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A simple path to asymptotics for the frontier of a branching Brownian motion

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Abstract We give proofs of two results about the position of the extremal particle in a branching Brownian motion, one concerning the median position and another the almost sure behaviour. Our methods are based on a many-to-two lemma which allows us to estimate the effect of the branching structure on the system by considering two dependent Bessel processes.

1 Introduction and main results

Kolmogorov *et al.* [13] proved that the extremal particle in a standard branching Brownian motion sits near $\sqrt{2}t$ at time t. Higher order corrections to this result were given by Bramson [3], and then almost sure fluctuations were proved by Hu and Shi [10]. These two remarkable papers, more than thirty years apart, provide results which reflect an extremely deep understanding of the underlying branching structure. This article grew out of a desire to know whether shorter or simpler proofs of these results exist.

We consider a branching Brownian motion (BBM) beginning with one particle at 0, which moves like a standard Brownian motion until an independent exponentially distributed time with parameter 1. At this time it dies and is replaced (in its current position) by two new particles, which — relative to their birth time and position — behave like independent copies of their parent, moving like Brownian motions and branching at rate 1 into two copies of themselves. Let $N(t)$ be the set of all particles alive at time t, and if $v \in N(t)$ then let $X_v(t)$ be the position of v at time t. If $v \in N(t)$ and $s < t$, then let $X_v(s)$ be the position of the unique ancestor of v that was alive at time s. Define $M_t = \max_{v\in N(t)} X_v(t)$.

1.1 Bramson's result on the distribution of M_t

Define

$$u(t,x) = \mathbb{P}\left(M_t \le x\right).$$

This function u satisfies the Fisher-Kolmogorov-Petrovski-Piscounov (FKPP) equation

$$u_t = \frac{1}{2}u_{xx} + u^2 - u,$$

(with Heaviside initial condition) which has been studied for many years both analytically and probabilistically: see for example Kolmogorov *et al.* [13], Fisher [5], Skorohod [18], McKean [15], Bramson [3, 4], Neveu [16], Uchiyama [19], Aronson and Weinburger [2], Karpelevich *et al.* [11], Harris [8], Kyprianou [14], Harris *et al.* [7]. In particular (see [13]) u converges to a *travelling wave*: that is, there exist functions m of t and w of x such that

$$u(t, m(t)+x) \to w(x)$$

uniformly in x as $t \to \infty$.

We would like to offer a proof of the following result which is much shorter and simpler than the original proof by Bramson [3]:

Theorem 1 (Bramson, 1978). *The centering term $m(t)$ satisfies*

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O(1)$$

as $t \to \infty$.

As Bramson notes in [3], "an immediate frontal assault using moment estimates, but ignoring the branching structure of the process, will fail." That is, let $G(t)$ be the number of particles near $m(t)$ at time t. If some particle has large position at time $s < t$ then many particles are likely to have large position at time t, and this means that the moments of $G(t)$ are misleading. For this reason, instead of estimating $G(t)$ directly, we estimate $H(t)$, the number of particles near $m(t)$ that have remained below $m(t)s/t$ for all times $s < t$. It is not difficult to guess that particles behaving in this way look like Bessel-3 processes below the line $m(t)s/t$, $s \in [0,t]$. Essentially our proof simply takes advantage of this observation[1].

For the lower bound Bramson develops and applies very accurate estimates for Brownian bridges to calculate the second moment of the number of particles in some set (which is something like a more complicated version of $H(t)$). We instead use a change of measure which allows us to apply basic estimates on Bessel processes. This is only possible thanks to a general many-to-two lemma developed in [9].

For the upper bound (Proposition 9) we apply the first moment method to the same quantity $H(t)$. However we must then estimate $G(t) - H(t)$. Here we borrow the outline of an idea from Bramson, which we use to give a straightforward estimate of the probability that $G(t) - H(t)$ is non-zero.

1.2 Hu and Shi's result on the paths of M_t

Having established Bramson's result on the centering term $m(t)$, we move on to the almost sure behaviour of M_t. We prove the following result, which is the analogue of a result for quite general branching random walks given by Hu and Shi [10].

Theorem 2. *The maximum M_t satisfies*

$$\frac{M_t - \sqrt{2}t}{\log t} \longrightarrow -\frac{3}{2\sqrt{2}} \quad \textit{in probability}$$

and

$$\liminf_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{3}{2\sqrt{2}} \quad \textit{almost surely.} \tag{1}$$

However,

$$\limsup_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{1}{2\sqrt{2}} \quad \textit{almost surely.} \tag{2}$$

This result says that although the extremal particle looks like $m(t)$ for most times t, occasionally a particle will travel much further. Technically the theorem as stated here is a new result as Hu and Shi considered only discrete-time branching random walks, but it would not take too much effort to derive it from their work. We proceed instead by applying the estimates developed in the proof of Theorem 1 along with the Borel-Cantelli lemma and exponential tightness of Brownian motion. Only the lower bound in (2) requires a significant amount of extra work, and for that we take an approach similar to that of Hu and Shi in [10]. They noticed that although the probability that a particle has position much bigger than $m(t)$ at a fixed time t is very small, the probability that there exists a time s between (say) n and $2n$ such that a particle has position much bigger than $m(s)$ at time s is actually quite large.

[1] The reader may deduce from this description that we shall, as part of our proof, calculate estimates for branching Brownian motion with absorption. This model was originally studied by Kesten [12] with the initial intention of investigating the maximal displacement in a BBM.

1.3 Extensions to other models

We note that although we consider only the simplest possible BBM, with binary branching at fixed rate 1, our methods can be applied to rather more general models. There is however one important necessary condition, that *the mean and variance of the number of particles born at a branching event must be finite.* This is simply due to the fact that we apply a second moment method.

Addario-Berry and Reed [1] (in their Theorem 3) proved an analogue of Bramson's result (our Theorem 1) for a wide class of branching random walks. It is possible that our methods could be adapted to extend their result to the case where the birth distribution is not almost surely bounded. Since the purpose of this paper is to provide short and simple proofs to two sophisticated results, and the generality of branching random walks introduces various technical complications, we do not carry out this work here.

2 Bessel-3 processes

We recall some very basic properties of Bessel-3 processes, and then do much of the dirty work of Theorem 1 and Proposition 13 (which is the most difficult part of Theorem 2) by calculating the expectation of two functionals of two dependent Bessel-3 processes. These calculations, in Lemmas 3 and 4, are not motivated until later in the article, but we include them here as they are simply facts about Bessel processes and do not contribute a great deal to the main ideas of the proofs.

