Quasiconvexity equals rank-one convexity for isotropic sets of 2x2 matrices

Sebastian Heinz

submitted: August 16, 2011

Weierstraß-Institut
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: sebastian.heinz@wias-berlin.de

No. 1637
Berlin 2011

2010 Mathematics Subject Classification. 26B25 52A30.

Key words and phrases. Quasiconvexity, rank-one convexity, lamination convexity, isotropy.

Research supported by the DFG through FOR 797 Analysis and Computation of Microstructures in Finite Plasticity under Mie 459/5-2.
Abstract

Let $K$ be a given compact set of real $2 \times 2$ matrices that is isotropic, meaning invariant under the left and right action of the special orthogonal group. Then we show that the quasiconvex hull of $K$ coincides with the rank-one convex hull (and even with the lamination convex hull of order 2). In particular, there is no difference between quasiconvexity and rank-one convexity for $K$. This is a generalization of a known result for connected sets.

1 Introduction

We study quasiconvexity in the calculus of variations. Morrey [Mor52] introduced it as the essential property for functions in the context of sequentially weakly lower semicontinuity for multiple integrals. He also conjectured that quasiconvexity is a "non-local" property, which was later shown to be true by Kristensen [Kri99]. At the heart of Kristensen’s proof lies Šverák’s counterexample of a rank-one convex function that fails to be quasiconvex [Šve92]. However, this counterexample works only in the case of an underlying space $\mathbb{M}^{m \times n}$ with $m \geq 3$, $n \geq 2$. Müller [Mül99a] showed that rank-one convexity implies quasiconvexity on diagonal $2 \times 2$ matrices. The general situation in $\mathbb{M}^{2 \times 2}$ remains unknown.

Closely related to the quasiconvexity for functions is the corresponding concept for sets. Basically, quasiconvex sets are lower-level sets of quasiconvex continuous functions. We focus on quasiconvexity for isotropic sets in $\mathbb{M}^{2 \times 2}$ and prove the following result (see Theorem 7.2).

**Theorem (Equivalence).** Let $K \subseteq \mathbb{M}^{2 \times 2}$ be a given compact and isotropic set. Then $K$ is lamination convex if and only if $K$ is quasiconvex.

As long as the set $K$ is connected, there is even equivalence between lamination convexity and polycovexity. This was shown by Conti et al. [CDLMR03] and, before, by Cardaliaguet and Tahraoui [CT00, CT02a, CT02b] in the case when $K$ contains only matrices with non-negative determinant. Conti et al. [CDLMR03] give also an example of a disconnected $K$ that is lamination convex but not polycovex. In addition, we will characterize the structure of the quasiconvex hull of $K$. Our main result reads (see Theorem 7.3)

**Theorem (Characterization of $K^{qc}$).** Let $K \subseteq \mathbb{M}^{2 \times 2}$ be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2.

The paper is organized as follows:

In Section 2 we will fix the notations and recall definitions of the convexity notions that are used later on. Preliminaries can be found in Section 3 and 4. Then we refine a result by Conti et
al. [CDLMR03] for connected $K$ in Section 5. Section 6 is dedicated to the closed lamination convex hull $K^{\text{cl}}$ and its structure. The key observation is that the principle structure of $K^{\text{cl}}$ is already determined by the lamination convex hull of order one. In Section 7 we deal with the equivalence of lamination convexity and quasiconvexity. The main step is to show that what is disconnected in $K^{\text{cl}}$ remains so in $K^{\text{pc}}$. Then we apply a deep result by Faraco and Székelyhidi [FS08] saying that the quasiconvex hull for the support of a homogeneous gradient Young measure is connected.

## 2 Functions, measures, and hulls

We are going to recall some convexity notions that play an important role in this paper. Our focus lies on dimension 2. A detailed discussion, also for higher dimensions, can be found in Dacorogna [Dac89, 4.1], Ball [Bal77] and Müller [Mül99b].

We denote by $\mathbb{M}^{2\times2}$ the vector space of all real $2 \times 2$ matrices equipped with the Euclidean structure of $\mathbb{R}^4$. The corresponding matrix norm is denoted by $|.|$, the identity matrix by $I$. Let $f : \mathbb{M}^{2\times2} \rightarrow \mathbb{R}$ be a given continuous function. Then $f$ is convex if for every $A, B \in \mathbb{M}^{2\times2}$ we have

$$\forall \lambda \in [0, 1] \quad f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B). \quad (1)$$

The function $f$ is polyconvex if there exists a convex function $g : \mathbb{R}^5 \rightarrow \mathbb{R}$ such that for every $A \in \mathbb{M}^{2\times2}$ we have $f(A) = g(A, \text{det}(A))$, where $\text{det}(A)$ denotes the determinant of $A$. We will often use that for every real number $\alpha \in \mathbb{R}$ the function $\alpha \text{det}$ is polyconvex. The function $f$ is quasiconvex (in the sense of Morrey [Mor52]), if for every $A \in \mathbb{M}^{2\times2}$ and every smooth function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with compact support we have

$$0 \leq \int_{\mathbb{R}^2} (f(A + D\phi(x)) - f(A)) \, dx.$$

The function $f$ is rank-one convex if (1) holds for every $A, B \in \mathbb{M}^{2\times2}$ that are rank-one connected, meaning $A - B$ equals the tensor product $a \otimes b$ for some vectors $a, b \in \mathbb{R}^2$. Polyconvexity and rank-one convexity were introduced by Ball [Bal77].

With the help of the convexity notions for functions, we now define the convexity notions for sets. Let $K \subseteq \mathbb{M}^{2\times2}$ be a given set and $A \in \mathbb{M}^{2\times2}$ a matrix. Then $A$ lies in the polyconvex hull of $K$ and we write $A \in K^{\text{pc}}$ whenever $f(A) \leq \sup \{ f(B) \mid B \in K \}$ holds for every polyconvex function $f : \mathbb{M}^{2\times2} \rightarrow \mathbb{R}$. The set $K$ is called polyconvex whenever $K = K^{\text{pc}}$ holds. The quasiconvex hull and the rank-one convex hull as well as quasiconvexity and rank-one convexity for sets are defined correspondingly.

