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Catalytic branching processes via spine techniques and renewal theory

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Abstract In this article we contribute to the moment analysis of branching processes in catalytic media. The many-to-few lemma based on the spine technique is used to derive a system of (discrete space) partial differential equations for the number of particles in a variation of constants formulation. The long-time behavior is then deduced from renewal theorems and induction.

1. Introduction and Results

A classical subject of probability theory is the analysis of branching processes in discrete or continuous time, going back to the study of extinction of family names by Francis Galton. There have been many contributions to the area since, and we present here an application of a recent development in the probabilistic theory. We identify qualitatively different regimes for the longtime behaviour for moments of sizes of populations in a simple model of a branching Markov process in a catalytic environment. For this problem, which was first approached from an analytic point of view, the interplay of the two effects of spatial motion and branching can easily be tackled with the many-to-few lemma.

To give some background for the branching mechanism, we recall the discrete-time Galton-Watson process. Given a random variable X with law μ taking values in $\mathbb{N}$, the branching mechanism is modelled as follows: for a deterministic or random initial number $Z_0 \in \mathbb{N}$ of particles, one defines for $n = 1, 2, ...$

$$Z_{n+1} = \sum_{r=0}^{Z_n} X_r(n),$$

where all $X_r(n)$ are independent and distributed according to μ. Each particle in generation n is thought of as giving birth to a random number of particles according to μ, and these particles together form generation $n+1$. For the continuous-time analogue each particle carries an independent exponential clock of rate 1 and performs its breeding event once its clock rings.

It is well-known that a crucial quantity appearing in the analysis is $m = \mathbb{E}[X]$, the expected number of offspring particles. The process has positive chance of long-term survival if and only if $m > 1$. This is known as the supercritical case. The cases $m = 1$ (critical) and $m < 1$ (subcritical) also show qualitatively different behaviour in the rate at which the probability of survival decays. As this paper deals with the moment analysis of a spatial relative to this system, we mention the classical trichotomy for the moment asymptotics of Galton-Watson processes:

$$\lim_{t\to\infty} e^{-k(m-1)t}\mathbb{E}\big[Z_t^k\big] \in (0,\infty) \quad ,\forall k \in \mathbb{N}, \tag{1.1}$$

so that all moments increase to infinity if $m > 1$, stay bounded if $m = 1$, and decay to zero if $m < 1$.

In the present article we are interested in the moment asymptotics in a version of the Galton-Watson process for which a system of branching particles moves in space and particles branch only in the presence of a catalyst. More precisely, we start a particle ξ which moves on some countable set S according to a continuous-time Markov process with Q-matrix $\mathcal{A}$. This particle carries an exponential clock of rate 1 that only ticks if ξ is at the same site as the catalyst, which we assume for now sits at 0. (We will later give results for a moving catalyst.) If and when the clock rings, then the particle dies and is replaced in its position by a random number of offspring. This number is distributed according to some offspring distribution μ, and all newly born particles behave as independent copies of their parent: they move on S according to $\mathcal{A}$ and branch after an exponential rate 1 amount of time spent at 0.
The main objective of our study is the longtime behaviour of the moments

$$M^k(t,x,y) = \mathbb{E}\big[N_t(y)^k \,\big|\, \xi_0 = x\big] \qquad \text{and} \qquad M^k(t,x) = \mathbb{E}\big[N_t^k \,\big|\, \xi_0 = x\big],$$

where $N_t(y)$ is the number of particles alive at site y at time t and $N_t = \sum_{y\in\mathbb{Z}^d} N_t(y)$ is the total number of particles alive at time t.

Under the additional assumption that $\mathcal{A} = \Delta$ is the discrete Laplacian, the moment analysis was carried out in [AB00], [ABY98], [ABY98] via partial differential equations and Tauberian theorems. Our approach is more robust so that the assumptions on the underlying motion can be dropped.

To our knowledge the asymptotics of higher moments of the branching process in a catalytic environment were first considered analytically in [Y91] for the special case of a binary branching mechanism (i.e. μ is concentrated on 0 and 2 only) and particles performing simple random walks. In [ABY98], see also the Erratum [ABY98b], it was proposed (without rigorous proofs) to study the asymptotics for the moments via partial differential equations and Tauberian theorems under the assumptions

- **(A1)** irreducibility,
- **(A2)** spatial homogeneity,
- **(A3)** symmetry,
- **(A4)** finite variance of jump sizes

on the underlying spatial motion $\mathcal{A}$. Interestingly, the effects of the spatial motion and the branching mechanism are well separated in the partial differential equations allowing for a detailed analysis of their interplay. Those equations were derived rigorously under **(A1)** to **(A3)** in [AB00] via analytic arguments from partial differential equations for the generating functions $\mathbb{E}_x[e^{-zN_t(y)}]$ and $\mathbb{E}_x[e^{-zN_t}]$.

Our approach is different from the analytic approaches mentioned above: first we apply a combinatorial spine representation of [HR11] to derive sets of partial differential equations, in variation of constants form, for the kth moments of $N_t(y)$ and N_t. A set of combinatorial factors can be given a direct probabilistic explanation, whereas the same factors appear in [AB00] abstractly from Faà di Bruno's formula of differentiation.

To derive asymptotic results as announced in [ABY98], one might then follow their approach via spectral theory and Tauberian theorems applied to the partial differential equations. The price one has to pay is to assume additionally a property like **(A4)** as then the local central limit theorem yields the asymptotic behaviour for the transition probabilities which, via Tauberian theorems, can be translated to asymptotics for the moments.

We follow a different route: building upon ideas of [DS10], renewal theorems are applied to the variation of constants formula for the first moments. This, combined with an induction for higher moments, allows us to circumvent all assumptions **(A1)** to **(A4)**.

In order to avoid many pathological cases we do impose from now on the assumption

(A) the motion governed by $\mathcal{A}$ is irreducible.

