Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Emergence of rate-independent dissipation from viscous systems with wiggly energies

Alexander Mielke[1, 2]

submitted: 19 September 2011

Dedicated to Ingo Müller on the occasion of his 75th birthday

[1] Weierstraß Institut
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin
Germany

No. 1643
Berlin 2011

2010 *Mathematics Subject Classification.* 74N30, 74D10, 70K70.

Key words and phrases. Gamma convergence for evolution, De Giorgi formulation, rate-independent plasticity, viscous gradient flow, wiggly energy.

Research partially supported by European Research Council under ERC-2010-AdG 267802 *Analysis of Multiscale Systems Driven by Functionals*. The author is grateful for helpful discussions with Lev Truskinovsky and Giuseppe Savaré.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the passage from viscous system to rate-independent system in the limit of vanishing viscosity and for wiggly energies. Our new convergence approach is based on the $(\mathcal{R}, \mathcal{R}^*)$ formulation of De Giorgi, where we pass to the Γ limit in the dissipation functional. The difficulty is that the type of dissipation changes from a quadratic functional to one that is homogeneous of degree 1. The analysis uses the decomposition of the restoring force into a macroscopic part and a fluctuating part, where the latter is handled via homogenization.

1 Introduction

Thermodynamics has developed an extended theory to explain how viscous dissipation on a mesoscopic level can be understood via coarse graining from fluctuations in a microscopic systems, see e.g. [OnM53, Ött05] and the references therein. The Onsager principle states that the rate $\dot{z}$ of the state z changes according to the associated thermodynamic driving force $-\mathrm{D}\mathcal{F}(z)$ modified by a linear operator $K(z)$, which is symmetric (Onsager's symmetry relations) and positive definite. Thus, the system takes the form

$$\dot{z} = -K(z)\mathrm{D}\mathcal{F}(z) \quad \text{or equivalently} \quad G(z)\dot{z} = -\mathrm{D}_z\mathcal{F}(z)$$

where $G(z) = K(z)^{-1}$ plays the role of a viscosity matrix, which is again symmetric and positive definite. In mathematical terms these systems are called *gradient systems*, where G is Riemann's metric tensor.

Here we discuss the next level of coarse graining, where the above system is externally forced on a very slow time scale, i.e. we consider $\mathcal{E}(t,z) = \mathcal{F}(z) - \langle \ell(t), z\rangle$. Then, one expects the system to stay in metastable states (local equilibria) most of the time with fast transitions between the slow phases. In a wiggly energy landscape the gradient $\mathrm{D}_z\mathcal{E}(t,z)$ fluctuates and for all times there are many metastable states. Hence, on a coarse grained level one expects again a slow motion which is driven by dry friction, or more generally called rate-independent dissipation. Such phenomena also occur in the hysteretic behavior of shape-memory alloys (see e.g. [HuM93, ACJ96, BrS96, Mül98]), where the wiggly energy landscape arises through the discrete atomistic positions of the interfaces between different phases.

To formulate the question in a mathematically more precise manner, we consider a family of viscous problems in the form of gradient systems $(\mathcal{Z}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where $\mathcal{R}_\varepsilon(z,v) = \frac{1}{2}\langle G_\varepsilon(z)v, v\rangle$ is the dissipation potential and $\mathrm{D}_v\mathcal{R}_\varepsilon(z)v$ is the viscous friction force. Hence the system reads

$$0 = \mathrm{D}_{\dot{z}}\mathcal{R}_\varepsilon(z,\dot{z}) + \mathrm{D}_z\mathcal{E}_\varepsilon(t,z). \tag{1.1}$$

Here the time variable $t \in [0,T]$ is chosen such that it denotes the slow macroscopic time, meaning that $\partial_t\mathcal{E}_\varepsilon(t,z)$ is of order 1 in ε. On this time scale the viscosity will be small, i.e. $G_\varepsilon = O(\varepsilon^\alpha)$ for $\alpha > 0$, such that the viscous relaxation time is small as well.

The aim is to show that the solutions $z^\varepsilon : [0,T] \to \mathcal{Z}$ converge to solutions $z^0 : [0,T] \to \mathcal{Z}$ of a suitable limit system that is again a generalized gradient system, namely

$$0 \in \partial_{\dot z}\mathcal{R}(z,\dot z) + \mathrm{D}_z\mathcal{E}(t,z). \tag{1.2}$$

The convergence from viscous gradient systems (1.1) to a limit systems that is again a viscous gradient systems, is successfully investigated in many works, see e.g. [Ste08, SaS04, Ser10]. We are interested in the case where the nature of dissipation changes, namely when the limit dissipation is rate independent, and thus gives rise to hysteresis phenomena. Typical examples are dry friction in the surface of rough bodies, elastoplasticity, or phase transformation in shape-memory alloys [Mül89, HuM93, Mül98, MüS01]. Rate independence means that the dissipation potential $\mathcal{R}$ is positively homogeneous of degree 1:

$$\mathcal{R}(z,\gamma v) = \gamma\mathcal{R}(z,v) \quad \text{for all } \gamma > 0,\ z \in \mathcal{Z},\ \text{and } v \in \mathrm{T}_z\mathcal{Z}.$$

Thus, it is obvious that the quadratic potentials $\mathcal{R}_\varepsilon$ for the viscous model (1.1) cannot converge as functions to the corresponding $\mathcal{R}$ in the limit system (1.2). Nevertheless, we expect that in the energy balance

$$\mathcal{E}_\varepsilon(t,z_\varepsilon(t)) + \int_0^t \mathrm{D}_{\dot z}\mathcal{R}_\varepsilon(z_\varepsilon(s),\dot z_\varepsilon(s))\cdot\dot z_\varepsilon(s)\,\mathrm{d}s = \mathcal{E}_\varepsilon(0,z_\varepsilon(0)) + \int_0^t \partial_t\mathcal{E}(s,z_\varepsilon(s))\,\mathrm{d}s \tag{1.3}$$

the total dissipated energy converges to the dissipated energy in the limit model, namely

$$\lim_{\varepsilon\to 0}\int_0^t 2\mathcal{R}_\varepsilon(z_\varepsilon(s),\dot z_\varepsilon(s))\,\mathrm{d}s = \int_0^t \mathcal{R}(z(s),\dot z(s))\,\mathrm{d}s,$$

where we used $\partial_v\mathcal{R}_\varepsilon(z,v)\cdot v = h_\varepsilon\mathcal{R}_\varepsilon(z,v)$ with $h_\varepsilon = 2$ for $\varepsilon > 0$ and $\partial_v\mathcal{R}(z,v)\cdot v = h_0\mathcal{R}(z,v)$ with $h_0 = 1$, reflecting the degree h_ε of homogeneity.

In [Jam96, ACJ96] a simple toy problem was introduced for explaining the hysteresis in shape-memory alloys. In fact, a slight variant of this model was introduced much earlier by [Pra28], which is called Prandtl-Tomlinson model in [Pop10, Ch. 11], see Figure 1.1. This model was devised to describe hys-

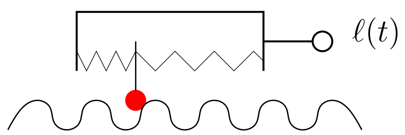

Figure 1.1: Hysteresis occurring in surface friction as explained in the Prandtl-Tomlinson model.

teresis and the kinetic relations for friction on a rough surface, which we call a wiggly surface. The differential equation takes the form

$$m\ddot z + d(\dot z - \dot\ell(t)) + k(z - \ell(t)) = a\cos(z/\varepsilon). \tag{1.4}$$

This is indeed similar to the model we are interested in. In fact, we consider the quasistatic case only where $m = 0$ and with small viscosity, i.e. $d = \varepsilon^\alpha$ for some $\alpha > 0$.

This paper is devoted to a new and more general approach to the limit analysis. Moreover, in Section 4 we will summarize the results in [MiT11], where a chain of $N = 1/\varepsilon$ of bistable, viscous elements was studied leading to a limit model of one-dimensional elastoplasticity. This work builds on the developments in [PuT00, PuT02, PuT05] and the abstract concepts of Γ convergence in rate-independent systems in [MRS08]. For a stochastic version of wiggly energies we refer to [Sul09].

Our main result is Theorem 3.2, which concerns a new convergence proof for the passage from systems with small viscosity and a wiggly energy to systems with rate-independent friction. Our approach is based on an energetic formulation for generalized gradient systems, the so-called $(\mathcal{R},\mathcal{R}^*)$ formulation of De Giorgi, see [DGMT80]. In particular, considering solutions $z_\varepsilon : [0,T] \to \mathbb{R}$ of (1.4) for $m = 0$ and $d = \varepsilon^\alpha$, we show that the solutions converge to a function $z : [0,T] \to R$ satisfying the limit equation

$$0 \in a\,\mathrm{Sign}(\dot z) + kz - k\ell(t). \tag{1.5}$$

While this convergence result was already established in [PuT02, PuT05], the main point here is to introduce a new and hopefully more flexible variational approach for passing to the limit in (1.1). This allows us to study a more general version of the model (1.4) with $m = 0$ or that of [Jam96], but still dealing with a scalar variable $z \in \mathcal{Z} = \mathbb{R}$. The functional $\mathcal{E}$ is given in the form

$$\mathcal{E}_\varepsilon(t,z) = \Phi(z) + \varepsilon W(z, z/\varepsilon) - \ell(t)z,$$

where Φ is a uniformly convex macroscopic potential, while W denotes the wiggly part of the energy, because we assume the periodicity $W(z,p) = W(z,p{+}1)$ for all $z,p \in \mathbb{R}$. The dissipation is assumed to be small namely $\mathcal{R}_\varepsilon(z,v) = \frac{\varepsilon^\alpha}{2}v^2$ for $\alpha > 0$. To pass to the limit we use De Giorgi $(\mathcal{R},\mathcal{R}^*)$ formulation, which reads here

$$\mathcal{E}_\varepsilon(T,z(T)) + \int_0^T \Big(\mathcal{R}_\varepsilon(z,\dot z) + \mathcal{R}^*_\varepsilon(z, -\mathrm{D}_z\mathcal{E}_\varepsilon(t,z))\Big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(0,z(0)) - \int_0^T \dot\ell(s)z(s)\,\mathrm{d}s,$$

where $\mathcal{R}^*_\varepsilon$ is the dual dissipation potential

$$\mathcal{R}^*_\varepsilon(z,\xi) = \sup\{\, \langle \xi, v\rangle - \mathcal{R}_\varepsilon(z,v) \mid v \in \mathrm{T}_z\mathcal{Z}\,\}.$$

It is surprising that this scalar estimate posed only for the final time T is equivalent to the evolution equation (1.1), see Proposition 2.1. Note that the energy balance (1.3) looks similar at a first glance, but it is significantly weaker as it does not involve the $(\mathcal{R},\mathcal{R}^*)$ duality.

