Weierstraß-Institut
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Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

An effective medium approach to the asymptotics of the statistical moments of the parabolic Anderson model and Lifshitz tails

Dedicated to Peter Stollmann on the occasion of his 50th birthday

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submitted: July 1, 2011

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No. 1623
Berlin 2011

2010 *Mathematics Subject Classification.* 60H25, 82B44, 82C44, 35J10, 35P20, 58J35 .

Key words and phrases. Random medium, random Schrödinger operators, heat equation with random potential, parabolic Anderson problem, large deviations, moment asymptotics, integrated density of states, Lifshitz tails.

Supported in part by DFG Forschergruppe 718„Analysis and stochastics in complex physical systems", and the DFG-SFB/TR12 „Symmetries and Universality in Mesoscopic Systems“ .

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Abstract

Originally introduced in solid state physics to model amorphous materials and alloys exhibiting disorder induced metal-insulator transitions, the Anderson model $H_\omega = -\Delta + V_\omega$ on $l^2(\mathbb{Z}^d)$ has become in mathematical physics as well as in probability theory a paradigmatic example for the relevance of disorder effects. Here Δ is the discrete Laplacian and $V_\omega = \{V_\omega(x) : x \in \mathbb{Z}^d\}$ is an i.i.d. random field taking values in $\mathbb{R}$.
A popular model in probability theory is the parabolic Anderson model (PAM), i.e. the discrete diffusion equation $\partial_t u(x,t) = -H_\omega u(x,t)$ on $\mathbb{Z}^d \times \mathbb{R}_+$, $u(x,0) = 1$, where random sources and sinks are modelled by the Anderson Hamiltonian. A characteristic property of the solutions of (PAM) is the occurrence of intermittency peaks in the large time limit. These intermittency peaks determine the thermodynamic observables extensively studied in the probabilistic literature using path integral methods and the theory of large deviations.
The rigorous study of the relation between the probabilistic approach to the parabolic Anderson model and the spectral theory of Anderson localization is at least mathematically less developed. We see our publication as a step in this direction. In particular we will prove an unified approach to the transition of the statistical moments $\langle u(0,t)\rangle$ and the integrated density of states from classical to quantum regime using an effective medium approach. As a by-product we will obtain a logarithmic correction in the traditional Lifshitz tail setting when V_ω satisfies a fat tail condition.

1 Introduction

The Anderson model is the family of discrete random Schrödinger operators $\{H_\omega\}$ defined by

$$H_\omega = -\Delta + V_\omega.$$

Here Δ is the discrete Laplacian on $l^2(\mathbb{Z}^d)$

$$[\Delta u](x) \ = \sum_{|x-y|=1} [u(y) - u(x)].$$

The random potential $\{V_\omega(x)\}_{x\in\mathbb{Z}^d}$ is a field of independent and identically distributed random variables with common distribution P_0. Denoting the expectation value by $\langle\ .\ \rangle$ we assume

$$G(t) := \log\langle\exp(-tV_\omega(0))\rangle \ < \ \infty \tag{1}$$

for all $t \geq 0$. $\{H_\omega\}$ is an ergodic family of self adjoint operators on $l^2(\mathbb{Z}^d)$. In many concrete situations exponential localization is proven at the bottom of the spectrum [15, 26, 30], i.e.

- dense point spectrum close to $\inf \sigma(H_\omega)$,
- exponentially decaying eigenfunctions.

The spectral analysis of $\{H_\omega\}$ is motivated by applications in solid state physics, e.g. localization phenomena, electrical resistance, low temperature physics, ... We refer to [19] and references therein.

The parabolic Anderson model (PAM) is the discrete diffusion equation with random sources and sinks:

$$\begin{aligned} \partial_t u(x,t) &= -H_\omega u(x,t) \qquad && (x,t) \in \mathbb{Z}^d \times [0,\infty), \\ u(x,0) &= 1 && x \in \mathbb{Z}^d. \end{aligned}$$

Assuming (1) the parabolic Anderson model has a.s. an unique, nonnegative solution given by the Feynman-Kac-representation [7]

$$u(x,t) \;=\; \mathbb{E}^x \left[\exp\left(-\int_0^t V_\omega(x_s) ds \right) \right]. \tag{2}$$

Here $\mathbb{E}^x[.]$ is the expectation value of the random walk in continuous time generated by $-\Delta$ starting in x. For $t \geq 0$ the random field $\{u(x,t) : x \in \mathbb{Z}^d\}$ is stationary, ergodic and mixing under translations. The moments $\langle u(0,t)^p \rangle$ and the correlation function are finite [7, 12].
Describing the large time diffusive behaviour of a classical particle in a random medium with traps the applications of the parabolic Anderson model are numerous. (PAM) is used as a linearised model of chemical reaction kinetics exhibiting macroscopic pattern formation in the spatial distribution of reagents, has interpretations in polymer physics and is used to describe population dynamics in an inhomogeneous environment modelling the availability of nutrients. For a very recent application of (PAM) in this biological setting as well as for a comprehensive summary of other interpretations, respectively interesting generalizations of (PAM) we refer to [20] and references therein, see also [12, 7, 10, 25].
In the limit $t \longrightarrow \infty$ the solution $u(x,t)$ shows a.s. a very strong spatial inhomogenity caused by very rare potential constellations. This phenomen is known in the probabilistic literature as intermittency and is described by asymptotic behaviour of the moments $\langle u(0,t)^p \rangle$ [7]. Assuming $V_\omega(x) \geq 0$ the first moment $\langle u(0,t) \rangle$ can be interpreted as the survival probability of a particle that is put randomly on $\mathbb{Z}^d$.

The intuitive link between the probabilistic and the spectral point of view is:

Shape of intermittency peaks	$\longleftrightarrow$	Localized eigenfunctions,
Local killing rate	$\longleftrightarrow$	Eigenvalues.

A quantity to formalize the intuitive link between the Anderson model and PAM is the the integrated density of states measure ν ([15, 17, 31] and references therein). Here we are interested in the integrated density of states (IDS) $N(E)$, i.e. the distribution function of ν

$$N(E) := \nu((-\infty, E]) = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \sharp\{\text{eigenvalues of } H_\Lambda^D \leq E\} \tag{3}$$

with

$$H_\Lambda^D \;=\; \chi_\Lambda H_\omega \chi_\Lambda. \tag{4}$$

The integrated density of states $N(E)$ is the fundamental quantity to study the thermodynamical properties of disordered systems. Moreover, $N(E)$ is used to prove localization properties of the system. In particular we are interested in Lifshitz tails, i.e. the behaviour of the IDS in the limit $E \searrow \inf\sigma(H_\omega)$. Assuming (1) the Laplace transform

$$\widehat{N}(t) := \int e^{-\lambda t}\, d\nu(\lambda) < \infty \qquad (t>0) \tag{5}$$

of ν exists [14] and has the Feynman–Kac representation ([4],[14])

$$\widehat{N}(t) = \langle \mathbb{E}^0 \left[\exp\left(-\int_0^t V_\omega(x_s)ds \right) \delta_0(x_t) \right] \rangle. \tag{6}$$

The first proof of Lifshitz behavior (for the Poisson model) was given by Donsker and Varadhan [6]. Starting from the Feynman–Kac representation their estimate of $\widehat{N}(t)$ in the limit $t \to \infty$ relied on an investigation of the "Wiener sausage" and the machinery of large deviations for Markov processes developed by these authors. To obtain information about the behavior of $N(E)$ for $E \searrow \inf\sigma(H_\omega)$ from the large t behavior of $\widehat{N}(t)$ one uses Tauberian theorems [3], see also Appendix 2. This technique was already used by Pastur [1, 27]. The behaviour of $\widehat{N}(t)$ in the limit $t \to \infty$. is also closely related to the long time behaviour of the moments $\langle u(t,0) \rangle$ of the parabolic Anderson model.

To formulate our main result Theorem 2, we remind the definition of regularly varying functions and of the de Haan class [3], see also Appendix 1.

Definition 1.

(i) A function $g > 0$ defined on some neighbourhood $[X, \infty)$ of infinity satisfying

$$g(\lambda t)/g(t) \overset{t\to\infty}{=} \lambda^\rho (1 + o(1))$$

for all $\lambda \geq 0$ is called regularly varying of index ρ. We write $g \in R_\rho$. If $\rho = 0$ then g is said to be slowly varying. If g varies regularly with index ρ, we have $g(t) = t^\rho g_0(t)$, $g_0 \in R_0$.

(ii) For $g \in R_\rho$ and $\lambda \in (0,1]$ the de Haan class Π_g is the class of functions $H : \mathbb{R} \to \mathbb{R}$ satisfying

$$H(t) - H(\lambda t) \overset{t\to\infty}{=} c_g h_\rho(\lambda) g(t)(1 + o(1)),$$

where $g \in R_\rho$ is called the auxilary function and c_g is the g-index.

Our main result estimates the Laplace transform $\widehat{N}(t)$ defined in (5) and the first moment $\langle u(t,0)\rangle$ in terms of two variational functionals. Here $u(t,0)$ is the solution of the parabolic Anderson model. The variational functional of the lower bound is given by

$$\chi_\ell^-(t) := 4d\sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right) + c_g h_\rho(\ell^{-d})g(t) \qquad \ell\in\mathbb{N} \quad (7)$$

and

$$\chi_\ell^+(t) := \begin{cases} \max_{1/2\le h\le 1}\min\left[2d(1-2\sqrt{1-h}), \gamma/2 + (1-h)c_g h_\rho(1-h)g(t)\right] & \ell = 1,\\ \gamma\sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right) + 4^{-1}c_g h_\rho(4\ell)^{-d})g(t) & \ell > 1,\end{cases} \quad (8)$$

$\ell\in\mathbb{N}$, $\gamma > 0$, is the corresponding variational functional of the upper bound.

Theorem 2. *Suppose $G(t) < \infty$, $t \ge 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g(t) \in R_\rho$, $\rho\in[-1,\infty)$, g-index c_g and $tg(t)\to\infty$ in the limit $t\to\infty$. Denote by $\widehat{N}(t)$ the Laplace transform of the integrated density of states and by $\langle u(t,0)\rangle$ the first moment of the solution of the parabolic Anderson model. Then with $\chi_\ell^\pm(t)$ as defined above*

$$G(t) - t\inf_{\ell\in\mathbb{N}}\chi_\ell^-(t)(1+o(1)) \le \log\widehat{N}(t) \le \log\langle u(t,0)\rangle \le G(t) - t\inf_{\ell\in\mathbb{N}}\chi_\ell^+(t)(1+o(1)). \quad (9)$$

Remark 3. Due to $\langle u(t,0)u(s,0)\rangle = \langle u(t+s,0)\rangle$ [7] Theorem 2 can also be used to estimate the higher moments of $u(t,0)$.

Theorem 2 is motivated by the theory of critical phenomena in statistical physics. Looked at from this angle the variational problem in (9) correponds to the minimization of the free energy. The ground state energy of the Dirichlet-Laplacian on $\Lambda_\ell = \Lambda_\ell(0) := \{x\in\mathbb{Z}^d : |x|_\infty \le \ell\}$ given by

$$\sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right) \approx \ell^{-2}$$

plays the role of an order parameter. The parameter ℓ corresponds to the extension of the Lifshitz ground state. The dependance on the random potential is encoded in the effective potential $G(\lambda t)/\lambda t$, $\lambda\in[0,1]$, respectively the deviation

$$S(\lambda,t) := G(t)/t - G(\lambda t)/\lambda t \ge 0 \quad (10)$$

to the maximal effective potential value $G(t)/t$. Denoting by $\ell^*(t)$ the optimal length defined by

$$\chi^\pm_{\ell^*(t)}(t) = \inf_{\ell\in\mathbb{N}}\chi_\ell^\pm(t)$$

we can distinguish two qualitatively different regimes in dependence on the single site distribution [26].