This assumption is not necessary, and the interested reader may easily reconstruct the additional cases from our proofs. It simply allows us to avoid many pathological examples. To state the main result, we denote the transition probabilities of $\mathcal{A}$ by

$$p_t(x,y) = \mathbb{P}_x(\xi_t = y)$$

and the Green function by

$$G_\infty(x,y) = \int_0^\infty p_t(x,y)\,dt.$$

Recall that, by irreducibility, the Green function is finite for all $x, y \in S$ if and only if $\mathcal{A}$ is transient.

Theorem 1. *Suppose μ has finite moments of all orders; then the following regimes occur for all integers $k \geq 1$:*

i) *If the branching mechanism is **subcritical**, then*

$$\lim_{t\to\infty} M^k(t,x) \in (0,1) \quad \Longleftrightarrow \quad \mathcal{A} \text{ is transient}$$
$$\lim_{t\to\infty} M^k(t,x) = 0 \quad \Longleftrightarrow \quad \mathcal{A} \text{ is recurrent},$$

and

$$\lim_{t\to\infty} M^k(t,x,y) = 0 \qquad \text{in all cases.}$$

ii) *If the branching mechanism is **critical**, then*

$$\lim_{t\to\infty} M^k(t,x) \in (0,\infty)$$

and

$$\lim_{t\to\infty} M^k(t,x,y) = 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} p_t(x,y) = 0,$$

$$\lim_{t\to\infty} M^k(t,x,y) \in (0,1] \quad \Longleftrightarrow \quad \lim_{t\to\infty} p_t(x,y) > 0.$$

iii) *If the branching mechanism is **supercritical**, then there is a critical constant*

$$\beta = \frac{1}{G_\infty(0,0)} + 1$$

such that

a) *for* $m < \beta$

$$\lim_{t\to\infty} M^k(t,x) \in (0,\infty)$$

and

$$\lim_{t\to\infty} M^k(t,x,y) = 0.$$

b) *for* $m = \beta$

$$\lim_{t\to\infty} M^k(t,x) = \infty$$

and

$$\lim_{t\to\infty} M^k(t,x,y) \begin{cases} \in (0,\infty) & : \int_0^\infty r p_r(0,0)\,dr < \infty \\ = \infty & : \int_0^\infty r p_r(0,0)\,dr = \infty \end{cases}$$

(in both cases the moments grow only subexponentially fast).

c) *for* $m > \beta$

$$\lim_{t\to\infty} e^{-kr(m)t} M^k(t,x) \in (0,\infty)$$

and

$$\lim_{t\to\infty} e^{-kr(m)t} M^k(t,x,y) \in (0,\infty),$$

where $r(m)$ *equals the unique solution* λ *to* $\int_0^\infty e^{-\lambda t} p_t(0,0)\,dt = \frac{1}{m-1}$.

The formulation of the theorem does not include the limiting constants. Indeed, the proofs give some of those (in an explicit form involving the transition probabilities p_t) in the supercritical regime but they seem to be of little use. The use of spectral theory for symmetric Q-matrix $\mathcal{A}$ allows one to derive the exponential growth rate $r(m)$ as the maximal eigenvalue of a Schrödinger operator with one-point potential and the appearing constants via the eigenfunctions. Our renewal theorem based proof gives the representation of $r(m)$ as the inverse of the Laplace transform of $p_t(0,0)$ at $1/(m-1)$ and the eigenfunction expressed via integrals of $p_t(0,0)$. As $p_t(0,0)$ is rarely known explicitly, the integral form of the constants is not very useful (apart from the trivial case of Example 1 below). Only in case iii) b) for $\int_0^\infty r p_r(0,0)\,dr = \infty$ are the proofs unable to give strong asymptotics. This is caused by the use of an infinite-mean renewal theorem which only gives asymptotic bounds up to an unknown factor between 1 and 2.
There is basically one example in which $p_t(0,0)$ is trivially known:

Example 1: For the trivial motion $\mathcal{A} = 0$, i.e. branching particles are fixed at the same site as the catalyst, the supercritical cases iii) a) and b) do not occur as $\mathcal{A}$ is trivially recurrent so that $\beta = 1$. Furthermore, in this example $p_t(0,0) = 1$ for all $t \geq 0$ so that $r(m) = m - 1$. In fact by examining the proof of Theorem 1 one recovers (1.1) with all constants.

The explicit representation for the exponential growth rate allows for a more careful comparison with the non-spatial case.

Corollary 1. *Let $r(m)$ be the exponential growth rate obtained in the supercritical case of Theorem 1. Then*

$$m \mapsto r(m)$$

is convex, with

$$r(0) = 0, \qquad r(m) \leq m - 1, \qquad \lim_{m\to\infty} \frac{r(m)}{(m-1)} = 1.$$

Proof. This follows from elementary manipulations of the defining equation for $r(m)$. □

Example 2: The most classical non-trivial example is to take $\mathcal{A} = \Delta$ on $\mathbb{Z}^d$ as studied in [ABY98]. In the recurrent case $d = 1, 2$, implying $\beta = 1$, our approach gives

$$M^k(t,x) \sim \begin{cases} 0 & : m < 1 \\ C & : m = 1 \\ Ce^{kr(m)t} & : m > 1, \end{cases}$$

$$M^k(t,x,y) \sim \begin{cases} 0 & : m < 1 \\ 0 & : m = 1 \\ Ce^{kr(m)t} & : m > 1. \end{cases}$$

In the transient case $d \geq 3$, implying $\beta > 1$, our proofs give

$$M^k(t,x) \sim \begin{cases} C & : m < \beta \\ Ct^k & : m = \beta, d = 3,4 \\ \infty & : d \geq 5 \\ Ce^{kr(m)t} & : m > \beta, \end{cases}$$

$$M^k(t,x,y) \sim \begin{cases} 0 & : m < \beta \\ \infty & : m = \beta, d = 3,4 \\ C & : m = \beta, d \geq 5 \\ Ce^{kr(m)t} & : m > \beta \end{cases}$$

with asymptotic equivalence to infinity interpreted as divergence to infinity. We will sketch in Remark 1 below how the additional property $\lim_{t\to\infty} p_t(x,y)t^{d/2} \in (0,\infty)$, due to the local central limit theorem, can be used to derive explicit asymptotic rates in all cases.

