Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Andronov–Hopf bifurcation of higher codimensions in a Liénard system

Klaus R. Schneider,[1] Alexander Grin[2]

submitted: August 5, 2011

[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin, Germany
E-Mail: Klaus.Schneider@wias-berlin.de

[2] Grodno State University
Ozheshko Street 22
230023 Grodno, Belarus
E-Mail: grin@grsu.by

No. 1634
Berlin 2011

2010 *Mathematics Subject Classification.* 34A34,34C05,34C23,37G15.

2008 *Physics and Astronomy Classification Scheme.* 05.45.Xt, 89.75.Kd.

Key words and phrases. degenerate Hopf bifurcation, codimension three, multiple limit cycles, bifurcation diagrams.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Consider a polynomial Liénard system depending on three parameters a, b, c and with the following properties: (i) The origin is the unique equilibrium for all parameters. (ii). If a crosses zero, then the origin changes its stability, and a limit cycle bifurcates from the equilibrium. We investigate analytically this bifurcation in dependence on the parameters b and c and establish the existence of families of limit cycles of multiplicity one, two and three bifurcating from the origin.

1 Introduction

Liènard systems play an important role as mathematical models in applied sciences, especially in biology, mechanics, electronics and chemistry. The existence of limit cycles (isolated periodic solutions) in such systems is a key problem for modeling rhythmic behavior. One approach to establish the existence of limit cycles consists in applying methods from bifurcation theory [1, 3, 5]. Another possibility is to use the Andronov-Hopf function [2].

In the following we consider the polynomial Liènard system

$$\begin{aligned} \frac{dx}{dt} &= ax + y - bx^3 + cx^5 - x^7, \\ \frac{dy}{dt} &= -x, \end{aligned} \tag{1.1}$$

where a, b, c are real parameters. It is easy to see that the origin is the unique equilibrium of (1.1) in the finite part of the phase plane for any parameter tuple (a, b, c). Its stability can be determined in general by the characteristic roots of the linearized system of (1.1) at the origin. In case of system (1.1) they read

$$\lambda_{1,2}(a) = \frac{1}{2}\Big(a \pm \sqrt{a^2 - 4}\Big). \tag{1.2}$$

Thus, the origin is an equilibrium of focus type for $a^2 < 4$, which is exponentially stable (exponentially unstable) for $-2 < a < 0 \quad (0 < a < 2)$ independent of the parameters b and c.
Our goal is to study the bifurcation of limit cycles from the origin with a as bifurcation parameter. The bifurcation of a limit cycle from an equilibrium of focus type in a planar system is called Andronov-Hopf bifurcation. It is well-known that the occurrence of Andronov-Hopf bifurcation is connected with an exchange of stability of the equilibrium. Hence, we can conclude from (1.2)

that the occurrence of Andronov-Hopf bifurcation in (1.1) requires that the parameter a crosses the value zero. We note that the transversality condition

$$\frac{dRe\lambda_{1,2}(a)}{da}_{|a=0} = \frac{1}{2} \neq 0$$

is fulfilled, that is, if a passes zero then a limit cycle of (1.1) bifurcates from the origin. Hence, all point of the parameter plane $a = 0$ are bifurcation points.
In what follows we investigate the influence of the parameters b and c on the Andronov-Hopf bifurcation in system (1.1). Especially, we look for subsets of the plane $a = 0$ corresponding to bifurcation points of higher codimension, that is, which are connected with the bifurcation of multiple limit cycles from the origin.
Our main results concerning the structure of bifurcation points related to Hopf bifurcation can be characterized as follows:

(i). If we assume that the parameters b and c in (1.1) are fixed, $b = b_0, c = c_0$, where we assume $b = b_0 \neq 0$, then we have the case of codimension one Andronov-Hopf bifurcation: When a passes the critical value zero, a family of simple limit cycles $\Gamma(a, b_0, c_0)$ bifurcates from the origin, where the amplitude r of $\Gamma(a, b_0, c_0)$ increases proportional to $\sqrt{|a|}$. The bifurcation direction (and the stability) depends on the sign of b_0 (see Fig.1 and Fig.2). The parameter c does not change qualitatively the bifurcation behavior. Thus, any point $(0, b_0, c_0)$ of the parameter plane $a = 0$ satisfying $b_0 \neq 0$ is an Andronov-Hopf bifurcation point of codimension one.

(ii). If we assume that the parameter c in (1.1) is fixed different from zero, that is $c = c_0 \neq 0$, then we are faced with the situation of codimension two Andronov-Hopf bifurcation of system (1.1): If the point (a, b) crosses the b-axis at a point $(0, b_0)$ with $b_0 \neq 0$, then we have the situation as in item (i), otherwise there exists in case $c_0 > 0$ $(c_0 < 0)$ a curve $\mathcal{M}_2(c_0)$ in the half-plane $a > 0$ (a curve $\mathcal{P}_{2,1}(c_0)$ in the half-plane $a < 0$) emanating from the origin, defined for $|a| + |b|$ sufficiently small, and which is connected with the Andronov-Hopf bifurcation of a limit cycle of multiplicity two (semistable limit cycle) from the origin (see Fig.5 and Fig.6).

(iii). If we do not put any restriction on the parameters a, b, c except to be small, then we have the situation of codimension three Andronov-Hopf bifurcation, that means, if the point (a, b, c) crosses the plane $a = 0$ at a point $(0, b_0, c_0)$ with $c_0 \neq 0$, then we have the situation as described in item (ii), that is, there is a surface $\mathcal{M}_2^+$ in the region $a > 0, b > 0, c > 0$ (a surface $\mathcal{P}_{2,1}^+$ in the region $a < 0, b < 0, c < 0$) containing the c-axis as boundary such that any curve $\mathcal{K}_2$ on these surfaces starting at a point on the c-axis describes a family of limit cycles with multiplicity two of system (1.1) bifurcating from the origin in the phase plane. If we denote the intersection of the surface $\mathcal{M}_2^+$ with the plane $c = c_0 > 0$ (of the surface $\mathcal{P}_{2,1}^+$ with the plane $c = c_0 < 0$) by $\mathcal{M}_2(c_0)$ ($\mathcal{P}_{2,1}(c_0)$), then we recover the corresponding bifurcation curves represented in Fig.5 and Fig.6, respectively.
In the region $c > 0$ there exists another surface $\mathcal{P}_{2,2}^+$ with the following properties:

(a). $\mathcal{P}^+_{2,2}$ and $\mathcal{M}^+_2$ have the common boundary $\mathcal{K}^+_3$ which is formed by a smooth curve emanating from the origin. The closure $\overline{\mathcal{P}^+_{2,2} \cup \mathcal{M}^+_2}$ represents a surface having a singular fold at $\mathcal{K}^+_3$.

(b). The origin is the unique boundary point of the surface $\mathcal{P}^+_{2,2}$ on the c-axis.

(c). There exists a unique curve $\mathcal{K}^+_3$ in the parameter space emanating from the origin and located in the region $c > 0$ such that for each point (a, b, c) on that curve system (1.1) has a limit cycle of multiplicity three whose amplitude tends to zero when the point (a, b, c) tends to zero.

(d). Any curve located on the surface $\mathcal{P}^+_{2,2}$ and starting at a point on the curve $\mathcal{K}^+_3$ is connected with the bifurcation a limit cycle of multiplicity two from a limit cycle of multiplicity three.

There are regions in the parameter space $c > 0$ having points on the c-axis as boundary points and to which there belong systems (1.1) with one, two or three simple limit cycles. Fig. 7 represents the local bifurcation diagram of system (1.1) generated by the intersection of the $c = c = c_0 > 0$ with the bifurcation surfaces $\mathcal{M}^+_2$ and $\mathcal{P}^+_{2,2}$. The corresponding bifurcation curves are denoted by $\mathcal{M}_2(c_0)$ and $\mathcal{P}_{2,2}(c_0)$, respectively. The point $K_3(c_0)$ represents the common boundary point of the curves $\mathcal{P}_{2,2}(c_0)$ and $\mathcal{M}_2(c_0)$; to $K_3(c_0)$ there corresponds a system (1.1) with a stable limit cycle of multiplicity three.

Although the obtained bifurcation results are of local nature they can be used as initial guess for numerical continuation methods.

The paper is organized as follows: In section 2 we reformulate system (1.1) by means of polar coordinates, derive results about the stability of the unique equilibrium and on the nonexistence of limit cycles of (1.1), and introduce the basic notation and tools (displacement function, Lyapunov numbers). Section 3 treats Andronov-Hopf bifurcation of codimensions one, two and three by means of the displacement function.

2 Preliminaries

We assume that the parameters a, b, c in system (1.1) belong to the ball

$$\mathcal{B}_1 := \{(a, b, c) \in R^3 : |a|^2 + |b|^2 + |c|^2 \leq 1\}$$

. Introducing polar coordinates $x = r\cos\varphi, y = r\sin\varphi$ we get from (1.1) the system

$$\begin{aligned}\frac{dr}{dt} &= ar\cos^2\varphi - br^3\cos^4\varphi + cr^5\cos^6\varphi - r^7\cos^8\varphi,\\ \frac{d\varphi}{dt} &= -\Big(1 + \frac{a}{2}\sin 2\varphi\Big) + r^2\sin\varphi\cos^3\varphi\,\Big(b - cr^2\cos^2\varphi + r^4\cos^4\varphi\Big).\end{aligned} \tag{2.1}$$

We note that $d\varphi/dt < 0$ for $(a, b, c) \in \mathcal{B}_1$ and for sufficiently small r.