The crucial point is then to pass to the limit in the dissipation integral. Suitable a priori estimates for the solutions z_ε are stated in Proposition 3.3 and proved in Appendix B. The necessary lower semicontinuity result is stated in Proposition 3.1 and proved in Appendix A using homogenization arguments as in [Bra02, Ch. 3]. In the limit we obtain the upper energy estimate

$$\mathcal{E}(T,z(T)) + \int_0^T \mathcal{M}\big(z(t),\dot z(t), -\mathrm{D}\mathcal{E}(t,z(t))\big)\,\mathrm{d}t \le \mathcal{E}(0,z(0)) + \int_0^T \partial_t\mathcal{E}(t,z(t))\,\mathrm{d}t,$$

where $\mathcal{M}(z,v,\xi)$ does not have the structure $\mathcal{R}(z,v){+}\mathcal{R}^*(z,\xi)$. Nevertheless, analyzing the structure of M we find the limit system $(\mathcal{Z},\mathcal{E},\mathcal{R})$ with

$$\mathcal{E}(t,z) = \Phi(z) - \ell(t)z \quad \text{and} \quad \mathcal{R}(z,v) = \begin{cases} \rho_+(z)v & \text{for } v \ge 0, \\ \rho_-(z)|v| & \text{for } v \le 0, \end{cases}$$

where $\rho_+(z) = \max\{\, \mathrm{D}_y W(z,y) \mid y \in \mathbb{R}\,\}$ and where $\rho_-(z) = -\min\{\, \mathrm{D}_y W(z,y) \mid y \in \mathbb{R}\,\}$.

As our main general conclusion, we find that $\mathcal{E}$ is the macroscopic (non-wiggly) part of the energy $\mathcal{E}_\varepsilon$, while the dissipation $\mathcal{R}$ is determined solely from the wiggly part W. The conclusion was obtained in [MiT11], where the vanishing-viscosity limit for a chain of N bistable springs is considered. We summarize the results in Section 4 and display the arising pseudo-elastic continuum model with a scalar internal phase indicator obeying a rate-independent evolution law. This model is well-known to describe the evolution of certain phase transformations shape-memory wires, see [Mül89, BrS96, Mül98, MüS01].

2 Multiscale limits for gradient systems

2.1 Generalized gradient systems

The mathematical modeling of dissipative systems was largely simplified by the concept of dissipative materials involving a dissipative potential for the rate of the internal variable z, see e.g. [HaN75, ZiW87]. Mathematically this leads to so-called generalized gradient systems $(\mathcal{Z}, \mathcal{E}, \mathcal{R})$, where $\mathcal{Z}$ is the state space, $\mathcal{E}$ is the possibly time-dependent energy functional, and $\mathcal{R}$ is the dissipation potential. As $-\mathrm{D}_z\mathcal{E}$ is the thermodynamic driving (restoring) force and $\mathrm{D}_v\mathcal{R}(z,v)$ the dissipative friction force, the force balance leads to the systems

$$\mathrm{D}_{\dot z}\mathcal{R}(z,\dot z) + \mathrm{D}_z\mathcal{E}(t,z) = 0 \in \mathrm{T}^*_z\mathcal{Z}. \tag{2.1}$$

In viscous systems $\mathcal{R}$ depends quadratically on v and the friction force is linear in v, namely $G(z)v$, see [OnM53, Ött05]. However, other cases may be relevant in cases far away from equilibrium. The friction force is then no longer uniquely defined but contained in the convex subdifferential

$$\partial_v\mathcal{R}(z,v) := \{\, \eta \in \mathrm{T}^*_{\mathcal{Z}} \mid \mathcal{R}(z,v{+}w) \geq \mathcal{R}(z,v) + \langle \eta, w\rangle \text{ for all } w \in \mathrm{T}_z\mathcal{Z}\,\}.$$

For instance, in viscoplasticity one may have $\mathcal{R}(z,v) = \sigma_{\text{yield}}(z)|A(z)v| + \frac{1}{2}\langle G(z)v, v\rangle$ and is led to the equation

$$0 = \sigma_{\text{yield}} A(z)^* \,\mathrm{Sign}(A(z)\dot z) + G(z)\dot z + \mathrm{D}_z\mathcal{E}(t,z).$$

The force balance (2.1) can be reformulated equivalently to a rate equation or a (rate of a) energy balance by using the Legendre equivalence for subdifferentials. If $\Psi : X \to \mathbb{R}$ is a convex functional, then the Legendre transform $\Psi^* : X^* \to \mathbb{R}$ is defined via $\Psi(\xi) = \sup\{\, \langle \xi, v\rangle - \Psi(v) \,|\, v \in X\,\}$ and, by definition, we have the Fenchel estimate

$$\Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle \quad \text{for all } v \in X \text{ and } \xi \in X^*. \tag{2.2}$$

Moreover, we have the Legendre-Fenchel equivalence

$$\xi \in \partial\Psi(v) \quad \Longleftrightarrow \quad v \in \partial\Psi^*(\xi) \quad \Longleftrightarrow \quad \Psi(v) + \Psi^*(\xi) \leq \langle \xi, v\rangle. \tag{2.3}$$

In the last condition the estimate "$\leq$" is sufficient to conclude equality, since "$\geq$" is automatic from (2.2).

Denoting by $\mathcal{R}^*(z,\cdot)$ the Legendre transform of $\mathcal{R}(z,\cdot)$, system (2.1) is equivalent to the rate equation

$$\dot z \in \partial\mathcal{R}^*(z, -\mathrm{D}_z\mathcal{E}(t,z)) \quad \text{for a.a. } t \in [0,T]. \tag{2.4}$$

Note that (2.1) and (2.4) ask that for almost all $t \in [0,T]$ an inclusion in a Banach space holds. Hence, it surprising the the following *upper energy estimate*, which is scalar and only for the final time T, is also equivalent. This formulation is also called the $(\mathcal{R},\mathcal{R}^*)$ formulation of De Giorgi, cf. [DGMT80].

Proposition 2.1 *If $z : [0,T] \to \mathcal{Z}$ satisfies the upper energy estimate*

$$\mathcal{E}(T,z(T)) + \int_0^T \Big(\mathcal{R}(z,\dot z) + \mathcal{R}^*(z,-\mathrm{D}_z\mathcal{E}(t,z))\Big)\,\mathrm{d}s \leq \mathcal{E}(0,z(0)) + \int_0^T \partial_s\mathcal{E}(s,z(s))\,\mathrm{d}s, \quad (2.5)$$

then it also satisfies (2.1) *and* (2.4).

Proof: The chain rule and the Fenchel estimate (2.2) give

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(s,z(s)) = \langle \mathrm{D}_z\mathcal{E}(s,z),\dot z\rangle + \partial_s\mathcal{E}(s,z) \leq -\mathcal{R}(z,\dot z) - \mathcal{R}^*(z,-\mathrm{D}_z\mathcal{E}(s,z)) + \partial_s\mathcal{E}(s,z). \quad (2.6)$$

We define the function $e(t) = \mathcal{E}(t,z(t)) + \int_0^t \mathcal{R}(z,\dot z) + \mathcal{R}^*(z,-\mathrm{D}_z\mathcal{E}(t,z))\,\mathrm{d}s - \mathcal{E}(0,z(0)) + \int_0^t \partial_s\mathcal{E}(s,z)\,\mathrm{d}s$. Integrating the above estimate for $s \in [t_0,t_1]$ for $0 \leq t_0 < t_1 \leq T$ we obtain easily obtain the monotonicity $e(t_2) \geq e(t_1) \geq e(0) = 0$. Since the upper energy estimate (2.5) means $e(T) \leq e(0) = 0$, we conclude $e(t) = 0$ for all $t \in [0,T]$. Thus, we must have equality in (2.6) for a.a. $t \in [0,T]$, which allows us to apply (2.3). Hence, (2.1) and (2.4) hold. ■

2.2 Multiscale passage

We consider a family of generalized gradient systems $(\mathcal{Z},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ and are interested to find a limit system $(\mathcal{Z},\mathcal{E},\mathcal{R})$ such that the solutions z_ε converge to the solutions z of the limit system.

Definition 2.2 *We say that the systems $(\mathcal{Z},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ dynamically converges to the system $(\mathcal{Z},\mathcal{E},\mathcal{R})$, if the following holds:*

$$\left.\begin{array}{c} z_\varepsilon : [0,T] \to \mathcal{Z} \text{ solves } (\mathcal{Z},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon) \\ z_\varepsilon^0 \to z^0,\ \mathcal{E}_\varepsilon(0,z_\varepsilon^0) \to \mathcal{E}(0,z^0) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \forall\, t \in [0,T]\colon z_\varepsilon(t) \to z(t) \text{ and} \\ z : [0,T] \to \mathcal{Z} \text{ solves } (\mathcal{Z},\mathcal{E},\mathcal{R}). \end{array}\right. \quad (2.7)$$

The major task is to find conditions on the convergence of $(\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ towards $(\mathcal{E},\mathcal{R})$ which guarantees the dynamical convergence. There are several theories for such convergences if the nature of the dissipation stays the same. For classical gradient systems (with quadratic $\mathcal{R}_\varepsilon$) there is the rather general theory of Γ-convergence in [Ste08, SaS04, Ser10]. For purely rate-independent systems the theory developed in [MRS08] has found several applications in homogenization, dimension reduction and numerical convergence, see e.g. [RSZ09, Mie11].