A similar question to that in Theorem 1 is discussed in [GH06] for a corresponding parabolic Anderson model. Supposing that at $t = 0$ one branching particle starts at each site x of the lattice and the branching particles and a catalyst ζ moves according to simple random walks on $\mathbb{Z}^d$ with jump rate κ, the quenched first moment of the number of particles solves the random heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \kappa \Delta u(t,x) + W(t,x)u(t,x), & t \geq 0, x \in \mathbb{Z}^d, \\ u(0,x) = 1, & x \in \mathbb{Z}^d. \end{cases}$$

The random potential $W(t,x)$ equals 1 if at time t the catalytic particle is at site x and 0 otherwise. In contrast to the present article, they consider longtime asymptotics of the moments $\mathbb{E}^{\zeta}\left[u(t,x)^p\right]$; that is they first average over the branching before taking the moments and then averaging over the random environment.
Indeed, we can extend our renewal theory approach in this direction. Much of our proof of Theorem 1 works also when we allow the catalyst to move according to some random process. In fact we do not need to assume that ζ_t is a simple random walk; for our results it is sufficient to assume that the difference process $\xi_t - \zeta_t$ is a well-defined homogeneous Markov process. Nonetheless the heavy use of renewal theorems forces us to restrict the moment analysis in the supercritical case to the total number of particles.

Theorem 2. *Suppose μ has finite moments of all orders; suppose that the catalyst ζ_t moves in such a way that $\xi_t - \zeta_t$ is a well-defined homogeneous Markov process with generator $\mathcal{C}$. Then the following regimes occur for all integers $k \geq 1$:*

i) *If the branching mechanism is **subcritical**, then*

$$\lim_{t\to\infty} M^k(t,x) \in (0,1) \iff \mathcal{C} \text{ is transient}$$
$$\lim_{t\to\infty} M^k(t,x) = 0 \iff \mathcal{C} \text{ is recurrent,}$$

and

$$\lim_{t\to\infty} M^k(t,x,y) \in (0,1) \iff \mathcal{C} \text{ is transient and } \lim_{t\to\infty} p_t(x,y) > 0$$
$$\lim_{t\to\infty} M^k(t,x,y) = 0 \iff \text{otherwise.}$$

ii) *If the branching mechanism is **critical**, then*

$$\lim_{t\to\infty} M^k(t,x) \in (0,\infty)$$

and

$$\lim_{t\to\infty} M^k(t,x,y) = 0 \iff \lim_{t\to\infty} p_t(x,y) = 0,$$
$$\lim_{t\to\infty} M^k(t,x,y) \in (0,1] \iff \lim_{t\to\infty} p_t(x,y) > 0.$$

iii) *If the branching mechanism is **supercritical**, then there is a critical constant*

$$\beta = \frac{1}{\int_0^\infty \mathbb{P}_x(\xi_t = \zeta_t)dt} + 1$$

such that

a) *for $m < \beta$, $\lim_{t\to\infty} M^k(t,x) \in (0,\infty)$.*
b) *for $m = \beta$, $\lim_{t\to\infty} M^k(t,x) = \infty$ but $M^k(t,x)$ grows only subexponentially fast.*
c) *for $m > \beta$, $\lim_{t\to\infty} e^{-kr(m)t} M^k(t,x) \in (0,\infty)$ where $r(m)$ equals the unique solution λ to $\int_0^\infty e^{-\lambda t}\mathbb{P}_x(\xi_t = \zeta_t)\,dt = \frac{1}{m-1}$.*

2. Proofs

The key tool in our proofs will be the many-to-few lemma proved in [HR11] which relies on modern spine techniques. These emerged from work of Kurtz, Lyons, Pemantle and Peres in the mid-1990s [KLPP97, L97, LPP95]. The idea is that to understand certain functionals of branching processes, it is enough to carefully study the behaviour of one special particle, the *spine*. In particular very general *many-to-one* lemmas emerged, allowing one to easily calculate expectations of sums over particles like

$$\mathbb{E}\left[\sum_{v\in N_t} f(v)\right],$$

where $f(v)$ is some well-behaved functional of the behaviour of the particle v up to time t, and N_t here is viewed as the *set* of particles alive at time t, rather than the number. It will always be clear from the context which meaning for N_t is intended.
It is natural to ask whether similar results exist for higher moments of sums over N_t. This is the idea behind [HR11], wherein it turns out that to understand the kth moment one must consider a system of k particles. The k particles introduce complications compared to the single particle required for first moments, but this is still significantly simpler than controlling the behaviour of the potentially huge random number of particles in N_t. Our results here work equally well in the case where the catalyst is also moving within S according to a Markov process: as above we write ζ_t for the position of the catalyst at time t.

While we do not need to understand the full spine setup here, we shall require some explanation. For each $k \geq 0$ let $p_k = \mathbb{P}(X = k)$ and $m_k = \mathbb{E}[X^k]$, the kth moment of the offspring distribution (in particular $m_1 = m$). We define a new measure $\mathbb{Q} = \mathbb{Q}_x^k$, under which there are k distinguished lines of descent known as spines. The construction of $\mathbb{Q}$ relies on a carefully chosen change of measure, but we do not need to understand the full construction and instead refer to [HR11]. In order to use the technique, we simply have to understand the dynamics of the system under $\mathbb{Q}$. Under $\mathbb{Q}_x^k$ particles behave as follows:

- We begin with one particle at position x which (as well as its position) has a mark k. We think of a particle with mark j as carrying j spines.
- Whenever a particle with mark j, $j \geq 1$, spends an (independent) exponential time with parameter m_j in the same position as the catalyst, it dies and is replaced by a random number of new particles with law A_j.
- The probability of the event $\{A_j = a\}$ is $a^j p_a m_j^{-1}$. (This is the *jth size-biased* distribution relative to μ.)
- Given that a particles $v_1, \ldots, v_a$ are born, the j spines each choose a particle to follow independently and uniformly at random. Thus particle v_i has mark l with probability $a^{-l}(1-a^{-1})^{j-l}$, $l = 0, \ldots, j$, $i = 1, \ldots, a$. We also note that this means that there are always k spines amongst the particles alive; equivalently the sum of the marks over all particles alive always equals k.
- Particles with mark 0 are no longer of interest (in fact they behave just as under $\mathbb{P}$, branching at rate 1 when in the same position as the catalyst and giving birth to numbers of particles with law μ, but we will not need to use this).