Concerning the stability of the origin in the critical case $a = 0$ we get from (2.1):

Theorem 2.1 *In the critical case $a = 0$, under the conditions*

$$b \geq 0,\ c \leq 0 \tag{2.2}$$

the origin is a globally asymptotically stable equilibrium of system (1.1),
the origin is unstable if one of the conditions

$$\begin{aligned} &b < 0, \\ &b = 0,\ c > 0 \end{aligned} \tag{2.3}$$

is satisfied.

This theorem implies

Corollary 2.1 *Under the condition*

$$a \leq 0,\ b \geq 0,\ c \leq 0 \tag{2.4}$$

system (1.1) has no limit cycle.

We note that Corollary 2.1 can also be proved by applying Bendixson's criterion to system (1.1).

From Corollary 2.1 we get

Lemma 2.1 *To given $c \leq 0$ there is no bifurcation curve of system (1.1) located in the quadrant $a \leq 0, b \geq 0$ of the (a, b)-parameter plane related to the generation of a limit cycle.*

From (2.1) we can derive a similar result.

Lemma 2.2 *To given $c > 0$ there is no bifurcation curve of system (1.1) located in the region $a > 0, b \leq 0$ of the (a, b)-parameter plane related to the Andronov-Hopf bifurcation of a limit cycle.*

In what follows we introduce the so-called displacement function ψ which can be used to determine the number of limit cycles of system (1.1) bifurcating from the origin, their stability and multiplicity.

For this purpose we introduce the notation

$$k(\varphi, a) := -\Big(1 + \frac{a}{2} \sin 2\varphi\Big), \quad v(\varphi, a) := \frac{\sin\varphi \cos^3\varphi}{k(\varphi, a)}. \tag{2.5}$$

It holds

$$k(\varphi,0)\equiv -1,\ k(\varphi,a)<0 \text{ for } |a|<2,\ v(\varphi,0)\equiv -\sin\varphi\cos^3\varphi. \tag{2.6}$$

From (2.1) we obtain for $(a,b,c)\in\mathcal{B}_1$ and sufficiently small r

$$\begin{aligned}\frac{dr}{d\varphi} &= \frac{a\cos^2\varphi\, r - b\cos^4\varphi\, r^3 + c\cos^6\varphi\, r^5 - \cos^8\varphi\, r^7}{k(\varphi,a)\Big(1+v(\varphi,a)(b\,r^2 - c\cos^2\varphi\, r^4+\cos^4\varphi\, r^6)\Big)}\\ &= \Big[\frac{a\cos^2\varphi}{k(\varphi,a)}\,r - \frac{b\cos^4\varphi}{k(\varphi,a)}\,r^3 + \frac{c\cos^6\varphi}{k(\varphi,a)}\,r^5 - \frac{\cos^8\varphi}{k(\varphi,a)}\,r^7\Big]\\ &\times\Big[1-\Big(b\,v(\varphi,a)\,r^2 - cv(\varphi,a)\cos^2\varphi\, r^4 + v(\varphi,a)\cos^4\varphi\, r^6\Big)\\ &+\Big(b\,v(\varphi,a)\,r^2 - c\,v(\varphi,a)\cos^2\varphi\, r^4 + v(\varphi,a)\cos^4\varphi\, r^6\Big)^2\\ &-\Big(b\,v(\varphi,a)\,r^2 - c\,v(\varphi,a)\cos^2\varphi\, r^4 + v(\varphi,a)\cos^4\varphi\, r^6\Big)^3+\cdots-\cdots\Big].\end{aligned}\tag{2.7}$$

Hence, we can represent (2.7) in the form

$$\begin{aligned}\frac{dr}{d\varphi} =&k_1(\varphi,a,b,c)r + k_3(\varphi,a,b,c)r^3+\\ &+k_5(\varphi,a,b,c)r^5 + k_7(\varphi,a,b,c)r^7+\cdots,\end{aligned}\tag{2.8}$$

where the functions $k_i, i=1,3...$, are smooth (having derivatives of any order) in their arguments. The expressions for k_1, ..., k_7 can be found in Appendix A.
To the differential equation (2.8) we consider the initial value problem $r(0)=r_0$ and denote its solution by $r=g(\varphi,r_0,a,b,c)$. Under our assumptions we can represent it for sufficiently small r_0 in the form

$$\begin{aligned}r = g(\varphi,r_0,a,b,c) = l_1(\varphi,a,b,c)r_0 + l_3(\varphi,a,b,c)r_0^3+\\ +l_5(\varphi,a,b,c)r_0^5+l_7(\varphi,a,b,c)r_0^7+\cdots.\end{aligned}\tag{2.9}$$

Substituting (2.9) into (2.8) and taking into account the initial conditions

$$l_1(0,a,b,c)=1,\ l_j(0,a,b,c)=0 \quad\text{for}\quad j=3,5,7,\cdots \tag{2.10}$$

we get the initial value problems

$$\begin{aligned}\frac{dl_1}{d\varphi} =&k_1(\varphi,a)\,l_1, \quad l_1(0,a,b,c)=1,\\ \frac{dl_3}{d\varphi} =&k_1(\varphi,a)\,l_3 + k_3(\varphi,a,b)\,l_1^3, \quad l_3(0,a,b,c)=0,\\ \frac{dl_5}{d\varphi} =&k_1(\varphi,a)\,l_5 + 3k_3(\varphi,a,b)\,l_1^2 l_3 + k_5(\varphi,a,b,c)\,l_1^5,\\ &l_5(0,a,b,c)=0,\\ \frac{dl_7}{d\varphi} =&k_1(\varphi,a)\,l_7 + 3k_3(\varphi,a,b)(l_1l_3^2+l_1^2l_5)+\\ &+5k_5(\varphi,a,b,c)\,l_1^4l_3 + k_7(\varphi,a,b,c)\,l_1^7, \quad l_7(0,a,b,c)=0,\end{aligned}\tag{2.11}$$

where the functions $l_1, l_3, ...$ are smooth in their arguments. The expressions for $l_1, ..., l_7$ can be found in Appendix B.

Now we introduce the displacement function ψ by

$$\psi(r_0, a, b, c) := g(2\pi, r_0, a, b, c) - r_0. \tag{2.12}$$

We note that ψ is defined for $(a, b, c) \in \mathcal{B}_1$ and for sufficiently small $r_0 \geq 0$ and is smooth in all arguments.
It is obvious that a positive isolated root r_0^* of the displacement function ψ corresponds to a limit cycle Γ of system (1.1), where the multiplicity of Γ is defined by the multiplicity of r_0^*.
From (2.12) and (2.9) we get

$$\begin{aligned}\psi(r_0, a, b, c) =& \alpha_1(a, b, c)r_0 + \alpha_3(a, b, c)r_0^3 + \alpha_5(a, b, c)r_0^5 \\ &+ \alpha_7(a, b, c)r_0^7 + \cdots ,\end{aligned} \tag{2.13}$$

where the coefficients $\alpha_i(a, b, c)$ are defined by

$$\begin{aligned}&\alpha_1(a, b, c) \equiv \alpha_1(a) \equiv l_1(2\pi, a) - 1, \\ &\alpha_3(a, b, c) \equiv \alpha_3(a, b) \equiv l_3(2\pi, a, b), \\ &\alpha_i(a, b, c) \equiv l_i(2\pi, a, b, c) \quad \text{for} \quad i = 5, 7, \cdots ,\end{aligned} \tag{2.14}$$

that is, they are smooth in their arguments. The expressions for $\alpha_1, ..., \alpha_7$ can be found in Appendix C.

Definition 2.1 *The numbers $\alpha_{2k+1}(0, b, c), k = 0, 1, ...,$ are called the Lyapunov values of the equilibrium point $(x = 0, y = 0)$ of focus type of system (1.1) for $a = 0$.*

Now we return to the displacement function ψ. From (2.13) we get the following properties:

(i). To any given $(a, b, c) \in \mathcal{B}_1$, the equation

$$\psi(r_0, a, b, c) = 0 \tag{2.15}$$

has always the root $r_0 = 0$, which corresponds to the equilibrium point $(x = 0, y = 0)$ of system (1.1). A positive root r_0^* of finite multiplicity k corresponds to a limit cycle Γ of multiplicity k of system (1.1).

(ii). If for increasing r_0, ψ changes its sign at a positive root r_0^* from negative to positive (from positive to negative), then the corresponding limit cycle is unstable (asymptotically stable). If ψ does not change its sign, then the corresponding limit cycle is semistable.

From these properties we can conclude that the Andronov-Hopf bifurcation in system (1.1) is equivalent to the bifurcation of a positive root of equation (2.15) from the root $r_0 = 0$. Taking into account that in case of system (1.1) the displacement function ψ is according to (2.13) an odd function, then the bifurcation of a positive root of (2.15) from $r_0 = 0$ implies that at the same time a negative root bifurcates from $r_0 = 0$. Thus, the bifurcation of a positive root requires that $r_0 = 0$ is a root of at least multiplicity three. From (6.5) and (6.6) we obtain that $r_0 = 0$ is a root of multiplicity three for $a = 0$ and $b \neq 0$, from (6.5),(6.9) and (6.10) it follows that $r_0 = 0$ is a root of multiplicity five for $a = 0$, $b = 0$ and $c \neq 0$. Finally, we get from (6.11)

$$\alpha_7(0,0,0) = \int_0^{2\pi} \cos^8 \varphi d\varphi = \frac{35}{64}\pi \neq 0, \tag{2.16}$$

that is $r_0 = 0$ is a root of multiplicity seven for $a = b = c = 0$.

