Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

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submitted: 7th August 1995

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Preprint No. 164
Berlin 1995

The work was partially supported by the NSF grant DMS 9504513.

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A NOTE ON THE EXTREMALITY OF THE DISORDERED STATE FOR THE ISING MODEL ON THE BETHE LATTICE

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August 4, 1995

ABSTRACT. We give a simple proof that the limit Ising Gibbs measure with free boundary conditions on the Bethe lattice with the forward branching ratio $k \geq 2$ is extremal if and only if β is less or equal to the spin glass transition value, given by $\tanh(\beta_c^{SG}) = 1/\sqrt{k}$.

Key words: Bethe lattice, FK representation, Ising model

In this short note we prove the following:

Theorem 1. *For the Ising model on the Bethe lattice with the forward branching ratio $k \geq 2$ the limit infinite volume Gibbs state with free boundary conditions is extremal if and only if*

$$\tanh(\beta) \leq \tanh(\beta_c^{SG}) = 1/\sqrt{k}.$$

The role played by the spin glass transition value β_c^{SG} was analysed in details in the context of the Ising spin glass on the Bethe lattice with $k = 2$ in [CCST]. A modification of their method was used in [B] to prove the theorem above, but the latter paper contains a mistake. This mistake is claimed to be corrected in a recent article [BRZ]. We refer to the articles [CCST], [B] and [BRZ] for a thorough discussion of the underlying problem. A method of the proof we suggest here is different from those in [B] and [BRZ], seems to be much simpler from the computational point of view and, in a way, intrinsic for the model under consideration. However, as in [B] and [BRZ], the key idea of reductions to recursive estimates on second moments is inherited from the original paper [CCST].

So let $\mathcal{T} = (V, \mathcal{E})$ to denote the halfspace Bethe lattice with the forward branching ratio $k \geq 2$, where V and $\mathcal{E}$ are the corresponding vertex(or site) and edge(or bond) sets respectively. The root site of $\mathcal{T}$ will be always indexed by zero. For any finite connected subtree $\mathcal{T}_A = (A, \mathcal{E}_A)$ we define the Gibbs state on $\mathcal{T}_A$ with free boundary conditions at the inverse temperature $\beta > 0$ as a probability measure $\mathbb{P}_A^\beta$ on $\Omega_A = \{-1, 1\}^A$, which assigns weights

$$\exp\{ -\beta \sum_{<i,j>} x_i x_j \}$$

[1]The work was partially supported by the NSF grant DMS 9504513

to spin configurations $(x_i)_{i\in A} \in \Omega_A$. The summation above is over all unordered pairs of nearest neighbours $i,j :< i,j >\in \mathcal{E}_A$. In fact all $\mathbb{P}^\beta_A$ are relativizations on Ω_A of a certain probability measure $\mathbb{P}^\beta$ on $\Omega = \{-1,1\}^V$, which is precisely the limit infinite volume Gibbs measure with free boundary conditions, also referred to as the disordered state in the title of this note. It is the tree structure of the graph $\mathcal{T}$ which is entirely responsible for the latter assertion. This becomes transparent if one evokes the FK (Fortuin-Kasteleyn) representation of lattice ferromagnetic systems with pair interactions (see, for example, [ACCN] in general and [CCST] for the Bethe lattice case). In our situation everything boils down to the following representation of $\mathbb{P}^\beta_A$:

$$\text{Set} \quad p = \tanh(\beta).$$

Step1. Consider an independent Bernoulli percolation on $\mathcal{E}_A$, i.e. assign to each bond configuration $n_A \in \{0,1\}^{\mathcal{E}_A}$ the probability

$$\mathbb{Q}^p_A(n_A) = p^{\sum_{b\in\mathcal{E}_A} n_A(b)}(1-p)^{|\mathcal{E}_A|-\sum_{b\in\mathcal{E}_A} n_A(b)},$$

where $|\mathcal{E}_A|$ is the cardinality (number of edges) in $\mathcal{E}_A$.

Step2. Given a bond configuration n_A, two sites $i,j \in A$ are called connected if they are connected by the chain of open bonds in $\mathcal{E}_A$, i.e if $n_A = 1$ on all bonds from the (unique) chain leading from i to j. Thus, any (random) configuration n_A splits A into disjoint union of maximal connected components (or clusters). Now, in order to specify values of spins at various sites of A, paint independently each cluster of A into $+1$ or -1 with probability $1/2$ each.