If W_t, $t \geq 0$ is a Brownian motion in $\mathbb{R}^3$ then its modulus $|W_t|$, $t \geq 0$ is called a Bessel-3 process (or simply a Bessel process). Suppose that B_t is a Brownian motion in $\mathbb{R}$ started from $B_0 = x$ under a probability measure P_x; then $X_t := x^{-1} B_t \mathbb{1}_{\{B_s > 0 \ \forall s \leq t\}}$ is a non-negative unit-mean martingale under P_x. We may change measure by X_t, defining a new probability measure $\hat{P}_x$ via

$$\left.\frac{d\hat{P}_x}{dP_x}\right|_{\mathcal{F}_t} := X_t$$

(where $\mathcal{F}_t$ is the natural filtration of the Brownian motion B_t) and then B_t, $t \geq 0$ is a Bessel process under $\hat{P}_x$. The density of a Bessel process satisfies

$$\hat{P}_x(B_t \in dz) = \frac{z}{x\sqrt{2\pi t}} \left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t} \right) dz.$$

This and much more about Bessel processes can be found in many textbooks, for example Revuz and Yor [17].

We now claim that

$$e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t} \leq \frac{2xz}{t} \quad \forall\, x, z \geq 0,\ t > 0. \tag{3}$$

Indeed the derivative of the left-hand side with respect to z is

$$\frac{x}{t}\left(e^{-(z-x)^2/2t} + e^{-(z+x)^2/2t} \right) + \frac{z}{t}\left(e^{-(z+x)^2/2t} - e^{-(z-x)^2/2t} \right);$$

the first term above is no greater than $2x/t$, while the second is negative whenever $x, z \geq 0$.

We also choose and fix $\gamma \in (0, \sqrt{2})$ such that

$$e^{\delta} - e^{-\delta} = 2\sinh\delta \geq \sqrt{2}\delta \quad \text{for all} \quad \delta \in [0, 2\gamma];$$

then

$$\frac{z}{x\sqrt{2\pi t}}\left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\right) \geq e^{-z^2/2t - x^2/2t}\frac{z^2}{\sqrt{\pi t^3}} \tag{4}$$

whenever $zx \leq \gamma t$.

Now suppose that under $\hat{P}$ we have a time $\tau \in [0, \infty)$ and two Bessel processes Y_t^1 and Y_t^2, $t \geq 0$ such that

- $Y_0^1 = Y_0^2 = 1$;
- $Y_t^1 = Y_t^2$ for all $t \leq \tau$;
- $(Y_t^1 - Y_\tau^1,\ t > \tau)$ and $(Y_t^2 - Y_\tau^2,\ t > \tau)$ are independent given τ and Y_τ^1.

The following lemma does most of the hard work in proving the lower bound for Theorem 1.

Lemma 3. *Let*

$$\beta = \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log t}{t} + \frac{y}{t},$$

$$A_1 = \{1 \leq Y_t^1 \leq 2\} \quad \textit{and} \quad A_2 = \{1 \leq Y_t^2 \leq 2\}.$$

For all large t and $y \in [0, \sqrt{\gamma t}]$,

$$\int_0^t \hat{P}\left[Y_s^1 e^{-\frac{3\log t}{2t}s - \beta Y_s^1}\mathbb{1}_{A_1\cap A_2}\middle|\tau = s\right]ds \leq ct^{-3}$$

for some constant c not depending on t or y.

Proof. The idea here is that the probability that a Bessel process is near the origin at time t is approximately $t^{-3/2}$. If s is small, then we have two (almost) independent Bessel processes which must both be near the origin at time t, giving t^{-3}. If s is large, then we effectively have only one Bessel process, giving $t^{-3/2}$, but the $\exp(\frac{3\log t}{2t}s)$ gives us an extra $t^{-3/2}$. It then remains to check when s is neither large or small, but the above effects combine in the right way such that things turn out nicely then too. We apply in each case the basic estimate (3) on the Bessel density.

We first check the small s case: for large t,

$$\begin{aligned}\int_0^1 \hat{P}\left[Y_s^1 e^{-\frac{3\log t}{2t}s - \beta Y_s^1}\mathbb{1}_{A_1\cap A_2}\middle|\tau = s\right]ds &\leq \int_0^1 \hat{P}(A_1\cap A_2|\tau = s)ds\\ &\leq c_1\hat{P}(A_1)^2 \leq c_2 t^{-3}.\end{aligned}$$

Similarly for the large s case:

$$\int_{t-1}^t \hat{P}\left[Y_s^1 e^{-\frac{3\log t}{2t}s - \beta Y_s^1}\mathbb{1}_{A_1\cap A_2}\middle|\tau = s\right]ds \leq c_3 t^{-3/2}\hat{P}(A_1) \leq c_4 t^{-3}.$$

Finally the main case, when $s \in [1, t-1]$:

$$\begin{aligned}
&\int_1^{t-1} \hat{P}\left[Y_s^1 e^{-\frac{3\log t}{2t}s - \beta Y_s^1} \mathbb{1}_{A_1\cap A_2} \middle| \tau = s \right] ds \\
&\leq \int_1^{t-1} \int_0^\infty \frac{z^3}{s^{3/2}} e^{-\beta z - \frac{3\log t}{2t}s} \left(\int_1^2 \frac{2x^2}{\sqrt{2\pi(t-s)^3}} dx \right)^2 dz \ \ ds \\
&\leq c_5 \int_1^{t-1} \frac{e^{-\frac{3\log t}{2t}s}}{s^{3/2}(t-s)^3} \int_0^\infty z^3 e^{-z} dz \ \ ds \\
&\leq c_6 t^{-7/2} \int_{1/t}^{1-1/t} \frac{e^{-(\frac{3}{2}\log t)u}}{u^{3/2}(1-u)^3} du
\end{aligned}$$

and it is a simple task to check that the last integral above is bounded by $\sqrt{t}$ times a constant:

$$\begin{aligned}
&\int_{1/t}^{1-1/t} \frac{e^{-(\frac{3}{2}\log t)u}}{u^{3/2}(1-u)^3} du \\
&\leq c_7 \int_{1/t}^{1/2} u^{-3/2} du + c_8 t^{-3/2} \int_{1/t}^{1/2} s^{-3} e^{(\frac{3}{2}\log t)s} ds \\
&\leq c_9 t^{1/2} + c_{13} t^{-3/2} \int_{t^{-1}}^{t^{-1/6}} s^{-3} e^{(\frac{3}{2}\log t)s} ds + c_8 t^{-3/2} \int_{t^{-1/6}}^{1/2} s^{-3} e^{(\frac{3}{2}\log t)s} ds \\
&\leq c_9 t^{1/2} + c_{10} t^{-3/2} \int_{t^{-1}}^{t^{-1/6}} s^{-3} ds + c_8 t^{-1/4} \int_{t^{-1/6}}^{1/2} e^{(\frac{3}{2}\log t)s} ds \\
&\leq c_{11} t^{1/2}
\end{aligned}$$

as required. □

Our next lemma is very similar; it estimates a slightly different functional, which will appear in Proposition 13.