We will give an alternative characterization in the case of compact sets. Therefore, denote by $\mathcal{P}_0(\mathbb{M}^{2\times2})$ the set of all compactly supported probability measures that are defined over the Borel sets of $\mathbb{M}^{2\times2}$. Let $\nu \in \mathcal{P}_0(\mathbb{M}^{2\times2})$ be a given element. We write $\bar{\nu}$ for its mean value and $\text{supp}(\nu)$ for its support, meaning the compliment of the set $\cup \{ U \subseteq \mathbb{M}^{2\times2} \mid \nu(U) = 0 \land U \text{ open} \}$. In addition, let $f : \mathbb{M}^{2\times2} \rightarrow \mathbb{R}$ be a continuous function. Then the following pairing is finite and well-defined

$$\langle \nu, f \rangle = \int_{\mathbb{M}^{2\times2}} f(A) \, d\nu(A).$$
We define the sets $\mathcal{P}^{pc}$, $\mathcal{P}^{qc}$ and $\mathcal{P}^{rc}$. A probability measure $\nu \in \mathcal{P}_0(\mathbb{M}^{2 \times 2})$ lies in $\mathcal{P}^{pc}$ ($\mathcal{P}^{qc}$ or $\mathcal{P}^{rc}$) if and only if Jensen’s inequality $f(\bar{\nu}) \leq \langle \nu, f \rangle$ is fulfilled for every polyconvex (quasiconvex or rank-one convex) continuous function $f : \mathbb{M}^{2 \times 2} \to \mathbb{R}$. Kinderlehrer and Pedregal [KP91] show that every $\nu \in \mathcal{P}^{qc}$ is a homogenous gradient Young measure. Whereas every $\nu \in \mathcal{P}^{rc}$ is a laminate, see Pedregal [Ped93].

**Remark 2.1.** Let $K \in \mathbb{M}^{2 \times 2}$ be a given compact set. Then the set $K^{pc}$ coincides with $\{\bar{\nu} \mid \nu \in \mathcal{P}^{pc} \land \text{supp}(\nu) \subseteq K\}$ and $K^{qc}$ as well as $K^{rc}$ can be characterized in a corresponding way.

As in Müller and Šverák [MŠ96], $K$ is called lamination convex if for every rank-one connected $A, B \in K$ and every real number $\lambda \in [0, 1]$ we have that $\lambda A + (1-\lambda)B$ lies in $K$. The closed lamination convex hull $K^{clc}$ of all closed lamination convex subsets in $\mathbb{M}^{2 \times 2}$ that contain $K$. Note that $\{A, B\}^{clc}$ equals $\{\lambda A + (1-\lambda)B \mid \lambda \in [0, 1]\}$ and, hence, is a connected set if $A, B \in \mathbb{M}^{2 \times 2}$ are rank-one connected. Otherwise $\{A, B\}^{clc} = \{A, B\}$ is disconnected. Here we call a given set $S \subseteq \mathbb{M}^{2 \times 2}$ connected if there is no way to write $S$ as the union of two disjoint nonempty relatively-open subsets of $S$. Moreover, we set $K^{lc,1} = \bigcup \{A, B\}^{clc} \in \mathcal{A}_2(K)$ as well as $K^{lc,2} = (K^{lc,1})^{clc}$, which are called the lamination convex hulls of order one and two, respectively. We would like to remark that, in general, the set $K^{clc}$ and the lamination convex hull of $K$ is different as has been shown by Kolář [Kol03].

The previous definitions together with the hierarchy of convexity notions on the level of functions imply that

$$K \subseteq K^{lc,1} \subseteq K^{lc,2} \subseteq K^{clc} \subseteq K^{rc} \subseteq K^{qc} \subseteq K^{pc}.$$ 

Finally, we denote by $cc(K)$ the set of all connected components (meaning maximal connected subsets) of $K$.

### 3 Compatible isotropic sets

We give a characterization of compatible isotropic sets. The general result for $\mathbb{M}^{n \times n}$, $n \geq 1$, is due to Šilhavý [Šil01, Pro. 3.1]. In our case $\mathbb{M}^{2 \times 2}$, this was already done by Aubert and Tahraoui [AT87, Thé. 2.8], if only for matrices with non-negative determinant. The proofs of Lemma 3.1 and Lemma 3.2 are given for the convenience of the reader.

We call a set $M \subseteq \mathbb{M}^{2 \times 2}$ *isotropic* whenever it is invariant under the left and right action of the special orthogonal group $SO(2)$, meaning $M = M^{iso}$ where

$$M^{iso} = \{QAR \mid Q, R \in SO(2) \land A \in M\}.$$ 

Here we consider $SO(2)$ as a subset of $\mathbb{M}^{2 \times 2}$ so that the group action becomes just matrix multiplication. The following notation works well in the context of isotropic sets and has been used before by many authors. Let $A \in \mathbb{M}^{2 \times 2}$ be a given matrix, then we define $\lambda(A) = (\lambda_1(A), \lambda_2(A)) \in \mathbb{R}^2$ as the only pair of real numbers such that $\{||\lambda_1(A)||, \lambda_2(A)|| \}$ is the set
of singular values of $A$ and, in addition, $|λ_1(A)| ≤ λ_2(A)$ as well as $\det(A) = λ_1(A)λ_2(A)$ holds. In fact, we have that

$$\{A\}^{iso} = \{B\}^{iso} ⇔ λ(A) = λ(B) ⇔ (|A| = |B| \land \det(A) = \det(B)).$$

We say that two subsets $M_1, M_2 ⊆ M^{2×2}$ are **compatible** whenever there exist rank-one connected matrices $A_1 ∈ M_1$ and $A_2 ∈ M_2$. Otherwise $M_1$ and $M_2$ are called **incompatible**.