Quantum regime: If we assume that $G(t)/t \in \Pi_g$ with

$$g(t) \stackrel{t\to\infty}{\to} 0 \tag{11}$$

the asymptotics of $\langle u(t,0)\rangle$ and $\widehat{N}(t)$ are dominated by the energy form of the Laplacian, i.e. by the quantum kinetic energy, respectively the rate function of the occupation time measure in large deviation theory. As a consequence $\ell^*(t)$ will tend to infinity and from the physical point of view Lifshitz tails are expected [22], i.e. an asymptotic behaviour of the IDS like

$$N(E) \sim C_1 e^{-C_2(E-E_0)^{-d/2}}. \tag{12}$$

The content of the next Corollary is that this is only approximately correct.

Corollary 4. *Suppose $G(t) < \infty$, $t \geq 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g(t) = t^\rho g_0(t) \in R_\rho$, $\rho \in [-1,0]$, g-index c_g and $g_0 \in R_0$ s.t.*

$$g_0(tg_0(t)^{1/\rho})/g_0(t) \stackrel{t\to\infty}{\to} 1. \tag{13}$$

Furthermore assume that $g(t) \to 0$ and $tg(t) \to \infty$ in the limit $t \to \infty$. Then the asymptotic optimal length is given by

$$\ell^*(t) \stackrel{t\to\infty}{\sim} g(t)^{1/(d\rho-2)}$$

and there exists constants $C_1, C_2 > 0$, s.t

$$-C_1 t\ell^*(t)^{-2} \leq \log \widehat{N}(t) \leq \log\langle u(t,0)\rangle \leq -C_2 t\ell^*(t)^{-2}. \tag{14}$$

Suppose that $\rho \in [-1,0)$. Then the integrated density of states satisfies

$$-C_1 E^{-\frac{d}{2}+1+\rho^{-1}} g_0\left(CE^{(2-d\rho)/2\rho}\right)^{-1/\rho} \leq \log N(E) \leq -C_2 E^{-\frac{d}{2}+1+\rho^{-1}} g_0\left(CE^{(2-d\rho)/2\rho}\right)^{-1/\rho}. \tag{15}$$

with $C, C_1, C_2 > 0$ and $E \searrow 0$.

Choosing for example the uniform distribution in $[0,1]$ we have

$$\frac{G(t)}{t} - \frac{G(\lambda t)}{\lambda t} \stackrel{t\to\infty}{\sim} \frac{\log t}{\lambda t} \in R_{-1} \tag{16}$$

and (15) becomes

$$C_1 E^{-\frac{d}{2}} \log E \leq \log N(E) \leq C_2 E^{-\frac{d}{2}} \log E \tag{17}$$

in the limit $E \searrow 0$. Comparing (17) and the estimate

$$\lim_{E\searrow 0} \frac{\log\left(-\log(N(E))\right)}{\log(E)} = -\frac{d}{2}$$

proven with spectral theorectic methods in [16, 29] assuming the fat tail condition

$$\mathbb{P}[V_\omega(0) < E] \overset{E\searrow 0}{\sim} C E^k, \qquad k \in \mathbb{N}_0, \tag{18}$$

we obtain a logarithmic correction predicted in the physics literature [23, 28].
Assuming $\rho \in [1, 0)$ the assumptions of Corollary 4 corresponds in the probabilistic setting of [4] to the existence of a non-decreasing function $t \mapsto \alpha_t \in (0, \infty)$ and a function $\widetilde{G} \colon [0, \infty) \to (-\infty, 0]$, $\widetilde{G} \not\equiv 0$, such that

$$\lim_{t\to\infty} \frac{\alpha_t^{d+2}}{t} G\left(\frac{t}{\alpha_t^d}\, y\right) = \widetilde{G}(y), \qquad y \geq 0, \tag{19}$$

uniformly on compact sets in $(0, \infty)$ ($\rho = 0$ is discused in [11]). Condition (19) is satisfied if

$$\mathbb{P}[V_\omega(0) < E] \overset{E\searrow 0}{\sim} \exp(-C\, E^{-\frac{\rho+1}{\rho}}).$$

Using the Feynman-Kac-representaion (2) and the large deviation theory for path integrals it is proven in [4] that

$$\frac{1}{pt} \log\langle u(0,t)^p\rangle \overset{t\to\infty}{=} \frac{G\big(pt\, \alpha_{pt}^{-d}\big)}{pt\, \alpha_{pt}^{-d}} - \frac{1}{\alpha_{pt}^2}\big(\chi + o(1)\big), \tag{20}$$

with

$$\chi = \inf_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \int_{\mathbb{R}^d} |\nabla g(x)|^2\, dx - C\rho^{-1} \int_{\mathbb{R}^d} g(x)^{2(1+\rho)} - g(x)^2\, dx \Big\}, \tag{21}$$

An application of Tauber theory gives in the limit $E \searrow 0$

$$\log N(E) \;=\; C(\rho,\chi)\; E^{-\frac{d}{2}+\frac{1+\rho}{\rho}+o(1)}.$$

Finally let us discuss the almost bounded single site distributions, i.e. $G(t)/t \in \Pi_g$ with $g \in R_0$, $\lim_{t\to\infty} g(t) = 0$. This setting is again dominated by the kinetic energy and $\ell^*(t) \longrightarrow \infty$. The probabilistic counterpart of (14)

$$\frac{1}{pt} \log\langle u(t,0)^p\rangle = \frac{G\big(pt\, \alpha_{pt}^{-d}\big)}{pt\, \alpha_{pt}^{-d}} - \frac{1}{\alpha_{pt}^2}\big(\rho d(1 - \frac{1}{2}\log\frac{\rho}{\pi}) + o(1)\big), \tag{22}$$

as $t \longrightarrow \infty$ is proven in [11]. Here the scale function α_t is defined by the fixed point equation

$$g(t\alpha_t^{-d}) = \alpha_t^{-2}. \tag{23}$$

Furthermore we want to refer to [18], where in Theorem 1.5 for unbounded single site distributions satisfying $G(t)/t \in \Pi_g$ with $g \in R_0$, $\lim_{t\to\infty} g(t) = 0$ the asymptotics of the IDS in the limit $E \longrightarrow -\infty$ is proven.

Classical regime: Let us now consider the classical regime, i.e. $G(t)/t \in \Pi_g$ with

$$\liminf_{t\to\infty} g(t) > 0. \tag{24}$$

Then the quantum kinetic energy/occupation time measure and the effective medium are on the same scale, respectively $g(t)$ dominates the energy form of the Laplacian. As a consequence $\ell^*(t)$ stays finite in the limit $t \longrightarrow \infty$ and the IDS is given by the shifted rate function of the single site potential. Let us first discuss the asymptotics of the statistical moments and of $\widehat{N}(t)$.

Corollary 5. *Suppose $G(t) < \infty$, $t \geq 0$, $G(t)/t \in \Pi_g$ with auxilary function $g \in R_\rho$, $\rho \in [0,\infty)$ and g-index c_g, s.t. (24) is satisfied. With*

$$\chi_-^*(t) := \begin{cases} 1 & c_g g(t) \geq 2\pi^2, \\ 4c_g g(t) + c_g g(t) \log\left(2\pi^2/(c_g g(t))\right) & c_g g(t) < 2\pi^2 \end{cases}$$

and

$$\chi_+^*(t) := \begin{cases} 1 - 2(c_g g(t))^{-1/2} & c_g g(t) \geq 2e^{2d} + \gamma\pi^2/2d, \\ \min\left[\gamma/(4d), \frac{dc_g g(t)}{8} + \frac{dc_g g(t)}{8}\log\left(\frac{64\gamma\pi^2}{c_g g(t)}\right)\right] & c_g g(t) < 2e^{2d} + \gamma\pi^2/2d. \end{cases}$$

we have

$$G(t) - 2dt\chi_-^*(t)(1+o(1)) \leq \log \widehat{N}(t) \leq \log\langle u(t,0)\rangle \leq G(t) - 2dt\chi_+^*(t)(1+o(1)). \tag{25}$$

While $g(t)$ dominates the variational functional for $\rho > 0$, the slowly varying functions define the borderline between the quantum and the classical regime. The main objective of [8] is the double exponential distribution

$$\mathbb{P}[V_\omega(0) < E] \overset{E\to-\infty}{\sim} \exp\left(-e^{-E/c_g}\right) \tag{26}$$

with $G(t) = c_g t\log(c_g t) - c_g t + o(t)$, respectively $S(\lambda,t) = c_g \log(\lambda) + o(1)$. The probabilistic approach obtain (20) with

$$\chi = \min_{\substack{g\colon \mathbb{Z}^d\to\mathbb{R} \\ \sum g^2=1}} \left\{\frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|=1}} \left(g(x)-g(y)\right)^2 - \rho\sum_{x\in\mathbb{Z}^d} g^2(x)\log g^2(x)\right\}. \tag{27}$$

and $\alpha_t \sim 1/\sqrt{c_g} \in (0,\infty)$, i.e. the intermittency peaks are finite but nontrivial. As a consequence there are no Lifshitz tails. The IDS is the single site rate function

$$I(E) := \inf_{t>0}[Et + G(t)] \tag{28}$$

shifted by the constants $2d\chi_\pm^*$ encoding the size of the intermittency peaks.

Corollary 6. *Suppose $G(t) = c_g t\log(c_g t) - c_g t + o(t)$ and $\chi_\pm^*$ as in Corollary 5. Then*

$$-CI\left(E - 2d\chi_-^*\right)(1+o(1)) \;\leq\; \log N(E) \;\leq\; -I\left(E - 2d\chi_+^*\right)(1+o(1)). \tag{29}$$

Finally let us discuss single site distributions satisfying $G(t)/t \in \Pi_g$ with $g \in R_\rho$, $\rho > 0$, i.e. $\lim_{t\to\infty} g(t) = \infty$. An example is the Weibull distribution

$$\mathbb{P}[V_\omega(0) < E] \overset{E\to-\infty}{\sim} \exp\left(-C(-E)^\alpha\right),$$

$\alpha > 1$. The asymptotics of $\langle u(t,0)\rangle$ and $\widehat{N}(t)$ are dominated by $g(t)$. As a consequence we have $\ell^*(t) = 1$ and the asymptotic behaviour of the IDS is given by the maximal shifted single site rate function.

Corollary 7. *Suppose $G(t)/t \in \Pi_g$ with $g \in R_\rho$, $\rho > 0$. Then*

$$\log N(E) \;=\; -I\,(E-2d)\,(1+o(1)). \tag{30}$$

Corollary 7 corresponds to the results obtained in [9, 18].

To prepare the discussion of our strategy to prove Theorem 2 let us mention the key ideas of the spectral and the probabilistic argumentation.

Optimal-Fluctuation Method The core of the spectral theoretic approach is a rare region effect [32] predicted by Lifshitz [21, 22]. Let us assume that $V_\omega \geq 0$ and $\inf\sigma(H_\omega) = 0$. To find an eigenvalue smaller than $E > 0$, the uncertainty principle forces the potential V_ω to be smaller than E on a large set whose volume is of order $E^{-d/2}$. This is a very rare event and its probability is approximately

$$\mathbb{P}[\sharp\{x \in \Lambda : V(x) \leq E\} \geq E^{-d/2}] \approx e^{-C\,E^{-d/2}}. \tag{31}$$

Applying Dirichlet-Neumann-bracketing, perturbation theory or periodic approximation the heuristic argument above can be proven rigorously.