Lemma 4. *Let* $a_{s,t} = \frac{1}{2\sqrt{2}}\log s - \frac{1}{2\sqrt{2}}\frac{\log t}{t}s$*. If* $e \leq s \leq t \leq 2s$*, then*

$$\begin{aligned}
\int_0^s e^{-\frac{1}{2}\frac{\log t}{t}r} \hat{P}\left[\mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1\leq Y_t^1 \leq 2\}} Y_r^1 e^{-\beta_t Y_r^1} \middle| \tau = r \right] dr& \\
\leq c e^{-\frac{1}{2}\frac{\log t}{t}s} \left(\frac{1}{t^{5/2}} + \frac{1}{t^{3/2}(t-s+1)^{3/2}} \right)&
\end{aligned}$$

for some constant c *not depending on* s *or* t*.*

Proof. We approximate just as we did for Lemma 3. Essentially the $e^{-\beta_t Y_r^1}$ term means our initial Bessel process must be near the origin at time r; then two independent Bessel processes started from time r must be near the origin at times s and t respectively. This will give us a contribution of $r^{-3/2}(s-r)^{-3/2}(t-r)^{-3/2}$. Indeed for any $r \in [1, s-1]$, integrating out over Y_r^1,

$$\begin{aligned}
&\hat{P}\left[\mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1\leq Y_t^1 \leq 2\}} Y_r^1 e^{-\beta_t Y_r^1} \middle| \tau = r \right] \\
&\leq c_1 \int_0^\infty z e^{-\beta_t z} \frac{z^2}{r^{3/2}} \cdot \frac{1}{(s-r)^{3/2}} \cdot \frac{1}{(t-r)^{3/2}} \, dz \\
&\leq c_2 r^{-3/2}(s-r)^{-3/2}(t-r)^{-3/2}.
\end{aligned}$$

For $r \leq 1$ we are effectively asking two independent Bessel processes to be near the origin at times s and t, giving $s^{-3/2}t^{-3/2}$, and for $r \geq s-1$ we have just one Bessel process which must be near the origin at times s and t, giving $s^{-3/2}(t-s+1)^{-3/2}$. Thus (noting that $\log s \geq \frac{\log t}{t}s$ provided $s, t \geq e$)

$$\int_0^s e^{-\frac{1}{2}\frac{\log t}{t}r}\hat{P}\left[\mathbb{1}_{\{a_{s,t}+1\leq Y_s^1\leq a_{s,t}+2\}}\mathbb{1}_{\{1\leq Y_t^1\leq 2\}}Y_r^1 e^{-\beta_t Y_r^1}\,\middle|\,\tau = r\right]dr$$
$$\leq \frac{c_3}{s^{3/2}t^{3/2}} + c_4\int_1^{s-1}\frac{e^{-\frac{1}{2}\frac{\log t}{t}r}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}}dr + \frac{c_5 e^{-\frac{1}{2}\frac{\log t}{t}s}}{s^{3/2}(t-s+1)^{3/2}}.$$

Since s and t are of the same order and $e^{-\frac{1}{2}\frac{\log t}{t}s} \geq s^{-1/2}$ it remains to estimate the integral in the last line above — and we proceed again just as in Lemma 3, breaking the integral into three parts. For large r,

$$\int_1^{s/2}\frac{e^{-\frac{1}{2}\frac{\log t}{t}r}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}}dr \leq \frac{c_6}{s^{3/2}t^{3/2}},$$

for small r,

$$\int_{s-s/t^{1/4}}^{s-1}\frac{e^{-\frac{1}{2}\frac{\log t}{t}r}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}}dr \leq c_7\frac{e^{-\frac{1}{2}\frac{\log t}{t}s}}{s^{3/2}(t-s+1)^{3/2}},$$

and for intermediate r

$$\int_{s/2}^{s-s/t^{1/4}}\frac{e^{-\frac{1}{2}\frac{\log t}{t}r}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}}dr \leq c_8\frac{t^{3/4}}{s^{9/2}}\int_{s/2}^{s-s/t^{1/4}}e^{-\frac{1}{2}\frac{\log t}{t}r}dr$$
$$\leq c_9\frac{t^{7/4}}{s^{9/2}}e^{-\frac{1}{4}\frac{\log t}{t}s} \leq \frac{c_{10}}{t^{5/2}}e^{-\frac{1}{2}\frac{\log t}{t}s}$$

which completes the proof. □

3 The many-to-one and many-to-two lemmas

We mentioned in the introduction that we will attempt to count the number of particles remaining below a certain line and ending near $m(t)$, and that particles following such paths must look like Bessel processes below a line. In this section we qualify that heuristic.

3.1 The many-to-one lemma

It is well-known that the first moment of the number of particles in (a subset of) a branching process can be estimated via a many-to-one lemma. For our branching Brownian motion the number of particles at time t is approximately e^t, and to first order they behave independently so that the expected number satisfying a certain property is simply e^t times the probability that one particle (i.e. one Brownian motion) satisfies that property. More general formulations of this idea have been given over the years, notably by Hardy and Harris [6]. For our particular needs the following form will be most useful.

Let $g_t(\cdot)$ be a measurable functional of (t and) the path of a particle up to time t; so for example we might take

$$g_t(v) = t^2 e^{\int_0^t X_v(s)ds}.$$

Then

$$\mathbb{E}\left[\sum_{v\in N(t)} g_t(v)\right] = e^t E[g_t(\xi)]$$

where ξ_t, $t \geq 0$ is just a standard Brownian motion under P. Now fixing $\alpha > 0$ and $\beta \in \mathbb{R}$ and defining

$$\zeta(t) = \frac{1}{\alpha}(\alpha + \beta t - \xi_t)e^{\beta\xi_t - \beta^2 t/2}\mathbb{1}_{\{\xi_s < \alpha+\beta s \ \forall s \leq t\}},$$

the following lemma is a result of Girsanov's theorem and the knowledge of Bessel processes seen at the start of Section 2.

