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Calculation of ultrashort pulse propagation based on rational approximations for medium dispersion

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Abstract

Ultrashort optical pulses contain only a few optical cycles and exhibit broad spectra. Their carrier frequency is therefore not well defined and their description in terms of the standard slowly varying envelope approximation becomes questionable. Existing modeling approaches can be divided in two classes, namely generalized envelope equations, that stem from the nonlinear Schrödinger equation, and non-envelope equations which treat the field directly. Based on fundamental physical rules we will present an approach that effectively interpolates between these classes and provides a suitable setting for accurate and highly efficient numerical treatment of pulse propagation along nonlinear and dispersive optical media.

1 Introduction

Electromagnetic pulses with extreme short durations can nowadays routinely be produced in optical labs [16, 14, 21]. They are used for dispersion measurements, spectroscopy, and optical communication [3], as wide spectrum light sources [15], for testing of high-speed devices and tracing of chemical reactions [11]. More exotic applications include optical realization of black holes [22], optical transistors [12], optical clocks, and measurements of fundamental physical constants [17]. The award of one-half of the 2005 Nobel Prize in Physics to Hall and Hänsch indicates the significance and impact of this research.

Physics of ultrashort pulses creates new challenges also from the theoretical side. Technological applications will depend on mathematical modeling, since the underlying phenomena need to be described by suitable effective models which allow for an efficient numerical treatment. In the range of frequencies and pulse widths used nowadays, the common slowly varying envelope approximation is no longer valid for an adequate pulse description and new models have to be developed, tested, and implemented. Currently, such models either assume a very specific rational ansatz for the medium dispersion [9, 25] or represent the dispersion profile by a non-bounded polynomial function (third- and fourth-order nonlinear Schrödinger equations and their generalizations [10]). These models ignore some basic physical constraints (e.g., the relation between medium dispersion and dissipation) and introduce unacceptable errors for spectrally wide pulses [20]. There is a clear need for more general and flexible mathematical models.

2 Envelope Models

The traditional approach to the description of few-cycle optical pulses is the extension of the well-established theory of envelope equations. These models are related to the nonlinear Schrödinger

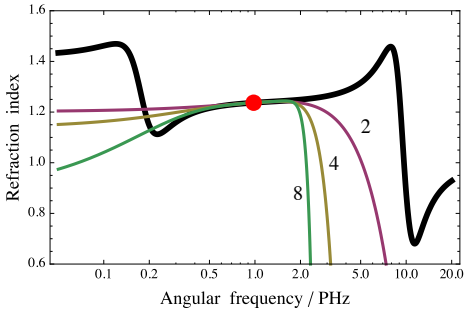

Figure 1: Refraction index of fluoride glass (fat black line) and truncations of the Taylor expansion (color lines) of the orders 2 (like in NSE), 4 and 8. Increase of the approximation order cannot resolve the divergence.

equation (NSE) for the electric field envelope and several generalizations of it [13], e.g.,

$$i\partial_z\psi + \sum_{m=2}^{M_{\max}} \frac{\beta_m}{m!}(i\partial_\tau)^m\psi + \gamma|\psi|^2\psi = 0, \tag{1}$$

where β_m and γ are dispersion and nonlinearity parameters, respectively. After the envelope $\psi(z,\tau)$ is calculated, the electric field can be obtained from the relation

$$E(z,t) = \frac{1}{2}\psi(z, t-\beta_1 z)e^{i(\beta_0 z-\omega_0 t)} + \text{c.c.}, \tag{2}$$

where $\tau = t - \beta_1 z$ is the retarded time. It is well known, that such envelope equations can possess soliton solutions [2], or at least long-term stable localized pulses that are of high interest for applications, e.g. for optical communication or for pulse shaping [1, 19]. The existence of such solitary solutions can be nicely interpreted in this context by a balance of dispersion and nonlinearity. In addition, there exists a vast amount of mathematical techniques as well as numerical techniques for solving such envelope equations, which makes them particular attractive in modeling of optical pulses.

