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Dispersion of nonlinear group velocity determines shortest envelope solitons

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Abstract

We demonstrate that a generalized nonlinear Schrödinger equation (NSE), that includes dispersion of the intensity-dependent group velocity, allows for exact solitary solutions. In the limit of a long pulse duration, these solutions naturally converge to a fundamental soliton of the standard NSE. In particular, the peak pulse intensity times squared pulse duration is constant. For short durations this scaling gets violated and a cusp of the envelope may be formed. The limiting singular solution determines then the shortest possible pulse duration and the largest possible peak power. We obtain these parameters explicitly in terms of the parameters of the generalized NSE.

1 Introduction

Optical solitons are waves localized either in space or time that are formed as a result of the interplay between nonlinearity and dispersion [9, 2]. A typical soliton, e.g., in an optical fiber, is often described by its complex envelope ψ which satisfies the slowly varying envelope approximation (SVEA). SVEA assumes that the pulse spectrum is concentrated around a well defined carrier frequency ω_0. Then, the dispersion is represented by the Taylor expansions around ω_0. The dispersion of the linear response given by the frequency-dependent wave vector $\beta(\omega)$ is then encoded within a discrete set $\beta_m = \beta^{(m)}(\omega_0)$ of expansion coefficients, where at least three lowest order terms with $m = 0, 1, 2$ are taken into account. For unidirectional propagation, the complex pulse envelope $\psi(z, t)$ is normally described in a co-moving frame $\psi = \psi(\zeta, \tau)$ with $\zeta = z$ and $\tau = t - \beta_1 z$, where β_1 is the inverse group velocity and τ is referred as retarded time. In the case of an instantaneous cubic nonlinearity, the envelope is governed by the nonlinear Srödinger equation (NSE)

$$i\partial_\zeta\psi + \frac{\beta_2}{2}(i\partial_\tau)^2\psi + \gamma|\psi|^2\psi = 0, \tag{1}$$

where β_2 is the group velocity dispersion (GVD) and the parameter γ is determined by the linear bulk dispersion, fiber geometry (effective fiber area), and nonlinear susceptibility of the third order [1]. The envelope is usually scaled in the way that $|\psi|^2$ represents the pulse power. For the focusing ($\gamma > 0$) nonlinearity, bright solitary solutions appear in the domain of negative dispersion ($\beta_2 < 0$).

The model given by the NSE does not impose any restrictions on soliton duration: two-fold decrease of the duration simply means four-fold increase of the peak power. It is the SVEA that lacks precision for shorter pulse durations and broader pulse spectra. That is why Eq. (1) is commonly replaced by a more accurate generalized NSE for short pulses. In what follows, we are primarily interested in solitary solutions of generalized pulse propagation equations and therefore we deliberately exclude linear losses and Raman scattering.

The simplest generalization of Eq. (1) is to use a larger number of purely real dispersion parameters β_m. The term $\frac{1}{2}\beta_2(i\partial_\tau)^2\psi$ then must be replaced by the so-called dispersion operator

$$\hat{\mathfrak{D}}\psi = \sum_{m=2}^{M_{\text{max}}} \frac{\beta_m}{m!}(i\partial_\tau)^m\psi, \tag{2}$$

which covers the behavior of $\beta(\omega)$ in a larger frequency domain. Still, even an infinite number of parameters β_m does not guarantee the necessary convergence [13, 3].

Another important generalization of the NSE is to introduce dispersion into the nonlinear term in Eq. (1). Such effective dispersion eliminates some of the deficiencies of the SVEA. It naturally appears in description of short pulses even in an ideal Kerr media with the non-dispersive instantaneous nonlinear response and in the absence of Raman scattering. E.g., if the group velocity $1/\beta_1$ and the phase velocity ω_0/β_0 are similar, one can derive the following generalized NSE [6, 7]

$$i\partial_\zeta\psi + \hat{\mathfrak{D}}\psi + \gamma(1+\omega_0^{-1}i\partial_\tau)|\psi|^2\psi = 0, \tag{3}$$

where the derivative of the nonlinear term describes the nonlinear GVD, i.e., an intensity-dependend contribution to the group velocity. The resulting self-steepening effect is a key factor in extension of the envelope-based generalized NSE towards the single-cycle regime.