For a particle v, we let $X_v(t)$ be its position at time t and B_v be its mark (the number of spines it is carrying). Let σ_v be the time of its birth and τ_v the time of its death, and define $\sigma_v(t) = \sigma_v \wedge t$ and $\tau_v(t) = \tau_v \wedge t$. Let χ_t^i be the current position of the ith spine. We call the collection of particles that have carried at least one spine up to time t the *skeleton* at time t, and write $\text{skel}(t)$. Figure 1 gives an impression of the skeleton at the start of the process.

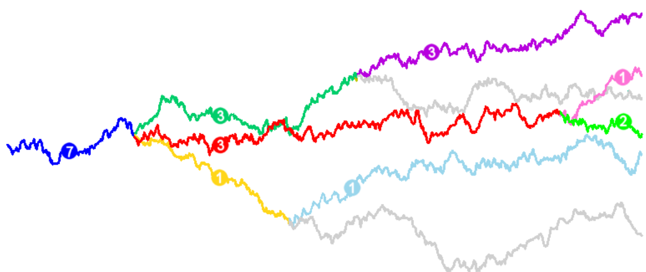

FIGURE 1. An impression of the start of the process: each particle in the skeleton is a different colour, and particles not in the skeleton are drawn in pale grey. The circles show the number of spines being carried by each particle in the skeleton.

A much more general form of the following lemma was proved in [HR11].

Lemma 1 (Many-to-few). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable. Then, for any $k \geq 1$,*

$$\mathbb{E}\left[\sum_{v_1,\ldots,v_k \in N_t} f(X_{v_1}(t)) \cdots f(X_{v_k}(t))\right] = \mathbb{Q}^k\left[f(\chi_t^1)\cdots f(\chi_t^k) \prod_{v \in \text{skel}(t)} \exp\left((m_{B_v} - 1)\int_{\sigma_v(t)}^{\tau_v(t)} \mathbb{1}_{\zeta_s}(X_v(s))ds\right)\right].$$

Clearly if we take $f \equiv 1$, then the left hand side is simply the kth moment of the number of particles alive at time t. The lemma is useful since the right-hand side depends on at most k particles at a time, rather than the arbitrarily large random number of particles on the left-hand side.

Having introduced the spine technique, we can now proceed with the proof of Theorem 1. We first use Lemma 1 for the case $k = 1$, which is simply the many-to-one lemma, to deduce two convenient representations for the first moments: a Feynman-Kac expression and a variation of constants formula. Indeed, the exponential expression equally works for other random potentials and, hence, is well known for instance in the parabolic Anderson model literature. More interestingly, the variation of constants representation is most useful in the case of a one-point potential: it simplifies to a renewal type equation. Understanding when those are proper renewal equations replaces the spectral theoretic arguments of [ABY98] and explains the different cases appearing in Theorem 1.

Lemma 2. *For any catalyst $\zeta_t, t \geq 0$, the first moments can be expressed as*

$$M^1(t,x) = \mathbb{E}_x\big[e^{(m-1)\int_0^t \mathbb{1}_{\zeta_r}(\xi_r)\,dr}\big], \tag{2.1}$$

$$M^1(t,x,y) = \mathbb{E}_x\big[e^{(m-1)\int_0^t \mathbb{1}_{\zeta_r}(\xi_r)\,dr}\mathbb{1}_y(\xi_t)\big], \tag{2.2}$$

where ξ_t is a single particle moving with Q-matrix $\mathcal{A}$. In the case of the fixed catalyst $\zeta_t = 0$ these quantities fulfill

$$M^1(t,x) = 1 + (m-1)p_t(x,0) * M^1(t,0), \tag{2.3}$$

$$M^1(t,x,y) = p_t(x,y) + (m-1)p_t(x,0) * M^1(t,0,y), \tag{2.4}$$

where $*$ denotes ordinary convolution in t. Equation (2.3) also holds whenever $\xi_t - \zeta_t$ is a homogeneous Markov process.

Before giving the proof let us briefly mention why the renewal type equations occur naturally, at least in the case of the static catalyst $\zeta_t = 0$. The Feynman-Kac representation can be proved in various ways; we derive it simply from the many-to-few lemma. The Feynman-Kac formula then leads naturally to solutions of discrete-space heat equations with one-point potential:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \mathcal{A}u(t,x) + (m-1)\mathbb{1}_0(x)u(t,x) \\ u(0,x) = \mathbb{1}_y(x) \end{cases}.$$

Applying the variation of constants formula for solutions gives

$$\begin{aligned} u(t,x) &= P_t u(0,x) + \int_0^t P_{t-s}(m-1)\mathbb{1}_0(x)u(s,x)\,ds \\ &= p_t(x,y) + (m-1)\int_0^t p_{t-s}(x,0)u(s,x)\,ds, \end{aligned}$$

where P_t is the semigroup corresponding to $\mathcal{A}$, i.e. $P_t f(x) = \mathbb{E}_x[f(\xi_t)]$. As we do not want (and do not need) to refer to analytic results we now give a direct simple proof for these representations.