However, the NSE, as well as its generalizations, and the very definition of the envelope (2) are based on truncated Taylor expansions of the propagation constant $\beta(\omega)$ around some reference frequency ω_0

$$\beta(\omega_0+\Omega) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!}\Omega^m. \tag{3}$$

The principal problem with the Taylor expansion is that $\beta(\omega)$ may posses resonances, i.e., singularities for some complex values of ω, as opposed by all truncations of Eq. (3). In other words, any Taylor expansion starts to diverge when $|\Omega|$ approaches the convergence radius, and this divergence cannot be suppressed by successive increase of the expansion order. This general phenomenon is illustrated in Fig. 1 for a refraction index generated by a bi-Lorentzian model for $\epsilon(\omega)$ for fluoride glass.

In particular, the Taylor expansion based generalized NSE can become invalid for ultrashort few-cycle optical pulses, whenever the pulse spectrum widens up too much. The problem becomes

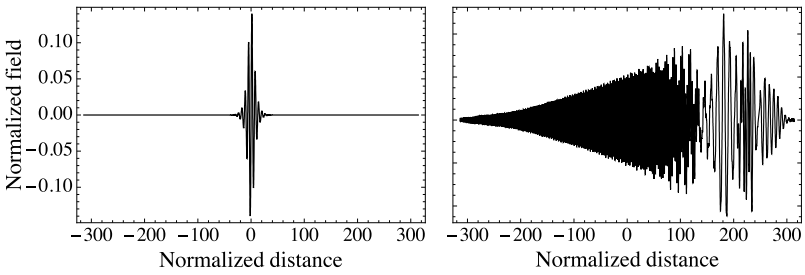

Figure 2: Left: electric field of the initial pulse. Right: the same after 10 ps propagation in a bulk Kerr medium with the refraction index shown in Fig. 1. For further details see [8].

especially evident by consideration of the asymptotic behavior of $\beta(\omega)$, i.e., $\beta(\omega) \to \omega/c$ for $\omega \to \infty$, as opposed by the behavior of polynomial approximations (3).

Another difficulty is related to the derivation of Eq. (1). Typically one has to assume that the evolution of the envelope $\psi(z,\tau)$ is slow as compared to $\exp i(\beta_0 z - \omega_0 t)$ factor in Eq. (2). An example is given in Fig. 2, where it is difficult to decide whether the envelope approximation is valid or not already after 10 ps of propagation time.

3 Short Pulse Equations

The alternative approach is to skip envelope formulations by treating the electric field directly, guiding into the class of short-pulse equations (SPE) [18, 7, 23]. Typically such models assume a reasonably simplified approximation for the medium dispersion and an instantaneous cubic nonlinearity. As an example, the approximate dispersion relation $\epsilon(\omega) = \bar{\varepsilon}(1 - \mu^2\omega_0^2/\omega^2)$ with two fit parameters $\bar{\varepsilon}$ and μ and a suitable reference frequency ω_0 for the refractive index in the transparency window (see Fig. 3) guides to the following SPE [18, 7]:

$$\partial_z U + \int_{-\infty}^{\tau} U(z,\tau')d\tau' + U^2\partial_\tau U = 0, \tag{4}$$

where the normalized electric field $U = U(z,\tau)$ depends on retarded time and propagation distance. For this equation also exact soliton solutions have been found [24]. Therefore there is substantiated hope to find solitons also in a wider class of SPE's. In particular, a more general rational fit $\epsilon(\omega) = \bar{\varepsilon}\,(1 - \mu^2\omega_0^2/\omega^2 + \nu^2\omega^2/\omega_0^2)$ is especially useful near the zero dispersion wavelength, where interesting effects by the transition between anomalous and normal dispersion are expected [8]. Here, the only possible solitary solution is a solitary breather that exhibits some intrinsic dynamics [24, 25, 7]. By shortening the pulse width the envelope of this breather evolves into a cusp that represents the shortest pulse which can propagate in such a medium [7]. Interestingly, such cusp-solitons can also be found in some cases of the generalized NSE [4].

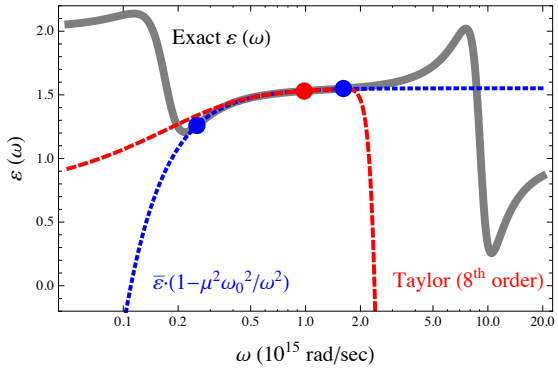

Figure 3: Material response dispersion of fluoride glass (fat grey line) and a simple rational approximation to it (blue dashed line). For comparison also the Taylor expansion of 8th order (red dashed line) is displayed [7].