Definition 2.2 *The equilibrium point $(x = 0, y = 0)$ of system (1.1) with $a = 0$ is called to have cyclicity l $(l \geq 1)$ if it holds*

$$\alpha_1(0) = 0, ..., \alpha_{2l-1}(0,b,c) = 0, \alpha_{2l+1}(0,b,c) \neq 0.$$

From our calculations above we get the following result

Theorem 2.2 *The equilibrium point $(x = 0, y = 0)$ of system (1.1) has cyclicity 1 in case $a = 0, b \neq 0$, it has cyclicity 2 in case $a = b = 0, c \neq 0$, and it has cyclicity 3 in case $a = b = c = 0$. Thus, system (1.1) has not more than three small amplitude limit cycles for sufficiently small $|a| + |b| + |c|$.*

In the following section we study Andronov-Hopf bifurcation for system (1.1) by using the displacement function ψ.

3 Andronov-Hopf bifurcation scenarios of the Liénard system (1.1)

As we mentioned in the section before, Andronov-Hopf bifurcation of system (1.1) is equivalent to the bifurcation of positive roots of equation (2.15) from the root $r_0 = 0$. For the sequel we write the displacement function ψ in the form

$$\psi(r_0, a, b, c) \equiv r_0\tilde{\psi}(r_0, a, b, c).$$

Setting $z = r_0^2$ the function $\tilde{\psi}$ reads

$$\tilde{\psi}(z, a, b, c) = \alpha_1(a) + \alpha_3(a, b)z + \alpha_5(a, b, c)z^2 + \alpha_7(a, b, c)z^3 + \cdots . \tag{3.1}$$

In what follows we look for small positive roots $z(a, b, c)$ of the equation

$$\alpha_1(a) + \alpha_3(a, b)z + \alpha_5(a, b, c)z^2 + \alpha_7(a, b, c)z^3 + \cdots = 0 \tag{3.2}$$

satisfying

$$z(a, b, c) \to 0 \quad \text{as } a \text{ tends to 0.}$$

We recall that any small positive root $z(a, b, c)$ of multiplicity k of equation (3.2) corresponds to a small amplitude limit cycle of multiplicity k of system (1.1).
The follwing properties of the functions α_1, α_3 and α_5 can be proved by using the appendices A-C.

Lemma 3.1 *There is a small positive number ν_0 such that*

$$\alpha_1(a) \neq 0, \quad sign\,\alpha_1(a) = -sign\,a \quad \textit{for} \quad 0 < |a| \leq \nu_0.$$

Lemma 3.2 *Assume $b = b_0 \neq 0$ and $|a| \leq \nu_0$. Then it holds*

$$\alpha_3(a, b_0) \neq 0, \quad sign\,\alpha_3(a, b_0) = sign\,b_0.$$

Lemma 3.3 *Assume $c = c_0 \neq 0$ and $|a| \leq \nu_0$. Then there is a monotone increasing function $\varrho : R^+ \to R^+$ with $\varrho(0) = 0$ such that for $|b| \leq \varrho(c_0)$*

$$\alpha_5(a, b, c_0) \neq 0, \quad sign\,\alpha_3(a, b, c_0) = -sign\,c_0.$$

3.1 Codimension 1 bifurcation

In this subsection we consider system (1.1) for $(a, b, c) \in \mathcal{B}_1$ under the assumptions that the bifurcation parameter a has a small modulus and that the parameters b and c are fixed, that is, we suppose

$$|a| \leq \nu_0, \quad b = b_0 \neq 0, \quad c = c_0. \tag{3.3}$$

Since we are interested in *small* positive roots of equation (3.2), we replace this equation under the assumption (3.3) by the truncated equation

$$\alpha_1(a) + \alpha_3(a, b_0)z \equiv \alpha_1(a) + b_0 F_1(a) z = 0. \tag{3.4}$$

If we suppose

$$-\alpha_1(a) b_0 > 0,$$

which is by Lemma 3.1 equivalent to

$$ab_0 > 0, \tag{3.5}$$

then equation (3.4) has for sufficiently small a a unique small positive simple root $\tilde{z}_1^+(a, b_0)$ defined by

$$\tilde{z}_1^+(a, b_0) = -\frac{\alpha_1(a)}{b_0 F_1(a)}.$$

Taking into account (6.10) we get the asymptotic relation

$$\tilde{z}_1^+(a, b_0) = \frac{4a}{3b_0} + O(a^2).$$

By means of the implicit function theorem we may confirm that also the equation (3.2) with $b = b_0 \neq 0, c = c_0$ has a simple positive root $z_1^+(a, b_0)$ with the same asymptotic representation

$$z_1^+(a, b_0, c_0) = \frac{4a}{3b_0} + O(a^2).$$

Hence, under the assumption that a has a sufficiently small modulus and satisfies the relation (3.5), the displacement function ψ has a simple positive root $r_1^+(a, b_0, c_0)$ with the asymptotic representation

$$r_1^+(a, b_0, c_0) = +\sqrt{\frac{4a}{3b_0}} + O(a) \tag{3.6}$$

and we have the result:

Theorem 3.1 *Consider system (1.1) in case $b = b_0 \neq 0, c = c_0$. Then, a simple limit cycle $\Gamma_1(a, b_0, c_0)$ of system (1.1) bifurcates from the origin when the parameter a crosses zero. In case $b_0 < 0\,(b_0 > 0)$, $\Gamma_1(a, b_0, c_0)$ exists for sufficiently small negative (positive) a and is unstable (asymptotically stable). Its amplitude satisfies (3.6) as a tends to zero.*

The corresponding bifurcation diagrams are represented in Fig.1 and Fig.2.

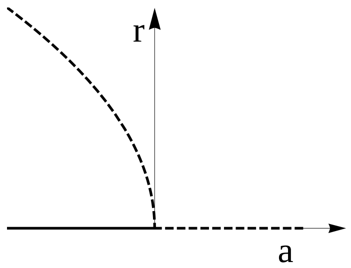

Fig.1. Subcritical Hopf bifurcation

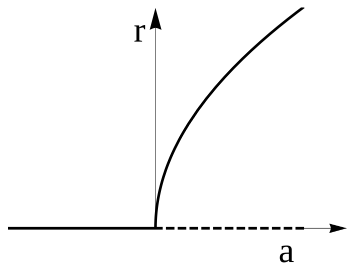

Fig.2.Supercritical Hopf bifurcation

3.2 Codimension 2 bifurcation

In this subsection we consider system (1.1) under the assumptions that a and b are parameters with small modulus and that the parameter c is fixed and different from 0, that is we assume

$$|a| \leq \nu_0, \quad |b| \ll 1, \quad c = c_0 \neq 0. \tag{3.7}$$

Since we look for small roots of equation (3.2), we replace this equation under the assumption (3.7) by the truncated equation

$$\psi_5(z, a, b, c) := \alpha_1(a) + \alpha_3(a, b)z + \alpha_5(a, b, c_0)z^2 = 0. \tag{3.8}$$

For $|b| \leq \varrho(c_0)$ (see Lemma 3.3), the solutions $\tilde{z}_{+,-}(a, b, c_0)$ of this equation read

$$\tilde{z}_{+,-}(a, b, c_0) = \frac{1}{2\alpha_5(a, b, c_0)}\Big(- \alpha_3(a, b) \pm \sqrt{\alpha_3^2(a, b) - 4\alpha_1(a)\alpha_5(a, b, c_0)}\Big). \tag{3.9}$$

In case

$$a = 0,\ b = b_0 \neq 0,\ |b_0| \leq \varrho(c_0)$$

formula (3.9) provides by (6.5), (6.9) and (6.10) the existence of two simple roots,

$$\tilde{z}_+(0, b_0, c_0) = 0, \quad \tilde{z}_-(0, b_0, c_0) = -\frac{\alpha_3(0, b_0)}{\alpha_5(0, b_0, c_0)} = -\frac{24b_0}{27\pi b_0^2 - 20c_0} \neq 0.$$

Therefore, in that case, formula (3.9) can be used to study the generation of a simple positive root $\tilde{z}_1^+(a, b_0, c_0)$ of equation (3.8) from the zero root $\tilde{z}_+(0, b_0, c_0)$, that is, to verify the Andronov-Hopf bifurcation scenario described in the subsection before.

In case $a = b = 0, c = c_0 \neq 0$, (3.9) yields the double root $\tilde{z}_2(0, 0, c_0) = 0$.

We are interested in deriving conditions such that equation (3.8) has a small positive double root $\tilde{z}_2^+(a, b, c_0)$ which tends to zero as the parameter a tends to zero. We recall that by Lemma 3.3 to given $c_0 \neq 0$ the relation $\alpha_5(a, b, c_0) \neq 0$ holds for $|a| \leq \nu_0, |b| \leq \varrho(c_0)$. Under the assumption

$$\alpha_3^2(a, b) = 4\alpha_1(a)\alpha_5(a, b, c_0) \tag{3.10}$$

equation (3.8) has the unique double root

$$\tilde{z}_2(a, b, c_0) = -\frac{\alpha_3(a, b)}{2\alpha_5(a, b, c_0)}. \tag{3.11}$$

This double root is positive only under the additional condition

$$\alpha_3(a, b)\alpha_5(a, b, c_0) < 0. \tag{3.12}$$

By (6.5), (6.6), Lemma 3.2 and Lemma 3.3 this inequality is equivalent to the condition

$$bc_0 > 0. \tag{3.13}$$

From (3.10) and (6.3) we get the relation

$$ac_0 > 0 \tag{3.14}$$

which is necessary for the existence of a double root of equation (3.8).
Using the relation (3.10) we may represent (3.11) in the form

$$\tilde{z}_2^2(a, b, c_0) = \frac{\alpha_1(a)}{\alpha_5(a, b, c_0)} \tag{3.15}$$

from which we obtain that the root $\tilde{z}_2(a, b, c_0)$, if it exists, tends to zero as a tends to zero.