**Lemma 3.1.** Let $A ∈ M^{2×2}$ be a given matrix. Then $\{A\}^{iso}$ and $SO(2)$ are compatible if and only if $|λ_1(A)| ≤ 1 ≤ λ_2(A)$ holds.

**Proof.** Assume that $|λ_1(A)| ≤ 1 ≤ λ_2(A)$. Then the following matrices are rank-one connected: $I ∈ SO(2)$ and

$$I + \begin{pmatrix} λ_1(A)λ_2(A) − 1 & \sqrt{(1 − λ_1(A)^2)(λ_2(A)^2 − 1)} \\ 0 & 0 \end{pmatrix} ∈ \{A\}^{iso}.$$ 

Now assume that $\{A\}^{iso}$ and $SO(2)$ are compatible. Then there exist vectors $a, b ∈ \mathbb{R}^2$ and a matrix $C ∈ \{A\}^{iso}$ such that $C = I + a ⊗ b$. We know that $\det(C) = 1 + \langle a, b \rangle$ and $|C|^2 = 2 + 2\langle a, b \rangle + |a|^2|b|^2$. Together with the Cauchy-Schwarz inequality, we obtain the estimate $|C|^2 − \det(C)^2 ≥ 0$. This implies that

$$λ_1(A)^2 + λ_2(A)^2 − λ_1(A)^2λ_2(A)^2 − 1 = (1 − λ_1(A)^2)(λ_2(A)^2 − 1) ≥ 0.$$ 

Hence, we must have $|λ_1(A)| ≤ 1 ≤ λ_2(A)$. \qed

**Lemma 3.2.** Let $A, B ∈ \mathbb{R}^{2×2}$ be given matrices. Then $\{A\}^{iso}$ and $\{B\}^{iso}$ are compatible if and only if $|λ_1(A)| ≤ λ_2(B)$ and, at the same time, $|λ_1(B)| ≤ λ_2(A)$.

**Proof.** Clearly, the lemma is true for $\det(A) = \det(B) = 0$. By symmetry, we can and we will assume that $\det(B) > 0$ for the rest of the proof. If necessary, we replace $A$ and $B$ by $−A$ and $−B$, respectively. In particular, we then have $0 < λ_1(B)$.

First, we start with $|λ_1(A)| ≤ λ_2(B)$ and $|λ_1(B)| ≤ λ_2(A)$. Then we obtain the inequality $|λ_1(A)/λ_2(B)| ≤ 1 ≤ λ_2(A)/λ_1(B)$. By Lemma 3.1, we conclude that the sets $\{C\}^{iso}$ and $SO(2)$ are compatible where $C = \text{diag}(λ_1(A)/λ_2(B), λ_2(A)/λ_1(B))$. Hence, there exist a rotation $R ∈ SO(2)$ and vectors $a, b ∈ \mathbb{R}^2$ such that $R + a ⊗ b = C$. If we multiply both sides from the right by $\text{diag}(λ_2(B), λ_1(B))$, we get

$$R \text{diag}(λ_2(B), λ_1(B)) + a ⊗ b = \text{diag}(λ_1(A), λ_2(A)).$$ 

This shows that $\{A\}^{iso}$ and $\{B\}^{iso}$ are compatible.

Second, we start with $\{A\}^{iso}$ and $\{B\}^{iso}$ being compatible. Then we can write

$$R \text{diag}(λ_2(B), λ_1(B)) + a ⊗ b = \text{diag}(λ_1(A), λ_2(A))Q$$

for some rotations $R, Q ∈ SO(2)$ and vectors $a, b ∈ \mathbb{R}^2$. Multiplying both sides from the right by $\text{diag}(1/λ_2(B), 1/λ_1(B))$, we see that $SO(2)$ and the set $\{D\}^{iso}$ are compatible where

$$D = \text{diag}(λ_1(A), λ_2(A))Q \text{diag}(1/λ_2(B), 1/λ_1(B)).$$ (2)
Hence, by Lemma 3.1, we must have $|\lambda_1(D)| \leq 1 \leq \lambda_2(D)$. This implies, in particular, that we can fix a vector $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$ such that $|Dx_0| = 1$.

The rest of the proof is by contradiction. Suppose that $|\lambda_1(A)| > \lambda_2(B)$. In view of (2), we obtain the inequality $|Dx_0| \geq |\lambda_1(A)|/\lambda_2(B) > 1$. Now suppose that $|\lambda_1(B)| > \lambda_2(A)$. Then we have $|Dx_0| \leq \lambda_2(A)/|\lambda_1(B)| < 1$. In both cases, we get a contradiction to the choice of $x_0$.

In Figure 1(a), you see a given set $\{A\}^{iso}$ and the region of all $\{B\}^{iso}$ such that $\{A\}^{iso}$ and $\{B\}^{iso}$ are compatible.

The lemma and remark are taken from Conti et al. [CDLMR03, Lem. 2.2, Rem. 2].

**Lemma 3.3.** Let $c \in \mathbb{R}\setminus\{0\}$ be a real number. Then the functions $\varphi_c^\pm : \mathbb{M}^{2 \times 2} \to \mathbb{R}$ given by

$$
\varphi_c^\pm(A) = \lambda_2(A) \pm \lambda_1(A) - \det(A)/c
$$

are polyconvex. The same holds for the functions $\varphi_0^\pm = -\det$.

**Proof.** The lemma follows from the convexity of the functions $\lambda_2 \pm \lambda_1$, which in turn is proved by the explicit computation

$$
\lambda_2(A) \pm \lambda_1(A) = \sqrt{|A|^2 \pm 2 \det(A)} = \sqrt{(A_{11} \pm A_{22})^2 + (A_{21} \mp A_{12})^2}.
$$

The functions $-\det/c$ as well as $-\det$ are polyconvex by definition.