Path-integral and large deviation techniques The probabilistic methods combine the Feynman-Kac-representation of $\langle u(0,t)\rangle$ and $\widehat{N}(t)$ and large deviations techniques for path integrals. Informally the key idea is to express $\langle u(t,0)\rangle$ by means of local times of random walks on $\mathbb{Z}^d$

$$l_t(x) \;:=\; \int_0^t 1_{x_s=x} ds \qquad\qquad x \in \mathbb{Z}^d, t \geq 0, \tag{32}$$

respectively the occupation time measure $L_t := l_t(x)/t$ and to average w.r.t. the random potential

$$\begin{aligned}
\langle u(x,t)\rangle \;&=\; \langle \mathbb{E}^x[\exp(\sum_{x\in\mathbb{Z}^d} V_\omega(x) l_t(x))]\rangle \\
&=\; \mathbb{E}^x[\exp(\sum_{x\in\mathbb{Z}^d} G(l_t(x)))] \\
&=\; \mathbb{E}^x[\exp G(t) + t \sum_{x\in\mathbb{Z}^d} \frac{1}{t}\,[G(L_t(x)t) - L_t(x)G(t)])].
\end{aligned}$$

The next step is to represent the expectation value above as a Laplace integral for the occupation time measure, to apply large deviation principles and Varadhan's Lemma to obtain

$$\begin{aligned}
&\langle u(x,t)\rangle \\
&=\; \exp(G(t)+o(t)) \int_{\mathfrak{M}_1(\mathbb{Z}^d)} e^{-t\,\sum_{x\in\mathbb{Z}^d}\,\frac{1}{t}[G(L_t(x)t)-L_t(x)G(t)]}\;\mathbb{P}_0[L_T \in d\eta] \\
&=\; \exp\big(G(t) - t\,\alpha(t)^{-2}\chi(1+o(1))\big)
\end{aligned}$$

as $t \to \infty$. Nevertheless a rigorous implementation of the argument above is nontrivial (a good guess of α_t is necessary) and has to be proven in four steps:

- making the space finite (but still time-dependent),
- using a Fourier expansion and scaling properties,
- removing the time-dependence of the box (compactification),
- applying the large deviation arguments.

As a consequence of the technical problems resulting from the mathematical implementation at least up to now an unified approach treating all single site distributions at once does not exist.

The strategy used in the proof below combines ideas from spectral and probability theory. The toehold proven in Section 4 is to restrict the estimates of $\langle u(x,t)\rangle$ and $\widehat{N}(t)$ to a cube $\Lambda = \Lambda(t)$ of time-dependent side length $L = L(t)$ and to study

$$\langle \exp\left(-t\ E_1\left(H_\Lambda^D\right)\right)\rangle \ = \ \langle \sup_{p\in\mathfrak{M}_1(\Lambda)} \exp\left(-t\ \left[\left(\sqrt{p}| - \Delta_\Lambda^D\sqrt{p}\right) + \left(\sqrt{p}|V_\omega\sqrt{p}\right)\right]\right)\rangle. \tag{33}$$

with $E_1(H_\Lambda^D) = \inf\sigma(H_\Lambda^D)$ and the set of probability measures with support contained in Λ

$$\mathfrak{M}_1(\Lambda) := \{p \in \mathfrak{M}_1(\mathbb{Z}^d) : \operatorname{supp} p \subset \Lambda\}. \tag{34}$$

In (33) two competing effects are coupled:

- High productivity in $\left(\sqrt{p}|V_\omega\sqrt{p}\right)$ with respect to V_ω

  versus

  Small probability of extreme productive values of the potential

- High productivity in $\left(\sqrt{p}|V_\omega\sqrt{p}\right)$ with respect to p

  versus

  Small probability of the occupation time encoded in $\left(\sqrt{p}| - \Delta_\Lambda^D\sqrt{p}\right)$.

To prove Theorem 2 we want to use, that the optimal p balancing between the two competing effects above satisfies

- p is concentrated on a small cube $\Lambda_\ell \subset \Lambda$,
- p is relatively uniform on Λ_ℓ.

The proof of the lower bound of Theorem 2 in Section 2 is elementary. We can interchange the supremum and the expectation value in (33) and obtain an effective medium problem. By restricting to a subset $\mathfrak{D} \subset \mathfrak{M}_1(\Lambda)$ and optimizing with respect to $\mathfrak{D}$ we obtain the lower bound.

The upper bound of (33) proven in Section 3 is slightly more difficult. We have to control all $p \in \mathfrak{M}_1(\Lambda)$ and it is not possible to interchange the supremum and the expectation value in (33). The first step in the proof is the classification of $p \in \mathfrak{M}_1(\Lambda)$ in Definition 13 below. From the spectral theoretic point of view Definition 13 corresponds to a classification with respect to the kinetic energy while from the stochastic point of view the classification is a down to earth variant of the contraction principle concerning the asymptotic probability of the occupation time measure p. The problem is then to estimate

$$\langle \sup_{p \in \mathfrak{F}(\ell)} \exp\left(t \sum_{x \in \Lambda} p(x) V(x) \right) \rangle \tag{35}$$

solved by an effective medium theory. Finally in Section 4 we prove that all occuring error terms are negligible compared to the first correction of $G(t)$ given by $t \inf_{\ell \in \mathbb{N}} \chi_\ell^{\pm}(t)$.

Let us summarize the current state of research. We discussed quite at lot of publications based on probabilistic and on spectral theoretic methods. The spectral approach is close to the physical intuition, but the assumptions are restrictive. Moreover important aspects of the phenomenology get lost. This concerns the dependence of the IDS on the single site distribution, e.g. the logarithmic correction for fat tail distributions. The probabilistic approach deals all single site distributions and obtain sharp asymptotics. Nevertheless an unified approach systematically explaining the relevant effects like the transition from the quantum to the classical regime does not exist. Symptomatically the probabilistic publications are motivated by spectral theorectic heuristics, while the goal of the formal proof consists in guessing a good scale function, s.t. a large deviation principle can be applied. The intrinsic motivation of the scaling remains unclear. While not obtaining the sharp asymptotics at least partially Theorem 2 resolves some of the questions discussed above. The problem is discussed in an unified setting. The distinction between quantum and classical regime is a natural consequence of the variational description. The elaborated large deviation techniques for path integral measures as well as the preparations to apply them are avoided. Finally an important motivation for our approach is understanding the structural relation between the probabilistic and spectral theoretic methods.

2 The lower bound

Assuming the hypotheses of Theorem 2 we want to prove in this section the following lower bound.

Proposition 8. *With $S(.,.)$ as in (10) we define*

$$\chi_\ell^-(t) = 4d \sin^2\left(\frac{\pi}{2} \frac{1}{\ell+1}\right) + S(\ell^{-d}, t).$$

Then

$$\langle u(t,0)\rangle \ \geq\ \widehat{N}(t) \geq\ \exp\left(G(t) - t\inf_{\ell\in\mathbb{N}}\chi_\ell^-(t)\right)(1+o(1)).$$

Lemma 9 below is strongly influenced by the probabilistic strategy of commutating the expectation values of the random potential and of the random walk to obtain an effective description.

Lemma 9. *Denote by $E_1(H_\Lambda^D) = \inf\sigma(H_\Lambda^D)$ the ground state energy of H_Λ^D defined in (4), $\Lambda = \Lambda_l(0) := \{x\in\mathbb{Z}^d : |x|_\infty \leq l\}$. Then*

$$\langle\exp\left(-t\ E_1\left(H_\Lambda^D\right)\right)\rangle\ \geq\ \exp\left(G(t) - t\inf_{p\in\mathfrak{M}_1(\Lambda)}\left((\sqrt{p}| - \Delta_\Lambda^D\sqrt{p})\ +\ \sum_{x\in\Lambda}p(x)S(p(x),t)\right)\right).$$

Proof.

$$\begin{aligned}
&\langle\exp\left(-t\ E_1\left(H_\Lambda^D\right)\right)\rangle \\
&\quad\geq \sup_{p\in\mathfrak{M}_1(\Lambda)}\langle\exp\left(-t\ \left(\sqrt{p}| - \Delta_\Lambda^D\sqrt{p}\right) + \left(\sqrt{p}|V_\omega\sqrt{p}\right)\right)\rangle \\
&\quad= \sup_{p\in\mathfrak{M}_1(\Lambda)}\exp\left(-t\ \left(\sqrt{p}| - \Delta_\Lambda^D\sqrt{p}\right)\right)\prod_{x\in\Lambda}\langle\exp\left(-tp(x)V_\omega(x)\right)\rangle \\
&\quad= \sup_{p\in\mathfrak{M}_1(\Lambda)}\exp\left(-t\ (\sqrt{p}| - \Delta_\Lambda^D\sqrt{p})\ +\ \sum_{x\in\Lambda}G(p(x)t)\right) \\
&\quad= \exp\left(G(t) - t\inf_{p\in\mathfrak{M}_1(\Lambda)}\left((\sqrt{p}| - \Delta_\Lambda^D\sqrt{p})\ +\ \sum_{x\in\Lambda}p(x)S(p(x),t)\right)\right).
\end{aligned}$$

□

The next problem is to distribute the probability mass $\|p\|_1 = 1$ of the occupation time measure $p\in\mathfrak{M}_1(\Lambda)$ s.t. the competition between diffusion and particle creation encoded in

$$(\sqrt{p}| - \Delta_\Lambda^D\sqrt{p})\ +\ \sum_{x\in\Lambda}p(x)S(p(x),t) \tag{36}$$

is minimized. We have to balance between:

- The energy form of the discrete Laplacian

  $$(\sqrt{p}|\Delta_\Lambda^D\sqrt{p}), \tag{37}$$

  i.e. the rate function of the occupation time measure encoded in $p\in\mathfrak{M}_1(\Lambda)$ [12]. If $t > 0$ is large, it is much more likely that the local time is smeared over a large region than being localized in small subset of Λ.

- The second term in (36) expresses the opposite effect. Due to convexity of $G(t)$ we have

$$\sum_{x\in\Lambda} G(p(x)t) \;\le\; \sum_{x\in\Lambda} p(x)G(t) \;=\; G(t), \tag{38}$$

respectively

$$\sum_{x\in\Lambda} p(x)S(p(x),t) \;\ge\; 0 \tag{39}$$

for general $p \in \mathfrak{M}_1(\Lambda)$ and equality if there is a $x \in \Lambda$ s.t. $p = \delta_x$. The function $S(\lambda, t)$ measures the deviation of the productivity of $p \in \mathfrak{M}_1(\Lambda)$ compared to the maximal productivity given by $G(t)$.

To deal the competition between diffusion and particle creation s.t. the lower and upper bound are in good agreement we restrict $\mathfrak{M}_1(\Lambda)$ to the subset $\mathfrak{D} \subset \mathfrak{M}_1(\Lambda)$ defined below.

Definition 10. Denote by $-\Delta^D_{\Lambda_\ell}$, $1 \le \ell \le l$, the restriction of the discrete Laplacian to the box $\Lambda_\ell = \Lambda_\ell(0) := \{x \in \mathbb{Z}^d : |x|_\infty \le \ell\}$, by

$$\begin{aligned} &\phi_\ell : \Lambda_\ell \to [0,\infty) \\ &\phi_\ell(x) = \prod_{j=1}^{d} \left(\frac{2}{\ell+1}\right)^{\frac{1}{2}} \sin\left(\frac{x_j \pi}{\ell+1}\right) \end{aligned}$$

its ground state and by

$$E_1\left(-\Delta^D_{\Lambda_\ell}\right) \;=\; 4d \sin^2\left(\frac{\pi}{2}\,\frac{1}{\ell+1}\right)$$

its ground state energy. Then,

$$\mathfrak{D} := \{\phi_\ell^2 :\; 1 \le \ell \le l\}.$$

The set $\mathfrak{D}$ contains relatively uniformly distributed prototypes of occupation time measures localized in a small volume. Inserting these intermittency peak candidates in Lemma 9 we obtain the following estimate.