If the group- and phase-velocities differ, one faces a more complex dispersion in the nonlinear term, i.e., it becomes nonlocal in time. Equation (3) is then replaced with [11, 12, 5]

$$i\partial_\zeta\psi + \hat{\mathfrak{D}}\psi + \gamma\frac{n(\omega_0)}{\omega_0}\frac{\omega_0+i\partial_\tau}{n(\omega_0+i\partial_\tau)}|\psi|^2\psi = 0, \tag{4}$$

where $n(\omega) = c\beta(\omega)/\omega$ is the refractive index. If dispersion of the nonlinear susceptibility and the effective fiber area cannot be ignored, an even more complicated operator appears in the nonlinear term in (4). Such a nonlocal nonlinear term has either (i) to be evaluated in the frequency domain or (ii) to be approximated in the spirit of Taylor expansion, analogous to Eq. (2).

In what follows, we take the second point of view and expand the nonlocal term in Eq. (4) up to the second order. After proper rescaling of the space, time, and field variables, we obtain the following model equation

$$i\partial_z\psi + \frac{1}{2}\partial_t^2\psi + \left(1 + i\sigma\partial_t - \frac{\mu}{2}\partial_t^2\right)|\psi|^2\psi = 0, \tag{5}$$

where from now on z, t, and ψ refer to rescaled variables and σ and μ are dimensionless parameters. The higher-order linear dispersion is neglected. In this work, we are primarily interested in the influence of the μ-term on the localized solitary solutions of Eq. (5). Such solutions will be first obtained for $\sigma = 0$ and then extended to $\sigma \neq 0$.

Without loss of generality, we assume that $|\psi|^2$ takes its maximum value at $t = 0$, which is referred to as peak power P_0. Localized solutions must decay at large $|t|$. Solitons, in particular, are expected to decay exponentially, $\psi \sim \exp(-|t|/t_0)$ for $t \to \pm\infty$. The quantity t_0 is the pulse duration. This parameter and the peak power are mutually related as it will be shown below. In particular, we will parametrize solitons by t_0 and look for the corresponding pulse shape and P_0.

The classical NSE (1) is recovered from Eq. (5) for $\sigma = \mu = 0$. The so called fundamental soliton solution is given by

$$\psi = \frac{1/t_0}{\cosh t/t_0} e^{iz/(2t_0^2)}. \tag{6}$$

For this solution,

$$P_0 t_0^2 = 1. \tag{7}$$

When $\sigma \neq 0$ but $\mu = 0$, Eq. (5) corresponds to the generalized NSE (3). Solitary solutions of Eq. (3) were recently found by adopting a universal Lax pair technique [8]. The shape of a direct generalization of the fundamental soliton solution reads

$$|\psi|^2 = \frac{2/t_0^2}{1 + \sqrt{1 + (\sigma/t_0)^2} \cosh(2t/t_0)}. \tag{8}$$

It reduces to (6) for $\sigma \ll t_0$. Furthermore,

$$P_0 t_0^2 = \frac{2}{1 + \sqrt{1 + (\sigma/t_0)^2}} < 1, \tag{9}$$

such that for the same pulse duration t_0, the resulting peak power P_0 appears to be smaller than that of the fundamental soliton (7). Note, that both (6) and (8) formally allow for an arbitrarily short pulse duration and an arbitrarily high peak power.

In what follows, we demonstrate that Eq. (5) allows for exact solitary solutions, even for $\mu \neq 0$. With increase of the pulse duration, these solutions receive $1/\cosh$ shape and are undistinguishable from the fundamental NSE soliton (6). A principal new feature is observed for short durations: the value of $P_0 t_0^2$ increases above unity and in some cases a cusp of the envelope $|\psi|$ may develop. This limiting singular solution determines the shortest possible pulse duration and the highest possible peak power. We explicitly obtain these pulse characteristics in terms of μ and σ.

