac{u_\infty}{2\rho_\infty} \quad \text{strongly in } L^2(0,T;L^1(\Omega)).$$

Since $\mu_n \to \mu_\infty$ weakly star in $L^\infty(0,T;H)$, we conclude that $u_\infty/(2\rho_\infty) = \mu_\infty$, i.e., that $u_\infty(t) = 2\mu_s\rho_\omega$ for a.a. $t \in (0,T)$. Next, the first weak convergence (4.7) also implies weak convergence in $C^0([0,T];V^*)$; in particular, $u_n(0)$ converges to $u_\infty(0) = 2\mu_s\rho_\omega$ weakly in V^*. On the other hand, by assumption $\mu(t_n) \to \mu_\omega$ weakly in H and, due to the already mentioned strong convergence $\rho_n \to \rho_\infty$, $\rho(t_n) \to \rho_\omega$ strongly in H. We infer that $u_n(0) = 2\mu(t_n)\rho(t_n)$ converges to $2\mu_\omega\rho_\omega$ weakly in $L^1(\Omega)$. By comparison, we conclude that $2\mu_s\rho_\omega = 2\mu_\omega\rho_\omega$, i.e., that $\mu_s = \mu_\omega$. □

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