Proof of Lemma 2. To prove (2.1) and (2.2) we apply the easiest case of Lemma 1: we choose $k = 1$ and $f \equiv 1$ (resp. $f(z) = \mathbb{1}_y(z)$ for (2.2)). Since there is exactly one spine at all times, the skeleton reduces to a single line of descent. Hence $m_{B_v} - 1 = m - 1$ and the integrals in the product combine to become a single integral along the path of the spine up to time t. Thus

$$M^1(t,x) = \mathbb{Q}_x\big[e^{(m-1)\int_0^t \mathbb{1}_{\zeta_r}(\xi_r)\,dr}\big] \quad \text{and} \quad M^1(t,x,y) = \mathbb{Q}_x\big[e^{(m-1)\int_0^t \mathbb{1}_{\zeta_r}(\xi_r)\,dr}\mathbb{1}_y(\xi_t)\big]$$

which is what we claimed but with expectations taken under $\mathbb{Q}$ rather than the original measure $\mathbb{P}$. However we note that the motion of the single spine is the same (it has Q-matrix $\mathcal{A}$) under both $\mathbb{P}$ and $\mathbb{Q}$, so (since — given the position of the catalyst — the right hand sides above depend only on the position of the spine) we may simply replace $\mathbb{Q}$ with $\mathbb{P}$, giving (2.1) and (2.2).
The variation of constants formulas can now be derived from the Feynman-Kac formulas in the case of the static catalyst $\zeta_t = 0$. We only prove the second identity, as the first can be proved similarly (even when $\xi_t - \zeta_t$ is any homogeneous Markov process - one simply replaces position 0

with ζ_t and applies homogeneity at the final step). We use the exponential series to get

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{(m-1)\int_0^t \mathbb{1}_0(\xi_r)\,dr}\mathbb{1}_y(\xi_t)\right]\\
&=\mathbb{E}_x\left[\sum_{n=0}^{\infty}\frac{(m-1)^n}{n!}\left(\int_0^t \mathbb{1}_0(\xi_r)\,dr\right)^n\mathbb{1}_y(\xi_t)\right]\\
&=\mathbb{P}_x(\xi_t=y)+\mathbb{E}_x\left[\sum_{n=1}^{\infty}\frac{(m-1)^n}{n!}\int_0^t\cdots\int_0^t \mathbb{1}_0(\xi_{r_1})\cdots\mathbb{1}_0(\xi_{r_n})\,dr_n\ldots dr_1\mathbb{1}_y(\xi_t)\right]\\
&=p_t(x,y)+\mathbb{E}_x\left[\sum_{n=1}^{\infty}(m-1)^n\int_0^t\int_{r_1}^t\cdots\int_{r_{n-1}}^t \mathbb{1}_0(\xi_{r_1})\cdots\mathbb{1}_0(\xi_{r_n})\,dr_n\ldots dr_2dr_1\mathbb{1}_y(\xi_t)\right].
\end{aligned}
$$

The last step is justified by the fact that the function that is integrated is symmetric in all arguments and, thus, it suffices to integrate over a simplex. We can exchange sum and expectation and obtain that the last expression equals

$$p_t(x,y)+(m-1)\int_0^t\sum_{n=1}^{\infty}(m-1)^{n-1}\int_{r_1}^t\cdots\int_{r_{n-1}}^t \mathbb{P}_x[\xi_{r_1}=0,\ldots,\xi_{r_n}=0]\,dr_n\ldots dr_2dr_1.$$

Due to the Markov property, the last expression equals

$$p_t(x,y)+(m-1)\int_0^t p_{r_1}(x,0)\sum_{n=1}^{\infty}(m-1)^{n-1}\int_{r_1}^t\cdots\int_{r_{n-1}}^t \mathbb{P}_0[\xi_{r_2-r_1}=0,\ldots,\xi_{r_n-r_1}=0]\,dr_n\ldots dr_2dr_1$$

and can be rewritten as

$$p_t(x,y)+(m-1)\int_0^t p_{r_1}(x,0)\left(\sum_{n=1}^{\infty}(m-1)^{n-1}\int_0^{t-r_1}\cdots\int_{r_{n-1}}^{t-r_1}\mathbb{P}_0[\xi_{r_2}=0,\ldots,\xi_{r_n}=0]\,dr_n\ldots dr_2\right)dr_1.$$

Using the same line of arguments backwards for the term in parentheses, the assertion follows. □

Having derived variation of constants formulas, there are different ways to analyze the asymptotics of the first moments. Assuming more regularity for the transition probablities, this can be done as sketched in the next remark.

Remark 1. *Taking Laplace transforms $\mathcal{L}$ in t, one can transform (2.3), and similarly (2.4), into the algebraic equation*

$$\mathcal{L}M^1(\lambda,x)=\frac{1}{\lambda}+(m-1)\mathcal{L}M^1(\lambda,0)\mathcal{L}p_\lambda(x,0)\qquad,\lambda>0,$$

which can be solved explicitly to obtain

$$\mathcal{L}M^1(\lambda,x)=\frac{1}{\lambda(1-(m-1)\mathcal{L}p_\lambda(x,0))}\qquad,\lambda>0. \tag{2.5}$$

Assuming the asymptotics of $p_t(x,0)$ are known for t tending to infinity (and are sufficiently regular), the asymptotics of $\mathcal{L}p_\lambda(x,0)$ for λ tending to zero can be deduced from Tauberian theorems. Hence, from Equation (2.5) one can then deduce the asymptotics of $\mathcal{L}M^1(\lambda,x)$ as λ tends to zero. This, using Tauberian theorems in the reverse direction, allows one to deduce the asymptotics of $M^1(t,x)$ for t tending to infinity.
Unfortunately, to make this approach work, ultimate monotonicity and asymptotics of the type $p_t(x,0)\sim Ct^{-\alpha}$ are needed. This motivated the authors of [ABY98] *to assume **(A4)** so that by the local central limit theorem*

$$p_t(x,0)\sim\left(\frac{d}{2\pi}\right)^{d/2}t^{-d/2}.$$

As we did not assume any regularity for p_t, the aforementioned approach fails in general. We instead use an approach based on renewal theorems recently seen in [DS10].