After performing both steps above we end up with a probability distribution on Ω_A. The important fact is that this measure happens to be precisely $\mathbb{P}^\beta_A$.

The above two-step procedure can be, using some labelling algorithm to avoid ambiguities, equally applied to construct probability measures $\mathbb{P}^\beta_A$ for infinite connected subsets $A \subseteq V$, in particular for V itself. Thus, let $\mathbb{Q}^p$ to denote the independent Bernoulli percolation measure on $\{0,1\}^{\mathcal{E}}$ and $\mathbb{P}^\beta$ to denote the corresponding measure on Ω. Clearly, $\mathbb{P}^\beta$ possesses the relativization property claimed above, i.e

$$\mathbb{P}^\beta(x|_A \in \bullet) = \mathbb{P}^\beta_A(\bullet), \tag{1}$$

where $x|_A$ is the restriction of the configuration $x \in \Omega$ to A. Equally clear is that $\mathbb{P}^\beta$ is the thermodynamic limit of the finite volume Gibbs states with free boundary conditions.

Consequently, many questions about $\mathbb{P}^\beta$ and $\mathbb{P}^\beta_A$ admit a natural percolation interpretation. A basic example is provided by the following computation:

For any finite subset $A \subset V$ set

$$x_A = \prod_{j\in A} x_j.$$

Then, if $|A|$ is odd,

(2) $$< x_A >^\beta = 0 ,$$

where $< \bullet >^\beta$ is the expectation with respect to $\mathbb{P}^\beta$.

If $|A|$ is even, let us say that a configuration $n \in \{0,1\}^{\mathcal{E}}$ splits A evenly, if there is even number of vertices of A in each maximal connected component of n. Then,

(3) $$< x_A >^\beta = \mathbb{Q}^p(n : n \text{ splits } A \text{ evenly})$$

Formulas (2) and (3) are immediate consequences of the FK representation and (1). Indeed, let $\mathcal{T}(A) = (V(A), \mathcal{E}(A))$ be the minimal connected subtree which spans A. Then, by (1),

$$< x_A >^\beta = < x_A >^\beta_{V(A)},$$

where $< \bullet >^\beta_{V(A)}$ is the expectation with respect to $\mathbb{P}^\beta_{V(A)}$. Simple combinatoric arguments, then, imply:

$$< x_A >^\beta_{V(A)} = 0 ,$$

in the odd case, and

$$< x_A >^\beta_{V(A)} = \mathbb{Q}^p_{\mathcal{E}(A)}(n : n \text{ splits } A \text{ evenly})$$

in the even case. Since $\mathbb{Q}^p_{\mathcal{E}(A)}$ is the relativization of $\mathbb{Q}^p$, (3) follows.

Another example of how the percolation approach works is provided by the following:

Proposition 2. *Let two disjoint finite subsets $A, B \subset V$ have edge disjoint minimal spanning trees, i.e $\mathcal{E}(A) \cap \mathcal{E}(B) = \emptyset$.*

a) If both $|A|$ and $|B|$ are even, then

(4) $$< x_A x_B >^\beta = < x_A >^\beta < x_B >^\beta .$$

b) If both $|A|$ and $|B|$ are odd, then for any site j, which lies on the unique chain connecting $V(A)$ to $V(B)$,

(5) $$< x_A x_B >^\beta = < x_A x_j >^\beta < x_B x_j >^\beta .$$

Proof: Both formulas are consequences of (2) and (3) above and independence relations for Bernoulli percolation.

We now turn to the proof of Theorem 1. Let x_0 to denote the value of the spin at the root site of $\mathcal{T}$. Also let B_N be the set of sites at distance N from the root, where the distance $d(i,j)$ between two sites i and j is defined to be the number of edges in the unique chain connecting those two sites. Finally, let $\mathcal{F}_N$ be the σ-algebra generated by the spin configurations from $\{-1,1\}^{B_N}$. Note that $\mathbb{P}^\beta$ is extremal, i.e $\mathbb{P}^\beta$ has a trivial tail σ-field, if and only if,

(6) $$\lim_{N\to\infty} Var^\beta(\mathbb{E}^\beta(x_0|\mathcal{F}_N)) = 0 ,$$

where all the expectations are computed with respect to $\mathbb{P}^\beta$. Since $< x_0 >^\beta = 0$, we shall identify $\mathcal{F}_N$ with the Eucledian space of all $\mathbb{P}^\beta$-zero mean functions