Taking into account (6.5) and (6.9) we can rewrite equation (3.10) as

$$b^2\Big(F_1^2(a) - 4\alpha_1(a)\big(F_2(a) + F_3(a)\big)\Big) = 4c_0\alpha_1(a)F_4(a). \tag{3.16}$$

The solutions $\tilde{b}^\pm(a, c_0)$ of the quadratic equation (3.16) can be represented in the form

$$\tilde{b}^\pm(a, c_0) = \pm\sqrt{\frac{40c_0}{9}a} + O(a). \tag{3.17}$$

Therefore, we have the result

Lemma 3.4 *Let c_0 be any number from the interval $(0,1)$. Then there exists a unique curve $\tilde{\mathcal{M}}_2(c_0)$ in a neighborhood of the origin in the (a,b)-parameter plane emanating from the origin, located in the half plane $a \geq 0$ and with the asymptotic representation*

$$b = \tilde{b}^+(a, c_0) = +\sqrt{\frac{40c_0}{9}a} + O(a) \quad \textit{as} \quad a \to +0$$

such that for any point $(a,b) \in \tilde{\mathcal{M}}_2(c_0)$ equation (3.8) has a positive double root $\tilde{z}_2^+(a, c_0) := \tilde{z}_2^+(a, \tilde{b}^+(a, c_0), c_0)$ with the asymptotic representation

$$\tilde{z}_2^+(a, c_0) = \sqrt{\frac{8a}{5c_0}} + O(a) \quad \textit{as} \quad a \to +0. \tag{3.18}$$

For $c = c_0 > 0$, the bifurcation curves related to the positive roots of the polynomial $\psi_5(z,a,b,c)$ are represented in Fig.3: the b-axis and the curve $\mathcal{M}_2(c_0)$.

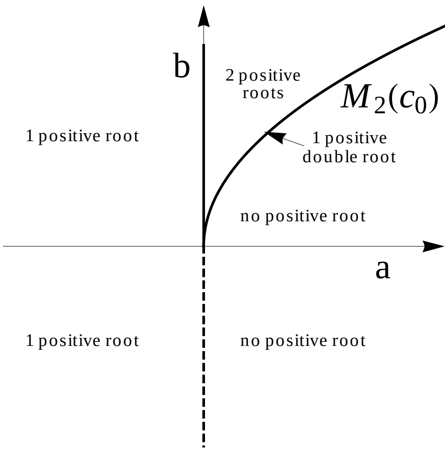

Fig.3.Bifurcation diagram
of the positive roots of
$\psi_5(z,a,b,c_0)$ in case $c_0 > 0$

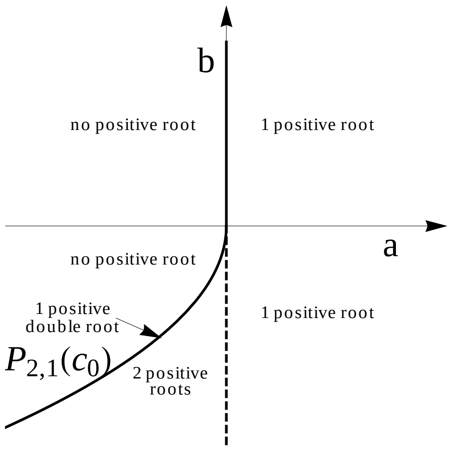

Fig.4. Bifurcation diagram
of the positive roots of
$\psi_5(z,a,b,c_0)$ in case $c_0 < 0$

Analogously, it holds

Lemma 3.5 *Let c_0 be any number from the interval $(-1,0)$. Then there exists a unique curve $\tilde{\mathcal{P}}_{2,1}(c_0)$ in a neighborhood of the (a,b)-parameter plane emanating from the origin, located in the half plane $a \leq 0$ and with the asymptotic representation*

$$b = \tilde{b}^-(a, c_0) = -\sqrt{\frac{40c_0}{9}a} + O(a) \quad \textit{as} \quad a \to -0$$

such that for any point $(a,b) \in \tilde{\mathcal{P}}_{2,1}(c_0)$ equation (3.8) has a positive double root $\tilde{z}_2^-(a, c_0) := \tilde{z}_2^-(a, \tilde{b}^-(a, c_0), c_0)$ with the asymptotic representation

$$\tilde{z}_2^-(a, c_0) = \sqrt{\frac{8a}{5c_0}} + O(a) \quad \textit{as} \quad a \to -0. \tag{3.19}$$

The bifurcation diagram of the positive roots of the polynomial $\psi_5(z,a,b,c)$ for $c = c_0 < 0$ is represented in Fig.4: the b-axis and the curve $P_{2,1}(c_0)$.

In what follows we describe an approach to prove that there exist a curve $\mathcal{M}_2(c_0)$ in an $O(a)$-neighborhood of $\tilde{\mathcal{M}}_2(c_0)$ and emanating from the origin such that for (a,b) on these curves the equation (3.2) with $c = c_0 \neq 0$ has a small positive double root corresponding to a limit cycle $\Gamma_2(a,b,c_0)$ of multiplicity two of system (1.1) with $c = c_0 \neq 0$ which tends to the origin if (a,b) tends to the origin on these curves.
We recall that $(\tilde{z}_2^+(a,c_0), \tilde{b}^+(a,c_0))$ is a solution of the system

$$\begin{aligned} \alpha_1(a) + \alpha_3(a,b)z + \alpha_5(a,b,c_0)z^2 = 0, \\ \alpha_3(a,b) + 2\alpha_5(a,b,c_0)z = 0. \end{aligned} \tag{3.20}$$

Moreover, we can verify that for $a \neq 0$ this solution is a simple solution. For sufficiently small a, $(\tilde{z}_2^+(a,c_0), \tilde{b}^+(a,c_0))$ represents an approximate solution of the system

$$\begin{aligned} \alpha_1(a) + \alpha_3(a,b)z + \alpha_5(a,b,c_0)z^2 + \alpha_7(a,b,c)z^3 + ... = 0, \\ \alpha_3(a,b) + 2\alpha_5(a,b,c_0)z + 3\alpha_7(a,b,c)z^2 + ... = 0. \end{aligned} \tag{3.21}$$

By means of a general theorem due to L.W. Kantorowitsch (see [4], page 752) we can prove that there exists a solution $(z_2^+(a,c_0), b^+(a,c_0))$ with $z_2^+(a,c_0) > 0$ of system (3.21) with the same asymptotic behavior as $(\tilde{z}_2^+(a,c_0), \tilde{b}^+(a,c_0))$. Therefore, we have the result.

Theorem 3.2 *Let c_0 be any number from the interval $(0,1)$. Then there are three curves in a sufficiently small neighborhood of the origin in the (a,b)-parameter plane connected with the bifurcation of a limit cycle of system (1.1):*

(i). The positive b-axis is connected with the bifurcation of a simple stable limit cycle from the origin when the point (a,b) crosses the b-axis for increasing a.

(ii). The curve $\mathcal{M}_2(c_0)$ located in the region $a > 0, b > 0$ is connected with the bifurcation of a semistable limit cycle $\Gamma_2(a,b,c_0)$ of multiplicity two from the origin surrounding an unstable equilibrium. If (a,b) crosses the curve $\mathcal{M}_2(c_0)$ for increasing b, then two limit cycles bifurcate from the semistable limit cycle $\Gamma_2(a,b,c_0)$.

(iii). The negative b-axis is connected with the bifurcation of a simple unstable limit cycle from the origin when the point (a,b) crosses the b-axis for decreasing a.

Fig.5 shows the bifurcation diagram corresponding to Theorem 3.2.

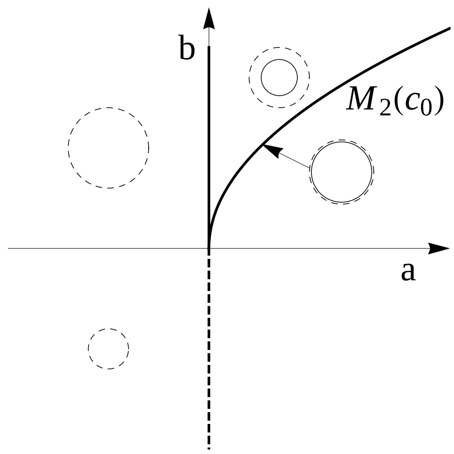

Fig.5. Codimension two bifurcation diagram of system (1.1) for fixed $c = c_0 > 0$

Analogously it holds

Theorem 3.3 *Let c_0 be any number from the interval $(-1, 0)$. Then there are three curves in a sufficiently small neighborhood of the origin in the (a, b)-parameter plane connected with the bifurcation of a limit cycle of system (1.1):*

(i). The positive b-axis is connected with the bifurcation of a simple stable limit cycle from the origin when the point (a, b) crosses the b-axis for increasing a.