**Remark 3.4.** Let $A \in \mathbb{M}^{2 \times 2}$ be given. Consider the matrices $A_+, A_- \in \mathbb{M}^{2 \times 2}$ defined via

$$
A_\pm = \begin{pmatrix}
|\det(A)|^{1/2} & \pm \sqrt{|A|^2 - 2|\det(A)|} \\
0 & |\det(A)|^{-1/2} \det(A)
\end{pmatrix}.
$$

The matrices $A_+$ and $A_-$ are rank-one connected and $A_+, A_- \in \{A\}^{iso}$ holds. Thus, the matrix $(A_+ + A_-)/2 = \text{diag}(|\det(A)|^{1/2}, |\det(A)|^{-1/2} \det(A))$ as well as every other matrix $B \in \mathbb{M}^{2 \times 2}$ with $\det(A) = \det(B)$ and $\lambda_2(B) \leq \lambda_2(A)$ lies in $(\{A\}^{iso})^{lc,1}$.

### 4 Lamination convex sets

We will introduce the sets $L_\alpha^\pm, L_\beta^0, \Delta_+(\alpha, \beta)$ and $\Delta_0(\beta)$. With the help of these sets, the proof of our results is becoming much simpler.

The following lemma can be used to construct compact lamination convex sets.

**Lemma 4.1.** Let $\alpha, \beta \geq 0$ be given real numbers. Then the following three sets are closed, isotropic and lamination convex

$$
L_\alpha^\pm = \{A \in \mathbb{M}^{2 \times 2} \mid \alpha \leq \pm \lambda_1(A)\}, \quad L_\beta^0 = \{A \in \mathbb{M}^{2 \times 2} \mid \lambda_2(A) \leq \beta\}.
$$
Proof. By definition, the sets $L_{a}^{+}, L_{a}^{-}$ and $L_{a}^{0}$ are closed as well as isotropic. The set $L_{a}^{0}$ is even convex, in fact, we have that $L_{a}^{0} = \{A \in \mathbb{M}^{2\times2} \mid \|A\|_{\ast} \leq \beta\}$ where $\|\cdot\|_{\ast}$ denotes the spectral norm. Since for $L_{a}^{-}$ we can exploit the fact $L_{a}^{-} = -L_{a}^{+}$, it remains to show that $L_{a}^{-}$ is lamination convex. Suppose that this is not the case. Then there exist rank-one connected $\alpha$, $\beta$ such that $\det(\mu A_{1} + (1-\mu)A_{2}) < 0$. On the one hand, since $\alpha_{0}$ is a singular value of the matrix $\mu A_{1} + (1-\mu)A_{2}$, there exist a normalized vector $x_{0} \in \mathbb{R}^{2}$ with $|x_{0}| = 1$ and a rotation $R \in \text{SO}(2)$ such that $x_{0}^{T}R(\mu A_{1} + (1-\mu)A_{2})x_{0} = \alpha_{0}$.

On the other hand, we know that $|x_{0}^{T}A_{i}x_{0}| \geq \lambda_{1}(A_{i}) \geq \alpha$ for $i = 1, 2$. We conclude that $x_{0}^{T}R_{1}A_{1}x_{0}$ and $x_{0}^{T}R_{2}A_{2}x_{0}$ have different signs. Hence, we can fix a real number $\mu_{0} \in [0, 1]$ such that $x_{0}^{T}R(\mu_{0}A_{1} + (1-\mu_{0})A_{2})x_{0} = 0$ and $\det(\mu_{0}A_{1} + (1-\mu_{0})A_{2}) = 0$. This forms a contradiction, since the function $-\det$ is rank-one convex (even polyconvex) and $-\det(A_{i}) \leq -\alpha^{2} < 0$ holds for $i = 1, 2$.

For given non-negative real numbers $\alpha, \beta \geq 0$ we consider the following isotropic and compact (possibly empty) sets

$$\triangle_{\pm}(\alpha, \beta) = \{A \in \mathbb{M}^{2\times2} \mid \alpha \leq \pm \lambda_{1}(A) \land \lambda_{2}(A) \leq \beta\},$$

$$\triangle_{0}(\beta) = \{A \in \mathbb{M}^{2\times2} \mid \lambda_{2}(A) \leq \beta\}.$$

We collect some properties of these sets.

Lemma 4.2. The sets $\triangle_{\pm}(\alpha, \beta)$, $\triangle_{-}(\alpha, \beta)$ as well as $\triangle_{0}(\beta)$ are compact, isotropic and lamination convex. Consider the matrices $A_{1}^{\pm} = \text{diag}(\pm \alpha, \alpha)$, $A_{2}^{\pm} = \text{diag}(\pm \alpha, \beta)$ and $A_{3}^{\pm} = \text{diag}(\pm \beta, \beta)$. Then we have $\triangle_{\pm}(\alpha, \beta) = (\{A_{1}^{\pm}\}_{\text{iso}} \cup \{A_{2}^{\pm}\}_{\text{iso}} \cup \{A_{3}^{\pm}\}_{\text{iso}})_{\text{clc}}$ as well as $\triangle_{0}(\beta) = (\{A_{3}^{-}\}_{\text{iso}} \cup \{A_{3}^{+}\}_{\text{iso}})_{\text{clc}}$.

Proof. The sets $\triangle_{\pm}(\alpha, \beta)$, $\triangle_{-}(\alpha, \beta)$ as well as $\triangle_{0}(\beta)$ can be written as the intersection of $L_{a}^{0}$, $L_{\alpha}^{0}$ and $L_{\beta}^{0}$ from Lemma 4.1, which implies the first part. The second part exploits that $\{A_{1}^{\pm}\}_{\text{iso}}$ and $\{A_{2}^{\pm}\}_{\text{iso}}$, $\{A_{2}^{\pm}\}_{\text{iso}}$ and $\{A_{3}^{\pm}\}_{\text{iso}}$ as well as $\{A_{3}^{-}\}_{\text{iso}}$ and $\{A_{3}^{+}\}_{\text{iso}}$ are compatible, see Lemma 3.2.