Proposition 11. *Denote by $E_1(H^D_\Lambda) = \inf \sigma(H^D_\Lambda)$ the ground state energy of H^D_Λ. Then,*

$$\langle \exp\left(-t\; E_1\left(H^D_\Lambda\right)\right)\rangle \;\ge\; \exp\left(G(t) - t \inf_{1\le\ell\le l} \chi^-_\ell(t)\right).$$

Proof. Lemma 9 and Definition 10 yield that

$$\begin{aligned} &\langle \exp\left(-t\; E_1\left(H^D_\Lambda\right)\right)\rangle \\ &\ge\; \exp\left(G(t) - t \inf_{p\in\mathfrak{D}}\left((\sqrt{p}|\Delta^D_\Lambda\sqrt{p}) \;+\; \sum_{x\in\Lambda} p(x)S(p(x),t)\right)\right). \end{aligned}$$

In particular choosing $\sqrt{p} = \phi_\ell$ with $\phi_\ell : \Lambda_\ell \to [0, \infty)$ and $E_1\left(-\Delta^D_{\Lambda_\ell}\right)$ as in Definition 10, we have

$$(\phi_\ell | -\Delta^D_\Lambda \phi_\ell) \;=\; E_1\left(-\Delta^D_\ell\right) \|\phi_\ell\|_2^2 \;=\; 4d \sin^2\left(\frac{\pi}{2}\,\frac{1}{\ell+1}\right).$$

Furthermore due to the convexity of $G(t)$ and Jensen inequality we can estimate for $p \in \mathfrak{M}_1(\Lambda_l)$

$$\sum_{x\in\Lambda} p(x) S(p(x), t) \;\leq\; G(t)/t - \ell^d G\left(\ell^{-d} t \sum_{x\in\Lambda_\ell} p(x)\right)/t = \; S(\ell^{-d}, t),$$

respectively

$$\langle \exp\left(-t\; E_1\left(H^D_\Lambda\right)\right)\rangle \;\geq\; \exp\left(G(t) - t \inf_{1\leq \ell \leq l} \chi^-_\ell(t)\right).$$

□

Proof of Proposition 8. Combining the Feynman-Kac representation of $\langle u(t,0)\rangle$ in (2) and $\widehat{N}(t)$ in (6) $\langle u(t,0) \geq \widehat{N}(t)$ is obvious. With the hitting time defined by $\tau_\Lambda(x_s) := \inf_{s\geq 0}\left[x_s \in \Lambda^c\right]$. and averaging with respect to Λ due to ergodicity, we obtain the lower bound

$$\begin{aligned}\widehat{N}(t) \geq \;& |\Lambda|^{-1} \sum_{x\in\Lambda} \langle\, \mathbb{E}^x\left[\exp\left(-\int_0^t V_\omega(x_s) ds\right)\; \delta_x(x_t)\; \mathbf{1}_{\tau_\Lambda > t}\right]\rangle \\ \geq \;& |\Lambda|^{-1}\, \langle \exp\left(-t E_1\left(H^D_\Lambda\right)\right)\rangle.\end{aligned}$$

Choosing l adequate and applying Proposition 11 we obtain

$$\widehat{N}(t) \geq \; \exp\left(G(t) - t \inf_{1\leq\ell\leq l} \chi^-_\ell(t)(1 + o(1))\right).$$

□

3 The upper bound on a box

In this Section we assume that the hypotheses of Theorem 2 are satisfied. We want to prove the following upper bound.

Proposition 12. *Denote by $E_1(H^D_\Lambda)$ the principal eigenvalue of the Hamiltonian H^D_Λ on $\Lambda = \Lambda_l(0)$, $l \in \mathbb{N}$ defined in (4). With the Faber-Krahn constant c_{FK}, $\gamma = c_{FK}/(12\pi)^2$,*

$$\chi^+_1(t) := \max_{0.5 < h \leq 1} \min\left[2d(1 - 2\sqrt{1-h}), \gamma/2 + (1-h) S((1-h), t))\right]$$

and

$$\chi_\ell^+(t) = \gamma \sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right) + S((4\ell)^{-d}, t)/4,$$

$\ell \geq 2$, *we have*

$$\langle \exp(-t\ E_1(H_\Lambda^D)) \rangle \ \ \leq \ \exp\left(G(t) - t \inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) + C|\Lambda|\right).$$

Trying to transfer the proof of Proposition 8 to Proposition 12 we observe two difficulties. We have to control each $p \in \mathfrak{M}_1(\mathbb{Z}^d)$ and it is not possible to interchange in (33) the expectation value and the supremum. To solve the first problem we restrict to the cube $\Lambda = \Lambda_l(0) := \{x \in \mathbb{Z}^d : |x|_\infty \leq l\}$ with $l \in \mathbb{N}$ to be choosen in the next section and define the following classification of $p \in \mathfrak{M}_1(\Lambda)$.

Definition 13 (Classification of the occupation time measures)**.**
With γ as in Proposition 12 we define

$$\mathfrak{F}(1) := \Big\{p \in \mathfrak{M}_1(\Lambda) : \gamma \sin^2\left(\frac{\pi}{4}\right) < \left(\sqrt{p}| - \Delta_\Lambda^D \sqrt{p}\right) \leq 2d\Big)\},$$

and, if $\ell \geq 2$,

$$\mathfrak{F}(\ell) := \left\{p \in \mathfrak{F}(1)^c : \gamma \sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right) < \left(\sqrt{p}| - \Delta_\Lambda^D \sqrt{p}\right) \leq \gamma \sin^2\left(\frac{\pi}{2\ell}\right)\right\}.$$

The classification of $p \in \mathfrak{M}_1(\Lambda)$ in Definition 13 is reminiscent of the contraction principle in large deviation theory [12]. The energy form of the Laplacian plays the role of the projection map while $\mathfrak{F}(\ell)$ correponds to the resulting value set.
We have to deal the case $\ell = 1$ separately to prove the asymptotics for the classical regime, i.e.

$$\langle u(t, 0) \rangle = \exp(G(t) - 2dt(1 + o(1))).$$

Moreover the argument below is prototypical for general ℓ.

Definition 14. Fix $1 \geq h > 0.5$. We define

$$\mathfrak{F}_h(1) := \{p \in \mathfrak{F}(1) : \max_{x\in\Lambda} p(x) \geq h\},$$
$$\mathfrak{F}_h^c(1) := \mathfrak{F}(1) \setminus \mathfrak{F}_h(1).$$

Let us first deal the situation when $p \in \mathfrak{F}(1)$ is "almost" a δ-peak.

Lemma 15.

$$\langle \sup_{p\in\mathfrak{F}_h(1)} \exp\left(t\ [(\ \sqrt{p}\ |\ \Delta_\Lambda^D \sqrt{p}\ ) + (\ \sqrt{p}\ |\ V_\omega \sqrt{p}\ )]\right)\rangle \leq |\Lambda| \exp\left(G(t) - 2dt(1 - 2\sqrt{1-h})\right).$$

Proof. Define x_1 by $p(x_1) = \max_{x\in\Lambda} p(x)$. Then

$$(\sqrt{p} \mid -\Delta_\Lambda^D \sqrt{p}) \geq \sum_{|x_1-x|=1} \left(\sqrt{p(x_1)} - \sqrt{p(x)}\right)^2 \geq 2d(1 - 2\sqrt{1-h}),$$

respectively

$$\langle \sup_{p\in\mathfrak{F}_h(1)} \exp\left(-t\,[(\sqrt{p} \mid \Delta_\Lambda^D \sqrt{p}) + (\sqrt{p} \mid V_\omega \sqrt{p})]\right)\rangle \;\leq\; |\Lambda| \exp\left(G(t) - 2dt(1 - 2\sqrt{1-h})\right).$$

□

Lemma 16.

$$\langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-t\,[(\sqrt{p} \mid \Delta_\Lambda^D \sqrt{p}) + (\sqrt{p} \mid V_\omega \sqrt{p})]\right)\rangle$$
$$\leq \exp\left(G(t) - \gamma t/2 - (1-h)tS((1-h),t)\right).$$

Proof. Define x_1, x_2 by

$$V_\omega(x_1) = \min_{x\in\Lambda} V_\omega(x),$$
$$V_\omega(x_2) = \min_{x\in\Lambda\setminus\{x_1\}} V_\omega(x).$$

Then

$$\langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-\sum_{x\in\Lambda} p(x)V_\omega(x)t\right)\rangle$$
$$\leq \sum_{y_1\in\Lambda} \langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-V_\omega(y_1)t - \sum_{x\in\Lambda} p(x)t\left(V_\omega(x) - V_\omega(y_1)\right)\right) : y_1 = x_1\rangle$$
$$\leq \sum_{\substack{y_1\in\Lambda \\ y_2\in\Lambda\setminus\{y_1\}}} \langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-V_\omega(y_1)t - (1-p(y_1))t\left(V_\omega(y_2) - V_\omega(y_1)\right)\right) : y_i = x_i, i = 1,2\rangle$$
$$\leq \sum_{\substack{y_1\in\Lambda \\ y_2\in\Lambda\setminus\{y_1\}}} \langle \exp\left(-hV_\omega(y_1)t - (1-h)tV_\omega(y_2)\right)\rangle$$
$$\leq |\Lambda|^2 \exp(G(ht) + G((1-h)t))$$
$$\leq |\Lambda|^2 \exp(G(t) - (1-h)tS((1-h),t)),$$

respectively

$$\langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-t\,[(\sqrt{p} \mid \Delta_\Lambda^D \sqrt{p}) + (\sqrt{p} \mid V_\omega \sqrt{p})]\right)\rangle$$
$$\leq \exp\left(-\gamma t/2\right)\,\langle \sup_{p\in\mathfrak{F}_h^c(1)} \exp\left(-t\,(\sqrt{p} \mid V_\omega \sqrt{p})\right)\rangle$$
$$\leq |\Lambda|^2 \exp\left(G(t) - \gamma t/2 - (1-h)tS((1-h),t)\right).$$

□

Combining Lemma 15 and 16 we have

Corollary 17. *With $\chi_1^+(t)$ defined in Proposition 12 we have*

$$\langle \sup_{p\in\mathfrak{F}(1)} \exp\left(-t\,[(\sqrt{p}\,|\,\Delta_\Lambda^D\sqrt{p}\,) + (\sqrt{p}\,|\,V_\omega\sqrt{p}\,)]\right)\rangle \le |\Lambda|^2 \exp\left(G(t) - t\chi_1^+(t)\right).$$

We now discuss $\ell \ge 2$.