(ii). The curve $\mathcal{P}_{2,1}(c_0)$ located in the region $a < 0, b < 0$ is connected with the bifurcation of a semistable limit cycle $\Gamma_2(a, b, c_0)$ of multiplicity two from the origin surrounding a stable equilibrium. If (a, b) crosses the curve $\mathcal{P}_{2,1}(c_0)$ for decreasing b, then two limit cycles bifurcate from the semistable limit cycle $\Gamma_2(a, b, c_0)$.

(iii). The negative b-axis is related with the bifurcation of a simple unstable limit cycle from the origin when the point (a, b) crosses the b-axis for decreasing a.

The bifurcation diagram corresponding to Theorem 3.3 is represented in Fig.6 (see next page).

3.3 Codimension 3 bifurcation

In this subsection we consider system (1.1) under the assumption that the parameter tupel (a, b, c) belongs to the ball $\mathcal{B}_{\varepsilon_0}$ with radius ε_0 centered at the origin

$$(a, b, c) \in \mathcal{B}_{\varepsilon_0} := \{(a, b, c) \in R^3 : a^2 + b^2 + c^2 \leq \varepsilon_0^2, 0 < \varepsilon_0 \ll 1\}. \tag{3.22}$$

Taking into account (2.16) there is a sufficiently small positive number ε_0 such that

$$\alpha_7(a, b, c) \geq \pi/4 \quad \text{for} \quad (a, b, c) \in \mathcal{B}_{\varepsilon_0}. \tag{3.23}$$

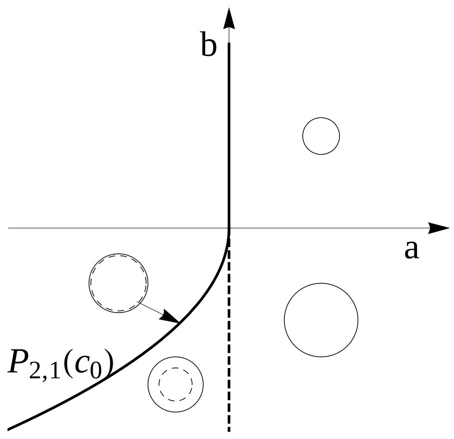

Fig.6. Codimension two bifurcation diagram of system (1.1) for fixed $c = c_0 < 0$

Since we are interested in small positive roots of equation (3.2) we replace this equation in $\mathcal{B}_{\varepsilon_0}$ by the truncated equation

$$\psi_7(z,a,b,c) := \alpha_1(a) + \alpha_3(a,b)z + \alpha_5(a,b,c)z^2 + \alpha_7(a,b,c)z^3 = 0. \tag{3.24}$$

In case $a = b = c = 0$, by (6.3), (6.5), (6.9) and (3.23), $z = 0$ is a root of multiplicity three of equation (3.24), and also of equation (3.2).

For the study of the number of roots $\tilde{z}$ of the cubic polynomial $\psi_7(z,a,b,c)$ in dependence on (a,b,c) it is useful to introduce the new variable y by

$$y = z + \frac{\alpha_5(a,b,c)}{3\alpha_7(a,b,c)}.$$

By this way, equation (3.24) is transformed into the equation

$$y^3 + 3p(a,b,c)y + 2q(a,b,c) = 0, \tag{3.25}$$

where

$$\begin{aligned} 2q(a,b,c) &:= \frac{2\alpha_5^3(a,b,c)}{27\alpha_7^3(a,b,c)} - \frac{\alpha_5(a,b,c)\alpha_3(a,b)}{3\alpha_7^2(a,b,c)} + \frac{\alpha_1(a)}{\alpha_7(a,b,c)}, \\ 3p(a,b,c) &:= \frac{3\alpha_7(a,b,c)\alpha_3(a,b) - \alpha_5^2(a,b,c)}{3\alpha_7^2(a,b,c)}. \end{aligned} \tag{3.26}$$

The following lemma follows from known results.

Lemma 3.6 *If the system*

$$p(a,b,c) = 0, \quad q(a,b,c) = 0 \tag{3.27}$$

has a solution $(\bar a,\bar b,\bar c)\in\mathcal{B}_{\varepsilon_0}$, then for $a=\bar a, b=\bar b, c=\bar c$ equation (3.24) has a unique real root $\tilde z_3(\bar a,\bar b,\bar c)$ of multiplicity 3. If the equation

$$p^3(a,b,c)+q^2(a,b,c)=0 \tag{3.28}$$

has a solution $(\bar a,\bar b,\bar c)\in\mathcal{B}_{\varepsilon_0}$ satisfying $p(\bar a,\bar b,\bar c)\neq 0$, then for $a=\bar a, b=\bar b, c=\bar c$ equation (3.24) has a unique real root $\tilde z_2(\bar a,\bar b,\bar c)$ of multiplicity 2.

As mentioned above, we have

$$p(0,0,0)=0=q(0,0,0), \tag{3.29}$$

that is, equation (3.24) has for $a=b=c=0$ a root of multiplicity 3. Now we prove

Theorem 3.4 *In the parameter space there exists a (local) curve $\tilde{\mathcal{K}}_3$ passing the origin such that for $(a,b,c)\in\tilde{\mathcal{K}}_3$ the equation (3.24) has a root $z_3(a,b,c)$ with multiplicity 3.*

Proof. Let s be a placeholder function of (a,b,c). In what follows we use the notation $(s)^0=s(0,0,0)$. Taking into account the relations (6.3), (6.5), (6.9) and (6.10) we have

$$\left(\frac{\partial p}{\partial a}\right)^0=0,\ \left(\frac{\partial p}{\partial b}\right)^0=\frac{16}{35},\ \left(\frac{\partial q}{\partial a}\right)^0=-\frac{32}{35},\ \left(\frac{\partial p}{\partial b}\right)^0=0.$$

Thus, the matrix

$$\begin{pmatrix} \left(\frac{\partial p}{\partial a}\right)^0 & \left(\frac{\partial p}{\partial b}\right)^0 \\ \left(\frac{\partial q}{\partial a}\right)^0 & \left(\frac{\partial q}{\partial b}\right)^0 \end{pmatrix}$$

is invertible. Taking into account the relations (3.27), (3.29), then by the implicit function theorem there are a sufficiently small interval I_3 containing the origin and functions: $\tilde a_3,\tilde b_3 : I_3\to R$ such that for $(a,b,c)\in\tilde{\mathcal{K}}_3 := \{(\tilde a_3(c),\tilde b_3(c),c)), c\in I_3\}$ the equation (3.24) has a root $z_3(a,b,c)$ with multiplicity 3. $\square$

From Lemma 3.6 we can conclude that if equation (3.24) has a root $z^*(a,b,c)$ of multiplicity two or three, then the corresponding parameter tupel (a,b,c) satisfies the equation (3.28) which is equivalent to

$$\begin{aligned}\Delta(a,b,c) &:= 4\alpha_3^3(a,b)\alpha_7(a,b,c)+27\alpha_1^2(a)\alpha_7^2(a,b,c)\\ &-18\alpha_1(a)\alpha_3(a,b)\alpha_5(a,b,c)\alpha_7(a,b,c)\\ &-\alpha_3^2(a,b)\alpha_5^2(a,b,c)+4\alpha_5^3(a,b,c)\alpha_1(a)=0.\end{aligned} \tag{3.30}$$

For the sequel we introduce the surface $\tilde{\mathcal{S}}$ defined by

$$\tilde{\mathcal{S}} := \{(a,b,c)\in\mathcal{B}_{\varepsilon_0} : \Delta(a,b,c)=0\}.$$

It follows from (3.28) that on $\tilde{\mathcal{S}}$ the relation $p(a,b,c)\le 0$ holds, that is we have

$$\alpha_5^2(a,b,c)\ge 3\alpha_3(a,b)\alpha_7(a,b,c)$$

on $\tilde{\mathcal{S}}$. Furthermore, we get from (3.30) and (6.10) that $\tilde{\mathcal{S}}$ contains all points $(0,0,c)$ with $(0,0,c) \in \mathcal{B}_{\varepsilon_0}$ (a piece of the c-axis). Moreover, the the curve $\tilde{\mathcal{K}}_3$ is located on $\tilde{\mathcal{S}}$.

Any multiple root of (3.24) is a root of the equation

$$\frac{\partial \psi_7}{\partial z}(z,a,b,c) \equiv \alpha_3(a,b) + 2\alpha_5(a,b,c)z + 3\alpha_7(a,b,c)z^2 = 0, \tag{3.31}$$

whose roots $\tilde{z}_2^{\pm}(a,b,c)$ read

$$\tilde{z}_2^{\pm}(a,b,c) = \frac{1}{3\alpha_7(a,b,c)}\Big(-\alpha_5(a,b,c) \pm \sqrt{\alpha_5^2(a,b,c) - 3\alpha_3(a,b)\alpha_7(a,b,c)}\Big). \tag{3.32}$$

It is clear that for a given parameter tupel $(a,b,c) \in \tilde{\mathcal{S}}$ satisfying $p(a,b,c) < 0$, the cubic polynomial $\psi_7(z,a,b,c)$ has either at $\tilde{z}_2^+(a,b,c)$ or at $\tilde{z}_2^-(a,b,c)$ a double root. In case that $\tilde{z}_2^-(a,b,c)$ ($\tilde{z}_2^+(a,b,c)$) is a double root it can be easily verified that $\psi_7(z,a,b,c)$ has a maximum at $z = \tilde{z}_2^-(a,b,c)$ (a minimum at $z = \tilde{z}_2^+(a,b,c)$).