Let $Z \subseteq \mathbb{M}^{2\times2}$ be a given compact and isotropic set. Using the pair $\sigma(Z) = (\sigma_{1}(Z), \sigma_{2}(Z))$ given by $\sigma_{1}(Z) = \min\{|\lambda_{1}(A)| \mid A \in Z\}$ and $\sigma_{2}(Z) = \max\{|\lambda_{2}(A)| \mid A \in Z\}$, we define the set $Z^{\triangle} \subseteq \mathbb{M}^{2\times2}$ (see Figure 1(b)) via

$$Z^{\triangle} = \begin{cases} \triangle_{\pm}(\sigma(Z)) & \text{if } \forall A \in Z \pm \lambda_{1}(A) > 0 \\ \triangle_{0}(\sigma(Z)) & \text{otherwise.} \end{cases} \quad (3)$$

In view of Lemma 4.2, we obtain $Z \subseteq Z^{\text{clc}} \subseteq Z^{\triangle}$.
5 A refinement for the connected case

Conti et al. [CDLMR03] show that polyconvexity and lamination convexity are the same for isotropic compact subsets of $\mathbb{M}^{2\times 2}$ that are connected. Their idea can be used to prove a bit more. In order to see that, we will sketch their proof and give the details where minor changes are necessary.

**Theorem 5.1.** Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given isotropic and compact set and $Z \in cc(K^{lc,1})$ a connected component. Then $Z^{lc,1}$ is polyconvex.

**Proof.** Let $Z \in cc(K^{lc,1})$ be an arbitrary but fixed connected component. Then we have

$$Z^{lc,1} \supseteq \{ B \in \mathbb{M}^{2\times 2} \mid \exists C \in Z^{lc,1} \ det(B) = det(C) \land |\lambda_1(B)| = \lambda_2(B) \}. \tag{4}$$

In fact, set $d_1 = \min \{ \det(B) \mid B \in Z \}$ and $d_2 = \max \{ \det(B) \mid B \in Z \}$. By definition, the set $\{ B \in \mathbb{M}^{2\times 2} \mid d_1 \leq \det(B) \leq d_2 \}$ is polyconvex and, hence, $Z^{lc,1}$ is a subset of it. The connectedness of $Z$ together with Remark 3.4 implies that every matrix $B \in \mathbb{M}^{2\times 2}$ with $d_1 \leq \det(B) \leq d_2$ and $|\lambda_1(B)| = \lambda_2(B)$ lies in $Z^{lc,1}$.

We show that for every matrix $A \in \mathbb{M}^{2\times 2} \setminus Z^{lc,1}$ there is a polyconvex function $\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R}$ that separates $A$ from $Z^{lc,1}$, meaning $\varphi(A) > \max \{ \varphi(B) \mid B \in Z^{lc,1} \}$. In order to do that, we follow Conti et al. [CDLMR03]. They show that it is sufficient to check every $A \in \mathbb{M}^{2\times 2} \setminus Z^{lc,1}$ such that $A = \mathrm{diag}(\sigma_1, \sigma_2)$ holds for some real numbers $0 \leq \sigma_1 \leq \sigma_2$. Fix such a matrix $A$. If $\sigma_1 = \sigma_2$ holds, the set $\{ B \in \mathbb{M}^{2\times 2} \mid \det(B) = \det(A) \}$ does not intersect $Z^{lc,1}$. Otherwise (4) yields that $A$ must lie in $Z^{lc,1}$, a contradiction. Thus, the connectedness of $Z^{lc,1}$ implies that we can either put $\varphi = \det$ or $\varphi = -\det$ and are done.

Assume that $\sigma_2 > \sigma_1$. Given a real number $c \in [\sigma_2, \sigma_2]$, they consider the level set

$$L_c = \begin{cases} \{ B \in \mathbb{M}^{2\times 2} \mid \varphi_c^-(B) = \varphi_c^-(A) \} & \text{for } c \in [\sigma_2, \sigma_1] \\ \{ B \in \mathbb{M}^{2\times 2} \mid \varphi_c^+(B) = \varphi_c^+(A) \} & \text{for } c \in [\sigma_1, \sigma_2], \end{cases} \tag{5}$$

see Lemma 3.3 for the definition of $\varphi_c^\pm$. They show that there exists a polyconvex $\varphi$ that separates $A$ from $Z^{lc,1}$ whenever at least one of the $L_c$ does not intersect $Z^{lc,1}$. In fact, by a nice
argument, they can reduce this further. Let \( \tilde{Z} \subseteq \mathbb{M}^{2 \times 2} \) be any compact, connected and isotropic set. They prove that there exists one \( L_c \) that does not intersect \( \tilde{Z} \) if for every \( c \in [-\sigma_2, \sigma_2] \) at least one of the sets \( L_c^* \cap \tilde{Z} \) and \( L_c^* \cap \tilde{Z} \) is empty, where \( L_c^* \) and \( L_c^* \) are the connected components of \( L_c \backslash \{A\} \). This can be used for \( \tilde{Z} = Z^{lc,1} \). Fix \( c \in [-\sigma_2, \sigma_2] \) and suppose that both sets \( L_c^* \cap Z^{lc,1} \) and \( L_c^* \cap Z^{lc,1} \) are non-empty. Then there show that \( A \) must lie in \( \{B, C\}^{lc,1} \) for some rank-one connected matrices \( B \in L_c^* \cap Z^{lc,1} \) and \( C \in L_c^* \cap Z^{lc,1} \). This forms a contradiction as long as \( Z^{lc,1} \) is lamination convex and, hence, completes their proof.