Definition 18. Let $h > 0$ and $p \in \mathfrak{M}_1(\Lambda)$. The level set is denoted by

$$U_h(p) := \{x \in \Lambda : p(x) \ge h\}\,.$$

For $\ell \ge 2$, $n \in \mathbb{N}_0$ we define

$$\mathfrak{F}(\ell, n) := \{p \in \mathfrak{F}(\ell) : |U_h(p)| = n\}\,.$$

The first step to prove an analogue of Corollary 17 for $\mathfrak{F}(\ell, n)$ is the following generalization of

$$\sum_{x\in\Lambda} p(x)W(x) \;\ge\; \min_{x\in\Lambda} W(x) \sum_{x\in\Lambda} p(x) \;=\; \min_{x\in\Lambda} W(x).$$

Lemma 19. *Let $\{W(x_j)\}_{j=1,..,|\Lambda|}$ be an arrangement of $W : \Lambda \to [0,\infty)$ according to the size, that is*

$$W(x_1) = \min_{x\in\Lambda} W(x),$$
$$W(x_j) = \min_{x\in\Lambda\setminus\{x_1,..x_{j-1}\}} W(x),$$

$p \in \mathfrak{F}(\ell, n)$, and suppose that

$$\lfloor h^{-1}\|p\chi_{U_h^c(p)}\|\rfloor := \max\{m \in \mathbb{N}_0 : m \le h^{-1}\|p\chi_{U_h^c(p)}\|\}.$$

Then,

$$\sum_{x\in U_h^c(p)} p(x)W(x) \;\ge\; \begin{cases} 0 & \|p\chi_{U_h^c(p)}\|_1 < h, \\ h\sum_{j=1}^{\lfloor h^{-1}\|p\chi_{U_h^c(p)}\|_1\rfloor} W(x_{n+j}) & \textit{otherwise.} \end{cases}$$

Proof. We only discuss $\|p\chi_{U_h^c(p)}\|_1 \ge h$. With $\{x_j\}$ as defined above, $\|\chi_{U_h^c(p)}p\|_1 \le h|U_h^c(p)|$ and $J = n + j$ we have

$$\sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} h \;\le\; \|\chi_{U_h^c(p)}p\|_1 \;=\; \sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} p(x_J) + \sum_{j=\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor+1}^{|U_h^c(p)|} p(x_J),$$

respectively

$$\sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} (h - p(x_J)) \;\le\; \sum_{j=\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor+1}^{|U_h^c(p)|} p(x_J).$$

As a consequence of $W:\Lambda\to[0,\infty)$ we have

$$
\begin{aligned}
&\sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} W(x_J)\,(h-p(x_J)) \\
&\qquad \leq W(x_{n+\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor}) \sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} (h-p(x_J)) \\
&\qquad \leq W(x_{n+\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor}) \sum_{j=\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor+1}^{|U_h^c(p)|} p(x_J) \\
&\qquad \leq \sum_{j=\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor+1}^{|U_h^c(p)|} p(x_J)W(x_J).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{x\in U_h^c(p)} p(x)W(x) \\
&\quad = h \sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} W(x_J) \\
&\quad\quad - \sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} W(x_J)\,(h-p(x_J)) + \sum_{j=\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor+1}^{|U_h^c(p)|} p(x_J)W(x_J) \\
&\quad \geq h \sum_{j=1}^{\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} W(x_J).
\end{aligned}
$$

□

The second ingredient to prove an analogue of Corollary 17 is a lower bound of $\|\chi_{U_h^c(p)}p\|_1$ for $p\in\mathfrak{F}(\ell,n)$ proven in Corollary 22. To attain this goal some intermediate steps are necessary.

Lemma 20. *Suppose $p\in\mathfrak{M}_1(\Lambda)$, $h>0$. Define $p_h:\mathbb{Z}^d\to\mathbb{R}$ by*

$$
p_h(x) := \begin{cases} (\sqrt{p(x)}-\sqrt{h})^2 & \text{if } x\in U_h(p), \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
\left(\sqrt{p}\middle| -\Delta_\Lambda^D\sqrt{p}\right) \;\geq\; \left(\sqrt{p_h}\middle| -\Delta_{U_h(p)}^D\sqrt{p_h}\right).
$$

Proof. As a consequence of the definition of $\mathfrak{M}_1(\Lambda)$ in (34) we have

$$\begin{aligned}
&\big(\sqrt{p}| - \Delta_\Lambda^D \sqrt{p}\big) \\
&= \frac{1}{2} \sum_{\substack{x,y\in\Lambda,\\ |x-y|=1}} (\sqrt{p(x)} - \sqrt{p(y)})^2 + \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda^c,\\ |x-y|=1}} (\sqrt{p(x)} - \sqrt{p(y)})^2 \\
&\geq \frac{1}{2} \sum_{\substack{x,y\in U_h(p),\\ |x-y|=1}} (\sqrt{p(x)} - \sqrt{p(y)})^2 + \sum_{x\in U_h(p)} \sum_{\substack{y\in U_h^c(p),\\ |x-y|=1}} (\sqrt{p(x)} - \sqrt{p(y)})^2 \\
&\geq \frac{1}{2} \sum_{\substack{x,y\in U_h(p),\\ |x-y|=1}} (\sqrt{p_h(x)} - \sqrt{p_h(y)})^2 + \sum_{x\in U_h(p)} \sum_{\substack{y\in U_h^c(p),\\ |x-y|=1}} (\sqrt{p(x)} - \sqrt{h})^2 \\
&= \big(\sqrt{p_h}| - \Delta_{U_h(p)}^D \sqrt{p_h}\big) .
\end{aligned}$$

□

Proposition 21 (Faber-Krahn-inequality[5]). *Suppose $U \subset \mathbb{Z}^d$, $|U| < \infty$. Then there is a constant $0 < c_{FK} \leq 2d$, s.t. the principal eigenvalue $E_1(-\Delta_U^D)$ of the discrete Dirichlet-Laplacian $-\Delta_U^D$ is bounded below by*

$$E_1(-\Delta_U^D) \geq c_{FK}|U|^{-2/d}.$$

As a consequence of the Faber-Krahn-inequality we find a lower bound of the probability mass outside the level set.

Corollary 22. *Suppose that $\ell \geq 2$, $p \in \mathfrak{F}(\ell, n)$ and $\gamma = c_{FK}/(12\pi)^2$. Then*

$$\|\chi_{U_h^c(p)}p\|_1 \geq 1 - \left(\left(\frac{\gamma\pi^2}{4c_{FK}}\right)^{1/2} n^{1/d}\ell^{-1} + (hn)^{1/2}\right)^2 .$$

Proof. By Definition 13, Lemma 20 and Proposition 21 we have

$$\begin{aligned}
\gamma \sin^2\left(\frac{\pi}{2\ell}\right) &\geq \big(\sqrt{p}| - \Delta_\Lambda^D \sqrt{p}\big) \\
&\geq \big(\sqrt{p_h}| - \Delta_{U_h(p)}^D \sqrt{p_h}\big) \\
&\geq E_1(-\Delta_{U_h(p)}^D)\|\sqrt{p_h}\|_2^2 \\
&\geq c_{FK}n^{-2/d}\|\sqrt{p_h}\|_2^2,
\end{aligned}$$

respectively

$$\|p_h\|_1 \leq \frac{\gamma}{c_{FK}} n^{2/d} \sin^2\left(\frac{\pi}{2\ell}\right) \leq \frac{\gamma\pi^2}{4c_{FK}}\ell^{-2}n^{2/d}$$

and

$$\begin{aligned}\|\chi_{U_h(p)}p\|_1 &\leq \|\chi_{U_h(p)}[\sqrt{p}-\sqrt{h}+\sqrt{h}]\|_2^2\\ &\leq \left(\|\chi_{U_h(p)}[\sqrt{p}-\sqrt{h}]\|_2+\|\chi_{U_h(p)}\sqrt{h}]\|_2\right)^2\\ &\leq \left(\left(\frac{\gamma\pi^2}{4c_{FK}}\right)^{1/2} n^{1/d}\ell^{-1}+(hn)^{1/2}\right)^2,\end{aligned}$$

respectively

$$\|\chi_{U_h^c(p)}p\|_1 \;=\; 1-\|\chi_{U_h(p)}p\|_1 \;\geq\; 1-\left(\left(\frac{\gamma\pi^2}{4c_{FK}}\right)^{1/2} n^{1/d}\ell^{-1}+(hn)^{1/2}\right)^2.$$

□

In the next Lemma we choose the height h in dependance on ℓ.

Lemma 23. *Suppose that* $\ell \geq 2$, $h=(4\ell)^{-d}$ *and* $p \in \mathfrak{F}(\ell,n)$, $n \leq h^{-1}=(4\ell)^d$. *Then*

$$2 \leq N_\ell := (4\ell)^d/2-1 \leq n+\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor,$$

and

$$h(N_\ell-1) \geq 1/4.$$

Proof. First observe

$$h(N_\ell-1)=1/2-2h \geq 1/4.$$

and $nh = 4^{-d}n\ell^{-d} \leq 1$, respectively $n^{1/d}\ell^{-1} \leq 4$. Combining the assumptions of Lemma 23, Corollary 22 and $\gamma = c_{FK}/(12\pi)^2$ defined in Proposition 12 we get

$$\begin{aligned}nh+h\lceil h^{-1}\|\chi_{U_h^c(p)}p\|_1\rceil &\geq nh+\|\chi_{U_h^c(p)}p\|_1\\ &\geq 1-\left(\left(\frac{\gamma\pi^2}{4c_{FK}}\right)^{1/2} n^{1/d}\ell^{-1}+(hn)^{1/2}\right)^2+nh\\ &\geq 1-\frac{\gamma\pi^2}{4c_{FK}}n^{2/d}\ell^{-2}-2\left(\frac{\gamma\pi^2}{4c_{FK}}\right)^{1/2} n^{1/d}\ell^{-1}\\ &\geq 1-\frac{\gamma\pi^2}{c_{FK}}4-4\left(\frac{\gamma\pi^2}{c_{FK}}\right)^{1/2}\\ &\geq 1/2,\end{aligned}$$

respectively

$$n+\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor \;\geq\; N_\ell \geq 2.$$

□

The next proposition generalizes the argument given in the proof of Lemma 16.

Proposition 24. *Suppose $\ell \geq 2$ and N_ℓ as defined in Lemma 23. Then*

$$\langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp\left(-t\sum_{x\in\Lambda} p(x)V_\omega(x)\right)\rangle \leq |\Lambda|\binom{|\Lambda|}{N_\ell - 1}\exp\left(G(t) - 4^{-1}tS((4\ell)^{-d},t)\right).$$

Proof. Assume $\{V_j^*\}_{j=1,..,|\Lambda|}$ is an arrangement of $V:\Lambda\to\mathbb{R}$ according to the size, that is

$$\begin{aligned} V_1^* &= V_\omega(x_1) = \min_{x\in\Lambda} V_\omega(x), \\ V_j^* &= V_\omega(x_j) = \min_{x\in\Lambda\setminus\{x_1,..x_{j-1}\}} V_\omega(x). \end{aligned}$$

and $\{p_i^*\}_{i=1,..,|\Lambda|}$ is defined by

$$\begin{aligned} p_1^* &= p(y_1) = \max_{y\in\Lambda} p(y), \\ p_i^* &= p(y_i) = \max_{y\in\Lambda\setminus\{y_1,..y_{i-1}\}} p(y). \end{aligned}$$

An elementary induction argument gives

$$\sum_{x\in\Lambda} p(x)V_\omega(x) \geq \sum_{j=1}^{|\Lambda|} p_j^* V_j^*,$$

respectively

$$\langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp\left(-t\sum_{x\in\Lambda} p(x)V_\omega(x)\right)\rangle \leq \langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp\left(-t\sum_{j=1}^{|\Lambda|} p_j^* V_j^*\right)\rangle.$$