Obviously, the convergence $\mathcal{R}_\varepsilon \to \mathcal{R}$ is not available in our case, because $\mathcal{R}_\varepsilon$ is quadratic, whereas $\mathcal{R}$ is 1-homogeneous. So we need to develop a new theory allowing for the change of the structure of the dissipation. We will do this by using De Giorgi's $(\mathcal{R},\mathcal{R}^*)$ formulation of Proposition 2.1, now given in the form

$$\mathcal{E}_\varepsilon(T,z_\varepsilon(T)) + \boldsymbol{D}_\varepsilon(z_\varepsilon) \leq \mathcal{E}_\varepsilon(0,z_\varepsilon(0)) + \int_0^T \partial_s\mathcal{E}_\varepsilon(s,z_\varepsilon(s))\,\mathrm{d}s, \quad (2.8)$$

where the dissipation functional $\boldsymbol{D}_\varepsilon$ is given in the form

$$\boldsymbol{D}_\varepsilon(z) = \int_0^T M_\varepsilon(t, z(t), \dot z(t))\,\mathrm{d}t \ \text{ with } \ M_\varepsilon(t,z,v) = \mathcal{R}_\varepsilon(z,v) + \mathcal{R}_\varepsilon^*(z, -\mathrm{D}\mathcal{E}_\varepsilon(t,z)).$$

For our limit passage we will construct a limit functional $\mathcal{E}$ and a limit functional $\boldsymbol{D}$ in the form $\int_0^T M(t,z,\dot z)\,\mathrm{d}t$ such that for all $t \in [0,T]$ the following convergences hold

$$\widetilde z_\varepsilon \to \widetilde z \quad \Longrightarrow \quad \mathcal{E}(t,\widetilde z) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,\widetilde z_\varepsilon) \text{ and } \partial_t \mathcal{E}_\varepsilon(t,\widetilde z_\varepsilon) \to \partial_t\mathcal{E}(t,\widetilde z), \tag{2.9a}$$

$$\widetilde z_\varepsilon \to \widetilde z \text{ in } \mathrm{C}^0([0,T];\mathcal{Z}) \quad \Longrightarrow \quad \boldsymbol{D}(\widetilde z) \le \liminf_{\varepsilon\to 0} \boldsymbol{D}_\varepsilon(\widetilde z_\varepsilon). \tag{2.9b}$$

Obviously, these convergences are enough to pass to the limit $\varepsilon \to 0$ in (2.8). However, to have a counterpart of Proposition 2.1 we need more structure on the integrand M of $\boldsymbol{D}$.

In our application we will show that $\boldsymbol{D}$ can be represented in the form

$$\boldsymbol{D}(z) = \int_0^T \mathcal{M}(z(t), \dot z(t)), -\mathrm{D}\mathcal{E}(t, z(t))\,\mathrm{d}t, \tag{2.10a}$$

where $\mathcal{M}$ may no longer have the structure $\mathcal{R}{+}\mathcal{R}^*$. For more general cases we refer to Remark 2.4. In all cases, the crucial condition inherited from $\mathcal{M}_\varepsilon$ will be

$$\mathcal{M}(z,v,\xi) \ge \langle \xi, v\rangle \quad \text{for all } z,v,\xi. \tag{2.10b}$$

Recall that this lower estimate is holds for $M_\varepsilon(z,v,\xi) = \mathcal{R}(z,v){+}\mathcal{R}_\varepsilon^*(z,\xi)$ because of (2.2). Having this, we can define the contact set (cf. [MRS11]) via

$$\mathcal{C}_\mathcal{M}(z) = \{\, (v,\xi) \mid \langle \xi, v\rangle = \mathcal{M}(z,v,\xi) < \infty \,\}. \tag{2.10c}$$

Then, the arguments in the proof of Proposition 2.1 involving the chain rule show that the limit z always has to satisfy $(\dot z(t), -\mathrm{D}\mathcal{E}(t,z(t)) \in \mathcal{C}_\mathcal{M}(z(t))$. Hence, it is sufficient to find a limiting dissipation potential $\mathcal{R}$ such that for all z we have

$$\mathcal{C}_\mathcal{M}(z) = \mathcal{C}_{\mathcal{R}+\mathcal{R}^*}(z) := \{\, (v,\xi) \mid \langle \xi, v\rangle = \mathcal{R}(z,v){+}\mathcal{R}^*(z,\xi) < \infty \,\}. \tag{2.10d}$$

Note that this relation is weaker than $\mathcal{M}(z,v,\xi) \ge \mathcal{R}(z,v){+}\mathcal{R}^*(z,\xi)$, and the structure of $\mathcal{R}$ might be different from those of $\mathcal{R}_\varepsilon$. In our application $\mathcal{R}_\varepsilon(z,\cdot)$ will be quadratic, while $\mathcal{R}(z,\cdot)$ is 1-homogeneous. We are led to the following convergence result.

Theorem 2.3 *Let the solutions $z_\varepsilon : [0,T] \to \mathcal{Z}$ of $(\mathcal{Z}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon>0}$ satisfy* (2.9)*,*

$$z_\varepsilon \to z \text{ in } \mathrm{C}^0([0,T];\mathcal{Z}), \ \text{ and } \ \mathcal{E}_\varepsilon(0, z_\varepsilon(0)) \to \mathcal{E}(0, z(0)).$$

Moreover, assume that $\boldsymbol{D}$, $\mathcal{M}$, $\mathcal{R}$ satisfy (2.10)*, then $z : [0,T] \to \mathcal{Z}$ is a solution of the generalized gradient system $(\mathcal{Z}, \mathcal{E}, \mathcal{R})$.*

Proof: We first observe that z satisfies the upper energy estimate (2.5). Comparing with the integrated version of the chain rule we obtain

$$\int_0^T \mathcal{M}(z, \dot z, -\mathrm{D}\mathcal{E}(t,z)) + \langle \mathrm{D}\mathcal{E}(t,z), \dot z\rangle\,\mathrm{d}t \le 0.$$

Using (2.10b) we conclude that the integrand vanishes a.e. in $[0,T]$. This implies that $(\dot z, -\mathrm{D}\mathcal{E}(t,z))$ lies in $\mathcal{C}_{\mathcal{R}+\mathcal{R}*}$ a.e. in $[0,T]$. Now the Fenchel equivalence (2.3) yields

$$0 \in \partial_{\dot z}\mathcal{R}(z,\dot z) + \mathrm{D}\mathcal{E}(t,z) \quad \text{or} \quad \dot z \in \partial_\xi \mathcal{R}^*(z, -\mathrm{D}\mathcal{E}(t,z)),$$

which is the desired result. ∎

Remark 2.4 *In general, the dissipation functionals* $\boldsymbol{D}_\varepsilon$ *will have a* Γ-$\liminf$ *of the form* $\boldsymbol{D}(z) = \int_0^T M(t,z(t),\dot z(t))\,\mathrm{d}t$*, see e.g. [AM*11]. Thus, we have lost the control over the derivatives* ξ*. Then, the crucial condition* (2.10b) *needs to be replaced by*

$$M(t,z,v) \geq -\langle \mathrm{D}\mathcal{E}(t,z), v\rangle \quad \text{for all } t,z,v. \tag{2.11}$$

It is an open question to provide good sufficient conditions guaranteeing that this condition holds automatically, but it is usually easy to check for concrete cases.

Clearly, $M(t,z,\cdot)$ *will be convex and lower semicontinuous. If additionally* M *can be written in the form*

$$M(t,z,v) = \mathcal{R}(z,v) + m(t,z)$$

for a dissipation potential $\mathcal{R}$*, then* (2.11) *implies* $m(t,z) \geq \mathcal{R}^*(z,-\mathrm{D}\mathcal{E}(t,z))$*. Hence, we obtain the upper energy estimate*

$$\begin{aligned}&\mathcal{E}(T,z(T)) + \int_0^T \Big(\mathcal{R}(z(t),\dot z(t)) + \mathcal{R}^*(z(t), -\mathrm{D}\mathcal{E}(t,z(t)))\Big)\,\mathrm{d}t \\ &\leq \mathcal{E}(T,z(T)) + \int_0^T M(t,z(t),\dot z(t))\,\mathrm{d}t \;\leq\; \mathcal{E}(0,z(0)) + \int_0^T \partial_t\mathcal{E}(t,z(t))\,\mathrm{d}t\end{aligned}$$

and conclude, as in Proposition 2.1, that z *is a solution of the generalized gradient system* $(\mathcal{Z},\mathcal{E},\mathcal{R})$*.*

3 Limit passage in the wiggly energy model

We now turn our attention to a slightly more general version of the particular model introduced in [Jam96] and further analyzed in [PuT05]. It is a viscous gradient system $(\mathcal{Z},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with

$$\mathcal{Z} = \mathbb{R}, \quad \mathcal{E}_\varepsilon(t,z) = \Phi(z) + \varepsilon W(z, z/\varepsilon) - \ell(t)z, \quad \mathcal{R}_\varepsilon(z,\dot z) = \frac{\varepsilon^\alpha}{2}\dot z^2.$$

Here $\Phi \in \mathrm{C}^2(\mathbb{R})$ and $W \in \mathrm{C}^2(\mathbb{R}^2)$ denote the macroscopic part and the wiggly part of the energy, while $\ell \in \mathrm{C}^1([0,T])$ is the loading. For Φ we assume uniform convexity $\mathrm{D}^2\Phi(z) \geq \phi_0 > 0$. The wiggly energy is assumed to be nontrivially periodic with period 1 in the second variable. In particular, we assume

$$\rho_+(z) := \max\{\,\mathrm{D}_y W(z,y) \mid y \in \mathbb{R}\,\} > 0,\ \rho_-(z) := \min\{\,\mathrm{D}_y W(z,y) \mid y \in \mathbb{R}\,\} < 0. \tag{3.1a}$$

Figure 3.1 shows that the wiggles in the energy are not seen macroscopically while the restoring force $\mathrm{D}_z\mathcal{E}_\varepsilon$ is strongly oscillating.

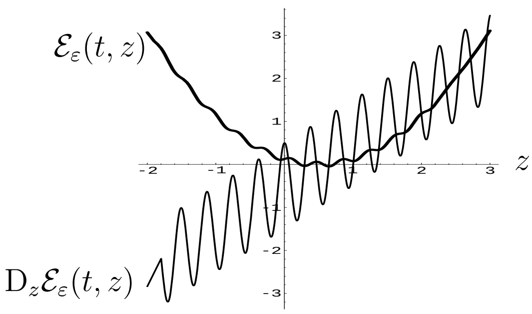

Figure 3.1: The wiggly energy functional $\mathcal{E}_\varepsilon$ and its derivative $\mathrm{D}_z\mathcal{E}_\varepsilon$.