Lemma 5 (Many-to-one lemma).

$$\mathbb{E}\left[\sum_{v\in N(t)} g_t(v)\right] = e^t E[g_t(\xi)] = e^t \mathbb{Q}\left[\frac{1}{\zeta(t)} g_t(\xi)\right]$$

where under $\mathbb{Q}$, $\alpha + \beta t - \xi_t$, $t \geq 0$ *is a Bessel process.*

3.2 The many-to-two lemma

We can use the many-to-one lemma to calculate expectations of numbers of particles with certain properties. However, as outlined in the introduction, we would like to apply second moment methods. Thus we will need a many-to-two lemma, which — just as the many-to-one lemma reduces calculating first moments to the expectation of functionals of just one particle — will reduce calculating second moments to the expectation of functionals of two, necessarily dependent, particles. This is a natural idea and has been around to some extent for many years; indeed Bramson uses a very basic many-to-two lemma in [3]. However, just as we used a non-trivial measure change in developing our many-to-one lemma above, we would like a more refined many-to-two lemma involving Bessel processes. We will not give a proof here — as Bramson says, "a rigorous verification of [even the most basic version] is quite messy" — and refer to [9] which gives a quite general formulation, of which our lemma is a special case. The idea is that calculating second moments is akin to choosing two "typical" particles at random from a set; these particles followed the same path up to the death of their most recent ancestor, and then evolved independently. The many-to-two lemma reflects this heuristic.

Suppose that under $\mathbb{Q}$, as well as the process ξ_t seen in Section 3.1, we have two processes ξ^1_t and ξ^2_t, $t \geq 0$ and a time $T \in [0, \infty)$ such that

- $\alpha + \beta t - \xi^1_t$ and $\alpha + \beta t - \xi^2_t$ are Bessel processes started from α;
- $\xi^1_t = \xi^2_t$ for all $t \leq T$;
- $(\xi^1_t - \xi^1_T, t > T)$ and $(\xi^2_t - \xi^2_T, t > T)$ are independent given T and ξ^1_T.

Define

$$\zeta^i(t) = \frac{1}{\alpha}(\alpha + \beta t - \xi_t^i)e^{\beta\xi_t^i - \beta^2 t/2}\mathbb{1}_{\{\xi_s^i < \alpha + \beta s \ \forall s \leq t\}}$$

for $i = 1, 2$ and $t \geq 0$.

Lemma 6 (Many-to-two lemma)**.** *Let $g_t(\cdot)$ and $h_t(\cdot)$ be two measurable functionals of t and the path of a particle up to time t, as in Section 3.1. Then*

$$\mathbb{E}\left[\sum_{u,v\in N(t)} g_t(u)h_t(v)\right]$$
$$= e^t\mathbb{Q}\left[\frac{1}{\zeta^1(t)}g_t(\xi^1)h_t(\xi^1)\right] + \int_0^t 2e^{2t-s}\mathbb{Q}\left[\frac{\zeta^1(s)}{\zeta^1(t)\zeta^2(t)}g_t(\xi^1)h_t(\xi^2)\middle| T = s\right] ds.$$

As mentioned above, the dependence between the two Bessel processes reflects the dependence structure of the BBM: any pair of particles (u, v) in the BBM are entirely dependent until their most recent common ancestor, and completely independent thereafter. The first term on the right-hand side of the many-to-two lemma takes account of the possibility that the Bessel processes have not yet split (which corresponds to the event that u and v are in fact the same particle) and otherwise the second term integrates out the split time T of the two Bessel processes (which corresponds to integrating out the last time at which the most recent common ancestor of u and v was alive).

4 Proof of Theorem 1

For $t > 0$ set (as in Section 2)

$$\beta = \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log t}{t} + \frac{y}{t}.$$

Now define

$$H_\alpha(t) = \#\left\{u \in N(t) : X_u(s) \leq \alpha + \beta s \ \ \forall s \leq t, \ \ \beta t - 1 \leq X_u(t) \leq \beta t\right\}.$$

As outlined in the introduction, we shall show that the first two moments of $H_\alpha(t)$ give an accurate picture of the probability that there is a particle near βt at time t. We begin by calculating the first moment.

For $i = 1, 2$, $t > 0$ and $s \geq 0$, let

$$B_i = \left\{\beta t - 1 \leq \xi_t^i \leq \beta t\right\}$$

and recall that we defined

$$\zeta^i(s) = \frac{1}{\alpha}(\beta s + \alpha - \xi_s^i)\exp\left(\beta\xi_s^i - \frac{1}{2}\beta^2 s\right)\mathbb{1}_{\{\xi_r^i \leq \beta r + \alpha \ \forall r \leq s\}}.$$

We write $f(y, t) \sim g(y, t)$ if $cf \leq g \leq c'f$ for some strictly positive constants c and c' not depending on any of the parameters t, y, α.

Lemma 7. *For any $\alpha \geq 1$,*

$$\mathbb{E}[H_\alpha(t)] \sim \alpha^2 e^{-\sqrt{2}y}$$

for all $t \geq 1$, $y \in \mathbb{R}$ and $\alpha \in [1, \sqrt{\gamma t}]$.

Proof. For large t,

$$\begin{aligned}\mathbb{E}[H_\alpha(t)] = e^t\mathbb{Q}\left[\frac{1}{\zeta^1(t)}\mathbb{1}_{B_1}\right] &= e^t\mathbb{Q}\left[\frac{\alpha}{\beta t+\alpha-\xi_t}e^{-\beta\xi_t^1+\beta^2 t/2}\mathbb{1}_{B_1}\right]\\ &\sim e^{t-\beta^2 t/2}\mathbb{Q}(B_1)\\ &\sim t^{3/2}e^{-\sqrt{2}y}\mathbb{Q}(\alpha\le\beta t+\alpha-\xi_t\le\alpha+1).\end{aligned}$$

Now, $\beta t+\alpha-\xi_t$ is a Bessel process started from α under $\mathbb{Q}$, so by (3) and (4)

$$\mathbb{Q}(\alpha\le\beta t+\alpha-\xi_t\le\alpha+1)\sim\int_\alpha^{\alpha+1}\frac{z^2}{t^{3/2}}dz\sim\alpha^2 t^{-3/2}$$

which gives the result. □

We now prove a lower bound for $m(t)$ by calculating the second moment of $H_1(t)$.