In our case, we use the following argument. We still have \( A \in \{B, C\}^{lc,1} \subseteq Z^{lc,2} \) and conclude that \( d_1 \leq \det(A) \leq d_2 \). Connectedness of \( Z \) implies that there exists a matrix \( A' \in Z \) with \( \det(A') = \det(A) \). We know that \( A \notin Z^{lc,1} \) holds and, hence, we conclude that \( \lambda_2(A') < \lambda_2(A) \) by Remark 3.4. A simple computation shows that

\[
\forall c \in [-\sigma_2, \sigma_2] \quad \varphi^\pm(A) \geq \varphi^\pm(A').
\]  

If for every \( c \in [-\sigma_2, \sigma_2] \) at least one of the sets \( L_c^* \cap Z \) and \( L_c^* \cap Z \) is empty, then we use the above argument for \( \tilde{Z} = Z \). Hence, we can fix a real number \( c \in [-\sigma_2, \sigma_2] \) such that \( L_c \) does not intersect \( \tilde{Z} \). Let \( \varphi \in \{\varphi^+, \varphi^-\} \) be the function that defines \( L_c \) in (5). Then connectedness of \( Z \) implies that either \( \varphi(A) < \min \{\varphi(B) \mid B \in Z\} \) or \( \varphi(A) > \max \{\varphi(B) \mid B \in Z\} \). In view of (6), the second alternative must hold, meaning \( \varphi \) separates \( A \) from \( Z \). Polyconvexity of \( \varphi \) implies that \( \varphi \) also separates \( A \) from \( Z^{lc,1} \) and we are done. Now if there exists a real number \( c \in [-\sigma_2, \sigma_2] \) such that both sets \( L_c^* \cap Z \) and \( L_c^* \cap Z \) are non-empty, then, as before, there exist \( B \in L_c^* \cap Z \) and \( C \in L_c^* \cap Z \) such that \( A \) lies in \( \{B, C\}^{lc,1} \). But \( \{B, C\}^{lc,1} \) is contained in \( Z^{lc,1} \) and, hence, we must have \( A \in Z^{lc,1} \), a contradiction. \( \square \)

6 Closed lamination convex hull

We are going to characterize the closed lamination convex hull of an isotropic and compact set of \( 2 \times 2 \) matrices. The key ingredients are the following two lemmas. The first shows that the laminates of order one fully describe the topology of the closed lamination convex hull.

**Lemma 6.1.** Let \( K \subseteq \mathbb{M}^{2 \times 2} \) be compact and isotropic. Let \( Z_1, Z_2 \in cc(K^{lc,1}) \) be arbitrary but fixed connected components with \( Z_1 \neq Z_2 \). Then \( (Z_1)^{\Delta} \) and \( (Z_2)^{\Delta} \) are incompatible and so are \( (Z_1)^{lc} \) and \( (Z_2)^{lc} \) as well as \( Z_1 \) and \( Z_2 \).

**Proof.** Since \( K \) is compact, so are the sets \( K^{lc,1} \), \( Z_1 \) and \( Z_2 \). The compact and isotropic sets given via \( K_i = Z_i \cap K \) fulfill \( (K_i)^{lc,1} = (Z_i)^{lc,1} \) for \( i = 1, 2 \). In addition, the sets \( K_1 \) and \( K_2 \) are incompatible. Otherwise there exist rank-one connected matrices \( B_1 \in K_1 \) and \( B_2 \in K_2 \) such that \( \{B_1, B_2\}^{lc} \) connects \( Z_1 \) and \( Z_2 \), which forms a contradiction.

We know that \( Z_i \subseteq (Z_i)^{lc} \subseteq (Z_i)^{\Delta} \) as well as \( (K_i)^{lc} \subseteq (K_i)^{\Delta} \) and, hence, \( (K_i)^{\Delta} = (Z_i)^{\Delta} \) holds for \( i = 1, 2 \). It suffices to prove that \( (K_1)^{\Delta} \) and \( (K_2)^{\Delta} \) are incompatible. Without loss of generality, we set \( \sigma_2(K_1) \leq \sigma_2(K_2) \). We distinguish two cases. First, suppose that \( \sigma_1(K_2) > \sigma_2(K_1) \). Then, by Lemma 3.2, the sets \( (K_1)^{\Delta} \) and \( (K_2)^{\Delta} \) are incompatible. Second, suppose that \( \sigma_1(K_2) \leq \sigma_2(K_1) \). Fix matrices \( B_1 \in K_1 \) and \( B_2, B_2' \in K_2 \) such that \( \lambda_2(B_1) = \)}
\[ \sigma_2(K_1), |\lambda_1(B_2)| = \sigma_1(K_2) \text{ and } \lambda_2(B_2') = \sigma_2(K_2). \] Since \( K_1 \) and \( K_2 \) are incompatible, so are \( \{B_1\} \) and \( \{B_2\} \) as well as \( \{B_1\} \) and \( \{B_2'\} \). We conclude that
\[ |\lambda_1(B_2)| \leq \lambda_2(B_2) < |\lambda_1(B_1)| \leq \lambda_2(B_1) < |\lambda_1(B_2')| \leq \lambda_2(B_2'). \]

But then the set \( K_2 \) decomposes into at least two incompatible subsets and, hence, \( Z_2 \) is not connected. This is a contradiction. \( \square \)

The next lemma gives a candidate for the closed lamination convex hull.

**Lemma 6.2.** Let \( K \subseteq \mathbb{M}^{2 \times 2} \) be a given compact and isotropic set. Then the set \( T = \bigcup \{ Z_{\text{clc}} \mid Z \in \text{cc}(K_{\text{lc}}) \} \) is compact, lamination convex and contains \( K \).