Suppose $h = (4\ell)^{-d}$ and $N_\ell \geq 2$ as in Lemma 23. Observing $V_j^* - V_1^* \geq 0$, $p_j^* \geq h$, $j = 1,..,n$, Lemma 19 and Lemma 23 give for $p \in \mathfrak{F}(\ell,n)$

$$\begin{aligned} \sum_{j=1}^{|\Lambda|} p_j^* V_j^* &= V_1^* + \sum_{j=1}^{n} p_j^*(V_j^* - V_1^*) + \sum_{j=n+1}^{|\Lambda|} p_j^*(V_j^* - V_1^*) \\ &\geq V_1^* + h \sum_{j=1}^{n+\lfloor h^{-1}\|\chi_{U_h^c(p)}p\|_1\rfloor} (V_j^* - V_1^*) \\ &\geq V_1^* + h \sum_{j=1}^{N_\ell}(V_j^* - V_1^*) \\ &= (1 - h(N_\ell - 1))V_1^* + h\sum_{j=2}^{N_\ell} V_j^*, \end{aligned}$$

respectively with Lemma 23

$$
\begin{aligned}
&\langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp\left(-t\sum_{x\in\Lambda} p(x)V_\omega(x)\right)\rangle \\
&\leq \langle \exp(-(1-h(N_\ell-1))V_1^* - h\sum_{j=2}^{N_\ell} V_j^*)\rangle. \\
&= \sum_{x\in\Lambda}\sum_{\substack{W\subset\Lambda,\\ |W|=N_\ell-1}} \langle \exp(-(1-h(N_\ell-1))V_\omega(x) - h\sum_{y\in W} V_\omega(y)) : x = x_1^*, W = \{x_2^*,..,x_{N_\ell}^*\}\rangle. \\
&\leq \sum_{x\in\Lambda}\sum_{\substack{W\subset\Lambda,\\ |W|=N_\ell-1}} \langle \exp(-(1-h(N_\ell-1))V_\omega(x) - h\sum_{y\in W} V_\omega(y))\rangle. \\
&= |\Lambda|\binom{|\Lambda|}{N_\ell-1}\langle \exp(-(1-h(N_\ell-1))tV_\omega(0))\rangle \prod_{j=2}^{N_\ell}\langle\exp(-htV_\omega(0))\rangle.
\end{aligned}
$$

Integration with respect to the single site potential and Lemma 23 give

$$
\begin{aligned}
\langle \sup_{p\in\mathfrak{F}(\ell,n)} & \exp\left(-t\sum_{x\in\Lambda} p(x)V_\omega(x)\right)\rangle \\
&\leq |\Lambda|\binom{|\Lambda|}{N_\ell-1}\exp\left(G((1-h(N_\ell-1))t) + (N_\ell-1)G(th)\right) \\
&\leq |\Lambda|\binom{|\Lambda|}{N_\ell-1}\exp\left(G(t) - h(N_\ell-1)tS(h,t)\right) \\
&\leq |\Lambda|\binom{|\Lambda|}{N_\ell-1}\exp\left(G(t) - tS(h,t)/4\right).
\end{aligned}
$$

□

Proof of Proposition 12. As a consequence of the Faber-Krahn-inequality exists $\tilde{l} = cl$ s.t. $\mathfrak{M}_1(\Lambda) \subset \bigcup_{\ell=1}^{\tilde{l}} \mathfrak{F}(\ell)$. Choosing $h = (4\ell)^{-d}$ as in Proposition 24 we observe $|U_h(p)| \leq (4\ell)^d \leq (4\tilde{l})^d$. Corollary 17 gives

$$
\begin{aligned}
&\langle\exp(-t\, E_1(H_\Lambda^D))\rangle \\
&\quad\leq \langle \sup_{p\in\mathfrak{F}(1)} \exp\left(-t\,[(\sqrt{p}\,|\,-\Delta_\Lambda^D\sqrt{p}\,) + (\,\sqrt{p}\,|\,V_\omega\sqrt{p}\,)]\right)\rangle \\
&\qquad + \sum_{\ell=2}^{\tilde{l}}\sum_{n=0}^{(4\ell)^d}\langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp\left(-t\,[(\sqrt{p}\,|\,-\Delta_\Lambda^D\sqrt{p}\,) + (\,\sqrt{p}\,|\,V_\omega\sqrt{p}\,)]\right)\rangle \\
&\quad\leq |\Lambda|^2\exp\left(G(t) - t\chi_1^+(t)\right) \\
&\qquad + \sum_{\ell=2}^{\tilde{l}}\exp\left(-\gamma\sin^2\left(\frac{\pi}{2}\frac{1}{\ell+1}\right)\right)\sum_{n=0}^{(4\ell)^d}\langle \sup_{p\in\mathfrak{F}(\ell,n)} \exp(-t\,(\,\sqrt{p}\,|\,V_\omega\sqrt{p}\,))\rangle.
\end{aligned}
$$

As a consequence of Proposition 24 we obtain

$$\langle \exp(-t\ E_1(H^D_{\Lambda,\omega}))\rangle$$

$$\leq\ |\Lambda|^2 \exp\left(G(t) - t\chi_1^+(t)\right) + C|\Lambda| \sum_{\ell=2}^{\tilde{l}} \binom{|\Lambda|}{N_\ell - 1} \exp\left(G(t) - t\chi_\ell^+(t)\right)$$

$$\leq\ |\Lambda|^2 \exp\left(G(t) - t\chi_1^+(t)\right) + |\Lambda|^2\ \exp\left(G(t) - t\inf_{\ell\geq 2} \chi_\ell^+(t)\right) \sum_{\ell=2}^{\tilde{l}} \binom{|\Lambda|}{N_\ell - 1}$$

$$\leq \exp\left(G(t) - t\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) + C|\Lambda|\right).$$

□

4 The proof of Theorem 2

To prove Theorem 2 we have to choose the side length $l = l(t)$ of the cube $\Lambda = \Lambda_l(0)$ s.t. the error in Proposition 12 and the error resulting from boundary conditions are negligible, that is

$$C_1 l(t)^d + C_2 t l(t)^{-2} \overset{t\to\infty}{=} o\left(t \inf_{\ell\in\mathbb{N}} \chi_\ell^+(t)\right).$$

To solve this problem we have to compute $\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t)$ in dependance on the single site distribution, respectively the corresponding cumulant generating function. In Theorem 2 we assume $G(t) < \infty$, $t \geq 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g \in R_\rho$, $\rho \in [-1, \infty)$ and g-index c_g, i.e. we have

$$S(\lambda, t) = \frac{G(t)}{t} - \frac{G(\lambda t)}{\lambda t} \overset{t\to\infty}{=} c_g h_\rho(\lambda) g(t)(1 + o(1)),$$

$\lambda \in (0, 1]$. As discussed in [3], p. 128, $h_\rho(\lambda)$, $\lambda \in (0, 1]$ has the representation

$$h_\rho(\lambda) = \int_\lambda^1 u^{\rho-1} du + o(1) = \begin{cases} -\log(\lambda) & \rho = 0, \\ (1 - \lambda^\rho)/\rho & \rho \neq 0, \end{cases} \tag{40}$$

i.e. to estimate $\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t)$ we have to distinguish three cases. Let us first discuss single site distributions exhibiting Lifshitz tail behaviour.

Lemma 25. *Suppose $G(t) < \infty$, $t \geq 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g \in R_\rho$, $\rho \in [-1, 0)$ and $tg(t) \to \infty$ in the limit $t \to \infty$. Denote the optimal length by*

$$\ell^*(t) := g(t)^{1/(d\rho - 2)}.$$

Then

$$C_1 \ell^*(t)^{-2} \leq \inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) \leq \inf_{\ell\in\mathbb{N}} \chi_\ell^-(t) \leq C_2 \ell^*(t)^{-2},$$

$C_1, C_2 > 0$, t sufficiently large.

Proof. To prove the lower bound compute the corresponding continuous minimization problem. As a consequence of $\ell^*(t) \to \infty$ the upper bound is given by

$$\inf_{\ell\in\mathbb{N}} \chi_\ell^-(t) \le 4d\sin^2\left(\frac{\pi}{2}\frac{1}{\lceil \ell^*(t)\rceil + 1}\right) + S(\lceil \ell^*(t)\rceil^{-d}, t) \le C_2 \ell^*(t)^{-2}.$$

□

In the next lemma we discuss $\rho = 0$ defining the borderline between the classical and the quantum regime. It contains the single peak case ($\ell^*(t) = 1$), the double exponential distribution ($\ell^*(t) \sim 1/\sqrt{c_g}$) and the almost bounded single site distributions ($\ell^*(t) \to \infty$, $t \to \infty$).

Lemma 26. *Suppose $G(t) < \infty$, $t \ge 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g \in R_0$. Then*

$$\inf_{\ell\in\mathbb{N}} \chi_\ell^-(t) \le \begin{cases} 2d & c_g g(t) \ge 2\pi^2, \\ 8dc_g g(t) + 2dc_g g(t)\log\left(2\pi^2/(c_g g(t))\right) & c_g g(t) < 2\pi^2 \end{cases} \tag{41}$$

and

$$\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) \ge \begin{cases} 2d - 4d(c_g g(t))^{-1/2} & c_g g(t) \ge 2e^{2d} + \gamma\pi^2/2d, \\ \min\left[\gamma/2, \frac{dc_g g(t)}{8} + \frac{dc_g g(t)}{8}\log\left(\frac{64\gamma\pi^2}{c_g g(t)}\right)\right] & c_g g(t) < 2e^{2d} + \gamma\pi^2/2d. \end{cases} \tag{42}$$

Proof. Approximating $\sin(x)$ and choosing

$$\ell^*(t) = \max\left[1, \left(\frac{2\pi^2}{c_g g(t)}\right)^{1/2}\right]$$

we obtain (41). The starting point to prove (42) is

$$\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) \ge \min\left(\chi_1^+(t), \inf_{\ell\in[2,\infty)}\left[\frac{\pi^2\gamma}{10}\ell^{-2} + \frac{d}{4}c_g g(t)\log(4\ell)\right]\right). \tag{43}$$

Computing the infinum with

$$\ell^*(t) = \max\left[2, \left(\frac{4\pi^2\gamma}{5dc_g g(t)}\right)^{1/2}\right], \tag{44}$$

the lower bound (42) is then a consequence of some elementary, but painful calculations. □

Finally let us discuss the one-peak case

Lemma 27. *Suppose $G(t) < \infty$, $t \ge 0$, $G(t)/t \in \Pi_g$ with auxilary function $g \in R_\rho$, $\rho \in (0,\infty)$. Then $\ell^*(t) = 1$ and*

$$2d = \inf_{\ell\in\mathbb{N}} \chi_\ell^-(t) \ge \inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) \ge 2d\left(1 - 2g(t)^{-1/2}\right) \tag{45}$$

for t sufficiently large.

Proof. By assumption we have

$$S(\lambda,t) \; = c_g\rho^{-1}(1-\lambda^\rho)g(t)+o(1) \overset{t\to\infty}{\to} \infty.$$

Choosing $h = 1-1/g(t) \geq 1/2$ we obtain (45). □

Corollary 28. *Suppose $G(t) < \infty$, $t \geq 0$ and $G(t)/t \in \Pi_g$ with auxilary function $g(t) = t^\rho g_0(t) \in R_\rho$, $\rho \in [-1,\infty)$, $g_0 \in R_0$ and g-index c_g. Furthermore if $\rho = -1$ we assume that $g_0(t) \overset{t\to\infty}{\to} \infty$. Defining $l = l(t) := \lceil \alpha(t)\ell^*(t)\rceil$ with $\ell^*(t)$ as Lemma 25 -27 and*

$$\alpha(t) = \begin{cases} g_0(t)^{\frac{1}{d(d+2)}} & \rho = -1, \\ t^{-\frac{1+\rho}{d(d\rho-2)}} & \rho \in (-1,0), \\ t^{1/2d} & \rho \in [0,\infty) \end{cases}$$

we have

$$K_1 l^d + K_2 t l^{-2} = o\left(t \inf_{\ell\in\mathbb{N}} \chi_\ell^+(t)\right). \tag{46}$$

Proof. As a consequence of $\inf_{\ell\in\mathbb{N}} \chi_\ell^+(t) \geq 0$ we can estimate

$$0 \leq \left|\frac{K_1 l^d + K_2 t l^{-2}}{t\inf_{\ell\in\mathbb{N}}\chi_\ell^+(t)}\right| \leq \frac{K_1 l^d + K_2 t l^{-2}}{C_1 t \ell^*(t)^{-2}} \leq K\alpha(t)^d t^{-1}\ell^*(t)^{d+2} + K\alpha(t)^{-2}.$$

The case $\rho \in [0,\infty)$ is obvious. Suppose $\rho \in [-1,0)$. With $\alpha(t) \to \infty$ and $\ell^*(t) = g(t)^{1/(d\rho-2)}$ we have

$$0 \leq \left|\frac{K_1 l^d + K_2 t l^{-2}}{t\inf_{\ell\in\mathbb{N}}\chi_\ell^+(t)}\right| \leq K\alpha(t)^d t^{-\frac{d+2}{d\rho-2}-1} g_0(t)^{\frac{d+2}{d\rho-2}} + K\alpha(t)^{-2} \to 0.$$

□

The next lemma is a slight modification of Proposition 4.4 in [4] to deal the bounded and the unbounded setting simultaneously.