$f : \{-1,1\}^{B_N} \to \mathbb{R}$ equipped with the scalar product $< \bullet >^\beta$. $\mathcal{F}_N$ is, then, spanned by the linear independent family $\{x_A\}_{A\subset B_N}$. We also use $\mathcal{F}_N^+$ to denote the positive cone spanned by this family. To facilitate notations let us use x_N to denote the restriction of a configuration x on $\{-1,1\}^{B_N}$ (instead of $x|_{B_N}$ above).Define:

$$g_N(x_N) \;=\; \mathbb{E}^\beta(x_0|\mathcal{F}_N) \;=\; Proj|_{\mathcal{F}_N} x_0.$$

Because of the self similarity of the measure $\mathbb{P}^\beta$, we may consider g_N to be just a function on $\{-1,1\}^{k^N}$ and use it unambiguously to denote the projection of any spin to the σ-algebra generated by its N-th generation of descendants. Here we say that i belongs to the N-th generation of descendants of j, if $d(i,j)=N$ and j lies on the unique chain leading from i to zero.

There are k branches emanating from the root site. Let us denote by x_N^l the restriction of a configuration $x_N \in \{-1,1\}^{B_N}$ to the l-th branch. Similarly, each subset $A \subseteq B_N$ can be splitted into the disjoint union

$$A \;=\; \cup_1^k A_l,$$

where A_l contains those sites of A which lie on the l-th branch. We may, then, rewrite x_A as

$$x_A \;=\; \prod_1^k x_{A_l}^l.$$

Finally, define $\mathcal{F}_N^l$ to be the subspace of $\mathcal{F}_N$, which is spanned by the polynomials $\prod_1^k x_{A_l}^l$ with $|A_l|$ odd. Obviously,

$$\mathcal{F}_N \;=\; \bigoplus_1^k \mathcal{F}_N^l.$$

Proposition 3. *For every l; $l \;=\; 1,...,k$,*

$$Proj|_{\mathcal{F}_N^l} g_N(x_N) \;=\; p g_{N-1}(x_N^l). \tag{7}$$

Proof: If $|A_l|$ is odd, then by (4) and (5),

$$< x_0 \prod_m x_{A_m}^m >^\beta \;=\; < x_1^l x_{A_l}^l >^\beta < x_0 x_1^l >^\beta < \prod_{m\neq l} x_{A_m}^m >^\beta \;=$$

$$=\; p < g_{N-1}(x_N^l) x_{A_l}^l >^\beta < \prod_{m\neq l} x_{A_m}^m >^\beta \;=\; p < g_{N-1}(x_N^l) \prod_m x_{A_m}^m >^\beta .$$

The above proposition provides a gateway for a recursive estimate on

$$\|g_N\|^2 \stackrel{def}{=} Var^\beta(\mathbb{E}^\beta(x_0|\mathcal{F}_N)).$$

Another crucial property of g_N can be formulated as follows:

Proposition 4. *For each $N \geq 0$ the projection $g_N(x_N)$ belongs to the positive cone $\mathcal{F}_N^+$.*

Proof: The claim follows by induction. We have to show that for each $M \in \mathbb{N}$; if $u \in \mathcal{F}_M$ and $< x_A u >^\beta \leq 0 \; \forall x_A \in \mathcal{F}_M^+$, then

$$(8) \qquad < g_M(x_M)u >^\beta \leq 0$$

also. This is, of course, straightforward for $M = 0$. Assume that the induction assumption holds for $M = N-1$ and suppose that (9) is no longer true for $M = N$, i.e that one can find $u \in \mathcal{F}_N$, such that $< g_N u >^\beta > 0$, whereas $< x_A u >^\beta \leq 0, \; \forall A \subseteq B_N$. By the symmetry considerations we may assume from the beginning that $u \in \mathcal{F}_N^1$, i.e that

$$u = \sum_{|A_1|-\text{odd}} \; \sum_{|A_2|+\ldots+|A_k|-\text{even}} a_{A_1 A_2 \ldots A_k} \prod_1^k x_{A_l}^l .$$