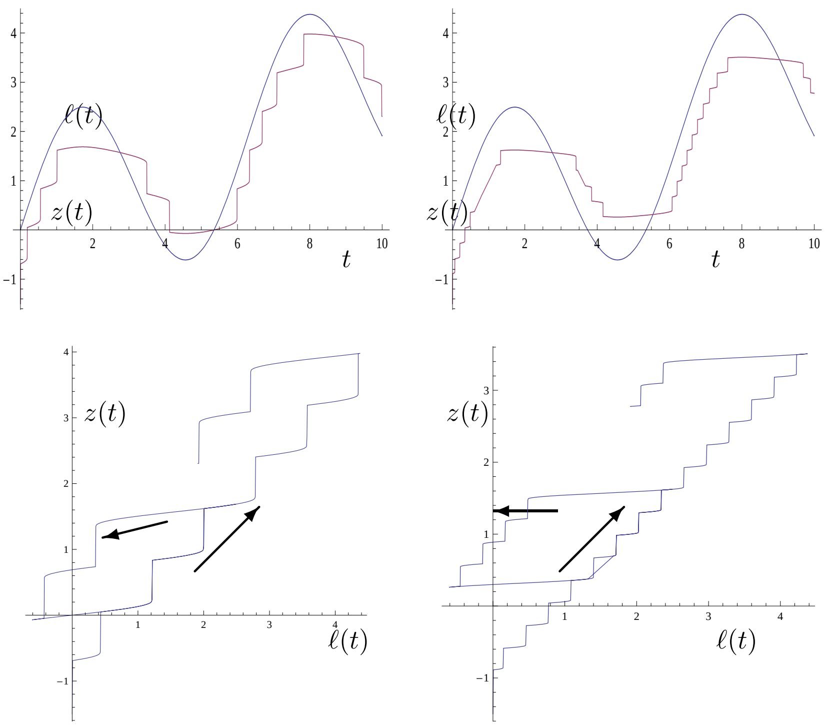

Figure 3.2: Simulations for $\mathcal{E}_\varepsilon(t,z) = \frac{1}{2}z^2 + \varepsilon\cos(\frac{z}{\varepsilon}) - (2\sin t + 0.3t)z$ and $\mathcal{R}(\dot z) = \dot z^2/2000$ for $\varepsilon = 0.125$ (left) and $\varepsilon = 0.05$ (right), respectively.

The ODE describing the evolution is given explicitly as follows:

$$\varepsilon^\alpha \dot z = -\Phi'(z) - \mathrm{D}_y W(z, \tfrac{z}{\varepsilon}) - \varepsilon \mathrm{D}_z W(z, \tfrac{z}{\varepsilon}) + \ell(t). \tag{3.2}$$

The dynamics of the solutions is displayed in Figure 3.2. The rate-independent hysteretic behavior is already nicely established for moderate $\varepsilon > 0$. Figures 3.1 and 3.2 also match diagrams in [MüS01] for the modeling of *pseudo-elasticity*, where $N = 1/\varepsilon$ plays the role of interfaces in a shape-memory wire.

To study the limiting behavior for $\varepsilon \to 0$ we use the approach described in Section 2.2, which is based on the equivalent $(\mathcal{R}, \mathcal{R}^*)$ formulation in terms of upper energy estimate (2.5). The integrand $\mathcal{M}_\varepsilon$ of the dissipation functional $\boldsymbol{D}_\varepsilon(z) = \int_0^T \mathcal{M}_\varepsilon(z, \dot z, -\mathrm{D}\mathcal{E}(t,z))\,\mathrm{d}t$ now has the simple form

$$\mathcal{M}_\varepsilon(z,v,\xi) = \frac{\varepsilon^\alpha}{2}v^2 + \frac{1}{2\varepsilon^\alpha}|\xi|^2.$$

The trick for controlling the limit passage is to decompose the restoring force along a solution $z_\varepsilon : [0,T] \to \mathbb{R}$ in a specific way, namely

$$\begin{aligned} &\xi_\varepsilon(t) := -\mathrm{D}\mathcal{E}_\varepsilon(t, z_\varepsilon(t)) = \eta_\varepsilon(t) - w_\varepsilon(t) \quad \text{with} \\ &\eta_\varepsilon(t) = -\Phi'(z_\varepsilon(t)) - \varepsilon \mathrm{D}_z W(z_\varepsilon(t), z_\varepsilon(t)/\varepsilon) + \ell(t) \text{ and} \\ &w_\varepsilon(t) = \mathrm{D}_y W(t_\varepsilon(t), z_\varepsilon(t)/\varepsilon). \end{aligned} \tag{3.3}$$

Here the macroscopic part η_ε will converge uniformly to $-\Phi'(z(t)) + \ell(t)$ if z_ε converges uniformly to z. This will be used in the following key result, where the wiggly part will is controlled by a homogenization argument as in [Bra02, Sect. 3].

Proposition 3.1 *Let $z_\varepsilon, z \in \mathrm{W}^{1,1}([0,T])$ and $\eta_\varepsilon, \xi \in \mathrm{C}^0([0,T])$ be such that*

$$z_\varepsilon \to z \text{ and } \eta_\varepsilon \to \eta \quad \text{in } \mathrm{C}^0([0,T]).$$

Let W and $\mathcal{M}_\varepsilon$ be given as above with $\alpha > 0$. Then,

$$\liminf_{\varepsilon \to 0} \int_0^T \mathcal{M}_\varepsilon(z_\varepsilon, \dot z_\varepsilon, \eta_\varepsilon - \mathrm{D}_y W(z_\varepsilon, z_\varepsilon/\varepsilon))\,\mathrm{d}t \geq \int_0^T \mathcal{M}(z, \dot z, \eta)\,\mathrm{d}t, \tag{3.4}$$

where the limit function $\mathcal{M}$ is given by

$$\begin{aligned} &\mathcal{M}(z,v,\xi) = |v| K(z,\xi) + \chi_{[\rho_-(z),\rho_+(z)]}(\xi), \\ &K(z,\xi) = \int_{y=0}^1 |\xi + \mathrm{D}_y W(z,y)|\,\mathrm{d}y \quad \text{and } \chi_A(\xi) = \begin{cases} 0 & \text{for } \xi \in A, \\ \infty & \text{for } \xi \notin A. \end{cases} \end{aligned}$$

The proof of this result is the content of Appendix A.

We can now construct the *contact set* $\mathcal{C}_\mathcal{M}$ by using the structure of $K(z,\xi)$. Since the range of $\mathrm{D}_y W(z,\cdot)$ is $[\rho_-(z), \rho_+(z)]$ we have $K(z,\xi) = |\xi|$ outside of this interval, while $K(z,\xi) > |\xi|$ for $\xi \in]\rho_-(z), \rho_+(z)[$. Thus, we conclude $\mathcal{M}(z,v,\xi) \geq |v||\xi| \geq v\xi$ as desired. Moreover, to have the equality $\mathcal{M}(z,v,\xi) = v\xi$ we need

$$\xi \in [\rho_-(z), \rho_+(z)] \quad \text{and} \quad |v| K(z,\xi) = v\xi,$$

which implies the representation

$$\mathcal{C}_\mathcal{M}(z) = \{\, (v,\xi) \in \mathbb{R} \times [\rho_-(z), \rho_+(z)] \mid v = 0 \text{ or } \big(\pm v > 0 \text{ and } \xi = \rho_\pm(z)\big) \,\}.$$

Defining the 1-homogenous dissipation potential $\mathcal{R}$ and its dual $\mathcal{R}^*$ via

$$\mathcal{R}(z,v) := \begin{cases} \rho_+(z) v & \text{for } v \geq 0, \\ \rho_-(z) v & \text{for } v \leq 0, \end{cases} \quad \Longrightarrow \quad \mathcal{R}^*(z,\xi) = \chi_{[\rho_-(z),\rho_+(z)]}(\xi), \tag{3.5}$$

we easily find the same contact sets, i.e.

$$\mathcal{C}_\mathcal{M}(z) = \mathcal{C}_{\mathcal{R}+\mathcal{R}^*}(z) = \{\, (v,\xi) \in \mathbb{R}^2 \mid \xi v = \mathcal{R}(z,v) + \mathcal{R}^*(z,\xi) < \infty \,\}. \tag{3.6}$$

Thus, Theorem 2.3 leads us to the final convergence result for the wiggly-energy model in its small viscosity limit, see Theorem 3.2. The limit system is the generalized gradient system $(\mathbb{R},\mathcal{E},\mathcal{R})$ with $\mathcal{E}(t,z)=\Phi(z)-\ell(t)z$ and $\mathcal{R}$ from (3.5). The corresponding ODE is the differential inclusion

$$0 \in \partial_{\dot z}\mathcal{R}(z,\dot z)+\mathrm{D}_z\mathcal{E}(t,z). \tag{3.7}$$

However, to achieve the program described above, we need one further essential condition, which is called joint-convexity condition in [Mie05, MiR07]. This condition reads

$$\inf\{\,\Phi''(z)\mid z\in\mathbb{R}\,\}=:\phi_2>w_2:=\sup\{\,|\mathrm{D}_z\mathrm{D}_yW(z,y)|\mid z,y\in\mathbb{R}\,\} \tag{3.8}$$

and implies that the functions ρ_+ and ρ_- may not vary too fast. Obviously this condition is satisfied if Φ is uniformly convex (i.e. $\phi_2>0$) and W does not depends z (i.e. $w_2=0$) as in [Jam96, ACJ96, PuT05]. A natural consequence of this condition is

$$|\rho_+(z_1)-\rho_+(z_2)|\le w_2|z_1-z_2| \quad \text{and} \quad |\rho_-(z_1)-\rho_-(z_2)|\le w_2|z_1-z_2| \tag{3.9}$$

for all $z_1,z_2\in\mathbb{R}$. To see this, fix z_1 and z_2 and choose y_j with $\mathrm{D}_yW(z_j,y_j)=\rho_+(z_j)$. Then, $\rho_+(z_1)=\mathrm{D}_yW(z_1,y_1)\ge\mathrm{D}_yW(z_1,y_2)\ge -w_2|z_1-z_2|+\mathrm{D}_yW(z_2,y_2)=-w_2|z_1-z_2|+\rho_+(z_2)$. Interchanging z_1 and z_2 the estimate for ρ_+ follows, and ρ_- works analogously.

To formulate our main convergence result we introduce the functions $\widetilde z_\pm:[0,T]\to\mathbb{R}$ via the conditions

$$\Phi'(\widetilde z_+(t))+\rho_-(\widetilde z_+(t))=\ell(t) \quad \text{and} \quad \Phi'(\widetilde z_-(t))+\rho_+(\widetilde z_-(t))=\ell(t). \tag{3.10}$$

Condition (3.9) shows that $\widetilde z_+$ and $\widetilde z_-$ are uniquely defined and are Lipschitz continuous with $\mathrm{Lip}(\widetilde z_\pm)\le \mathrm{Lip}(\ell)/(\phi_2-w_2)$.