If we substitute the expressions for $\tilde{z}_2^{\pm}(a,b,c)$ from (3.32) into the equation (3.24) we get the relations

$$\begin{aligned}
&\Delta^+(a,b,c) := \psi_7(\tilde{z}_2^+(a,b,c),a,b,c)\\
&= 3\alpha_7(a,b,c)\Big(9\alpha_1(a)\alpha_7(a,b,c) - \alpha_5(a,b,c)\alpha_3(a,b)\Big)\\
&+ 2\Big(\alpha_5^2(a,b,c) - 3\alpha_3(a,b)\alpha_7(a,b,c)\Big)\times\\
&\times \Big(\alpha_5(a,b,c) - \sqrt{\alpha_5^2(a,b,c) - 3\alpha_3(a,b)\alpha_7(a,b,c)}\Big) = 0,
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&\Delta^-(a,b,c) := \psi_7(\tilde{z}_2^-(a,b,c),a,b,c)\\
&= 3\alpha_7(a,b,c)\Big(9\alpha_1(a)\alpha_7(a,b,c) - \alpha_5(a,b,c)\alpha_3(a,b)\Big)\\
&+ 2\Big(\alpha_5^2(a,b,c) - 3\alpha_3(a,b)\alpha_7(a,b,c)\Big)\times\\
&\times \Big(\alpha_5(a,b,c) + \sqrt{\alpha_5^2(a,b,c) - 3\alpha_3(a,b)\alpha_7(a,b,c)}\Big) = 0.
\end{aligned} \tag{3.34}$$

Between Δ, Δ^+ and Δ^- there holds the following relation

$$27\alpha_7^2(a,b,c)\Delta(a,b,c) \equiv \Delta^+(a,b,c)\Delta^-(a,b,c). \tag{3.35}$$

If we introduce the surfaces $\tilde{\mathcal{P}}_2$ and $\tilde{\mathcal{M}}_2$ by

$$\tilde{\mathcal{P}}_2 := \{(a,b,c) \in \mathcal{B}_{\varepsilon_0} : \Delta^+(a,b,c) = 0, p(a,b,c) < 0\},$$

$$\tilde{\mathcal{M}}_2 := \{(a,b,c) \in \mathcal{B}_{\varepsilon_0} : \Delta^-(a,b,c) = 0, p(a,b,c) < 0\}$$

then the following relations can be verified

$$\tilde{\mathcal{K}}_3 := \overline{\tilde{\mathcal{P}}_2} \cap \overline{\tilde{\mathcal{M}}_2}, \quad \tilde{\mathcal{S}} = \tilde{\mathcal{P}}_2 \cup \tilde{\mathcal{K}}_3 \cup \tilde{\mathcal{M}}_2.$$

From our considerations above it follows that we are interested only in such subsurfaces $\tilde{\mathcal{P}}_2^+ \subset \tilde{\mathcal{P}}_2$ and $\tilde{\mathcal{M}}_2^+ \subset \tilde{\mathcal{M}}_2$ with the property that to any point (a, b, c) of the subsurfaces $\tilde{\mathcal{P}}_2^+$ and $\tilde{\mathcal{M}}_2^+$ there corresponds a polynomial $\psi_7(z, a, b, c)$ with a unique *positive* double root. This property guarantees the existence of a small limit cycle of multiplicity two of the original system (1.1).

From (3.32) we obtain that the double root $\tilde{z}_2^-(a, b, c)$ is positive if and only if the condition holds

$$-\alpha_5(a, b, c) > \sqrt{\alpha_5^2(a, b, c) - 3\alpha_3(a, b)\alpha_7(a, b, c)}. \tag{3.36}$$

It is easy to see that the inequality (3.36) is equivalent to the inequalities

$$\alpha_5(a, b, c) < 0, \quad \alpha_3(a, b) > 0. \tag{3.37}$$

Using Lemma 3.2 and Lemma 3.3, we can conclude that the subsurface $\tilde{\mathcal{M}}_2^+$ consists of all points of the surface $\tilde{\mathcal{M}}_2$ satisfying $c > 0, b > 0$.
From (3.34) we get that the subsurface $\tilde{\mathcal{M}}_2^+$ contains all points $(0, 0, c)$ with $0 < c \leq c_*$, where $|c_*|$ is sufficiently small. Thus, $\tilde{\mathcal{M}}_2^+$ contains a piece of the c-axis. This property implies that if we denote the intersection of $\tilde{\mathcal{M}}_2^+$ with the plane $c = c_0 > 0, 0 < c_0 < c_*$ by $\mathcal{M}_2^+(c_0)$ the codimension two bifurcation diagram Fig. 3 can be recovered.

From (3.32) we get that the double root $\tilde{z}_2^+(a, b, c)$ is positive if and only if the condition holds

$$\alpha_5(a, b, c) < \sqrt{\alpha_5^2(a, b, c) - 3\alpha_3(a, b)\alpha_7(a, b, c)}. \tag{3.38}$$

It can be easily verified that inequality (3.38) holds if the inequalities

$$\alpha_5(a, b, c) \geq 0, \quad \alpha_3(a, b) < 0 \tag{3.39}$$

are fulfilled. In that case the inequality $p(a, b, c) < 0$ is always valid. Using Lemma 3.2 and Lemma 3.3 we can conclude analogously to above that the subsurface $\tilde{\mathcal{P}}_2^+$ contains all points of the surface $\tilde{\mathcal{P}}_2$ satisfying $c < 0, b < 0$. We denote this part of the subsurface $\tilde{\mathcal{P}}_2^+$ by $\tilde{\mathcal{P}}_{2,1}^+$. Since $\tilde{\mathcal{P}}_{2,1}^+$ contains all points $(0, 0, c)$ with $c_{**} < c < 0$, where $|c_{**}|$ is sufficiently small, we can conclude that if we denote the intersection of $\tilde{\mathcal{P}}_{2,1}^+$ with the plane $c = c_0 < 0$ by $\mathcal{P}_{2,1}(c_0)$ then the bifurcation diagram in Fig.4 can be recovered.
If we consider those points of the surface $\tilde{\mathcal{P}}_2$ satisfying

$$\alpha_5(a, b, c) < 0, \tag{3.40}$$

then the inequality (3.38) is also valid. We denote this part of the surface $\tilde{\mathcal{P}}_2$ by $\tilde{\mathcal{P}}_{2,2}^+$. We note the following properties of the subsurface $\tilde{\mathcal{P}}_{2,2}^+$:

(i). By Lemma 3.3, $\tilde{\mathcal{P}}_{2,2}^+$ is located in the region $c > 0$.

(ii). The closure of $\tilde{\mathcal{P}}_{2,2}^+$ does not contain a piece of the c-axis.

(iii). $\overline{\tilde{\mathcal{P}}_{2,2}^+} \cap \overline{\tilde{\mathcal{M}}_2^+} = \tilde{\mathcal{K}}_3^+$.

Thus, we can conclude that any curve located on the subsurface $\overline{\tilde{\mathcal{P}}_{2,2}^+}$ and starting at a point on the curve $\tilde{\mathcal{K}}_3^+$ different from the origin is connected with the bifurcation of a positive double root of the equation $\psi_7(a,b,c) = 0$ from a positive root of multiplicity three of this equation.

We summarize our investigations.

Lemma 3.7 *For sufficiently small ε_0, in the ball $\mathcal{B}_{\varepsilon_0}$ there exist the surfaces $\tilde{\mathcal{M}}_2^+$, $\tilde{\mathcal{P}}_{2,1}^+$, $\tilde{\mathcal{P}}_{2,2}^+$, and the curve $\tilde{\mathcal{K}}_3^+$ with the following properties:*

(i). $\tilde{\mathcal{M}}_2^+$ is located in the region $c > 0, b > 0, a > 0$. To each point $(a,b,c) \in \tilde{\mathcal{M}}_2^+$ there corresponds a polynomial $\psi_7(z,a,b,c)$ having the positive double root $\tilde{z}_2^-(a,b,c)$ at which $\psi_7(z,a,b,c)$ has a maximum. The closure of $\tilde{\mathcal{M}}_2^+$ contains the part of the c-axis characterized by $0 \le c \le c_*$, where c_* is sufficiently small.

(ii). $\tilde{\mathcal{P}}_{2,1}^+$ is located in the region $c < 0, b < 0, a < 0$. To each point $(a,b,c) \in \tilde{\mathcal{P}}_{2,1}^+$ there corresponds a polynomial $\psi_7(z,a,b,c)$ having the positive double root $\tilde{z}_2^-(a,b,c)$ at which $\psi_7(z,a,b,c)$ has a minimum. The closure of $\tilde{\mathcal{P}}_{2,1}^+$ contains the part of the c-axis characterized by $-c_{**} \le c \le 0$, where c_{**} is sufficiently small.

(iii). $\tilde{\mathcal{P}}_{2,2}^+$ is located in the region of $c > 0$. To each point $(a,b,c) \in \tilde{\mathcal{P}}_{2,2}^+$ there corresponds a polynomial $\psi_7(z,a,b,c)$ having the positive double root $\tilde{z}_2^-(a,b,c)$ at which $\psi_7(z,a,b,c)$ has a minimum. The closure of $\tilde{\mathcal{P}}_{2,2}^+$ contains only the origin as unique point on the c-axis.

(iv). The smooth curve $\tilde{\mathcal{K}}_3^+$ is located in the region $a > 0, b > 0, c > 0$ and emanates from the origin. To each point $(a,b,c) \in \tilde{\mathcal{K}}_3^+$ there corresponds a polynomial $\psi_7(z,a,b,c)$ having the positive root $\tilde{z}_3^+(a,b,c)$ of multiplicity three.