Proof of Theorem 1 for M^1. Taking into account irreducibility and the Markov property of $\mathcal{A}$, we see that the property "$\int_0^\infty \mathbb{1}_0(\xi_r)\,dr = \infty$ almost surely" does not depend on the starting value ξ_0. If $\mathcal{A}$ is recurrent, we thus obtain from dominated convergence and (2.2) that $M^1(t,x)$ tends to zero for all $x \in \mathbb{Z}^d$ in regime i). If $\mathcal{A}$ is transient, then $M^1(t,x)$ converges to a constant since it is decreasing and bounded from below: fix s such that $P_x(\xi_r \neq 0 \;\forall r \geq s) > 0$, and note that

$$M^1(t,x) \geq \mathbb{E}_x[e^{(m-1)\int_0^t \mathbb{1}_0(\xi_r)dr}\mathbb{1}_{\{\xi_r \neq 0 \,\forall r \geq s\}}] \geq e^{(m-1)s}\mathbb{P}_x(\xi_r \neq 0 \;\forall r \geq s) > 0. \tag{2.6}$$

However,

$$M^1(t,x,y) = \mathbb{E}_x\left[e^{(m-1)\int_0^t \mathbb{1}_0(\xi_r)\,dr}\mathbb{1}_y(\xi_t)\right] \leq \mathbb{P}_x(\xi_t = y) \overset{t\to\infty}{\longrightarrow} 0,$$

so that regime i) is proved.

Regime ii) is trivial as here $M^1(t,x) = 1$ and $M^1(t,x,y) = p_t(x,y)$. Next, for regime iii) a) we exploit both the standard and the reverse Hölder inequality for $p > 1$:

$$M^1(t,x,y) \geq \mathbb{E}_x\left[e^{-(1/(p-1))(m-1)\int_0^t \mathbb{1}_0(\xi_r)\,dr}\right]^{-(p-1)} p_t(x,y)^p, \tag{2.7}$$

$$M^1(t,x,y) \leq \mathbb{E}_x\left[e^{p(m-1)\int_0^t \mathbb{1}_0(\xi_r)\,dr}\right]^{1/p} p_t(x,y)^{(p-1)/p}. \tag{2.8}$$

In the recurrent case $G_\infty(0,0) = \infty$ and thus $\beta = 1$, so this case has already been dealt with in regime ii). Hence we may assume that $\mathcal{A}$ is transient so that $\int_0^\infty \mathbb{1}_0(\xi_r)\,dr < \infty$ with positive probability. This shows that the expectation in the lower bound (2.7) converges to a finite constant. By assumption $m-1 < \beta$ so that there is $p > 1$ satisfying $p(m-1) < \beta$. With this choice of p, part 3) of Theorem 1 of [DS10] implies that also the expectation in the upper bound (2.8) converges to a finite constant. In total this shows that

$$Cp_t(x,y)^p \leq M^1(t,x,y) \leq C' p_t(x,y)^{(p-1)/p}$$

and the claim for $M^1(t,x,y)$ follows. For $M^1(t,x)$ we can directly refer to Theorem 1 of [DS10].

For regimes iii) b) and c) we give arguments based on renewal theorems. A closer look at the variation of constants formula (2.4) shows that only for $x = 0$, $M^1(t,x,y)$ occurs on both sides of the equation. Hence, we start with the case $x = 0$ and afterwards deduce the asymptotics for $x \neq 0$.

Let us begin with the simpler case iii) c). As mentioned above, in this case we may assume that $\mathcal{A}$ is transient so that $\int_0^\infty p_r(0,0)\,dr < \infty$. Hence, dominated convergence ensures that the equation $\int_0^\infty e^{-\lambda t}p_t(0,0)\,dt = 1/(m-1)$ has a unique positive root λ, which we call $r(m)$. The definition of $r(m)$ shows that $U(dt) := (m-1)e^{-r(m)t}p_t(0,0)\,dt$ is a probability measure on $[0,\infty)$ and furthermore $e^{-r(m)t}p_t(0,y)$ is directly Riemann integrable. Hence the classical renewal theorem (see page 349 of [F71]) can be applied to the (complete) renewal equation

$$f(t) = g(t) + f * U(t),$$

with $f(t) = e^{-r(m)t}M^1(t,0,y)$ and $g(t) = e^{-r(m)t}p_t(0,y)$. The renewal theorem implies that

$$\lim_{t\to\infty} f(t) = \frac{\int_0^\infty g(s)\,ds}{\int_0^\infty U((s,\infty))\,ds} \in (0,\infty) \tag{2.9}$$

so that the claim for $M^1(t,0,y)$ follows including the limiting constants.

For iii) b), we need to be more careful as the criticality implies that $(m-1)\int_0^\infty p_r(0,0)\,dr = 1$. Hence, the measure U as defined above is already a probability measure so that the variation of constants formula is indeed a proper renewal equation. The renewal measure U only has finite mean if additionally

$$\int_0^\infty r p_r(0,0)\,dr < \infty. \tag{2.10}$$

In the case of finite mean the claim follows as above from (2.9) without the exponential correction (i.e. $r(m) = 0$). Note that $p_t(0,y)$ is directly Riemann integrable as the case $\beta > 0$ implies that $\mathcal{A}$ is transient and $p_t(0,y)$ is decreasing.