Theorem 3.2 *Let $\Phi,\ W,\ \ell$, and $\mathcal{E}_\varepsilon$ be as described above and such that* (3.8) *holds. Then for every $\alpha>0$ we have the following convergence result. If $z_\varepsilon:[0,T]\to\mathbb{R}$ are solutions of* (3.2) *satisfying*

$$z_\varepsilon(0)\to z^0\in[\widetilde z_-(0),\widetilde z_+(0)],$$

then for $\varepsilon\to 0$ we have and the convergences

$$z_\varepsilon\to z \quad \text{in } \mathrm{C}^0([0,T]) \quad \text{and} \quad \int_{t_1}^{t_2}2\mathcal{R}_\varepsilon(\dot z_\varepsilon(t))\,\mathrm{d}t\to\int_{t_1}^{t_2}\mathcal{R}(\dot z(t))\,\mathrm{d}t$$

for $0\le t_1<t_2\le T$, where $z:[0,T]\to\mathbb{R}$ is the unique solution of the generalized gradient flow (3.7) *with $z(0)=z^0$.*

The discussions in [Mie05, Sect. 3.5+3.6] show that condition (3.9) is essential for (3.7) to have a unique Lipschitz continuous solution. We will need this to control the solutions z_ε in such a way that we are able to apply Proposition 3.1. The condition $z^0\in[\widetilde z_-(0),\widetilde z_+(0)]$ is also needed to have a solutions for the limit problem (3.7).

In fact, the dynamics of the limit system (3.7) is easily described by writing the different cases associated with the nonsmooth subdifferential $\partial_{\dot z}\mathcal{R}(z,\cdot)$. We have

$$\begin{aligned}
0=\rho_+(z)+\Phi'(z)-\ell(t) \;&\Leftrightarrow\; z=\widetilde z_-(t) && \Longrightarrow \quad \dot z\ge 0,\\
0\in\,]\rho_-(z),\rho_+(z)[\,+\Phi'(z)-\ell(t) \;&\Leftrightarrow\; z=\,]\widetilde z_-(t),\widetilde z_+(t)[&& \Longrightarrow \quad \dot z=0,\\
0=\rho_-(z)+\Phi'(z)-\ell(t) \;&\Leftrightarrow\; z=\widetilde z_+(t) && \Longrightarrow \quad \dot z\le 0,\\
0\notin[\rho_-(z),\rho_+(z)]+\Phi'(z)-\ell(t) \;&\Leftrightarrow\; z\notin[\widetilde z_-(t),\widetilde z_+(t)] && \quad \text{is forbidden.}
\end{aligned}$$

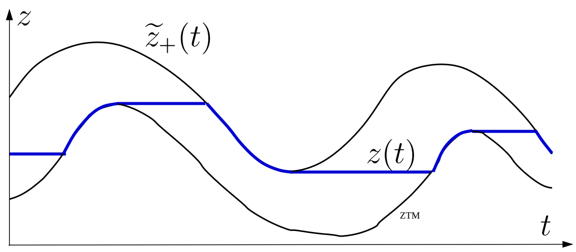

Figure 3.3: Solution $z(t)$ of (3.7) moves between $\widetilde{z}_-(t)$ and $\widetilde{z}_+(t)$.

Hence, the solution stays between the moving boundaries $\widetilde{z}_-(t)$ and $\widetilde{z}_+(t)$ in such a way that it is constant as long as possible, see Figure 3.3.

The following result shows that the solutions z_ε can be controlled in a suitable way.

Proposition 3.3 *Let $\Phi,\ W,\ \ell$, and $\mathcal{E}_\varepsilon$ be as described above such that* (3.8) *holds and let $\alpha > 0$. Then, there exists constants $C > 0$ and $c > 0$, such that the following holds: If the solutions $z_\varepsilon : [0,T] \to \mathbb{R}$ of* (3.2) *satisfy*

$$\sigma_\varepsilon := \mathrm{dist}\,\big(z_\varepsilon(0), [\widetilde{z}_-(0), \widetilde{z}_+(0)]\big) \to 0, \tag{3.11}$$

then for all $t, s \in [0,T]$ we have the estimates

$$|z_\varepsilon(t) - z_\varepsilon(s)| \le C\big(\sigma_\varepsilon + \varepsilon + \varepsilon^{\alpha/2} + |t{-}s|\big), \tag{3.12a}$$

$$\mathrm{dist}\,\big(z_\varepsilon(t), [\widetilde{z}_-(t), \widetilde{z}_+(t)]\big) \le C\big(\sigma_\varepsilon \mathrm{e}^{-ct/\varepsilon^\alpha} + \varepsilon + \varepsilon^{2\alpha}\big). \tag{3.12b}$$

We refer to Appendix B for the proof of this proposition. With these notations and the above propositions at hand, the proof of Theorem 3.2 can be completed as follows.

Proof of Theorem 3.2: The estimate (3.12a) gives equicontinuity of the family $(z_\varepsilon)_{\varepsilon\in]0,1[}$. Hence, the Arzelà-Ascoli theorem yields a subsequence $(z_{\varepsilon_j})_{j\in\mathbb{N}}$ with $\varepsilon_j \to 0$ and a limit $z \in \mathrm{C}^0([0,T])$ such that $z_{\varepsilon_j} \to z$ uniformly. Exploiting (3.12a) once again, we have the Lipschitz continuity $|z(t){-}z(s)| \le C|t{-}s|$.

Thus, we are able to apply Proposition 3.1 with η_ε defined in (3.3). Hence, the limit function z satisfies the upper energy estimate

$$\mathcal{E}(T, z(T)) + \int_0^T \mathcal{M}(z(t), \dot z(t), -\Phi'(z(t)){+}\ell(t))\,\mathrm{d}t = \mathcal{E}(0, z(0)) - \int_0^T \dot\ell z\,\mathrm{d}t.$$

By the definition of the contact sets, the relation $\mathcal{C}_\mathcal{M} = \mathcal{C}_{\mathcal{R}+\mathcal{R}^*}$ (cf. (3.6)), and the chain-rule argument of Proposition 2.1 we obtain

$$(\dot z(t), -\Phi'(z(t)){+}\ell(t)) \in \mathcal{C}_{\mathcal{R}+\mathcal{R}^*}(z(t)) \quad \text{for a.a. } t \in [0,T].$$

Hence, we have shown that z is the desired solution of (3.7) with $z(0) = z^0 = \lim_{\varepsilon\to0} z_\varepsilon(0)$. So far, we have established only the convergence of the subsequence $(z_{\varepsilon_j})_{j\in\mathbb{N}}$ only. However, because the solution z of (3.7) with $z(0) = z^0$ is unique (see [Mie05, MiR07], where we again use the joint convexity (3.8)), the whole family has to converge.

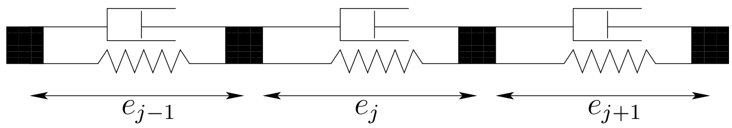

Figure 4.1: Viscoelastic chain with bistable springs.

To obtain the convergence of the dissipation integral we use that for $\varepsilon > 0$ all solutions z_ε satisfy $2\mathcal{R}_\varepsilon(\dot z_\varepsilon) = \mathcal{R}_\varepsilon(\dot z_\varepsilon)+\mathcal{R}^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(t,z_\varepsilon))$, Hence, for all $0 \le t_1 < t_2 \le T$ we have

$$\begin{aligned}\int_{t_1}^{t_2} 2\mathcal{R}_\varepsilon(\dot z_\varepsilon(t))\,\mathrm{d}t &= \int_{t_1}^{t_2} \big(\mathcal{R}_\varepsilon(\dot z_\varepsilon(t))+\mathcal{R}^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(t,z_\varepsilon))\big)\,\mathrm{d}t \\ &= \mathcal{E}_\varepsilon(t_1,z_\varepsilon(t_1)) - \mathcal{E}_\varepsilon(t_2,z_\varepsilon(t_2)) - \int_{t_1}^{t_2} \dot\ell(t)z_\varepsilon(t)\,\mathrm{d}t \\ &\to \mathcal{E}(t_1,z(t_1)) - \mathcal{E}(t_2,z(t_2)) - \int_{t_1}^{t_2} \dot\ell(t)z(t)\,\mathrm{d}t \;=\; \int_{t_1}^{t_2} \mathcal{R}(\dot z(t))\,\mathrm{d}t.\end{aligned}$$

Hence, Theorem 3.2 is established. ∎

We note that the convergence of the dissipated energy $2\mathcal{R}_\varepsilon(\dot z_\varepsilon)$ to its limit $\mathcal{R}(z,\dot z)$ is not uniform but only in the weak* sense of measures as indicated in Theorem 3.2. This is due to quite different microscopic behavior of $\dot z_\varepsilon$. The times for passing adjacent wiggles is proportional to ε, however the maximal speed during this transition is of order $\varepsilon^{-\alpha}$ as $\mathrm{D}_y W$ varies of order 1. In fact, $\dot z_\varepsilon$ is of order 1 most of the time with a short burst where $|\dot z_\varepsilon| \sim \varepsilon^{-\alpha}$ on an interval of a length of order $\varepsilon^{1+\alpha}$. This behavior is nicely displayed in Figure 3.2.

Thus, $\mathcal{R}_\varepsilon(\dot z_\varepsilon) = \varepsilon^\alpha \dot z_\varepsilon^2/2$ is of order ε^α most of the time with localized burst with size and length of the orders $\varepsilon^{-\alpha}$ and $\varepsilon^{1+\alpha}$, respectively. Only taking integrals over fixed intervals we see convergence to $\int_{t_1}^{t_2} \mathcal{R}(\dot z(t))\,\mathrm{d}t$.

4 Elastoplasticity arising from a chain of viscous, bistable springs

Another system with a wiggly energy is studied in [MiT11]. It is given in terms of a chain of N bistable springs with small viscous damping, see Figure 4.1. Denoting by e_j the strain in each spring we consider the coupled system

$$\left.\begin{array}{l}\nu\dot e_j = -\Phi'(e_j) + \mu_j^N + G(t,j/N) + \sigma(t) \quad \text{for } j=1,...,N;\\ \mathcal{C}_N((e_j)) := \frac{1}{N}\sum_{j=1}^N e_j = \ell(t).\end{array}\right\} \tag{4.1}$$

Here $\Phi : \mathbb{R} \to \mathbb{R}$ is a double-well potential, which is conveniently chose to be the biquadratic potential

$$\Phi_{\text{biq}}(e) := \frac{k}{2}\min\{(e{+}a)^2,(e{-}a)^2\}. \tag{4.2}$$

The coefficients μ_j^N are biases that are chosen independently and identically distributed according to a probability density $f \in \mathrm{L}^1([-\mu_*,\mu_*])$ with average 0.