The intersection of the surfaces $\tilde{\mathcal{M}}_2^+$ and $\tilde{\mathcal{P}}_{2,2}^+$ with the plane $c = c_0 > 0$ is represented in Fig.7 on the next page showing the bifurcation diagram concerning the zeros of the polynomial $\psi_7(z,a,b,c)$.

The intersection of the surface $\tilde{\mathcal{P}}_{2,1}^+$ with the plane $c = c_0 < 0$ yields a bifurcation diagram concerning the zeros of the polynomial $\psi_7(z,a,b,c)$ which coincides qualitatively with that one represented in Fig. 4.

We note that for the curve $\tilde{\mathcal{K}}_3^+$ the following asymptotic representation can be derived by means of (3.26), (6.5), (6.9), and (6.10)

$$\begin{aligned} b = \tilde{b}_3(a) &:= 35^{1/3}\, a^{2/3} + O(a^{5/3}) \quad \text{for} \quad 0 < a \ll 1, \\ c = \tilde{c}_3 &= \frac{6\,35^{2/3}}{5}\, a^{1/3} + O(a^{4/3}) \quad \text{for} \quad 0 < a \ll 1, \end{aligned} \tag{3.41}$$

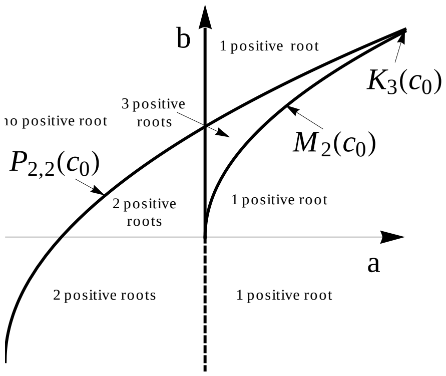

Fig.7. Intersection of the plane $c = c_0 > 0$ with the codimension three bifurcation surfaces of the positive roots of the polynomial $\psi_7(z, a, b, c)$

the corresponding positive root $\tilde{z}_3^+(a, b, c)$ of multiplicity three of the polynomial $\psi_7(z, a, b, c)$ has the representation

$$\tilde{z}_3^+(a, b, c) = \frac{16}{35^{1/3}}\, a^{1/3} + O(a^{4/3}) \quad \text{for} \quad 0 < a \ll 1.$$

Using the mentioned Theorem of Kantorowitsch, we may prove that there is a curve $\mathcal{K}_3^+$ with the same asymptotic representation as for the curve $\tilde{\mathcal{K}}_3^+$ such that for $(a, b, c) \in \mathcal{K}_3^+$ the polynomial $\psi_7(z, a, b, c)$ has a positive root $z_3^+(a, b, c)$ of multiplicity three with the same asymptotic representation as $\tilde{z}_3^+(a, b, c)$ and that there are surfaces $\mathcal{P}_{2,2}^+, \mathcal{P}_{2,1}^+$, and $\mathcal{M}_2^+$ such that Lemma 3.7 holds for these surfaces concerning the zeros of the function $\psi(z, a, b, c)$. Therefore, we have proved our main result

Theorem 3.5 *Let ε_0 and μ_0 be sufficiently small positive numbers. In the ball $\mathcal{B}_{\varepsilon_0}$ there are surfaces $\mathcal{P}_{2,2}^+, \mathcal{P}_{2,1}^+$, and $\mathcal{M}_2^+$ with the following properties:*

(i). *$\mathcal{M}_2^+$ is located in the region $a > 0, b > 0, c > 0$, where a piece of the c-axis including the origin belongs to the boundary of $\mathcal{M}_2^+$. To any point $(a, b, c) \in \mathcal{M}_2^+$ there corresponds a system (1.1) having in the neighborhood $\mathcal{N}_\varepsilon := \{(x, y) \in R^2 : x^2 + y^2 \le \mu_0^2\}$ a unique limit cycle of multiplicity two which is asymptotically orbitally stable with respect to its interior.*

(ii). *$\mathcal{P}_{2,2}^+$ is located in the region $c > 0$, where only the origin belongs to the boundary of $\mathcal{P}_{2,2}^+$. To any point $(a, b, c) \in \mathcal{P}_{2,2}^+$ there corresponds a system (1.1) having in $\mathcal{N}_{\mu_0}$ a unique limit cycle of multiplicity two which is asymptotically orbitally stable with respect to its exterior.*

(iii). *$\overline{\mathcal{P}_{2,2}^+} \cup \overline{\mathcal{M}_2^+} = \mathcal{K}_3^+$, where $\mathcal{K}_3^+$ is a smooth curve with the asymptotic representation (3.41) emanating from the origin. To any point $(a, b, c) \in \mathcal{K}_3^+$ there corresponds a system (1.1)*

having in $\mathcal{N}_{\mu_0}$ a unique orbitally asymptotically stable limit cycle $\Gamma_3(a, b, c)$ of multiplicity three, where $\Gamma_3(a, b, c)$ bifurcates from the origin.

(iv). $\mathcal{P}^+_{2,1}$ is located in the region $a < 0, b < 0, c < 0,$ where a piece of the c-axis including the origin belongs to the boundary of $\mathcal{P}^+_{2,1}$. To any point $(a, b, c) \in \mathcal{P}^+_{2,1}$ there corresponds a system (1.1) having in $\mathcal{N}_\varepsilon$ a unique limit cycle of multiplicity two which is asymptotically orbitally stable with respect to its interior.

The intersection of the plane $c = c_0 < 0$ with the codimension three bifurcation surfaces of system (1.1) coincides qualitatively with the codimension two bifurcation diagram of system (1.1) in Fig. 6, the intersection of the plane $c = c_0 > 0$ with the codimension three bifurcation surfaces of system (1.1) is represented in Fig. 8.

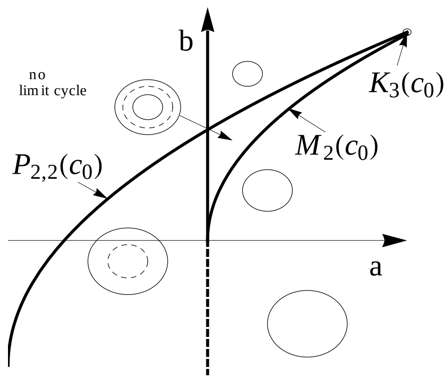

Fig.8. Intersection of the plane $c = c_0 > 0$ with the codimension three bifurcation surfaces of the positive roots of system (1.1)

Finally we note that the intersection of plane the $c = c_0 = 0$ with the codimension three bifurcation surfaces of system (1.1) coincides qualitatively with the bifurcation diagram in Fig. 6.

4 Appendix 1

From (2.8) and (2.7) we obtain

$$k_1(\varphi, a, b, c) \equiv k_1(\varphi, a) \equiv \frac{a\cos^2\varphi}{k(\varphi, a)}, \quad k_1(\varphi, 0) \equiv 0, \tag{4.1}$$

$$\begin{gathered} k_3(\varphi, a, b, c) \equiv k_3(\varphi, a, b) \equiv -b\,\frac{\cos^4\varphi}{k(\varphi, a)}\Big(1 + a\,\frac{\sin 2\varphi}{2k(\varphi, a)}\Big), \\ k_3(\varphi, 0, b) \equiv b\cos^4\varphi. \end{gathered} \tag{4.2}$$

$$
\begin{aligned}
k_5(\varphi,a,b,c) \equiv{}& \frac{c\cos^6\varphi}{k(\varphi,a)} + \frac{v(\varphi,a)(b^2+ac)\cos^4\varphi}{k(\varphi,a)} \\
&+ \frac{ab^2v^2(\varphi,a)\cos^2\varphi}{k(\varphi,a)}, \\
&k_5(\varphi,0,b,c) \equiv -c\cos^6\varphi + b^2\sin\varphi\cos^7\varphi,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
k_7(\varphi,a,b,c) \equiv{}& -\frac{\cos^8\varphi}{k(\varphi,a)} - \frac{v(\varphi,a)(2bc+a)\cos^6\varphi}{k(\varphi,a)} \\
&- \frac{v^2(\varphi,a)b(b^2+2ac)\cos^4\varphi}{k(\varphi,a)} - \frac{v^3(\varphi,a)ab^3\cos^2\varphi}{k(\varphi,a)}.
\end{aligned} \tag{4.4}
$$