2 Derivation

To derive solitary solutions we apply the following ansatz

$$\psi(z,t) = A(t) e^{iz/(2t_0^2)}, \tag{10}$$

as suggested by the fundamental soliton solution (6). Inserting (10) into (5) we obtain an equation for the complex soliton amplitude $A(t)$

$$A'' - \frac{A}{t_0^2} + 2|A|^2 A + 2i\sigma(|A|^2 A)' - \mu(|A|^2 A)'' = 0, \tag{11}$$

where derivatives with respect to t are denoted by a prime. Far from the soliton center Eq. (11) can be linearized. Then we immediately see that $A \sim \exp(-|t|/t_0)$. All solitary solutions of Eq. (11) asymptotically behave like the fundamental soliton (6). However, the peak power is generally different from $1/t_0^2$.

In contrast to the simplest NSE case, Eq. (11) does not allow for real valued solutions for $\sigma \neq 0$. Therefore, we introduce both the amplitude and the phase

$$A(t) = a(t)e^{i\Phi(t)},$$

where in accordance with our notations

$$a(0) = \sqrt{P_0}, \qquad a'(0) = 0.$$

We now rewrite Eq. (11) in the form

$$[(a - \mu a^3)e^{i\Phi}]'' e^{-i\Phi} - \frac{a}{t_0^2} + 2a^3 + 2i\sigma(a^3 e^{i\Phi})' e^{-i\Phi} = 0. \tag{12}$$

The imaginary part of this equation is

$$(a - \mu a^3)\Phi'' + 2(a - \mu a^3)'\Phi' + 6\sigma a^2 a' = 0.$$

It can be integrated once:

$$\Phi' = -\frac{\sigma}{2} \frac{3 - 2\mu a^2}{(1 - \mu a^2)^2} a^2, \tag{13}$$

where we imply a natural restriction: $\Phi' \to 0$ as $a \to 0$. Without loss of generality one can assume that the phase $\Phi(t \to -\infty) = 0$, then in general $\Phi(t \to +\infty) \neq 0$. The latter value can be found from Eq. (13) after the shape function $a(t)$ is determined.

Using Eq. (13), the real part of Eq. (12) can be transformed to the form

$$(a - \mu a^3)'' - \frac{a}{t_0^2} + 2a^3 + \frac{\sigma^2}{4} \frac{4(1 - \mu a^2)^2 - 1}{(1 - \mu a^2)^3} a^5 = 0.$$

We multiply the latter equation by $(1 - 3\mu a^2)a'$, integrate it once, and obtain

$$(1 - 3\mu a^2)^2 a'^2 - \left(1 - \frac{3}{2}\mu a^2\right) \frac{a^2}{t_0^2} + (1 - 2\mu a^2)a^4 + \frac{\sigma^2}{4} \frac{(1 - 2\mu a^2)^2}{(1 - \mu a^2)^2} a^6 = C, \tag{14}$$

where C is an integration constant. For a solitary solution with finite energy both $a(t), a'(t) \to 0$ at $t \to \pm\infty$, therefore $C = 0$.

Formally, all solutions of Eq. (14) can now be found in quadratures. In principle, the cusp solutions may appear if $\mu < 0$ and $P_0 \to 1/(3\mu)$ with the decrease of the pulse width t_0. That is how the limiting singular solitary solution appears. Properties of the solutions given by Eq. (14) are described in the next section.

3 Solitary solutions

We now consider Eq. (14) for $C = 0$ in more detail and describe localized solutions for $a(t)$, which is our main goal here.