If (2.10) fails, we need a renewal theorem for infinite mean variables. Iterating Equation (2.4) reveals the representation

$$M^1(t,0,y) = p_t(0,y) * \sum_{n\geq 0}(m-1)^n p_t(0,0)^{*n}, \tag{2.11}$$

where $*n$ denotes n-fold convolution in t and $p_t(0,y) * p_t(0,0)^{*0} = p_t(0,y)$. Note that convergence of the series is justified by

$$(m-1)^n p_t(0,0)^{*n} \leq \left((m-1)\int_0^t p_r(0,0)\,dr\right)^n$$

and the assumption on m. Lemma 1 of [E73] now implies that

$$\sum_{n\geq 0}(m-1)^n p_t(0,0)^{*n} \approx \frac{t}{(m-1)\int_0^t\int_s^\infty p_r(0,0)\,drds} \tag{2.12}$$

which tends to infinity as $(m-1)\int_s^\infty p_r(0,0)\,dr \to 0$ for $s\to\infty$ since we assumed that $(m-1)p_r(0,0)$ is a probability density in r. To derive from this observation the result for $M^1(t,0,y)$, note that the simple bound $p_t(0,y)\leq 1$ gives the upper bound

$$M^1(t,0,y) \leq \int_0^t \sum_{n\geq 0}(m-1)^n p_r(0,0)^{*n}\,dr. \tag{2.13}$$

For a lower bound, we use that due to irreducibility and continuity of $p_t(0,y)$ in t, there are $0<t_0<t_1$ and $\epsilon>0$ such that $p_t(0,y)>\epsilon$ for $t_0\leq t\leq t_1$. This shows that

$$M^1(t,0,y) \geq \epsilon\int_{t-t_0}^{t-t_1}\sum_{n\geq 0}(m-1)^n p_r(0,0)^{*n}\,dr. \tag{2.14}$$

Combined with (2.12) the lower and upper bounds directly prove the claim for $M^1(t,0,y)$.

It remains to deal with regime iii) b) and c) for $x\neq 0$. The results follow from the asymptotics of the convolutions as those do not vanish at infinity. But this can be deduced from simple upper and lower bounds similar to (2.13) and (2.14).

The asymptotic results for the expected total number of particles $M^1(t,x)$ follow from similar ideas: estimating as before

$$1+\epsilon\int_{t-t_0}^{t-t_1} M^1(r,0)\,dr \leq M^1(t,x) \leq 1+\int_0^t M^1(r,0)\,dr,$$

and applying case 2) of Theorem 1 of [DS10] to (2.1) with $x=0$, the result follows. □

Proof of Theorem 2 for M^1. Since the many-to-one result holds for *any* motion of the catalyst ζ_t, most of the proof of Theorem 2 proceeds just as for Theorem 1 with the location 0 replaced now by the location of the catalyst, ζ_t. We concentrate therefore on the points at which differences occur.

In case i), the lower bound for $M^1(t,x,y)$ when $\mathcal{C}$ is transient and $\mathbb{P}_x(\xi_t=y)\not\to 0$ proceeds via the same argument as the lower bound for $M^1(t,x)$ when $\mathcal{C}$ is transient. The rest of case i) and all of case ii) are just as in Theorem 1.

In case iii), the renewal equations can be solved only for $M^1(t,x)$. For $M^1(t,x,y)$, the probability of the event $\{\xi_t=y\}$ depends strongly on where the particle first meets the catalyst, which removes any possibility of applying renewal theory. For $M^1(t,x)$ this problem does not occur since the difference $\xi_t-\zeta_t$ is homogeneous, and the proof carries through without any problems. □

We now come to the crucial lemma of our paper. We use the the many-to-few lemma to reduce higher moments of N_t and $N_t(y)$ to the first moment. More precisely, a system of equations is derived that can be solved inductively once the first moment is known. This particular useful form is caused by the one-point catalyst. A similar system can be derived in the same manner in the deterministic case if the one-point potential is replaced by a n-point potential. However the case of a random n-point potential is much more delicate as the sources are "attracted" to the particles, destroying any chance of a renewal theory approach.

Lemma 3. *For $k \geq 2$, in the case of the fixed catalyst $\zeta_t = 0$ the kth moments fulfill*

$$M^k(t,x) = M^1(t,x) + M^1(t,x,0) * g_k\big((M^1(t,0), \cdots, M^{k-1}(t,0)\big), \tag{2.15}$$

$$M^k(t,x,y) = M^1(t,x,y) + M^1(t,x,0) * g_k\big(M^1(t,0,y), \cdots, M^{k-1}(t,0,y)\big), \tag{2.16}$$

where

$$g_k\big(M^1, ..., M^{k-1}\big) = \sum_{r=2}^{k} \mathbb{E}\left[\binom{X}{j}\right] \sum_{\substack{i_1,...,i_j>0 \\ i_1+...+i_j=k}} \frac{k!}{i_1! \cdots i_j!} M^{i_1} \cdots M^{i_j}.$$

Equation 2.15 also holds when the catalyst is random (provided that $\xi_t - \zeta_t$ is an inhomogeneous Markov process).

Proof. We shall only prove equation (2.15); the proof of equation (2.16) is almost identical, as is the extension to the case of the random catalyst. We recall the spine setup, and introduce some more notation. To begin with, all k spines are carried by the same particle ξ which branches at rate $m_k = \mathbb{E}[X^k]$ when at 0. Thus the k spines separate into two or more particles at rate $m_k - m$ when at 0 (since it is possible that at a birth event all k spines continue to follow the same particle, which happens at rate m). We consider what happens at this first "separation" time, and call it T.

Let $i_1, \ldots, i_j > 0$, $i_1 + \ldots + i_j = k$, and define $A_k(j; i_1, \ldots, i_j)$ to be the event that at a separation event, i_1 spines follow one particle, i_2 follow another, ..., and i_j follow another. The first particle splits into a new particles with probability $a^k p_a m_k^{-1}$ (see the definition of $\mathbb{Q}^k$). Then given that the first particle splits into a new particles, the probability that i_1 spines follow one particle, i_2 follow another, ..., and i_j follow another is

$$\frac{1}{a^k} \cdot \binom{a}{j} \cdot \frac{k!}{i_1! \cdots i_j!}$$

(the first factor is the probability of each spine making a particular choice from the a available; the second is the number of ways of choosing the j particles to assign the spines to; and the third is the number of ways of rearranging the spines amongst those j particles). Thus the probability of the event $A_k(j; i_1, \ldots, i_j)$ under $\mathbb{Q}^k$ is

$$\frac{1}{m_k} \mathbb{E}\left[\binom{a}{j}\right] \frac{k!}{i_1! \cdots i_j!}.$$

(Note that, as expected, this means that the total rate at which a separation event occurs is

$$m_k \cdot \frac{1}{m_k} \sum_{j=2}^{k} \mathbb{E}\left[\binom{X}{j}\right] \sum_{\substack{i_1,...,i_j>0 \\ i_1+...+i_j=k}} \frac{k!}{i_1! \cdots i_j!} = m_k - m$$

since the double sum is just the expected number of ways of assigning k things to X boxes without assigning them all to the same box.)