Proposition 8. *There exist t_0 and a constant $c\in(0,\infty)$ not depending on t or y such that*

$$\mathbb{P}(\exists u\in N(t): X_u(t)\ge\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t+y)\ge ce^{-\sqrt{2}y}$$

for all $y\in[0,\sqrt{\gamma t}]$ and $t\ge t_0$.

Proof. We saw in Lemma 7 that $\mathbb{E}[H_1(t)]\ge c'e^{-\sqrt{2}y}$; we shall now estimate the second moment of $H_1(t)$.

$$\begin{aligned}&\mathbb{E}[H_1(t)^2]\\ &= e^t\mathbb{Q}\left[\mathbb{1}_{B_1}\frac{1}{\zeta^1(t)}\right]+\int_0^t 2e^{2t-s}\mathbb{Q}\left[\mathbb{1}_{B_1\cap B_2}\frac{\zeta^1(s)}{\zeta^1(t)\zeta^2(t)}\middle| T=s\right]ds\\ &= \mathbb{E}[H_1(t)]\\ &\quad+2e^{2t}\int_0^t e^{-s}\mathbb{Q}\left[\frac{(\beta s+1-\xi_s)e^{\beta\xi_s-\beta^2 s/2}}{(\beta t+1-\xi_t^1)(\beta t+1-\xi_t^2)e^{\beta\xi_t^1+\beta\xi_t^2-\beta^2 t}}\mathbb{1}_{B_1\cap B_2}\middle| T=s\right]ds\\ &\le \mathbb{E}[H_1(t)]\\ &\quad+2e^{2t-\beta^2 t+2\beta}\int_0^t e^{-s}\mathbb{Q}\left[(\beta s+1-\xi_s^1)e^{\beta\xi_s^1-\beta^2 s/2}\mathbb{1}_{B_1\cap B_2}\middle| T=s\right]ds\\ &\le \mathbb{E}[H_1(t)]\\ &\quad+c_0t^3e^{-\sqrt{2}y}\int_0^t\mathbb{Q}\left[(\beta s+1-\xi_s^1)e^{-\frac{3}{2}\frac{\log t}{t}s-\beta(\beta s+1-\xi_s^1)}\mathbb{1}_{B_1\cap B_2}\middle| T=s\right]ds.\end{aligned}$$

Under $\mathbb{Q}$, $(\beta s+1-\xi_s^1, s\ge 0)$ and $(\beta s+1-\xi_s^2, s\ge 0)$ are Bessel processes starting from 1 that are equal up to T and independent (given T and ξ_T^1) after T. Thus, taking notation from Lemma 3 we have

$$\mathbb{E}[H_1(t)^2]\le\mathbb{E}[H_1(t)]+c_0t^3e^{-\sqrt{2}y}\int_0^t\hat{P}\left[Y_s^1e^{-\frac{3\log t}{2t}s-\beta_tY_s^1}\mathbb{1}_{A_1\cap A_2}\middle|\tau=s\right]ds.$$

But Lemma 3 tells us that the integral is at most a constant times t^{-3}, so for all large t and $y\in[0,\sqrt{\gamma t}]$

$$\mathbb{E}[H_1(t)^2]\le c\,\mathbb{E}[H_1(t)]$$

for some constant c not depending on y or t. We deduce that

$$\mathbb{P}(H_1(t) \neq 0) \geq \frac{\mathbb{E}[H_1(t)]^2}{\mathbb{E}[H_1(t)^2]} \geq c' e^{-\sqrt{2}y}$$

as required. □

For the upper bound on $m(t)$, we combine the first moment method for $H_\alpha(t)$ with an estimate of the probability that a particle ever moves too far from the origin.

Proposition 9. *There exist t_0 and and a constant $A \in (0,\infty)$ not depending on t or y such that*

$$\mathbb{P}\left(\exists u \in N(t) : X_u(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) \leq A(y+2)^4 e^{-\sqrt{2}y}$$

for all $y \in [0,\sqrt{t}]$ and $t \geq t_0$.

Proof. Recall from Lemma 7 that

$$\mathbb{P}(H_\alpha(t) \neq 0) \leq \mathbb{E}[H_\alpha(t)] \sim \alpha^2 e^{-\sqrt{2}y}.$$

Thus it remains to estimate how large we must choose α so that with high probability no particles ever go above $\beta s + \alpha$ for $s \in [0,t]$. To this end define

$$B = \{\exists u \in N(t), s \leq t : X_u(s) > \beta s + \alpha\}$$

and let

$$\Gamma = \#\{u \in N(t) : X_u(s) < \alpha + \beta s + 1 \;\; \forall s \leq t, \beta t - 1 \leq X_u(t) \leq \beta t + \alpha\}.$$

By similar calculations to those in Lemma 7 we easily see that

$$\mathbb{E}[\Gamma] \leq c(\alpha+1)^4 e^{-\sqrt{2}y}$$

for some constant c not depending on t, α or y. We claim that for $\alpha \geq y \geq 0$,

$$\mathbb{E}[\Gamma|B] \geq c'$$

for some constant $c' > 0$ also not depending on t, α or y; essentially if a particle has already reached $y + \beta s$ then it has done the hard work, and the usual cost $e^{-\sqrt{2}y}$ of reaching βt disappears. To see this, set

$$\tau = \inf\{s > 0 : \exists u \in N(s) \text{ with } X_u(s) > \alpha + \beta s\};$$

then

$$\mathbb{E}[\Gamma|B] = \frac{1}{\mathbb{P}(B)} \int_0^t \mathbb{E}[\Gamma|\tau = s]\mathbb{P}(\tau \in ds)$$

so to establish our claim it suffices to show that $\mathbb{E}[\Gamma|\tau = s]$ is larger than a constant not depending on s, t, α or y. On the event $\tau = s$, let v be the particle at position $\alpha + \beta s$ at time s. Let $\beta' = \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log t}{t}$, and let $N_v(r)$ be the set of descendants of particle v at time r, for $r \geq s$. Then, provided that $\alpha \geq y \geq 0$, on the event $\tau = s$

$$\Gamma \geq \#\{u \in N_v(t) : X_u(r) - X_u(s) \leq \beta'(r-s) + 1 \;\; \forall r \in [s,t], \\ \beta'(t-s) - 1 \leq X_u(t) - X_u(s) \leq \beta'(t-s)\}.$$