Lemma 29. *Suppose $L := L(t) = \lceil t\log(t)\rceil$. Then*

$$\langle \exp\left(-tE_1\left(H^D_{\Lambda_L}\right)\right)\rangle \;\leq\; \exp\left(G(t) - t\inf_{\ell\in\mathbb{N}}\chi_\ell^+(t)(1+o(1))\right).$$

Proof. Lemma 4.6 in [4] says, there is a constant $C > 0$ such that for every $l \in \mathbb{N}$, there is a function $\phi_l : \mathbb{Z}^d \to [0,\infty)$ with the following properties:

(i) ϕ_l is l-periodic in every component,

(ii) $\|\phi_l\|_\infty \leq C/l^2$,

(iii) For any potential $V:\mathbb{Z}^d\to[0,\infty)$ and any $L>l/2$ we can estimate

$$E_1\left((-\Delta+V+\phi_l)^D_{\Lambda_L}\right)\ \geq\ \min_{x\in\Lambda_{L+l}} E_1\left((-\Delta+V)^D_{\Lambda_l(x)}\right).$$

We define

$$\begin{aligned}\widetilde{V}_\omega\quad &: \mathbb{Z}^d\to[0,\infty),\\ \widetilde{V}_\omega(x) &= \begin{cases} V_\omega(x)-\min_{x\in\Lambda_{2L}}V_\omega(x) & x\in\Lambda_{2L},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

With $\widetilde{H}_\omega:=-\Delta+\widetilde{V}_\omega+\phi_l$ we obtain as a consequence of the result above

$$\begin{aligned}E_1\left(H^D_{\Lambda_L}\right) &= \inf_{\substack{f\in l_2(\Lambda_L)\\ \|f\|_2=1}}\left[(f|\widetilde{H}^D_{\Lambda_L}f)-(f|\phi_\ell f)\right]+\min_{x\in\Lambda_{2L}}V_\omega(x)\\ &\geq E_1\left(\widetilde{H}^D_{\Lambda_L}\right)-Cl^{-2}+\min_{x\in\Lambda_{2L}}V_\omega(x),\end{aligned}$$

respectively

$$\begin{aligned}&\langle\exp\left(-t\,E_1\left(H^D_{\Lambda_L}\right)\right)\rangle\\ &\quad\leq\exp(Cl^{-2}t)\langle\exp\left(-tE_1\left(\widetilde{H}^D_{\Lambda_L}\right)-t\min_{x\in\Lambda_{2L}}V_\omega(x)\right)\rangle\\ &\quad\leq\exp(Cl^{-2}t)\langle\max_{x\in\Lambda_{L+l}}\exp\left(-t\,E_1\left(H^D_{\Lambda_l(x)}\right)\right)\rangle\\ &\quad\leq\exp(Cl^{-2}t)\sum_{x\in\Lambda_{L+l}}\langle\exp\left(-t\,E_1\left(H^D_{\Lambda_l(x)}\right)\right)\rangle\\ &\quad\leq\ 2^d|\Lambda_L|\exp(Cl^{-2}t)\langle\exp\left(-t\,E_1\left(H^D_{\Lambda_l(0)}\right)\right)\rangle.\end{aligned}$$

Choosing $l=l(t)=\lceil\alpha(t)\ell^*(t)\rceil$ as in Corollary 28 we have $l(t)/2<L(t)$ for t sufficiently large. An application of Proposition 12 gives

$$\langle\exp\left(-t\,E_1\left(H^D_{\Lambda_L}\right)\right)\rangle\leq\ 3^d|\Lambda_L|\exp\left(G(t)-t\inf_{\ell\in\mathbb{N}}\chi^+_\ell(t)+C_1l(t)^d+C_2tl(t)^{-2}\right).$$

Lemma 29 is now a consequence of Corollary 28. □

Proof of the upper bound of Theorem 2: Choose $L=\lceil t\log(t)\rceil$. Then

$$\begin{aligned}&\langle u(t,0)\rangle=\langle\mathbb{E}^0\left[\exp\left(-\int_0^t V_\omega(x_s)ds\right)\right]\rangle\\ &=\langle\mathbb{E}^0\left[\exp\left(-\int_0^t V_\omega(x_s)ds\right)\mathbf{1}_{\tau_{\Lambda_L}\leq t}\right]\rangle+\langle\mathbb{E}^0\left[\exp\left(-\int_0^t V_\omega(x_s)ds\right)\mathbf{1}_{\tau_{\Lambda_L}>t}\right]\rangle.\end{aligned}$$

The second term can be estimated by

$$\langle \mathbb{E}^0 \left[\exp\left(\int_0^t V_\omega(x_s)ds \right) \mathbf{1}_{\tau_{\Lambda_L} > t} \right] \rangle \leq |\Lambda| \langle \exp\left(-t\, E_1\left(H_{\Lambda_L}^D\right)\right) \rangle.$$

Applying the estimate of the hitting probability

$$\mathbb{P}\left[\tau_L(x_t) \leq t\right] \leq 2^{d+1} \exp\left(-L\left(\log\left(L/(td)\right) - 1\right)\right)$$

([8], Lemma 2.5) we have

$$\begin{aligned} \langle \mathbb{E}^0 \left[\exp\left(\int_0^t V_\omega(x_s)ds \right) \mathbf{1}_{\tau_{\Lambda_L} \leq t} \right] \rangle &\leq \exp\left(G(t)\right) \mathbb{P}[\tau_{\Lambda_L} \leq t] \\ &\leq 2^{d+1} \exp\left(G(t) - t\log(t)\left(\log\left(\log(t)/d\right) - 1\right)\right). \end{aligned}$$

Finally as a consequence of Lemma 29 we have

$$\langle u(t,0) \rangle \leq \exp\left(G(t) - t \inf_{\ell \in \mathbb{N}} \chi_\ell^+(t)(1 + o(1)) \right)$$

in the limit $t \longrightarrow \infty$. □

Proof of Corollary 4 . Combining Theorem 2 and Lemma 25 we obtain

$$-C_1 t g(t)^{\frac{-2}{d\rho - 2}} \leq \log \widehat{N}(t) \leq \log \langle u(t,0) \rangle \leq -C_2 t g(t)^{\frac{-2}{d\rho-2}}.$$

Applying Corollary 31, i.e. the limit of oscillation version of de Bruijn's Tauberian theorem in Appendix 2, we obtain upper and lower bounds of the IDS in the limit $E \searrow 0$

$$C_1 \inf_{t>0} [Et - tg(t)^{\frac{-2}{d\rho-2}}](1 + o(1)) \leq \log N(E) \leq C_2 \inf_{t>0}[Et - tg(t)^{\frac{-2}{d\rho-2}}](1+o(1)). \tag{47}$$

Observing that

$$\frac{d}{dt} g(t) \overset{t\to\infty}{\sim} \rho g(t)/t, \tag{48}$$

([3], p.44) the minimizing time t^* of the Legendre transform satisfies with $C > 1$

$$g(t^*) = (t^*)^\rho g_0(t^*) \overset{E\searrow 0}{\sim} C E^{(2-d\rho)/2}. \tag{49}$$

As a consequence of (13) we can apply the inversion formula for regularly varying functions stated at the end of Appendix 1 and obtain

$$t^* \overset{E\searrow 0}{\sim} C E^{(2-d\rho)/2\rho} g_0 \left(C E^{(2-d\rho)/2\rho} \right)^{-1/\rho},$$

respectively

$$\begin{aligned} \inf_{t>0} \left[Et - tg(t)^{\frac{-2}{d\rho-2}} \right] &\overset{E\searrow 0}{\sim} C E^{(2-d\rho)/2\rho} g_0 \left(C E^{(2-d\rho)/2\rho} \right)^{-1/\rho} \left(E - g(t^*)^{\frac{-2}{d\rho-2}} \right) \\ &= -(C-1) E^{-\frac{d}{2}+1+\rho^{-1}} g_0 \left(C E^{(2-d\rho)/2\rho} \right)^{-1/\rho}. \end{aligned}$$

□

Proof of Corollary 5 -7 . Combining Theorem 2, Lemma 26 and Lemma 27 gives Corollary 5, i.e. the estimate

$$G(t) - 2dt\chi_-^*(t)(1+o(1)) \le \log \widehat{N}(t) \le \log\langle u(t,0)\rangle \le G(t) - 2dt\chi_+^*(t)(1+o(1)) \tag{50}$$

with

$$\chi_-^*(t) = \begin{cases} 1 & c_g g(t) \ge 2\pi^2, \\ 4c_g g(t) + c_g g(t)\log\left(2\pi^2/(c_g g(t))\right) & c_g g(t) < 2\pi^2 \end{cases}$$

and

$$\chi_+^*(t) = \begin{cases} 1 - 2(c_g g(t))^{-1/2} & c_g g(t) \ge 2e^{2d} + \gamma\pi^2/2d, \\ \min\left[\gamma/(4d), \frac{dc_g g(t)}{8} + \frac{dc_g g(t)}{8}\log\left(\frac{64\gamma\pi^2}{c_g g(t)}\right)\right] & c_g g(t) < 2e^{2d} + \gamma\pi^2/2d. \end{cases}$$

Suppose now $\rho > 0$. Then we have $g(t) \to \infty$ and $1 = \chi_-^*(t) \ge \chi_+^*(t) = 1 - o(1)$ in the limit $t \to \infty$. Applying Corollary 33 in Appendix 2, that is the limit of oscillation version of Kasahara's Tauberian theorem, we obtain in the limit $E \to -\infty$

$$\log N(E) = \inf_{t>0}[(E-2d)t + G(t)](1+o(1)) = -I(E-2d)(1+o(1)). \tag{51}$$

In the double exponential setting we have $c_g g(t) \to c_g$ and $\chi_-^*(t)$ as well as $\chi_+^*(t)$ converge to constants. Corollary 6 follows now from the analogue of Corollary 33 in the double exponential case proven in [24] (see also Appendix 2). □

Appendix 1: Regular varying functions

Regularly varying functions as introduced in Definition 1 are a generalization of $g(t) = t^\rho$. Their characteristic trait is

$$g(\lambda t)/g(t) \overset{t\to\infty}{=} \lambda^\rho(1+o(1))$$

for all $\lambda \ge 0$. Sometimes it is convenient to transfer attention from infinity to the origin. Thus if $g > 0$,

$$g(\lambda E)/g(E) \overset{E\searrow 0}{=} \lambda^\rho(1+o(1)),$$

we say g is regularly varying at the origin with index ρ, $g \in R_\rho(0+)$. This is equivalent to $g(1/E) \in R_{-\rho}$, [3], p.18. If $\rho = 0$, then g is said to be slowly varying. Regularly varying functions are a generalization of $g(t) = t^\rho$ in the sense that $g \in R_\rho$ implies $g(t) = t^\rho g_0(t)$ with $g_0 \in R_0$ [3], Theorem 1.4.1.