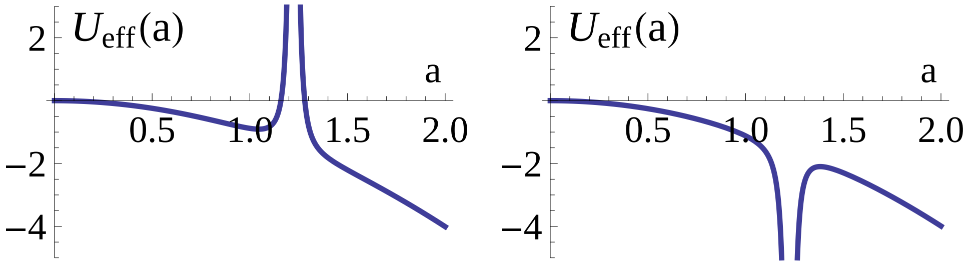

Figure 1: The effective potential in Eq. (16) for $\mu = 2/9$ such that $t_0^{\text{min}} = 1$ and for t_0 slightly above and slightly below t_0^{min}. Left: $t_0 = 1.01$. Right: $t_0 = 0.99$.

3.1 The case $\mu = 0$

The standard fundamental soliton is recovered from Eq. (14) when $\mu = 0$. Then Eq. (14) is simplified to the form

$$a'^2 - \frac{a^2}{t_0^2} + a^4 + \frac{\sigma^2}{4}a^6 = 0, \tag{15}$$

and its peak power is determined from the relation

$$\frac{\sigma^2}{4}P_0^2 + P_0 - \frac{1}{t_0^2} = 0,$$

which leads directly to Eq. (9). Introducing a new variable $\mathfrak{a} = 1/a^2 - t_0^2/2$, one can integrate (15) and obtain solitary solutions (6) and (8).

3.2 The case $\mu > 0$ and $\sigma = 0$

We start with the case $\mu > 0$ and put $\sigma = 0$ first. Equation (14) is then reduced to the following equation

$$a'^2 \underbrace{- \frac{1 - \frac{3}{2}\mu a^2}{(1 - 3\mu a^2)^2}\frac{a^2}{t_0^2} + \frac{1 - 2\mu a^2}{(1 - 3\mu a^2)^2}a^4}_{U_{\text{eff}}(a)} = 0. \tag{16}$$

The last two terms in this equation can be considered as an effective potential $U_{\text{eff}}(a)$. Then the trajectory $a = a(t)$ defined by this dynamical system belongs to the region $U_{\text{eff}}(a) \leq 0$. The peak power is determined from the condition $U_{\text{eff}}(\sqrt{P_0}) = 0$, that is, from the equation

$$P_0 t_0^2 = \frac{1 - \frac{3}{2}\mu P_0}{1 - 2\mu P_0}. \tag{17}$$

We now consider the latter equation for a fixed $\mu > 0$ and various pulse durations t_0. For a temporally wide pulse with $t_0^2 \gg \mu$ we obtain $P_0 = 1/t_0^2$, as it should be for the fundamental soliton. With decreasing t_0, the peak power exceeds $1/t_0^2$. Equation (17) yields physically

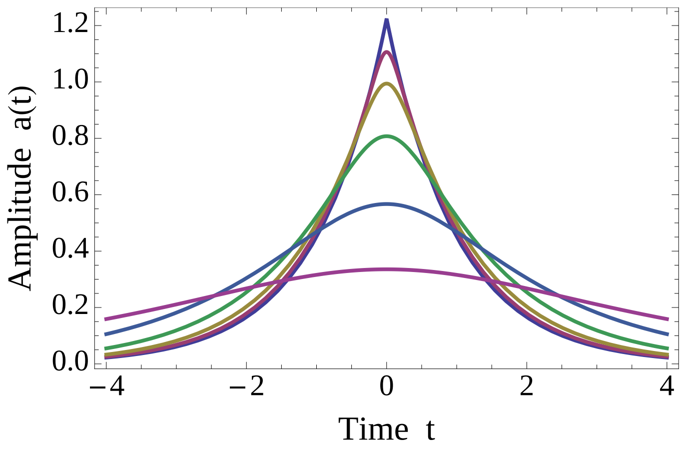

Figure 2: Illustrative solitary solutions of Eq. (16) for $\mu = 2/9$ such that $t_0^{\text{min}} = 1$ and different values of $t_0 \geq t_0^{\text{min}}$. From bottom to top $t_0 = 3.0,\ 1.8,\ 1.3,\ 1.1,\ 1.03$. The uppermost line is given by the limiting singular solution (21).