From (4.4) and (2.6) it follows

$$
k_7(\varphi,0,0,c) \equiv \cos^8\varphi.
$$

5 Appendix 2

Taking into account (4.1)-(4.3) we obtain from (2.11)

$$
\begin{aligned}
&l_1(\varphi,a,b,c) = l_1(\varphi,a) = \exp\int_0^\varphi k_1(\sigma,a)d\sigma, \quad l_1(\varphi,0)\equiv 1, \\
&\frac{\partial}{\partial a}l_1(\varphi,a)_{|a=0} = -\int_0^\varphi \cos^2\sigma d\sigma,\ \frac{\partial}{\partial a}l_1(2\pi,a)_{|a=0} = -\pi,
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&l_3(\varphi,a,b,c) = l_3(\varphi,a,b) = l_1(\varphi,a)\int_0^\varphi k_3(\sigma,a,b)\,l_1^2(\sigma,a)d\sigma \\
&= -b\,l_1(\varphi,a)\int_0^\varphi \frac{\cos^4\sigma\,l_1^2(\sigma,a)}{k(\sigma,a)}\Big(1+a\,\frac{\sin 2\sigma}{2k(\sigma,a)}\Big)d\sigma, \\
&l_3(\varphi,0,b) = b\int_0^\varphi \cos^4\sigma d\sigma,
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
&l_5(\varphi,a,b,c) = l_1(\varphi,a)\int_0^\varphi \Big(3k_3(\sigma,a,b)\,l_1(\sigma,a)l_3(\sigma,a,b)+ \\
&+ k_5(\varphi,a,b,c)\,l_1^4(\sigma,a)\Big)d\sigma,
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&l_7(\varphi,a,b,c) = l_1(\varphi,a)\int_0^\varphi \Big(k_7(\sigma,a,b,c)\,l_1^6(\sigma,a)+ \\
&+ 3k_3(\sigma,a,b)\big(l_1(\sigma,a)l_5(\sigma,a,b,c) + l_3^2(\sigma,a,b)\big)+ \\
&+ 5k_5(\sigma,a,b)\,l_1^3(\varphi,a)\,l_3(\sigma,a,b)\Big)d\sigma.
\end{aligned} \tag{5.4}
$$

6 Appendix 3

Using (2.14), (5.1), (4.1) and (2.5) we get

$$l_1(2\pi,a)=\exp\Big\{-a\int_0^{2\pi}\frac{\cos^2\varphi}{1+\frac{a}{2}\sin 2\varphi}d\varphi\Big\},\tag{6.1}$$

$$\alpha_1(a)=\exp\Big\{-a\int_0^{2\pi}\tfrac{\cos^2\varphi}{1+\frac{a}{2}\sin 2\varphi}d\varphi\Big\}-1.\tag{6.2}$$

Thus, it holds

$$\begin{gathered}\alpha_1(0)=0,\quad \alpha_1'(0)=-\int_0^{2\pi}\cos^2\varphi\,d\varphi=-\pi<0,\\ sign\ \alpha_1(a)=-sign\ a\quad\text{for sufficiently small } a\neq 0.\end{gathered}\tag{6.3}$$

By (2.14), (5.2) we have

$$\begin{aligned}\alpha_3(a,b)&=-b\,l_1(2\pi,a)\int_0^{2\pi}\frac{l_1^2(\varphi,a)}{k(\varphi,a)}\Big(\cos^4\varphi+a\,v(\varphi,a)\cos^2\varphi\Big)\,d\varphi\\ &=-b\,l_1(2\pi,a)\int_0^{2\pi}\frac{l_1^2(\varphi,a)\ \cos^4\varphi}{k(\varphi,a)}\Big(1+a\,\frac{\sin 2\varphi}{2k(\varphi,a)}\Big)\,d\varphi.\end{aligned}\tag{6.4}$$

Setting

$$F_1(a):=-l_1(2\pi,a)\int_0^{2\pi}\frac{l_1^2(\varphi,a)\ \cos^4\varphi}{k(\varphi,a)}\Big(1+a\,\frac{\sin 2\varphi}{2k(\varphi,a)}\Big)\,d\varphi$$

we can rewrite (6.4) in the form

$$\alpha_3(a,b)\equiv bF_1(a),\tag{6.5}$$

where by (5.1)

$$F_1(0)=\int_0^{2\pi}\cos^4\varphi\,d\varphi=\frac{3}{4}\pi>0.\tag{6.6}$$

From (2.14), (5.3) we obtain

$$\begin{aligned}&\alpha_5(a,b,c)=\\ &=l_1(2\pi,a)\int_0^{2\pi}\Big\{\Big[-\frac{\cos^4\varphi}{k(\varphi,a)}-\frac{av(\varphi,a)\cos^2\varphi}{k(\varphi,a)}\Big]3b\,l_1(\varphi,a)l_3(\varphi,a,b)\\ &+\Big[\frac{c\cos^6\varphi}{k(\varphi,a)}+\frac{v(\varphi,a)(b^2+ac)\cos^4\varphi}{k(\varphi,a)}+\frac{ab^2\,v^2(\varphi,a)\cos^2\varphi}{k(\varphi,a)}\Big]l_1^4(\varphi,a)\Big\}d\varphi\\ &=3b^2\,l_1(2\pi,a)\int_0^{2\pi}\frac{\cos^4\varphi\,l_1^2(\varphi,a)}{k(\varphi,a)}\int_0^{\varphi}\frac{\cos^4\sigma\,l_1^2(\sigma,a)}{k(\sigma,a)}\Big(1+a\,\frac{\sin 2\sigma}{2k(\sigma,a)}\Big)d\sigma\\ &\times\Big[1+\ a\frac{\sin 2\varphi}{2k(\varphi,a)}\Big]d\varphi\\ &+b^2\,l_1(2\pi,a)\int_0^{2\pi}\frac{v(\varphi,a)l_1^4(\varphi,a)\cos^4\varphi}{k(\varphi,a)}\Big[1+a\,\frac{\sin 2\varphi}{2k(\varphi,a)}\Big]d\varphi\\ &+c\,l_1(2\pi,a)\int_0^{2\pi}\frac{l_1^4(\varphi,a)\cos^6\varphi}{k(\varphi,a)}\Big[1+a\,\frac{\sin 2\varphi}{2k(\varphi,a)}\Big]d\varphi.\end{aligned}\tag{6.7}$$

Using (2.5) and putting

$$
\begin{aligned}
&F_2(a) := \\
&3l_1(2\pi, a)\int_0^{2\pi} \frac{\cos^4\varphi\, l_1^2(\varphi, a)}{k(\varphi, a)} \int_0^{\varphi} \frac{\cos^4\sigma l_1^2(\sigma, a)}{k(\sigma, a)}\Big(1 + a\,\frac{\sin 2\sigma}{2k(\sigma, a)}\Big)d\sigma \\
&\times \Big[1 + a\,\frac{\sin 2\varphi}{2k(\varphi, a)}\Big]d\varphi, \\
&F_3(a) := l_1(2\pi, a)\int_0^{2\pi} \frac{v(\varphi, a)\, l_1^4(\varphi, a)\cos^4\varphi}{k(\varphi, a)}\Big[1 + a\,\frac{\sin 2\varphi}{2k(\varphi, a)}\Big]d\varphi, \\
&F_4(a) := l_1(2\pi, a)\int_0^{2\pi} \frac{l_1^4(\varphi, a)\cos^6\varphi}{k(\varphi, a)}\Big[1 + a\,\frac{\sin 2\varphi}{2k(\varphi, a)}\Big]d\varphi
\end{aligned}
\tag{6.8}
$$

we can represent (6.7) in the form

$$
\alpha_5(a, b, c) \equiv b^2\,(F_2(a) + F_3(a)) + c\,F_4(a). \tag{6.9}
$$

From (6.3), (6.6), and (6.8) we obtain the asymptotic relations

$$
\begin{aligned}
&\alpha_1(a) = -\pi a + O(a^2), F_1(a) = \frac{3}{4}\pi + O(a), \\
&F_2(a) = \frac{27}{32}\pi^2 + O(a), F_3(a) = O(a), F_4(a) = -\frac{5}{8}\pi + O(a).
\end{aligned}
\tag{6.10}
$$

Finally, we get from (2.14), (5.4)

$$
\begin{aligned}
&\alpha_7(a, b, c) = l_1(2\pi, a)\int_0^{2\pi} \Big\{\Big[-\frac{\cos^8\varphi}{k(\varphi, a)} - \frac{v(\varphi, a)(2bc + a)\cos^6\varphi}{k(\varphi, a)} \\
&-\frac{v^2(\varphi, a)b(b^2 + 2ac)\cos^4\varphi}{k(\varphi, a)} - \frac{v^3(\varphi, a)ab^3\cos^2\varphi}{k(\varphi, a)}\Big]l_1^6(\varphi, a) + \\
&+\Big[-\frac{\cos^4\varphi}{k(\varphi, a)} - \frac{a\,v(\varphi, a)\cos^2\varphi}{k(\varphi, a)}\Big]3b\Big(l_1(\varphi, a)l_5(\varphi, a, b, c) + l_3^2(\varphi, a, b)\Big) + \\
&+5\Big[\frac{c\cos^6\varphi}{k(\varphi, a)} + \frac{v(\varphi, a)(b^2 + ac)\cos^4\varphi}{k(\varphi, a)} \\
&+\frac{ab^2\,v^2(\varphi, a)\cos^2\varphi}{k(\varphi, a)}\Big]l_1^3(\varphi, a)l_3(\varphi, a, b)\Big\}d\varphi.
\end{aligned}
\tag{6.11}
$$

References

[1] A. A. ANDRONOV, E. A. LEONTOVICH, I. I. GORDON AND A. G. MAIER, *Theory of Bifurcations of Dynamic Systems on a Plane*, Wiley & Sons N.Y., 1973.

[2] A. A. GRIN' AND L. A. CHERKAS, *Extrema of the Andronov-Hopf function of a polynomial Liénard system*, Differ. Equ. **41** (2005), 50–60.

[3] F. DUMORTIER, J. LLIBRE, AND J. C. ARTES, *Qualitative Theory of Planar Dynamical Systems*, Universitext, Springer-Verlag N.Y., 2006.

[4] L. P. KANTOROWITSCH AND G. P. AKILOV, [1964], *Funktionalanalysis in normierten Räumen*, Akademie-Verlag, Berlin, 1964.

[5] L. PERKO, *Differential Equations and Dynamical Systems*, 3rd Ed. Springer-Verlag N.Y., 2001.