However, for $j \geq 2$, *given* that we have a separation event, $A_k(j; i_1, \ldots, i_j)$ occurs with probability

$$\frac{1}{m_k} \mathbb{E}\left[\binom{X}{j}\right] \frac{k!}{i_1! \cdots i_j!} \left(\frac{m_k}{m_k - m}\right).$$

Write χ_t for the position of the particle carrying the k spines for $t \in [0, T)$, and define $\mathcal{F}_t$ to be the filtration containing all information (including about the spines) up to time t. Recall that the skeleton skel(t) is the tree generated by particles containing at least one spine up to time t; let skel($s; t$) similarly be the part of the skeleton falling between times s and t. Using the many-to-few lemma with $f = 1$, the fact that by definition before T all spines sit on the same particle and integrating out T, we obtain

$$\begin{aligned}
\mathbb{E}\big[N_t^k\big] &= \mathbb{Q}^k\left[\prod_{v\in\mathrm{skel}(t)} e^{(m_{B_v}-1)\int_{\sigma_v(t)}^{\tau_v(t)} \mathbb{1}_0(X_v(s))ds}\right] \\
&= \mathbb{Q}^k\left[e^{(m_k-1)\int_0^T \mathbb{1}_0(\chi_s)ds}\mathbb{1}_{\{T\le t\}}\mathbb{Q}^k\left[\prod_{v\in\mathrm{skel}(T;\,t)} e^{(m_{B_v}-1)\int_{\sigma_v(t)}^{\tau_v(t)} \mathbb{1}_0(X_v(s))ds}\middle|\mathcal{F}_T\right]\right] \\
&\qquad + \mathbb{Q}^k\left[e^{(m_k-1)\int_0^t \mathbb{1}_0(\chi_s)ds}\mathbb{1}_{\{T>t\}}\right] \\
&= \int_0^t \mathbb{Q}^k\Bigg[e^{(m_k-1)\int_0^u \mathbb{1}_0(\chi_s)ds}(m_k-m)\mathbb{1}_0(\chi_u)e^{-(m_k-m)\int_0^u \mathbb{1}_0(\chi_s)ds} \\
&\qquad \cdot \mathbb{Q}^k\left[\prod_{v\in\mathrm{skel}(u;\,t)} e^{(m_{B_v}-1)\int_{\sigma_v(t)}^{\tau_v(t)} \mathbb{1}_0(X_v(s))ds}\middle|\mathcal{F}_u; T=u\right]\Bigg]du \\
&\qquad + \mathbb{Q}^k\left[e^{(m_k-1)\int_0^t \mathbb{1}_0(\chi_s)ds}e^{-(m_k-m)\int_0^t \mathbb{1}_0(\chi_s)ds}\right].
\end{aligned}$$

To prove equation (2.16), the same arguments are used with $f = \mathbb{1}_y$ in place of $f = 1$. Now we split the sample space according to the distribution of the numbers of spines in the skeleton at time T. Since, given their positions and marks at time T, the particles in the skeleton behave independently, we may split the product up into j independent factors. Thus

$$\begin{aligned}
\mathbb{E}\big[N_t^k\big] &= \int_0^t \sum_{j=2}^k \sum_{\substack{i_1,\dots,i_j>0\\ i_1+\dots+i_j=k}} \mathbb{E}\left[\binom{X}{j}\right]\frac{k!}{i_1!\cdots i_j!}\mathbb{Q}^k\Bigg[e^{(m-1)\int_0^u \mathbb{1}_0(\chi_s)ds}\mathbb{1}_0(\chi_u) \\
&\qquad \cdot \prod_{l=1}^j \mathbb{Q}^{i_l}\left[\prod_{v\in\mathrm{skel}(t-u)} e^{(m_{B_v}-1)\int_{\sigma_v(t-u)}^{\tau_v(t-u)} \mathbb{1}_0(X_v(s))ds}\right]\Bigg]du \\
&\qquad + \mathbb{Q}^k\left[e^{(m-1)\int_0^t \mathbb{1}_0(\chi_s)ds}\right] \\
&= \int_0^t \sum_{j=2}^k \sum_{\substack{i_1,\dots,i_j>0\\ i_1+\dots+i_j=k}} \mathbb{E}\left[\binom{X}{j}\right]\frac{k!}{i_1!\cdots i_j!}\mathbb{E}_x\left[N_u(0)\right]\cdot\prod_{l=1}^j \mathbb{E}_0\left[N_{t-u}^{i_l}\right]du + \mathbb{E}_x\left[N_t\right],
\end{aligned}$$

where we have used the many-to-few lemma backwards with $f = \mathbb{1}_0$ (first expectation) and $f = 1$ (two last expectations) to obtain the last line. This is exactly the desired equation (2.15). For Equation (2.16) we again use $f = \mathbb{1}_y$ in place of $f = 1$ and copy the same lines of arguments. □

Remark 2. *The factors appearing in g_k are derived combinatorially from splitting the spines. In Lemma 3.1 of* [AB00] *they appeared from Faà di Bruno's differentiation formula.*

We can now finish the proof of our main result by induction.

Proof of Theorem 1 and Theorem 2 for M^k. As the righthand sides of (2.15) and (2.16) do not depend on the left-hand side, the results can now be easily deduced by induction starting with the asymptotic results for $M^1(t,x)$ and $M^1(t,x,y)$ derived above. □

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