Thus by Lemma 7, if $s \le t-1$ then (applying the strong Markov property)

$$\mathbb{E}[\Gamma|\tau = s] \ge c'.$$

If $s > t-1$ then $\mathbb{E}[\Gamma|\tau = s]$ is at least the probability that a single Brownian motion $B_r, r \ge 0$ remains within $[-1,1]$ for all $r \in [0,1]$, and satisfies $B_1 \in [-1,0]$. This establishes our claim, so for $\alpha \ge y \ge 0$

$$\mathbb{E}[\Gamma|B] \ge c' \quad \text{and} \quad \mathbb{E}[\Gamma] \le c(\alpha+1)^4 e^{-\sqrt{2}y}.$$

But then for $\alpha \ge y \ge 0$,

$$\mathbb{P}(B) \le \frac{\mathbb{E}[\Gamma]\mathbb{P}(B)}{\mathbb{E}[\Gamma \mathbb{1}_B]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma|B]} \le \frac{c}{c'}(\alpha+1)^4 e^{-\sqrt{2}y}.$$

Choosing $\alpha = y+1$, we have

$$\begin{aligned}\mathbb{P}\left(\exists u \in N(t) : X_u(t) \ge \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + y\right) &\le \mathbb{E}[H_{y+1}(t)] + \mathbb{P}(B)\\ &\le A(y+2)^4 e^{-\sqrt{2}y}\end{aligned}$$

as required. □

Proof of Theorem 1. As mentioned in the introduction, Kolmogorov *et al.* [13] showed that there exist functions $m(t)$ and $w(x)$ such that $u(t, m(t)+x) \to w(x)$ as $t \to \infty$. Clearly u is increasing in x. But we have shown that

$$ce^{-\sqrt{2}y} \le 1 - u(t, \sqrt{2}t - \tfrac{3}{2\sqrt{2}}\log t + y) \le A(y+2)^4 e^{-\sqrt{2}y}.$$

We deduce that $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O(1)$. □

5 Proof of Theorem 2

We proceed via a series of four results, each proving one of the upper or lower bounds in one of the statements (1) or (2).

Lemma 10. *The upper bound in (1) holds:*

$$\liminf_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} \le -\frac{3}{2\sqrt{2}} \quad \textit{almost surely.}$$

Proof. To rephrase the statement of the lemma, we show that for any $\varepsilon > 0$, there are arbitrarily large times such that there are no particles above $\sqrt{2}t - (3/2\sqrt{2} - \varepsilon)\log t$. Choose $R > 2/\varepsilon$, let $t_1 = 1$ and for $n > 1$ let $t_n = e^{Rt_{n-1}}$. Define

$$E_n = \{\exists u \in N(t_n) : X_u(t_n) > \sqrt{2}t_n - (\tfrac{3}{2\sqrt{2}} - \varepsilon)\log t_n\}$$

and

$$F_n = \{|N(t_n)| \le e^{2t_n},\ \ |X_u(t_n)| \le \sqrt{2}t_k \ \ \forall u \in N(t_k)\}.$$

We know that F_n happens for all large n, so it suffices to show that

$$\mathbb{P}\Big(\bigcap_{k\ge n}(E_k \cap F_k)\Big) \to 0 \quad \text{as } n \to \infty.$$

Now,

$$\mathbb{P}\Big(\bigcap_{k\geq n}(E_k\cap F_k)\Big) = \lim_{N\to\infty}\prod_{k=n}^{N}\mathbb{P}\Big(E_k\cap F_k\Big|\bigcap_{j=n}^{k-1}(E_j\cap F_j)\Big)$$

so we would like to show that the terms on the right-hand side are small. For a particle u, let E_n^u be the event that some descendant of u at time t_n has position larger than $\sqrt{2}t_n - \frac{3}{2\sqrt{2}} - \varepsilon)\log t_n$. Also let $s_n = t_n - t_{n-1}$. Then

$$\begin{aligned}
&\mathbb{P}\Big(E_k\cap F_k\Big|\bigcap_{j=n}^{k-1}(E_j\cap F_j)\Big)\\
&\leq \mathbb{P}\Big(E_k\Big|\bigcap_{j=n}^{k-1}(E_j\cap F_j)\Big)\\
&\leq \mathbb{P}\Big(\bigcup_{u\in N(t_{k-1})}E_k^u\Big|\bigcap_{j=n}^{k-1}(E_j\cap F_j)\Big)\\
&\leq e^{2t_{k-1}}\mathbb{P}\big(\exists u\in N(s_k): X_u(s_k) > \sqrt{2}s_k - \tfrac{3}{2\sqrt{2}}\log s_k + \tfrac{3}{2\sqrt{2}}\log(\tfrac{t_k - t_{k-1}}{t_k}) + \varepsilon\log t_k\big)\\
&\leq A(\log t_k + 2)^4 t_k^{2/R}\left(1-\frac{t_{k-1}}{t_k}\right)^{-3/2} t_k^{-\varepsilon}
\end{aligned}$$

where the last inequality used Proposition 9. Since we chose $R > 2/\varepsilon$, this is much smaller than 1 when k is large, as required. □

Lemma 11. *The upper bound in (2) holds:*

$$\limsup_{t\to\infty}\frac{M_t - \sqrt{2}t}{\log t} \leq -\frac{1}{2\sqrt{2}} \quad \textit{almost surely.}$$

Proof. We show that for large t and any $\varepsilon > 0$, there are no particles above $\sqrt{2}t - (1/2\sqrt{2} - 2\varepsilon)\log t$. By Proposition 9,

$$\begin{aligned}
&\mathbb{P}(\exists u\in N(t): X_u(t) > \sqrt{2}t - (\tfrac{1}{2\sqrt{2}} - \varepsilon)\log t)\\
&\leq A(\log t + 2)^4 e^{-\sqrt{2}(\frac{1}{\sqrt{2}}\log t + \varepsilon t)}\\
&\leq A(\log t + 2)^4 t^{-1-\varepsilon\sqrt{2}}.
\end{aligned}$$

Thus for any lattice times $t_n\to\infty$, by Borel-Cantelli

$$\mathbb{P}(\exists u\in N(t_n): X_u(t_n) > \sqrt{2}t_n - (\tfrac{1}{2\sqrt{2}} - \varepsilon)\log t_n \text{ for infinitely many } n) = 0.$$

It is now a simple exercise using the exponential tightness of Brownian motion and the fact that we may choose the times t_n arbitrarily close together to make sure that no particle can go above $\sqrt{2}t - (\frac{1}{2\sqrt{2}} - 2\varepsilon)\log t$ for any time t. □