Here the system is driven by the Dirichlet loading $\ell \in \mathrm{C}^1([0,T])$ prescribing the total length of the chain. The parameter σ is the Lagrange parameter associated with this constraint. Moreover, the function $G \in \mathrm{C}^1([0,T] \times [0,1])$ allows for a given time-dependent volume forcing. Using $\boldsymbol{e} = (e_1, ..., e_N)$ as a state vector, the system has the energy functional $\mathcal{E}_N$ and the dissipation potential $\mathcal{R}_N$:

$$\mathcal{E}_N(t, \boldsymbol{e}) = \frac{1}{N} \sum_{j=1}^{N} \Big(\Phi(e_j) - \mu_j^N e_j + G(t, j/N) e_j \Big) \quad \text{and} \quad \mathcal{R}_N(\boldsymbol{e}, \dot{\boldsymbol{e}}) = \frac{\nu}{2N} \sum_{j=1}^{N} \dot{e}_j^2.$$

The total system can now be written abstractly as a driven gradient system via

$$0 = \mathrm{D}_{\dot{\boldsymbol{e}}} \mathcal{R}_N(\boldsymbol{e}, \dot{\boldsymbol{e}}) + \mathrm{D}_{\boldsymbol{e}} \mathcal{E}_N(, \boldsymbol{e}) + \sigma(t) \mathrm{D} \mathcal{C}_N(\boldsymbol{e}), \quad \mathcal{C}_N(\boldsymbol{e}) = \ell(t).$$

In this system the small parameter is $\varepsilon = 1/N$, which is the ratio between the length of the springs and the total length. As we are interested in the case $\varepsilon \to 0$ or $N \to \infty$, the energy $\mathcal{E}_N$ becomes wiggly in the sense that there are many local minimizers for a given constraint $\mathcal{C}_N(\boldsymbol{e}) = \ell$, namely up to 2^N.

In [MiT11] the limit of particle number $N \to \infty$ and viscosity $\nu \to 0$ is studied by embedding the system into a spatially continuous setting for $\Omega =]0,1[$. For $\Phi = \Phi_{\text{biq}}$ we have two wells or phases for the springs, which we characterize by the phase indicators

$$z_j = \operatorname{sign}(e_j) \in \{-1, 0, 1\}.$$

The discrete variables z_j are the precursors of continuum plastic strain variables, which appear in the weak limit. To define the macroscopic averages we first need to introduce a spatial averaging operator.

We begin by embedding the solutions $\boldsymbol{e} \in \mathbb{R}^N$ into $\mathrm{L}^2(\Omega)$ via the characteristic functions

$$\chi_j^N = \chi_{](j-1)/N, j/N[} : x \mapsto \begin{cases} 1 & \text{for } x \in](j-1)/N, j/N[\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

This allows us to define the elastic strain fields $\overline{e}_N$ and a plastic strain field $\overline{p}_N$ via $(\overline{e}^N(t), \overline{p}^N(t)) := \mathcal{P}_N(\boldsymbol{e}^N(t))$ where

$$\mathcal{P}_N : \begin{cases} \mathbb{R}^N & \rightarrow & \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega), \\ \boldsymbol{e} = (e_j)_{j=1,\dots,N} & \mapsto & \Big(\sum_{j=1}^{N} e_j^N \chi_j^N \,,\, a \sum_{j=1}^{N} z_j^N \chi_j^N \Big) \end{cases} \tag{4.4}$$

Note that $\overline{e}$ and $\overline{p}$ are introduced in such a way that the relation

$$\Phi'(\overline{e}^N(t,x)) = k(\overline{e}^N(t,x) - \overline{p}^N(t,x))$$

holds. Thus, the nonlinear stress-strain relation is turned into a linear one, after using the nonlinear relation $e \mapsto \operatorname{sign}(e)$ for defining the plastic strain.

We now define the limit system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ via $\mathcal{Q} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$,

$$\mathcal{E}(\overline{e}, \overline{p}) = \int_\Omega \frac{k}{2} (\overline{e}(x) - \overline{p}(x))^2 + H(\overline{p}(x)) + G(t,x) \overline{e}(x) \, \mathrm{d}x, \quad \mathcal{R}(\dot{\overline{p}}) = \int_\Omega ka |\dot{\overline{p}}(x)| \, \mathrm{d}x.$$

Moreover, we have the constraint $\mathcal{C}(\overline{e}) := \int_\Omega \overline{e}(x)\,\mathrm{d}x = \ell(t)$, such that the limit equation reads

$$\begin{aligned} &0 \in \mathrm{D}_{\overline{e}}\mathcal{E}(\overline{e},\overline{p}) + \sigma(t)\mathrm{D}\mathcal{C} = k(\overline{e}{-}\overline{p}) + \sigma, \quad \mathcal{C}(\overline{e}) = \ell(t), \\ &0 \in \partial\mathcal{R}(\dot{\overline{p}}) + \mathrm{D}_{\overline{p}}\mathcal{E}(\overline{e},\overline{p}) = ka\,\mathrm{Sign}(\dot{\overline{p}}) + k(\overline{p}{-}\overline{e}) + \partial H(\overline{p}). \end{aligned} \tag{4.5}$$

Here the function $H : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is the convex hardening function induced by the random choices of the bias coefficients μ_j^N. From its distribution function $f \in \mathrm{L}^1([-\mu_*, \mu_*])$ we obtain H in the form

$$H(p) = 2aF^*\big((a{-}p)/(2a)\big) \in \mathbb{R} \cup \{\infty\} \quad \text{for } p \in \mathbb{R},$$

where the function F^* is obtained from f as follows. Define $F \in \mathrm{W}^{2,1}$ such that $F'' = f$ on $[-\mu_*, \mu_*]$ and $F'' = 0$ otherwise. Moreover, assume $F(\mu) = 0$ for $\mu \leq -\mu_*$. Then, F is convex and has a convex Legendre transform $F^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with $F^*(\xi) = \infty$ for $\xi \notin [0,1]$. Hence, we have $H(\overline{p}) = \infty$ for $|\overline{p}| \geq a$ in the general case. For the rectangular distribution $f(\mu) = 1/(2\mu_*)$ on $[-\mu_*, \mu_*]$ and 0 otherwise, we obtain $H(\overline{p}) = \mu_*(p^2{-}a^2/(2a)$.

The major result of [MiT11] is concerned with the limit $\varepsilon = 1/N \to 0$ and $\nu \to 0$ for small-viscosity system $(\mathcal{R}^N, \mathcal{E}_N, \mathcal{R}_{\nu,N})$ with the wiggly energy $\mathcal{E}_N$ under the constraint $\mathcal{C}_N$. To simplify the presentation here we choose the viscosity $\nu_N = 1/N^\alpha = \varepsilon^\alpha$ for a fixed $\alpha > 1$. In fact, we hope that the energetic method developed in the previous sections of this paper will allows us to treat the case $\alpha \in \,]0,1]$ as well. The following result is proved in [MiT11, Thm. 5.2] by completely different methods, namely by controlling the evolution of the viscous solutions $\boldsymbol{e}^N : [0,T] \to \mathbb{R}^n$ is a careful manner and uniform manner.

Theorem 4.1 *Assume $\nu_N = 1/N^\alpha$ for a fixed $\alpha > 1$. Consider the solutions $\mathbf{e}^N : [0,T] \to \mathbb{R}^N$ of the gradient system $(\mathcal{R}^N, \mathcal{E}_N, \mathcal{R}_{\nu_N,N})$, where the biases μ_j^N are chosen randomly (iid) according to the distribution f. Let the initial conditions $\mathbf{e}^N(0)$ satisfy $\mathbf{e}_j^N(0) < 0$ and*

$$\mathcal{P}_N(\boldsymbol{e}^N(0)) \rightharpoonup (\overline{e}^0, \overline{p}^0) \text{ in } \mathcal{Q} \quad \text{and} \quad \boldsymbol{E}^N(0, \boldsymbol{e}^N(0)) \to \mathcal{E}(0, \overline{e}^0, \overline{p}^0)$$

as $\varepsilon = 1/N \to 0$. Then, with probability 1 with respect to the random biases μ_j^N we have

$$\mathcal{P}_N(\boldsymbol{e}^N(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \text{ in } \mathcal{Q} \text{ for all } t \in [0,T] \text{ as } \varepsilon = 1/N \to 0,$$

where $(\overline{e}, \overline{p})$ is the unique solution of the pseudo-elastic system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ with constraint $\mathcal{C}$ and initial data $(\overline{e}(0), \overline{p}(0)) = (\overline{e}_0, \overline{p}_0)$.

A Proof of Proposition 3.1

In this section we give the full proof of the central liminf estimate that allows us to pass from the small viscosity limit to the rate-independent limit.

We derive a lower bound by estimating

$$\begin{aligned} \mathcal{M}_\varepsilon(z, v, \eta_\varepsilon{-}w_\varepsilon) &= \frac{\varepsilon^\alpha}{2}v^2 + \frac{1{-}\varepsilon^{\alpha/2}}{2\varepsilon^\alpha}(\eta_\varepsilon{-}w_\varepsilon)^2 + \frac{\varepsilon^{\alpha/2}}{2\varepsilon^\alpha}(\eta_\varepsilon{-}w_\varepsilon)^2 \\ &\geq (1{-}\varepsilon^{\alpha/2})|v|\big|\eta_\varepsilon{-}w_\varepsilon\big| + \frac{1}{2\varepsilon^{\alpha/2}}\,\mathrm{dist}(\eta_\varepsilon, [\rho_-(z_\varepsilon), \rho_+(z_\varepsilon)])^2 =: M_\varepsilon^{(1)} + M_\varepsilon^{(2)}, \end{aligned}$$

where we used $w_\varepsilon = \mathrm{D}_y W(z_\varepsilon, z_\varepsilon/\varepsilon) \in [\rho_-(z_\varepsilon), \rho_+(z_\varepsilon)]$. To treat the second term we simply observe the pointwise liminf estimate

$$(z_\varepsilon, \eta_\varepsilon) \to (z, \eta) \quad \Longrightarrow \quad \liminf_{\varepsilon\to 0} \frac{1}{2\varepsilon^{\alpha/2}} \operatorname{dist}(\eta_\varepsilon, [\rho_-(z_\varepsilon), \rho_+(z_\varepsilon)])^2 \geq \chi_{[\rho_-(z),\rho_+(z)]}(\eta).$$

Integration over $[0, T]$ and Fatou's lemma yield the first result:

$$\liminf_{\varepsilon\to 0} \int_0^T M_\varepsilon^{(2)}\,\mathrm{d}t = \liminf_{\varepsilon\to 0} \int_0^T \tfrac{\operatorname{dist}(\eta_\varepsilon, [\rho_-(z_\varepsilon), \rho_+(z_\varepsilon)])^2}{2\varepsilon^{\alpha/2}}\,\mathrm{d}t \geq \int_0^T \chi_{[\rho_-(z),\rho_+(z)]}(\eta)\,\mathrm{d}t. \tag{A.1}$$