**Proof.** By definition, \( T \) contains \( K \). We show that \( T \) is compact. Let \( A_1, A_2, \ldots \) be a given sequence in \( T \). Since \( T \) is a bounded set, we can and will assume that \( A_k \to A \in \mathbb{M}^{2 \times 2} \) holds for some matrix \( A \in \mathbb{M}^{2 \times 2} \). If necessary, we replace \( A_1, A_2, \ldots \) by a subsequence. Let \( Z_1, Z_2, \cdots \in \text{cc}(K_{\text{lc}}) \) be the sequence of connected components such that \( A_k \in (Z_k)_{\text{clc}} \subseteq (Z_k)_{\text{lc}} \) holds for every \( k = 1, 2, \ldots \). First, suppose that there exists a real number \( \epsilon > 0 \) such that for every \( k = 1, 2, \ldots \) we have \( |(Z_k)_{\text{lc}}| \geq \epsilon \) where \(|.|\) denotes the Lebesgue measure of a set. Boundedness of \( T \) implies that there exists a connected component \( Z_0 \in \text{cc}(K_{\text{lc}}) \) and a subsequence (not relabeled) such that \( Z_k = Z_0 \) for every \( k \). Since the set \( (Z_0)_{\text{clc}} \) is compact, \( A \) lies in \( (Z_0)_{\text{clc}} \subseteq T \). Second, suppose that there is no such \( \epsilon > 0 \) as before. Then there exists a subsequence (not relabeled) such that \( |(Z_k)_{\text{lc}}| \to 0 \) holds. In view of (3), this means that
\[ \sup \{|B_1 - B_2| \mid B_1 \in (Z_k)_{\text{lc}} \land B_2 \in Z_k \} \to 0. \]

We take any sequence \( A_1', A_2', \ldots \) in \( K_{\text{lc}} \) such that \( A_k' \in Z_k \) for every \( k = 1, 2, \ldots \). Then we must have \( A_k' \to A \) and, hence, compactness of \( K_{\text{lc}} \) implies that \( A \in K_{\text{lc}} \subseteq T \).

Now we show that \( T \) is lamination convex. Let \( A_1, A_2 \in T \) be given matrices. First, suppose that \( A_1, A_2 \in Z_{\text{clc}} \) for some \( Z \in \text{cc}(K_{\text{lc}}) \). Then we have \( \{A_1, A_2\}_{\text{clc}} \subseteq Z_{\text{clc}} \subseteq T \). Second, suppose that \( A_i \in Z_{\text{lc}} \) for \( i = 1, 2 \) such that \( Z_1, Z_2 \in \text{cc}(K_{\text{lc}}) \) and \( Z_1 \neq Z_2 \). We know from Lemma 6.1 that \( (Z_1)_{\text{lc}} \) and \( (Z_2)_{\text{lc}} \) are incompatible and so are \( \{A_1\} \) and \( \{A_2\} \). We conclude that \( \{A_1, A_2\}_{\text{clc}} = \{A_1, A_2\} \subseteq T \). \( \square \)

Finally, we are in the position to characterize the closed lamination convex hull.

**Theorem 6.3 (Characterization of \( K_{\text{clc}} \)).** Let \( K \subseteq \mathbb{M}^{2 \times 2} \) be compact and isotropic. Then its closed lamination convex hull is given by \( K_{\text{clc}} = K_{\text{lc}}^{c2} \).

**Proof.** Let \( T \subseteq \mathbb{M}^{2 \times 2} \) be as in Lemma 6.2. On the one hand, we know that \( K_{\text{clc}} = (K_{\text{lc}}^{c1})_{\text{clc}} \subseteq T \). On the other hand, we have shown in Lemma 6.2 that the set \( T \) is lamination convex, compact and contains \( K \). We conclude that \( K_{\text{clc}} = T \).

Let \( Z \in \text{cc}(K_{\text{lc}}) \) be a connected component. Then \( Z_{\text{lc}} \) is polyconvex as an application of Theorem 5.1. In particular, we have \( Z_{\text{lc}} = Z_{\text{lc}}^{c1} \) and, hence, \( K_{\text{clc}} \subseteq K_{\text{lc}}^{c2} \). Since the other inclusion holds by definition, we conclude that \( K_{\text{clc}} = K_{\text{lc}}^{c2} \). \( \square \)
7 Quasiconvex hull

We show the equivalence of quasiconvexity and laminaton convexity for isotropic compact subsets of $\mathbb{M}^{2 \times 2}$. We rely on a result by Faraco and Székelyhidi [FS08].

The next lemma deals with the case of two connected components.

**Lemma 7.1.** Let $S \subseteq \mathbb{M}^{2 \times 2}$ be a given compact set (not necessarily isotropic) and $0 \leq \beta_1 < \alpha_2 \leq \beta_2$ real numbers. If $S \subseteq \triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ holds and $S^{qc}$ is connected, then one of the sets $S \cap \triangle_0(\beta_1)$ and $S \cap \triangle_+(\alpha_2, \beta_2)$ must be empty.

**Proof.** By rescaling the matrix space $\mathbb{M}^{2 \times 2}$, if necessary, we can and we will assume that there exists a positive real number $\epsilon > 0$ such that

$$\beta_1 \leq 1 - \epsilon < 1 + \epsilon \leq \alpha_2 \leq \beta_2. \quad (7)$$

Lemma 3.3 implies that $f = \varphi_1 + 1$, with $f(A) = \lambda_1(A) + \lambda_2(A) - \det(A) - 1$, is a polyconvex function. In particular, the set $P = \{ A \in \mathbb{M}^{2 \times 2} \mid f(A) \leq -\epsilon^2 \}$ is polyconvex by definition. We are going to show that $\triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ is a subset of $P$. Consider the matrices $A_1 = \text{diag}(-\beta_1, \beta_1)$, $A_2 = \text{diag}(\beta_1, \beta_1)$, $A_3 = \text{diag}(\alpha_2, \alpha_2)$, $A_4 = \text{diag}(\alpha_2, \beta_2)$ and $A_5 = \text{diag}(\beta_2, \beta_2)$. Since, in addition, $P$ is isotropic, Lemma 4.2 implies that it is sufficient to show that $A_i \in P$ for $i = 1, \ldots, 5$. However this can be tested easily if we make use of (7) and the fact that for every $A \in \mathbb{M}^{2 \times 2}$ we have $f(A) = (1 - \lambda_1(A))(\lambda_2(A) - 1)$.