Lemma 12. *The lower bound in (1) holds:*

$$\liminf_{t\to\infty}\frac{M_t - \sqrt{2}t}{\log t} \geq -\frac{3}{2\sqrt{2}} \quad \textit{almost surely.}$$

Proof. We show that for large t and any $\varepsilon > 0$, there are always particles below $\sqrt{2}t - (\frac{3}{2\sqrt{2}} + 2\varepsilon)\log t$. Let

$$A_t = \{\nexists u \in N(t) : X_u(t) > \sqrt{2}t - (\tfrac{3}{2\sqrt{2}} + \varepsilon)\log t\}$$

and

$$B_t = \{|N(\log t)| \geq t^{1/2}, X_v(\log t) \geq -\sqrt{2}\log t \forall v \in N(\log t)\}.$$

Define $N(v;t)$ to be the set of descendants of particle v that are alive at time t. Let $l_t = t - \log t$. Then for all large t,

$$\begin{aligned}
&\mathbb{P}(A_t \cap B_t) \\
&\leq \mathbb{E}\left[\prod_{v\in N(\log t)} \mathbb{P}(\nexists u \in N(v;t) : X_u(t) > \sqrt{2}t - (\tfrac{3}{2\sqrt{2}} + \varepsilon)\log t | \mathcal{F}_{\log t})\mathbb{1}_{B_t}\right] \\
&\leq \mathbb{E}\left[\prod_{u\in N(\log t)} \mathbb{P}(\nexists u \in N(l_t) : X_u(l_t) > \sqrt{2}l_t - \tfrac{3}{2\sqrt{2}}\log l_t + \tfrac{3}{2\sqrt{2}}\log \tfrac{l_t}{t} + \varepsilon \log t)\mathbb{1}_{B_t}\right] \\
&\leq c^{\sqrt{t}}.
\end{aligned}$$

Thus by Borel-Cantelli, for any lattice times $t_n \to \infty$,

$$\mathbb{P}(A_{t_n} \cap B_{t_n} \text{ infinitely often}) = 0.$$

Since we know that almost surely for all large t, $|N(\log t)| \geq e^{\frac{1}{2}\log t} = t^{1/2}$ and $X_v(\log t) \geq -\sqrt{2}\log t$ for all $v \in N(\log t)$, we deduce that

$$\mathbb{P}(A_{t_n} \text{ infinitely often}) = 0.$$

Then it is again a simple task using the exponential tightness of Brownian motion to check that no particles can move further than $\varepsilon \log t$ between lattice times infinitely often (provided that we choose $t_n - t_{n-1}$ small enough). □

Proposition 13. *The lower bound in (2) holds:*

$$\limsup_{t\to\infty} \frac{M_t - \sqrt{2}t}{\log t} \geq -\frac{1}{2\sqrt{2}} \quad \textit{almost surely.}$$

Proof. This is similar to the proof of the lower bound in Theorem 1; it is effectively the same as the proof given by Hu and Shi [10], although again our Bessel changes of measure ease the calculations.

We let

$$\beta_t = \sqrt{2} - \frac{1}{2\sqrt{2}}\frac{\log t}{t}$$

and

$$V(t) = \{v \in N(t) : X_v(r) < \beta_t r + 1 \;\forall r \leq t, \;\; \beta_t t - 1 \leq X_v(t) \leq \beta_t t\}$$

and define

$$I_n = \int_n^{2n} \mathbb{1}_{\{V(t)\neq\emptyset\}} dt.$$

We estimate the first two moments of I_n. Immediately from our earlier lower bound on $\mathbb{P}(H_1(t) \neq 0)$ (from the proof of Proposition 8, taking $y = \frac{1}{\sqrt{2}} \log t$) we get

$$\mathbb{E}[I_n] = \int_n^{2n} \mathbb{P}(V(t) \neq \emptyset) dt \geq c \int_n^{2n} e^{-\sqrt{2} \cdot \frac{1}{\sqrt{2}} \log t} dt = c'.$$

Now,

$$\begin{aligned} \mathbb{E}[I_n^2] &= \mathbb{E}\left[\int_n^{2n} \int_n^{2n} \mathbb{1}_{\{V(s) \neq \emptyset\}} \mathbb{1}_{\{V(t) \neq \emptyset\}} \, ds \, dt\right] \\ &= 2 \int_n^{2n} \int_n^t \mathbb{P}(V(s) \neq \emptyset, V(t) \neq \emptyset) \, ds \, dt. \end{aligned}$$

But whenever $s \leq t$,

$$\mathbb{P}(V(s) \neq \emptyset, V(t) \neq \emptyset) \leq \mathbb{E}\big[|V(s)||V(t)|\big] = \mathbb{E}\big[|V(s)|\mathbb{E}\big[|V(t)|\big|\mathcal{F}_s\big]\big] \qquad (5)$$

and letting $N(u;t)$ be the set of descendants of particle u that are alive at time t,

$$\mathbb{E}\big[|V(t)|\big|\mathcal{F}_s\big] = \sum_{u \in N(s)} \mathbb{E}\left[\sum_{v \in N(u;t)} \mathbb{1}_{\{v \in V(t)\}} \middle| \mathcal{F}_s\right].$$

Now for any $s, t > 0$ let

$$A_t(s) = \{u \in N(s) : X_u(r) < \beta_t r + 1 \forall r \leq s\}$$

and

$$B_t(s) = \{u \in N(s) : \beta_t s - 1 \leq X_u(s) \leq \beta_t s\}.$$

Applying the many-to-one lemma, we have

$$\begin{aligned} &\mathbb{E}\left[\sum_{v \in N(u;t)} \mathbb{1}_{\{v \in V(t)\}} \middle| \mathcal{F}_s\right] \\ &= \mathbb{1}_{\{u \in A_t(s)\}} \mathbb{E}_{X_u(s) - \beta_t s}\left[\sum_{v \in N(t-s)} \mathbb{1}_{\{v \in A_t(t-s) \cap B_t(t-s)\}}\right] \\ &= \mathbb{1}_{\{u \in A_t(s)\}} e^{t-s} \mathbb{Q}_{X_u(s) - \beta_t s}\left[\frac{(-\xi_0 + 1)\mathbb{1}_{\{\xi_{t-s} \in B_t(t-s)\}}}{(\beta_t(t-s) - \xi_{t-s} + 1) e^{\beta_t(\xi_{t-s} - \xi_0) - \beta_t^2 (t-s)/2}}\right] \\ &\leq \mathbb{1}_{\{u \in A_t(s)\}} e^{t-s} (\beta_t s - X_u(s) + 1) e^{\beta_t X_u(s) - \beta_t^2 s} \mathbb{Q}_{X_u(s) - \beta_t s}\left[\frac{\mathbb{1}_{\{\xi_{t-s} \in B_t(t-s)\}}}{e^{\beta_t^2 (t-s) - \beta_t - \beta_t^2 (t-s)/2}}\right] \\ &\leq e^{-2s} t^{1/2} e^{\frac{1}{2} \frac{\log t}{t} s} \mathbb{1}_{\{u \in A_t(s)\}} (\beta_t s - X_u(s) + 1) e^{\beta_t X_u(s)} \mathbb{Q}\left(\xi_t \in B_t(t) \middle| \xi_s = X_u(s)\right) \end{aligned}$$