meaningful values for P_0 as long as

$$t_0 \geq t_0^{\text{min}} = 3\sqrt{\frac{\mu}{2}}. \tag{18}$$

This provides the shortest pulse duration and the largest peak power

$$P_0^{\text{max}} = \frac{1}{3\mu}, \qquad (P_0 t_0^2)_{\text{cusp}} = \frac{3}{2} \tag{19}$$

respectively. The scaling law holds for the shortest soliton, but with a different constant, cf. Eqs. (7) and (9). In particular, the peak power is 50% larger than that for the fundamental soliton with the same duration.

These results can also be explained analysing the effective potential in Eq. (16). For $t_0^2 \gg \mu > 0$ one can neglect all terms $\sim \mu a^2$ and obtain a standard double-well effective potential with the fundamental soliton solution (6). When t_0^2 decreases and approaches $9\mu/2$, the maximum value of a^2 approaches $1/(3\mu)$, such that the singular behavior of the effective potential cannot be longer ignored. Representative plots of $U_{\text{eff}}(a)$ for t_0 slightly above and slightly below the critical value (18) are shown in Fig. 1. Clearly, non-singular localized solutions of Eq. (16) do exist only in the first case. The smallest possible value of t_0 can also be derived by expanding the effective potential at the singularity point $a^2 = 1/(3\mu)$, namely

$$U_{\text{eff}}(a)\Bigg|_{a^2\to 1/(3\mu)} \to \frac{t_0^2 - 9\mu/2}{27\mu^2 t_0^2 (1-3\mu a^2)^2} + O(1), \tag{20}$$

which exhibits the sign change in the singularity shown in Fig. 1 in accordance with Eq. (18).

More accurate analysis of Eq. (16) shows that

$$a(t)\Bigg|_{t_0\to t_0^{\text{min}}} \to \frac{1}{\sqrt{3\mu}} \exp\left(-\frac{\sqrt{2}}{3\sqrt{\mu}}|t|\right), \tag{21}$$

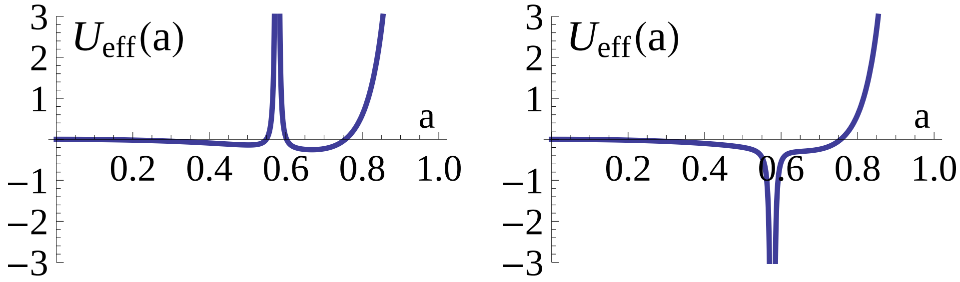

Figure 3: The effective potential in Eq. (22) for $\mu = 1$, $\sigma = 4$ such that $t_0^{\text{min}} = 1.5$ and for t_0 slightly above and slightly below t_0^{min}. Left: $t_0 = 1.52$. Right: $t_0 = 1.48$.

where the latter expression is the limiting soliton shape. It has a cusp at $t = 0$ which actually prevents the existence of solitons with exactly this or shorter time durations.

Numerical solutions of Eq. (16) for a fixed μ and several values of $t_0 \to t_0^{\text{min}}$ are shown in Fig. 2. When decreasing t_0, the soliton shape approaches the uppermost limiting $a(t)$ given by the singular solution (21).