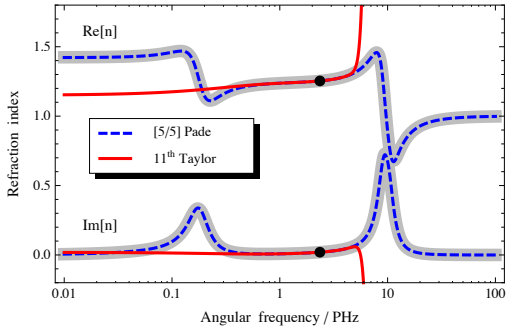

Figure 4: Dispersion of the complex refractive index for the same material as in Fig. 1 (fat grey). Red: Taylor expansion (11th order), blue dashed: [5/5] Padé approximation [5].

4 Rational Approximation

More generally, to describe ultrashort pulse evolution in terms of electric field, we employ a rational fit for the material dispersion, e.g. [5]:

$$n(\omega_0 + \Omega) \approx \frac{p_0 + p_1\Omega + \cdots + p_M\Omega^M}{1 + q_1\Omega + \cdots + q_N\Omega^N}. \tag{5}$$

A principle advantage of Eq. (5) over Eq. (3) is that it accounts for singularities in the complex ω-plane and that it approximates $n(\omega)$ in a considerably larger frequency domain, but locally still to the same precision as any Taylor expansion.

Moreover, by choosing polynomials of the same power $M = N$ in the numerator and denominator of Eq. (5) one additionally ensures that $n(\omega)$ remains bounded for $\omega \to \infty$, as it physically has to be, and as it is opposed by unbounded polynomial approximations. Even more, by proper locating the poles in the complex frequency domain, one can ensure causality, as a basic physical demand. This latter aspect is usually completely missing in envelope models. Among other things, this leads to a considerable reduction of the numerical stiffness when computing solu-

tions of the envelope equations. As a consequence, the resulting propagation equations become nonlocal in time, which expresses the delayed medium response. Structurally, the propagation equation becomes:

$$i\partial_z\Psi + (\omega_0 + i\partial_t)\left[\frac{1}{c}\,\hat{\mathfrak{D}}_n\Psi + \tilde{\gamma}\,\hat{\mathfrak{D}}_n^{-1}|\Psi|^2\Psi\right] = 0 \tag{6}$$

where the nonlocal operator $\hat{\mathfrak{D}}_n$ is defined as convolution with the inverse Fourier transform of the dispersion function (5). The electric field is calculated as a real part of $\Psi(z,t)e^{-i\omega_0 t}$, where ω_0 is just a suitable reference frequency within the pulse spectrum. One can even completely avoid the use of the reference frequency ω_0 by introducing an analytic signal for the electric field, as shown in [6]. An example for the the difference between the solutions of (6) and the envelope approach has been published in [5]. They become especially important for few-cycle pulses, and there already after short propagation distances.

5 Conclusion

We use the Padé approximat (Fig. 4) and more generally rational approximations to quantify the general dispersion operator in Eq. (6). The latter model is similar to non-envelope equations as it can be derived without the slowly-varying envelope approximation. On the other hand, Eq. (6) possesses the relatively simple structure of the generalized NSE and can be effectively addressed by adaptation of the existing numerical solvers.

References

[1] Agrawal, G.P.: Nonlinear Fiber Optics, 3rd edn. Optics and Photonics. Academic Press (2001)

[2] Akhmediev, N., Ankiewicz, A.: Solitons, *Optical and Quantum Electronics*, vol. 5. Chapman & Hall (1997)

[3] Akimoto, K.: Properties and applications of ultra-short electromagnetic mono- and sub-cycle waves. J. Phys. Soc. Jpn. **65**(7), 2020–2032 (1996)

[4] Amiranashvili, S., Bandelow, U., Akhemdiev, N.: Dispersion of nonlinear group velocity determines shortest envelope solitons. submitted **WIAS-preprint**(1639) (2011)