where for the last equality we used the fact that Bessel processes satisfy the Markov property. Substituting back into (5) and applying the many-to-two lemma we get

$$
\begin{aligned}
&\mathbb{P}(V(s)\neq\emptyset, V(t)\neq\emptyset)\\
&\leq \mathbb{E}\Bigg[\sum_{u,v\in N(s)} \mathbb{1}_{\{u\in V(s)\}} e^{-2s} t^{1/2} e^{\frac{1}{2}\frac{\log t}{t}s} \mathbb{1}_{\{v\in A_t(s)\}}(\beta_t s - X_v(s)+1)e^{\beta_t X_v(s)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{Q}\left(\xi_t\in B_t(t)\middle|\, \xi_s = X_v(s)\right)\Bigg]\\
&= e^s \mathbb{Q}\left[\frac{1}{\zeta^1(s)}\mathbb{1}_{\{\xi^1_s\in B_s(s)\}} e^{-2s} t^{1/2} e^{\frac{1}{2}\frac{\log t}{t}s}\zeta^1(s) e^{\beta_t^2 s/2}\mathbb{Q}\left(\xi^1_t\in B_t(t)\middle|\,\xi^1_s\right)\right]\\
&\quad + \int_0^s 2e^{2s-r}\mathbb{Q}\Bigg[\frac{\zeta^1(r)}{\zeta^1(s)\zeta^2(s)}\mathbb{1}_{\{\xi^1_s\in B_s(s)\}} e^{-2s} t^{1/2} e^{\frac{1}{2}\frac{\log t}{t}s}\zeta^2(s) e^{\beta_t^2 s/2}\\
&\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{Q}\left(\xi^2_t\in B_t(t)\middle|\,\xi^2_s\right)\Bigg| T(1,2)=r\Bigg] dr\\
&= t^{1/2}\mathbb{Q}\left(\xi^1_s\in B_s(s),\ \xi^1_t\in B_t(t)\right)\\
&\quad + 2t^{1/2}\int_0^s \mathbb{Q}\left[e^{-r}\frac{(\beta_t r-\xi^1_r+1)e^{\beta_t\xi^1_r-\beta_t^2 r/2}}{(\beta_t s-\xi^1_s+1)e^{\beta_t\xi^1_s-\beta_t^2 s/2}} e^s \mathbb{1}_{\{\xi^1_s\in B_s(s),\ \xi^2_t\in B_t(t)\}}\middle| T(1,2)=r\right] dr\\
&\leq t^{1/2}\mathbb{Q}\left(\xi^1_s\in B_s(s),\ \xi^1_t\in B_t(t)\right)\\
&\quad + 2e^{\sqrt{2}} t^{1/2} e^{\frac{1}{2}\frac{\log t}{t}s}\int_0^s e^{-\frac{1}{2}\frac{\log t}{t}r}\mathbb{Q}\Big[(\beta_t r-\xi^1_r+1)e^{-\beta_t(\beta_t r-\xi^1_r+1)}\\
&\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{1}_{\{\xi^1_s\in B_s(s),\ \xi^2_t\in B_t(t)\}}\Big| T(1,2)=r\Big] dr.
\end{aligned}
$$

We must now estimate the last line above. The $\mathbb{Q}(\cdot)$ part of the first term is the probability that a Bessel process is near the origin at time s, and then again at time t; so the first term is no bigger than a constant times $t^{1/2}s^{-3/2}(t-s+1)^{-3/2}$. Then using notation from Section 2, the expectation $\mathbb{Q}[\cdot]$ in the second term is

$$
\hat{P}\left[\mathbb{1}_{\{\frac{1}{2\sqrt{2}}\log s - \frac{1}{2\sqrt{2}}\frac{\log t}{t}s+1\leq Y^1_s\leq \frac{1}{2\sqrt{2}}\log s-\frac{1}{2\sqrt{2}}\frac{\log t}{t}s+2\}}\mathbb{1}_{\{1\leq Y^1_t\leq 2\}} Y^1_r e^{-\beta_t Y^1_r}\middle|\, \tau=r\right].
$$

Thus by Lemma 4,

$$
\mathbb{P}(V(s)\neq\emptyset, V(t)\neq\emptyset)\leq c_1(t^{-2}+t^{-1}(t-s+1)^{3/2})
$$

and hence

$$
\mathbb{E}[I_n^2]\leq 2c_1\int_n^{2n}\int_n^t (t^{-2}+t^{-1}(t-s+1)^{3/2})\, ds\, dt\leq c_2,
$$

so

$$
\mathbb{P}(I_n>0)\geq \mathbb{P}(I_n\geq \mathbb{E}[I_n]/2)\geq \frac{\mathbb{E}[I_n]^2}{4\mathbb{E}[I_n^2]}\geq c_3>0.
$$

When n is large, at time $2\delta\log n$ there are at least n^δ particles, all of which have position at least $-2\sqrt{2}\delta\log n$. By the above, the probability that none of these has a descendant that goes above $\sqrt{2}s-\frac{1}{2\sqrt{2}}\log s - 2\sqrt{2}\delta\log n$ for any s between $2\delta\log n+n$ and $2\delta\log n + 2n$ is no larger than

$$
(1-c_3)^{n^\delta}.
$$

The result follows by the Borel-Cantelli lemma since $\sum_n(1-c_3)^{n^\delta}<\infty$. □

Proof of Theorem 2. The convergence in probability is a trivial consequence of Theorem 1. The almost sure statements are given by combining Lemmas 10, 11 and 12 and Proposition 13. □

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