Thus, it remains to establish the liminf estimate for the first term $M_\varepsilon^{(1)}$, namely

$$\liminf_{\varepsilon\to 0} D_\varepsilon \geq \int_0^T |\dot z| K(z,\eta)\,\mathrm{d}t, \text{ where } D_\varepsilon = \int_0^T M_\varepsilon^{(1)}\,\mathrm{d}t = \int_0^T |\dot z_\varepsilon||\eta_\varepsilon(t) - \mathrm{D}_y W(z_\varepsilon, z_\varepsilon/\varepsilon)|\,\mathrm{d}t,$$

whenever z_ε and η_ε converge uniformly to z and η, respectively. Here we proceed analogously to [Bra02, Ch. 3]. For arbitrary $n \in \mathbb{N}$ fixed we obtain for all $\varepsilon \in\,]0, 1/n[$ the lower estimate

$$D_\varepsilon \geq \sum_{j=1}^n \int_{I_j^n} |\dot z_\varepsilon(t)| h_j^n(z_\varepsilon(t)/\varepsilon)\,\mathrm{d}t \quad \text{with } I_j^n = [\tfrac{j-1}{n}T, \tfrac{j}{n}T] \text{ and}$$
$$h_j^n(y) = \min\{\, |\eta_\varepsilon(s) - \mathrm{D}_y W(z_\varepsilon(s), y)| \mid s \in I_j^n,\ \varepsilon \in\,]0, 1/n[\,\}.$$

We assume first $z(\frac{j-1}{n}T) < z(\frac{j}{n}T)$, then for $\varepsilon \to 0$ we obtain

$$\int_{I_j^n} |\dot z_\varepsilon(t)| h_j^n(z_\varepsilon(t)/\varepsilon)\,\mathrm{d}t \geq \int_{z_\varepsilon(\frac{j-1}{n}T)}^{z_\varepsilon(\frac{j}{n}T)} h_j^n(z/\varepsilon)\,\mathrm{d}z \;\longrightarrow\; \int_0^1 h_j^n(y)\,\mathrm{d}y\, \big(z(\tfrac{j}{n}T) - z(\tfrac{j-1}{n}T)\big),$$

where we used periodicity of $h_j^n(\cdot)$. We argue similarly for $z(\frac{j-1}{n}T) \geq z(\frac{j}{n}T)$ and obtain

$$\liminf_{\varepsilon\to 0} \int_{I_j^n} |\dot z_\varepsilon(t)| h_j^n(z_\varepsilon(t)/\varepsilon)\,\mathrm{d}t \geq \int_0^1 h_j^n(y)\,\mathrm{d}y \big|z(\tfrac{j}{n}T) - z(\tfrac{j-1}{n}T)\big|.$$

Denoting by z_n the piecewise affine interpolant with $z_n(\frac{j}{n}T) = z(\frac{j}{n}T)$ and $k_n(t) = \int_0^1 h_j^n(y)\,\mathrm{d}y$ for $t \in I_j^n$, we obtain

$$\liminf_{\varepsilon\to 0} D_\varepsilon \geq \sum_{j=1}^n \int_0^1 h_j^n(y)\,\mathrm{d}y \big|z(\tfrac{j}{n}) - z(\tfrac{j-1}{n})\big| = \int_0^T k_n(t)|\dot z_n(t)|\,\mathrm{d}t. \tag{A.2}$$

Using $z \in W^{1,1}([0,T])$ we obtain $\dot z_n \to \dot z$ strongly in $\mathrm{L}^1([0,T])$. Moreover, the uniform convergence of $(z_\varepsilon, \eta_\varepsilon)$ to (z, η) yields the uniform convergence $k_n(t) \to K(z(t), \eta(t))$, where we use the continuity of the mapping

$$\mathbb{R}^2 \times [0,1] \ni (z, \eta, y) \;\mapsto\; |\eta - \mathrm{D}_y W(z, y)|.$$

Thus, passing to the limit $n \to \infty$ in the right-hand side of (A.2) we obtain the desired estimate $\liminf_{\varepsilon\to 0} \int_0^T M_\varepsilon^{(1)}\,\mathrm{d}t \geq \int_0^T |\dot z| K(z,\eta)\,\mathrm{d}t$.

Combining this with (A.1) Proposition 3.1 is proved.

B Proof of Proposition 3.3

Throughout this appendix we let $\beta = \min\{1, 2\alpha\}$ such that $\varepsilon^\beta = \max\{\varepsilon, \varepsilon^{2\alpha}\}$.

The proof consists of two parts. Part I is concerned with the solutions starting at time t in or near to $[\widetilde{z}_-(t), \widetilde{z}_+(t)]$. These solutions are shown to move in the interval $[t, t+\varepsilon^\beta]$ at most by $C\varepsilon^\beta$. Part II is concerned with the approach to the interval $[\widetilde{z}_-(t), \widetilde{z}_+(t)]$ if the solution starts outside.

Before we start, we emphasize that all solutions z_ε occurring in Proposition 3.3 lie in a bounded set $Z = [z_{\text{min}}, z_{\text{max}}] \subset \mathbb{R}$. This follows easily from the energy estimate $\mathcal{E}_\varepsilon(t, z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(0, z_\varepsilon(0)) - \int_0^t \dot{\ell}(s) z_\varepsilon(s)\, \mathrm{d}s$ and a Gronwall estimate. Thus, we can take suprema of continuous functions such as Φ and W and its derivatives up to order 2.

Lemma B.1 (Part I) *Under the assumptions of Proposition 3.3, for any $C_1 > 0$ there exists a C_2 such that for all $\varepsilon \in]0, 1[$ and all solutions z_ε of* (3.2) *we have the following implication. If for some $t_0 \in [0, T]$ we have*

$$\operatorname{dist}\big(z_\varepsilon(t_0), [\widetilde{z}_-(t_0), \widetilde{z}_+(t_0)]\big) \leq C_1 \varepsilon^\beta, \tag{B.1}$$

then $|z_\varepsilon(t) - z_\varepsilon(t_0)| \leq C_2 \varepsilon^\beta$ for all $t \in [t_0, t_0+\varepsilon^\beta] \cap [0, T]$.

Proof: We apply a blocking principle by showing that the solution cannot pass points ζ_- and ζ_+ with $z_\varepsilon(t_0) - C_2\varepsilon^\beta \leq \zeta_- < z_\varepsilon(t_0) < \zeta_+ \leq z_\varepsilon(t_0) + C_2\varepsilon^\beta$. Writing $g_\varepsilon(z) = -\phi'(z) - \varepsilon \mathrm{D}_z W(z, z/\varepsilon) - \mathrm{D}_y W(z, z/\varepsilon)$ it suffices to satisfy

$$\forall\, t \in [t_0, t_0+\delta] : \quad g_\varepsilon(\zeta_-) + \ell(t) \geq 0 \text{ and } g_\varepsilon(\zeta_+) + \ell(t) \leq 0.$$

We consider the case ζ_+ only, as ζ_- can be treated analogously.

We first choose y_+ such that $\mathrm{D}_y W(z_\varepsilon(t_0), y_+) = \rho_+(z_\varepsilon(t_0))$ and search for ζ_+ in the set $y_+ + \varepsilon\mathbb{Z}$. Using (B.1), the definition of $\widetilde{z}_-$ in (3.10), and $\Lambda := \mathrm{Lip}(\ell)$ we estimate $\ell(t)$ via

$$\ell(t) \leq \ell(t_0) + \Lambda(t - t_0) \leq \Phi'(\widetilde{z}_-(t_0)) + \rho_+(\widetilde{z}_-(t_0)) + \Lambda\varepsilon^\beta.$$

With this and $C_W^1 = \sup\{\, \mathrm{D}_z W(\zeta, y) \mid z \in Z,\ y \in \mathbb{R}\,\}$ and w_2 from (3.8) we proceed

$$\begin{aligned}
&g_\varepsilon(\zeta_+) + \ell(t) = -\Phi'(\zeta_+) - \varepsilon \mathrm{D}_z W(\zeta_+, y_+) - \mathrm{W}_y(\zeta_+, y_+) + \ell(t) \\
&\leq -\Phi'(\zeta_+) + \varepsilon C_W^1 - \mathrm{D}_y W(z_\varepsilon(t_0), y_+) + w_2(\zeta_+ - z_\varepsilon(t_0)) + \Phi'(\widetilde{z}_-(t_0)) + \rho_+(\widetilde{z}_-(t_0)) + \Lambda\varepsilon^\beta \\
&\leq -\Phi'(\zeta_+) + \Phi'(z_\varepsilon(t_0)) + w_2(\zeta_+ - z_\varepsilon(t_0)) + \varepsilon^\beta C_3 \ \leq\ -(\phi_2 - w_2)(\zeta_+ - z_\varepsilon(t_0)) + \varepsilon^\beta C_3,
\end{aligned}$$

where $C_3 = C_W^1 + \Lambda + \mathrm{Lip}(\Phi')C_1$. Because $\phi_2 - w_2 > 0$ by (3.8), we can define $\eta_+ = z_\varepsilon(t_0) + \varepsilon^\beta C_3/(\phi_2 - w_2)$ and choose $\zeta_+ \in y_+ + \varepsilon\mathbb{Z}$ as small as possible but satisfying $\zeta_+ \geq \eta_+$. Hence, the construction is finished with $C_2 = 1 + C_3/(\phi_2 - w_2)$.

Thus, with the similar construction of ζ_- we conclude that $z_\varepsilon(t)$ remains inside the interval $[\zeta_-, \zeta_+]$; and the assertion is established. ∎

Next we show that solutions $z_\varepsilon(t)$ outside the interval $[\widetilde{z}_-(t), \widetilde{z}_+(t)]$ are attracted back to this interval exponentially fast.

Lemma B.2 (Part II) *Under the assumptions of Proposition 3.3 there exists a constant $C_0 > 0$ such that $\delta_\varepsilon(t) = \mathrm{dist}(z_\varepsilon(t), [\widetilde{z}_-(t), \widetilde{z}_+(t)])$, where z_ε is a solution of* (3.2)*, satisfies*

$$\delta_\varepsilon(t) \leq \delta_\varepsilon(0)\mathrm{e}^{-\lambda_\varepsilon t} + \varepsilon^\beta C_0 \quad \textit{where } \lambda_\varepsilon = \tfrac{\phi_2 - w_2}{\varepsilon^\alpha} > 0. \tag{B.2}$$

Proof: We derive the differential inequality $\varepsilon^\alpha \dot{\delta}_\varepsilon \leq -(\phi_2 - w_2)\delta_\varepsilon + \max\{\varepsilon, \varepsilon^\alpha\}C$, which immediately implies the desired estimate for δ_ε with $C_0 = C/(\phi_2 - w_2)$.