One problem is the inversion of regularly varying functions. Theorem 1.5.12 in [3] says, if $g \in R_\rho$ with $\rho > 0$, then exists an asymptotic inverse $g^{-1} \in R_{1/\rho}$ with

$$g(g^{-1}(t)) \sim g^{-1}(g(t)) \sim t.$$

Up to asymptotic equivalence g^{-1} is unique. A corresponding result in the case $\rho < 0$ with $g^{-1} \in R_{1/\rho}(0+)$ can be deduced from Theorem 1.5.12. To obtain an explicit expression for the asymptotic inverse, we introduce the de Bruijn conjungate $g_0^{\#}$ ([3], p. 29), i.e the slowly varying functions $g_0 \in R_0$ satisfying

$$g_0(t) g_0^{\#}(t g_0(t)) \overset{t\to\infty}{\to} 1, \quad g_0^{\#}(t) g_0(t g_0^{\#}(t)) \overset{t\to\infty}{\to} 1.$$

Again, up to asymptotic equivalence $g_0^{\#}$ is unique. Suppose now that

$$g_0(t g_0(t)^{1/\rho})/g_0(t) \overset{t\to\infty}{\to} 1 \tag{52}$$

holds for some $\rho \neq 0$. As a consequence of Corollary 2.3.4 in [3] we have $(g_0^{1/\rho})^{\#} \sim g_0^{-1/\rho}$ and if $E \sim t^\rho g_0(t)$ then $t \sim E^{1/\rho} g_0(E^{1/\rho})^{-1/\rho}$ in the limit $t, E \to \infty$ if $\rho > 0$, respectively $t \to \infty$, $E \searrow 0$, $\rho < 0$. An example satisfying (52) is given by $g_0(t) = \log(t)$. The interested reader is encouraged to control the statements above for $g(t) = t^\rho \log(t)$.

Appendix 2: Tauber theory for Laplace transforms

In Appendix 2 we collect some results about Tauberian theorems. Given the asymptotic behaviour of the Laplace transform

$$\widehat{N}(t) := \int e^{-\lambda t} \, d\nu(\lambda) < \infty \tag{53}$$

in the limit $t \to \infty$, the problem is to reconstruct the behaviour of the distribution function $v(E) = \nu[E_0, E]$ in the limit $E \searrow E_0 = \sup\{E : v(E) = 0\}$. This is a very common problem in statistical physics, respectively probability theory, and the aim is a characterization of the asymptotics in terms of the Legendre transform.

$$\log\left(v(E)\right) \overset{E\searrow E_0}{\sim} \inf_{t>0} \left[Et + \log(\widehat{N}(t))\right] \tag{54}$$

(e.g. [26], Thm. 9.7). If the exponent of the Laplace transfrom is a regularly varying function, very explicit statements are possible. In the bounded setting one has de Bruijn's Tauberian theorem ([3], Thm. 4.12.9). The corresponding result in the unbounded setting is Kasahara's Tauberian theorem ([3], Thm. 4.12.7).

To apply de Bruijn's, respectively Kasahara's Tauberian theorem one has to know the exact asymptotics of $\widehat{N}(t)$. Sometimes this is a practical problem, because only upper and lower bounds of the Laplace transform are available. But if these bounds are in good agreement, we can apply the so called limit of oscillation theorems. We start by discussing the limit-of-oscillation version of de Bruijn's Tauberian theorem ([2], Thm. 0)

Theorem 30. *Let v be a measure on $(0,\infty)$ whose Laplace transform $\widehat{N}(t)$ satifies (53). By $v(E)$ we denote the distribution function of v. If $\alpha < -1$, $\psi \in R_\alpha(0+)$, put $\phi(\lambda) := \lambda\psi(\lambda) \in R_{\alpha+1}(0+)$. Suppose*

$$-B_1 \leq \liminf_{\lambda\searrow 0} \lambda \log \widehat{N}(\psi(\lambda)) \leq \limsup_{\lambda\searrow 0} \lambda \log \widehat{N}(\psi(\lambda)) \leq -B_2 \tag{55}$$

for $B_1, B_2 > 0$. *Then*

$$-C_1 \le \liminf_{\lambda \searrow 0} \lambda v(1/\phi(\lambda)) \le \limsup_{\lambda \searrow 0} \lambda v(1/\phi(\lambda)) \le -C_2 \quad (56)$$

with $C_1, C_2 > 0$.

Corollary 31. *Let* υ *be a measure on* $(0, \infty)$ *whose Laplace transform* $\widehat{N}(t)$ *satifies (53) and denote by* $v(E)$ *its distribution function. Suppose* $f \in R_\rho$*,* $0 < \rho < 1$ *and*

$$-B_1 f(t)(1 + o(1)) \le \log(\widehat{N}(t)) \le -B_2 f(t)(1 + o(1)) \qquad (t \to \infty) \quad (57)$$

for $B_1, B_2 > 0$. *Then*

$$C_1 \inf_{t>0}[Et - f(t)](1 + o(1)) \le \log(v(E)) \le C_2 \inf_{t>0}[Et - f(t)](1 + o(1)) \quad (E \searrow 0). \quad (58)$$

with $C_1, C_2 > 0$.

Proof. Again we encourage the interested reader to control the statements below for the special case $\widetilde{f}(t) = t^\rho$, $\rho \in (0, 1)$. With $\psi^{-1}(t) = 1/f(t) \in R_{-\rho}$ inequality (57) becomes

$$-B_1 \le \liminf_{t\to\infty} \psi^{-1}(t) \log \widehat{N}(t) \;\le\; \limsup_{t\to\infty} \psi^{-1}(t) \log \widehat{N}(t) \;\le\; -B_2,$$

respectively with $\lambda = 1/t$ and $\psi(\lambda) \in R_{-1/\rho}(0+)$

$$-B_1 \le \limsup_{\lambda \searrow 0} \lambda \log \widehat{N}(\psi(\lambda)) \;\le\; \liminf_{\lambda \searrow 0} \lambda \log \widehat{N}(\psi(\lambda)) \;\le\; -B_2..$$

As a consequence of Theorem 30 and $\phi(\lambda) = \lambda\psi(\lambda) \in R_{\frac{\rho-1}{\rho}}(0+)$ we have

$$-C_1 \le \liminf_{\lambda \searrow 0} \lambda v(1/\phi(\lambda)) \le \limsup_{\lambda \searrow 0} \lambda v(1/\phi(\lambda)) \le -C_2. \quad (59)$$

Setting $E = 1/\phi(\lambda)$ we obtain

$$-C_1/\phi^{-1}(1/E)(1 + o(1)) \;\le\; \log(v(E)) \;\le\; -C_2/\phi^{-1}(1/E)(1 + o(1)). \quad (60)$$

Let us now prove (58). Without restriction we can choose a differentiable version of $f(t)$ [3]. Starting with

$$t^* = \frac{\rho}{E} \frac{1}{\phi^{-1}(\rho/E)}$$

we have

$$\rho/E \sim \phi\left(\frac{\rho}{Et^*}\right) = \psi\left(\frac{\rho}{Et^*}\right) \frac{\rho}{Et^*}. \quad (61)$$

Inversion of ψ gives $\psi^{-1}(t^*) = \rho/(Et^*)$, respectively

$$0 = E - \frac{\rho}{t^*\psi^{-1}(t^*)} = E - \rho f(t^*)/t^* \overset{E\searrow 0}{\sim} E - \frac{d}{dt}f(t)\mid_{t=t^*} . \tag{62}$$

We obtain

$$\inf_{t>0}[Et - f(t)] \;=\; E\frac{\rho}{E}\frac{1}{\phi^{-1}(\rho/E)} - \frac{1}{\phi^{-1}(\rho/E)} \overset{E\searrow 0}{\sim} -(1-\rho)\rho^{\rho/(1-\rho)}/\phi^{-1}(1/E). \tag{63}$$

Combining (60) and (63) we end up with

$$\frac{C_1\rho^{\rho/(\rho-1)}}{1-\rho}\inf_{t>0}[Et - f(t)](1+o(1)) \le \log(v(E)) \le \frac{C_2\rho^{\rho/(\rho-1)}}{1-\rho}\inf_{t>0}[Et - f(t)](1+o(1)).$$

□

The corresponding result in the unbounded setting is the limit-of-oscillation version of Kasahara's Tauberian theorem, ([13], Thm. 1(ii)).

Theorem 32. *Let υ be a measure on $\mathbb{R}$ whose Laplace transform $\widehat{N}(t)$ satifies (53) and denote by $v(E)$ its distribution function. If $0 < \alpha < 1$, $\psi \in R_\alpha$, put $\phi(t) = t/\psi(t) \in R_{1-\alpha}$. Suppose*

$$B_1 \le \liminf_{t\to\infty}\; t^{-1}\log\widehat{N}(\psi(t)) \le \limsup_{t\to\infty}\; t^{-1}\log\widehat{N}(\psi(t)) \le B_2$$

for $B_1, B_2 > 0$. Then

$$C_1 \le \liminf_{E\to-\infty}\; E^{-1}\log v(-\phi(-E)) \le \limsup_{E\to-\infty}\; E^{-1}\log v(-\phi(-E)) \le C_2$$

with $C_1, C_2 > 0$.

Corollary 33. *Let υ be a measure on $\mathbb{R}$ whose Laplace transform $\widehat{N}(t)$ satifies (53). Denote by $v(E)$ the distribution function of υ. Suppose $f \in R_\rho$, $\rho > 1$ and*

$$B_1 f(t)(1+o(1)) \le \log(\widehat{N}(t))) \le B_2 f(t)(1+o(1)) \qquad (t\to\infty) \tag{64}$$

for $B_1, B_2 > 0$. Then

$$C_1\inf_{t>0}[Et + f(t)](1+o(1)) \le \log(v(E)) \le C_2\inf_{t>0}[Et + f(t)](1+o(1)) \quad (E\to-\infty). \tag{65}$$

Proof. Corollary 33 may be proved by a method closely analogous to that used in Corollary 31. □

Remark 34.

(i) The constants $C_1, C_2 > 0$ in the results above are explicetly computable. As a consequence if $B_1 = B_2$ then $C_1 = C_2$, i.e. we obtain de Bruijn's ([3], Thm. 4.12.9), respectively Kasahara's Tauberian theorem ([3], Thm. 4.12.7).

(ii) The key idea linking Laplace and Legendre transform is the concept of the relevant energy interval. In the asymptotic limit for every time t exists an energy $E = E(t)$, s.t. the behaviour of the Laplace transform is determined by a small energy interval around E. Using the Cramér-transform ([12], p.7) the idea above is used in [4] to prove de Bruijn's Tauberian theorem, repectively in [24] not aware of [13] and [2] to prove the corresponding limit of oscillation versions.

(iii) While in general no Tauber theorem exists if $\widehat{N}(t) \in R_1$ it is possible to transfer the limit of oscillation argument via Cramér-transform discussed in (ii) to the double exponential regime [24]. Suppose $B_1, B_2 > 0$, $f(t) = c_g t \log(c_g t) - c_g t$ and

$$B_1 f(t) \le \log \widehat{N}(t) \le B_2 f(t).$$

Then exists $C_1.C_2 > 0$ s.t

$$C_1 \inf_{t>0} \left[Et + f(t)\right](1 + o(1)) \le \log v(E) \le C_2 \inf_{t>0} \left[Et + f(t)\right](1 + o(1)). \quad (66)$$

Acknowledgements: The author thanks Werner Kirsch for a fruitful correspondence including many useful hints! We gratefully acknowledge financial support by the DFG-Forschergruppe 718 'Analysis and stochastics in complex physical systems', the DFG-SFB/TR12 'Symmetries and Universality in Mesoscopic Systems' and the DFG-Schwerpunktprogramm. on 'Interacting stochastic systems of high complexity'.

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