Then, by (4) and (7),

$$< g_N(x_N)u >^\beta = < x_0 u >^\beta = \sum_{|A_1|-\text{odd}} \tilde{a}_{A_1} < g_{N-1}(x_{N-1}^1) x_{A_1}^1 >^\beta ,$$

where

$$\tilde{a}_{A_1} = p \sum_{|A_2|+\ldots+|A_k| \text{ even}} a_{A_1 A_2 \ldots A_k} < \prod_2^k x_{A_l}^l >^\beta .$$

Consequently, setting $\tilde{u} = \tilde{u}(x_N^1) = p \sum_{|A_1|-\text{odd}} \tilde{a}_{A_1} x_{A_1}^1$, we obtain:

$$0 < < g_N(x_N)u >^\beta = < g_{N-1}(x_N^1)\tilde{u} >^\beta .$$

Because of the self similarity of $\mathbb{P}^\beta$, this contradicts the induction assumption. Indeed, for any x_A^1 with $|A|$ odd,

$$0 \geq < x_A^1 u >^\beta = < x_A^1 \tilde{u}(x_N^1) >^\beta ,$$

which means that $\tilde{u}(x_{N-1})$ provides an example of $\mathcal{F}_{N-1}$ function for which (8) fails with $M = N-1$.

Let us define a function (random variable) ξ_N via

$$(9) \qquad g_N = p \sum_{l=l}^k g_{N-1}(x_N^l) - \xi_N .$$

It follows immediately from Propositions 3 and 4 that $< \xi_N u >^\beta \geq 0$ for each $u \in \mathcal{F}_N^+$. In particular,

$$(10) \qquad < \xi_N g_N >^\beta \geq 0.$$

Therefore, multiplying both sides of (9) by g_N we obtain, in a view of (7), that

$$\|g_N\|^2 = kp^2 \|g_{N-1}\|^2 - < \xi_N g_N >^\beta \leq kp^2 \|g_{N-1}\|^2 ,$$

which implies the claim of the theorem for $kp^2 < 1$.

The assertion of the theorem in the critical case $kp^2 = 1$ follows from the recursive estimate on $||g_N||$ below:

$$||g_N||^2 \leq \frac{||g_{N-1}||^2}{1 + (k-1)p^2||g_{N-1}||^2}, \tag{11}$$

which is just a refinement of (10). Namely, we claim that for any $u \in \mathcal{F}_N^+$,

$$< \xi_N u >^\beta \geq (k-1)p^2||g_{N-1}||^2 < g_N u >^\beta . \tag{12}$$

In particular,

$$< \xi_N g_N >^\beta \geq (k-1)p^2||g_{N-1}||^2||g_N||^2,$$

and (11) follows.

Thus it remains to verify (12). It is enough to consider $u = x_A \in \mathcal{F}_N^1$. If $|A_q|$ is odd for some $q > 1$, i.e if x_A also belongs to some $\mathcal{F}_N^q$, then

$$< \xi_N x_A >^\beta = p\sum_2^k < g_{N-1}(x_N^l)x_A >^\beta \geq p < g_{N-1}(x_N^q)x_A >^\beta =$$

$$= < g_N x_A >^\beta \geq (k-1)p^2||g_{N-1}||^2 < g_N x_A >^\beta .$$

So let us assume that all $|A_l|$, $l > 1$, are even. Then,

$$< g_N x_A >^\beta = < x_0 x_{A_1}^1 >^\beta \prod_2^k < x_{A_l}^l >^\beta,$$

and for $l > 1$,

$$< g_{N-1}(x_N^l)x_A >^\beta = < x_0 x_{A_1}^1 >^\beta < x_0 g_{N-1}(x_N^l)x_{A_l} >^\beta \prod_{q \neq 1,l} < x_{A_q}^l >^\beta .$$

But,

$$< x_0 g_{N-1}(x_N^l)x_{A_l} >^\beta = p < x_1^l g_{N-1}(x_N^l)x_{A_l}^l >^\beta = p < g_{N_1}^2(x_N^l)x_{A_l}^l >^\beta$$

However, by the virtue of proposition 4, g_{N-1}^2 is a polynomial with nonnegative coefficients. Therefore, by the second Griffiths' inequality,

$$< g_{N-1}^2 x_{A_l}^l >^\beta \geq < g_{N-1}^2 >^\beta < x_{A_l}^l >^\beta,$$

and (12) follows.

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