To obtain the differential inclusion we distinguish three cases. In the case $z_\varepsilon(t) \in \,]\widetilde{z}_-(t), \widetilde{z}_+(t)[$ we have $\delta_\varepsilon = \dot{\delta}_\varepsilon = 0$ and the differential inclusion holds trivially. We will treat the case $z_\varepsilon(t) \geq \widetilde{z}_+(t)$, as the case $z_\varepsilon(t) \leq \widetilde{z}_-(t)$ is similar. Using (3.8) we find

$$\begin{aligned}
\varepsilon\dot{\delta}_\varepsilon &= \varepsilon^\alpha \dot{z}_\varepsilon - \varepsilon^\alpha \dot{\widetilde{z}}_+ \\
&\leq -\Phi'(z_\varepsilon) - \varepsilon \mathrm{D}_z W(z_\varepsilon, z_\varepsilon/\varepsilon) - \mathrm{D}_y W(z_\varepsilon, z_\varepsilon/\varepsilon) + \Phi'(\widetilde{z}_+) + \rho_-(\widetilde{z}_+) + \varepsilon^\alpha C_+ \\
&\leq -\big(\Phi'(z_\varepsilon) + \rho_-(z_\varepsilon) - \Phi'(\widetilde{z}_+) - \rho_-(\widetilde{z}_+)\big) + \varepsilon C_W^1 + \varepsilon^\alpha C_+ \\
&\leq -(\phi_2 - w_2)\delta_\varepsilon + \varepsilon C_W^1 + \varepsilon^\alpha C_+ \ \leq -(\phi_2 - w_2)\delta_\varepsilon + \max\{\varepsilon, \varepsilon^\alpha\}C,
\end{aligned}$$

where $C_W^1 = \sup |\mathrm{D}_x W|$, $C_+ = \sup |\dot{\widetilde{z}}_+|$, and $C = C_W^1 + C_+$. ∎

Clearly, this lemma provides the second estimate in Proposition 3.3, namely (3.12b). To derive the first estimate, i.e. (3.12a), we combine the two lemmas by taking C_0 from the second and choosing $C_1 = 2C_0$ in the second. If $\sigma_\varepsilon = \delta_\varepsilon(0) \leq \varepsilon^\beta C_0$ the estimate (B.2) shows that (B.1) holds for all t_0, which allows us to derive (3.12a) easily. Assuming $0 \leq s < t \leq T$ we let $J_\varepsilon = 1 + [|t - s|/\varepsilon^\beta] \in \mathbb{N}$ and $\tau = (t - s)/J_\varepsilon \leq \varepsilon^\beta$ and obtain

$$|z_\varepsilon(t) - z_\varepsilon(s)| \leq \textstyle\sum_{j=1}^{J_\varepsilon} |z_\varepsilon(s + j\tau) - z_\varepsilon(s + j\tau - \tau)| \leq C_2 J_\varepsilon \varepsilon^\beta \leq C_2(\varepsilon^\beta + (t - s)). \tag{B.3}$$

If $\sigma_\varepsilon = \delta_\varepsilon(0) > \varepsilon^\beta C_0$ we have a transient phase of length t_ε where $\delta_\varepsilon(t_\varepsilon) = \varepsilon^\beta C_0$. From (B.2) we see that $t_\varepsilon \leq C\varepsilon^{\alpha/2}$. Looking into the proof of Lemma B.2 we find that z_ε is monotone on the $[0, t_\varepsilon]$; e.g. if $z_\varepsilon(0) > \widetilde{z}_+(0) + \varepsilon^\beta C_0$ we have $\varepsilon^\alpha \dot{z}_\varepsilon \leq -(\phi_2 - w_2)\delta_\varepsilon + \varepsilon C_W^1 \leq -(\phi_2 - w_2)\varepsilon C_0 + \varepsilon C_W^1 \leq 0$ by the definition of $C_0 = (C_W^1 + C_+)/(\phi_2 - w_2)$. With a similar argument for the case $z_\varepsilon(0) \leq \widetilde{z}_-(0) - \varepsilon^\beta C_0$, we have the estimate

$$|z_\varepsilon(t) - z_\varepsilon(s)| \leq |z_\varepsilon(t_\varepsilon) - z_\varepsilon(0)| \leq \sigma_\varepsilon + C\varepsilon^{\alpha/2} \quad \text{for } s, t \in [0, t_\varepsilon].$$

Combining this with (B.3) on the interval $[t_\varepsilon, T]$ we obtain the desired result (3.12a). Hence, the proof of Proposition 3.3 is complete.

References

[ACJ96] R. ABEYARATNE, C.-H. CHU, and R. JAMES. Kinetics of materials with wiggly energies: theory and application to the evolution of twinning microstructures in a Cu-Al-Ni shape memory alloy. *Phil. Mag. A*, 73, 457–497, 1996.

[AM*11] S. ARNRICH, A. MIELKE, M. A. PELETIER, G. SAVARÉ, and M. VENERONI. Passing to the limit in a Wasserstein gradient flow: from diffusion to reaction. *Calc. Var. Part. Diff. Eqns.*, 2011. Accepted.

[BMR10] S. BARTELS, A. MIELKE, and T. ROUBÍČEK. Quasistatic small-strain plasticity in the limit of small hardening and its numerical approximation. *SIAM J. Numer. Anal.*, 2010. Submitted. WIAS preprint 1585.

[Bra02] A. BRAIDES. *Γ-Convergence for Beginners*. Oxford University Press, 2002.

[BrS96] M. BROKATE and J. SPREKELS. *Hysteresis and Phase Transitions*. Springer-Verlag, 1996.

[DGMT80] E. DE GIORGI, A. MARINO, and M. TOSQUES. Problems of evolution in metric spaces and maximal decreasing curve. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 68(3), 180–187, 1980.

[HaN75] B. HALPHEN and Q. S. NGUYEN. Sur les matériaux standards généralisés. *J. Mécanique*, 14, 39–63, 1975.

[HuM93] Y. HUO and I. MÜLLER. Nonequilibrium thermodynamics of pseudoelasticity. *Contin. Mech. Thermodyn.*, 5(3), 163–204, 1993.

[Jam96] R. D. JAMES. Hysteresis in phase transformations. In *ICIAM 95 (Hamburg, 1995)*, volume 87 of *Math. Res.*, pages 135–154. Akademie Verlag, Berlin, 1996.

[Mie05] A. MIELKE. Evolution in rate-independent systems (Ch. 6). In C. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559. Elsevier B.V., Amsterdam, 2005.

[Mie11] A. MIELKE. Generalized Prandtl-Ishlinskii operators arising from homogenization and dimension reduction. *Physica B*, 2011.

[MiR07] A. MIELKE and R. ROSSI. Existence and uniqueness results for a class of rate-independent hysteresis problems. *Math. Models Meth. Appl. Sci. (M^3AS)*, 17, 81–123, 2007.

[MiT11] A. MIELKE and L. TRUSKINOVSKY. From discrete visco-elasticity to continuum rate-independent plasticity: rigorous results. *Arch. Rational Mech. Anal.*, 2011. To appear. WIAS Preprint 1541.

[MRS08] A. MIELKE, T. ROUBÍČEK, and U. STEFANELLI. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Part. Diff. Eqns.*, 31, 387–416, 2008.

[MRS11] A. MIELKE, R. ROSSI, and G. SAVARÉ. BV solutions and viscosity approximations of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 2011. WIAS preprint 1451.

[Mül89] I. MÜLLER. On the size of the hysteresis in pseudoelasticity. *Contin. Mech. Thermodyn.*, 1(2), 125–142, 1989.

[Mül98] I. MÜLLER. Six lectures on shape memory. In *Boundaries, interfaces, and transitions (Banff, AB, 1995)*, volume 13 of *CRM Proc. Lecture Notes*, pages 125–161. Amer. Math. Soc., 1998.

[MüS01] I. MÜLLER and S. SEELECKE. Thermodynamic aspects of shape memory alloys. *Math. Comput. Modelling*, 34(12-13), 1307–1355, 2001.

[OnM53] L. ONSAGER and S. MACHLUP. Fluctuations and irreversible processes. *Phys. Rev.*, 91(6), 1505–1512, 1953.

[Ött05] H. C. ÖTTINGER. *Beyond Equilibrium Thermodynamics*. John Wiley, New Jersey, 2005.

[Pop10] V. L. POPOV. *Contact Mechanics and Friction*. Springer, 2010.

[Pra28] L. PRANDTL. Gedankenmodel zur kinetischen Theorie der festen Körper. *Z. Angew. Math. Mech.*, 8, 85–106, 1928.

[PuT00] G. PUGLISI and L. TRUSKINOVSKY. Mechanics of a discrete chain with bi-stable elements. *J. Mech. Phys. Solids*, 48(1), 1–27, 2000.

[PuT02] G. PUGLISI and L. TRUSKINOVSKY. Rate independent hysteresis in a bi-stable chain. *J. Mech. Phys. Solids*, 50(2), 165–187, 2002.

[PuT05] G. PUGLISI and L. TRUSKINOVSKY. Thermodynamics of rate-independent plasticity. *J. Mech. Phys. Solids*, 53, 655–679, 2005.

[RSZ09] T. ROUBÍČEK, L. SCARDIA, and C. ZANINI. Quasistatic delamination problem. *Contin. Mech. Thermodyn.*, 21, 223–235, 2009.

[SaS04] E. SANDIER and S. SERFATY. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, LVII, 1627–1672, 2004.

[Ser10] S. SERFATY. Gamma-convergence of gradient flows on Hilbert spaces and metric spaces and applications. *Preprint Jussieu*, 2010.

[Ste08] U. STEFANELLI. The Brezis-Ekeland principle for doubly nonlinear equations. *SIAM J. Control Optim.*, 47(3), 1615–1642, 2008.

[Sul09] T. J. SULLIVAN. *Analysis of Gradient Descents in Random Energies and Heat Baths*. PhD thesis, Dept. of Mathematics, University of Warwick, 2009.

[ZiW87] H. ZIEGLER and C. WEHRLI. The derivation of constitutive relations from the free energy and the dissipation function. In *Advances in Applied Mechanics, Vol. 25*, pages 183–237. Academic Press, Orlando, FL, 1987.