3.3 The case $\mu > 0$ and $\sigma \neq 0$

We now consider $a(t)$ for $\sigma \neq 0$. Again $C = 0$ in Eq. (14), which still has the form $a'^2 + U_{\text{eff}}(a) = 0$ but with a more complicated effective potential

$$U_{\text{eff}}(a) = -\frac{1 - \frac{3}{2}\mu a^2}{(1 - 3\mu a^2)^2}\frac{a^2}{t_0^2} + \frac{1 - 2\mu a^2}{(1 - 3\mu a^2)^2}a^4 + \frac{\sigma^2}{4}\frac{(1 - 2\mu a^2)^2}{(1 - 3\mu a^2)^2(1 - \mu a^2)^2}a^6. \tag{22}$$

Analysis of the soliton behavior is now more cumbersome, but the basic features are similar to those described in the previous section. For given μ and σ, the solitary solutions exist as long as $t_0 \geq t_0^{\text{min}}$. Two exemplary plots of $U_{\text{eff}}(a)$ for t_0 slightly above and slightly below the critical value are shown in Fig. 3. Evidently, a non-singular soliton exists only in the first case. This critical value of t_0 can be found from the equation for the peak power, cf. Eq. (17)

$$P_0 t_0^2 \left[1 + \frac{\sigma^2}{4}\frac{1 - 2\mu P_0}{(1 - \mu P_0)^2}P_0\right] = \frac{1 - \frac{3}{2}\mu P_0}{1 - 2\mu P_0}. \tag{23}$$

As in the previous section, the fundamental soliton corresponds to the limiting case $P_0 \to 1/t_0^2$ for $t_0^2 \gg \mu$. This gives us the possibility to trace the behavior of the peak power with decreasing t_0 for fixed μ and σ. Solitary solutions exist as long as $P_0 \leq P_0^{\text{max}}$ and $t_0 \geq t_0^{\text{min}}$ with

$$P_0^{\text{max}} = \frac{1}{3\mu}, \qquad t_0^{\text{min}} = 3\sqrt{\frac{\mu/2}{1 + \sigma^2/(16\mu)}}, \tag{24}$$

cf. Eqs. (18) and (19). In particular, Eq. (19) is replaced with

$$(P_0 t_0^2)_{\text{cusp}} = \frac{3/2}{1 + \sigma^2/(16\mu)}. \tag{25}$$

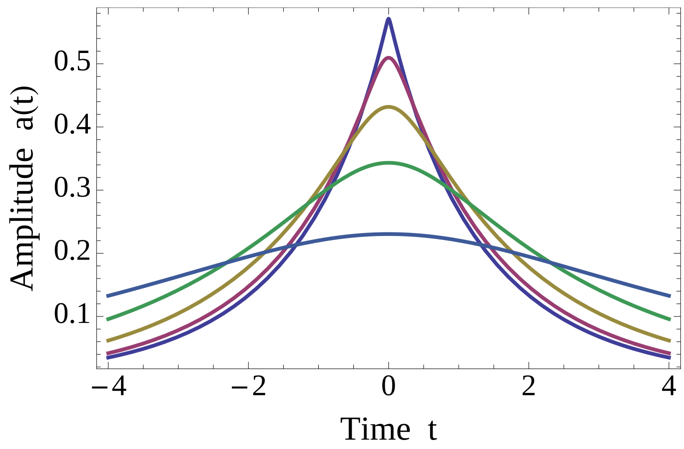

Figure 4: Illustrative solitary solutions of Eq. (14) for $C = 0$, $\mu = 1$, $\sigma = 4$ such that $t_0^{\text{min}} = 3/2$ and different values of $t_0 \geq t_0^{\text{min}}$. From bottom to top $t_0 = 4.0,\ 2.5,\ 1.9,\ 1.6,\ 1.501$. The uppermost line is close to the limiting singular solution.

Typical shapes of solitary solutions when t_0 is decreasing and approaching t_0^{min} are shown in Fig. 4.