We have shown that $S \subseteq \triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ is a subset of $P$. Since the set $P$ is polyconvex, the quasiconvex hull $S^{qc}$ is also contained in $P$. Yet the identity matrix lies not in $P$. As a consequence of Remark 3.4, for every matrix $A \in S^{qc}$ we must have $\det(A) \neq 1$. We know that $S^{qc}$ is connected and, in addition, $\det < 1$ holds in $\triangle_0(\beta_1)$ and $\det > 1$ in $\triangle_+(\alpha_2, \beta_2)$. Hence, one of the sets $S \cap \triangle_0(\beta_1)$ and $S \cap \triangle_+(\alpha_2, \beta_2)$ must be empty.

Now we are going to prove our result about the equivalence of laminaton convexity and quasiconvexity.

**Theorem 7.2** (Equivalence). Let $K \subseteq \mathbb{M}^{2 \times 2}$ be a given compact and isotropic set. Then $K$ is laminaton convex if and only if $K$ is quasiconvex.

**Proof.** We only have to show one implication. Assume that $K$ is laminaton convex and let $\nu \in \mathcal{P}^{qc}$ be a fixed homogenous gradient Young measure with support $S = \text{supp}(\nu) \subseteq K$.

By Remark 2.1, we need to show that $\bar{\nu} \in K$. Let $Z$ be the set of all connected components $Z \in \text{cc}(K)$ such that $Z \cap S$ is non-empty. First, suppose that there exists only one such connected component, meaning $Z = \{Z\}$. Since $Z$ is isotropic, laminaton convex, compact and connected, Theorem 5.1 implies that $Z$ is quasiconvex (even polyconvex). Hence, $\bar{\nu}$ must lie in $Z \subseteq K$.

Second, suppose that $S$ is distributed over more than one connected component. By compactness arguments, we can fix $Z_1, Z_2 \subseteq Z$ that are extremal in the following sense. For every
\( Z \in \mathcal{Z} \) we have \( \sigma_2(Z_1) \leq \sigma_2(Z) \leq \sigma_2(Z_2) \). Up to symmetry, there are only three different cases: \((Z_2)^\Delta = \Delta_+ (\alpha_2, \beta_2)\) and either \((Z_1)\Delta = \Delta_0 (\beta_1)\), \((Z_1)\Delta = \Delta_+ (\alpha_1, \beta_1)\) or \((Z_1)\Delta = \Delta_- (\alpha_1, \beta_1)\) for some reals \(0 \leq \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2\).

We fix real numbers \(\beta, \alpha \in \mathbb{R}\) such that \(\beta_1 < \beta < \alpha < \alpha_2\) holds as well as

\[
S \subseteq \Delta_0 (\beta) \cap \Delta_+ (\alpha, \beta_2).
\] (8)

In order to see that this can be done, let \(\epsilon > 0\) be a given real number. Recall that \(K^{lc,1} = K\) holds and, hence, Lemma 6.1 implies that elements in \(\mathcal{Z}\) are pairwise incompatible. We fix \(\beta(\epsilon), \alpha(\epsilon) \in \mathbb{R}\) such that \(\alpha_2 - \epsilon < \beta(\epsilon) < \alpha(\epsilon) < \alpha_2\) holds and, in addition, for every \(Z \in \mathcal{Z}\) we have either \(\sigma_2(Z) < \beta(\epsilon)\) or \(\sigma_1(Z) > \alpha(\epsilon)\). Suppose that \(\beta(\epsilon)\) and \(\alpha(\epsilon)\) fail to fulfill (8) for every \(\epsilon > 0\). Then there must be a sequence \(A_1, A_2, \ldots\) in \(S\) such that \(\lambda_1(A_k) < 0\) holds for every \(k = 1, 2, \ldots\) and \(\lambda_2(A_k) \to \alpha_2\). By compactness of the set \(S\), we can fix a cluster point \(A_0 \in S\) of this sequence. On the one hand, we know that \(\lambda_1(A_0) \leq 0\) and, hence, \(A_0 \notin (Z_2)^\Delta = \Delta_+ (\alpha_2, \beta_2)\). On the other hand, \(\lambda_2(A_0) = \alpha_2\) implies that \(\{A_0\}\) and \((Z_2)^\Delta\) are compatible. By Lemma 6.1, we must have \((Z_0)^\Delta = (Z_2)^\Delta\) where \(Z_0\) is given by \(A_0 \in Z_0 \in \mathcal{Z}\). This is a contradiction.

A result by Faraco and Székelyhidi [FS08, Cor. 3] implies that \(S^{qc}\) is connected. As a consequence of Lemma 7.1, one of the sets \(S \cap \Delta_0 (\beta)\) and \(S \cap \Delta_+ (\alpha, \beta_2)\) must be empty. This forms a contradiction. Hence, it is impossible that \(S\) is distributed over more than one connected component.

Our next result can be used to compute the quasiconvex hull.

**Theorem 7.3** (Characterization of \(K^{qc}\)). Let \(K \subseteq M^{2 \times 2}\) be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2.

**Proof.** Clearly, the set \(K^{lc}\) is compact, isotropic and lamination convex. Theorem 6.3 and Theorem 7.2 imply that \(K^{lc,2} = K^{lc} = K^{qc}\) holds. \(\square\)

**References**