[5] Amiranashvili, S., Bandelow, U., Mielke, A.: Pade approximant for refractive index and nonlocal envelope equations. Opt. Commun. **283**, 480–485 (2010)

[6] Amiranashvili, S., Demircan, A.: Hamiltonian structure of propagation equations for ultrashort optical pulses. Physical Review A **82**, 013,812 (2010)

[7] Amiranashvili, S., Vladimirov, A.G., Bandelow, U.: Solitary wave solutions for few-cycle optical pulses. Phys. Rev. A **77**, 063,821 (2008)

[8] Amiranashvili, S., Vladimirov, A.G., Bandelow, U.: A model equation for ultrashort optical pulses around the zero dispersion frequency. Europ. Phys. J. D **58**, 219–226 (2010)

[9] Bespalov, V., Kozlov, S., Shpolyanskiy, Y., Walmsley, I.: Simplified field wave equations for the nonlinear propagation of extremely short light pulses. Phys. Rev. A **66**(1), 013,811 (2002)

[10] Brabec, T., Krausz, F.: Nonlinear optical pulse propagation in the single-cycle regime. Phys. Rev. Lett. **78**(17), 3282–3285 (1997)

[11] Brabec, T., Krausz, F.: Intense few-cycle laser fields: Frontiers of nonlinear optics. Rev. Mod. Phys. **72**(2), 545–591 (2000)

[12] Demircan, A., Amiranashvili, S., Steinmeyer, G.: Controlling light by light with an optical event horizon. Physical Review Letters **106**(16), 163,901 (2011). DOI 10.1103/PhysRevLett.106.163901

[13] Demircan, A., Bandelow, U.: Analysis of the interplay between soliton fission and modulation instability in supercontinuum generation. Appl. Phys. B **86**(1), 31–39 (2007)

[14] Drescher, M., Hentschel, M., Kienberger, R., Tempea, G., Spielmann, C., Reider, G.A., Corkum, P.B., Krausz, F.: X-ray pulses approaching the attosecond frontier. Science **291**(5510), 1923–1927 (2001)

[15] Dudley, J., Genty, G., Coen, S.: Supercontinuum generation in photonic crystal fiber. Rev. Mod. Phys. **78**(4), 1135–1183 (2006)

[16] Hentschel, M., Kienberger, R., Spielmann, C., Reider, G.A., Milosevic, N., Brabec, T., Corkum, P., Heinzmann, U., Drescher, M., Krausz, F.: Attosecond metrology. Nature **414**(6863), 509–513 (2001)

[17] Jones, D.J., Diddams, S.A., Ranka, J.K., Stentz, A.J., Windeler, R.S., Hall, J.L., Cundiff, S.T.: Carrier-envelope phase control of femtosecond mode-locked lasers and direct optical frequency synthesis. Science **288**(5466), 635–639 (2000)

[18] Kozlov, S.A., Sazonov, S.V.: Nonlinear propagation of optical pulses of a few oscillations duration in dielectric media. JETP **84**(2), 221–228 (1997)

[19] Mitschke, F.: Fiber Optics. Springer (2010)

[20] Oughstun, K., Xiao, H.: Failure of the quasimonochromatic approximation for ultrashort pulse propagation in a dispersive, attenuative medium. Phys. Rev. Lett. **78**(4), 642–645 (1997)

[21] Paul, P.M., Toma, E.S., Breger, P., Mullot, G., Augé, F., Balcou, P., Muller, H.G., Agostini, P.: Observation of a train of attosecond pulses from high harmonic generation. Science **292**(5522), 1689–1692 (2001)

[22] Philbin, T.G., Kuklewicz, C., Robertson, S., Hill, S., Konig, F., Leonhardt, U.: Fiber-optical analog of the event horizont. Science **319**(5868), 1367–1370 (2008)

[23] Pietrzyk, M.E., Kanattšikov, I., Bandelow, U.: On the propagation of vector ultra-short pulses. J. Nonlin. Mathematical Physics **15**(2), 162–170 (2008)

[24] Sakovich, A., Sakovich, S.: The short pulse equation is integrable. J. Phys. Soc. Jpn. **74**(1), 239–241 (2005)

[25] Skobelev, S.A., Kartashov, D.V., Kim, A.V.: Few-optical-cycle solitons and pulse self-compression in a Kerr medium. Phys. Rev. Lett. **99**(20), 203,902 (2007)