3.4 The case $\mu < 0$

In this case, the behavior of solitary solutions of the generalized NSE (14) is qualitatively similar to that of the standard NSE (1). Namely, the effective potential in Eq. (14) is a regular function for all possible soliton shapes and the pulse duration parameter t_0 can be arbitrarily small. For extremely short solitons with $t_0 \to 0$ the peak power is determined by the relation

$$P_0 t_0^2 = \frac{3|\mu|}{2\sigma^2 + 4|\mu|} < 1.$$

In particular, the peak intensity is smaller then that of the fundamental soliton.

4 Conclusions

We investigated how dispersion of the nonlinear term in the generalized NSE (5) affects fundamental solitons. The first order dispersion ($\sigma \neq 0$ but $\mu = 0$) causes reduction of the peak intensity of a soliton. Similar decrease of the peak intensity is observed if the second order dispersion term is present and $\mu < 0$. In all cases, the pulse duration parameter t_0 can take arbitrarily small values. New effects are observed when the second order dispersion is present and simultaneously $\mu > 0$. In this case, the peak intensity is larger than that of the fundamental soliton. Soliton duration is bounded from below $t_0 > t_0^{\text{min}}$, with the limiting value t_0^{min} being given by Eq. (25). The limiting soliton has a characteristic cusp profile. Such singular profiles have recently been observed also for the nonenvelope pulse propagation equations, see [14, 10, 4].

Our results suggest that cusp formation may be a universal mechanism responsible for the appearance of the limiting solitons. This cusp formation prevents existence of solitons with exactly limiting or shorter durations.

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References

[1] G. P. Agrawal. *Nonlinear Fiber Optics*. Academic, New York, 4 edition, 2007.

[2] N. Akhmediev and A. Ankiewicz. *Solitons: Nonlinear Pulses and Beams*. Chapman and Hall, 1997.

[3] Sh. Amiranashvili, U. Bandelow, and A. Mielke. Padé approximant for refractive index and nonlocal envelope equations. *Opt. Comm.*, 283(3):480–485, 2010.

[4] Sh. Amiranashvili, A. G. Vladimirov, and U. Bandelow. Solitary-wave solutions for few-cycle optical pulses. *Phys. Rev. A*, 77(6):063821, 2008.

[5] Sh. Amiranashvili, A. G. Vladimirov, and U. Bandelow. A model equation for ultrashort optical pulses around the zero dispersion frequency. *Eur. Phys. J. D*, 58(2):219–226, 2010.

[6] T. Brabec and F. Krausz. Nonlinear optical pulse propagation in the single-cycle regime. *Phys. Rev. Lett.*, 78(17):3282–3285, 1997.

[7] J. M. Dudley, G. Genty, and S. Coen. Supercontinuum generation in photonic crystal fiber. *Rev. Mod. Phys.*, 78(4):1135–1183, 2006.

[8] S. H. Han and Q.-H. Park. Effect of self-steepening on optical solitons in a continuous wave background. *Phys. Rev. E*, 83(6):066601, 2011.

[9] A. Hasegawa. *Optical Solitons in Fibers*. Springer, 1980.

[10] A. V. Kim, S. A. Skobelev, D. Anderson, T. Hansson, and M. Lisak. Extreme nonlinear optics in a Kerr medium: Exact soliton solutions for a few cycles. *Phys. Rev. A*, 77(4):043823, 2008.

[11] P. Kinsler and G. H. C. New. Few-cycle pulse propagation. *Phys. Rev. A*, 67(2):023813, 2003.

[12] P. Kinsler and G. H. C. New. Few-cycle soliton propagation. *Phys. Rev. A*, 69(1):013805, 2004.

[13] K. E. Oughstun and H. Xiao. Failure of the quasimonochromatic approximation for ultra-short pulse propagation in a dispersive, attenuative medium. *Phys. Rev. Lett.*, 78(4):642–645, 1997.

[14] S. A. Skobelev, D. V. Kartashov, and A. V. Kim. Few-optical-cycle solitons and pulse self-compression in a Kerr medium. *Phys. Rev. Lett.*, 99(20):203902, 2007.