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A fast interface solver for the biharmonic Dirichlet problem on polygonal domains

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Abstract

In this paper we propose and analyze an efficient discretization scheme for the boundary reduction of the biharmonic Dirichlet problem on convex polygonal domains. First we study mapping properties of biharmonic Poincaré-Steklov operators. We show that the biharmonic Dirichlet problem can be reduced to the solution of a harmonic Dirichlet problem and of an equation with the restriction of the Poincaré-Steklov operator. We then propose a mixed FE discretization (by linear elements) of this equation which admits efficient preconditioning and matrix compression resulting in the complexity $\log \varepsilon^{-1} O(N \log^q N)$. Here N is the number of degrees of freedom on the underlying boundary, $\varepsilon > 0$ is an error reduction factor, $q = 2$ or $q = 3$ for rectangular or polygonal boundaries, respectively. As a consequence an asymptotically optimal iterative interface solver for boundary reductions of the biharmonic Dirichlet problem on convex polygonal domains is derived. A numerical example confirms the theory.

1 Introduction

In this paper we derive an efficient discretization method for solving the Dirichlet problem of the biharmonic equation

$$\begin{aligned} \Delta^2 v = 0 \qquad \text{in } \Omega \,, \\ v|_\Gamma = \phi|_\Gamma \,,\ \partial_n v|_\Gamma = \partial_n \phi|_\Gamma \,, \end{aligned} \tag{1.1}$$

where Ω is a convex polygonal domain with boundary Γ and ϕ is a sufficiently smooth function on a neighbourhood of Γ. We describe a fast interface solver with the complexity of the order $\log \varepsilon^{-1} O(N \log^q N)$. Here N is the number of degrees of freedom on the underlying boundary, $\varepsilon > 0$ is an error reduction factor, $q = 2$ in the case of a rectangle and $q = 3$ for a convex polygonal domain. The approach is based on asymptotically optimal algorithms for fast computations with the discrete Poincaré-Steklov operators for the bi–Laplacian on convex polygons.

There exists a large bibliography on approximation methods for biharmonic problems, here we mention only the papers [1, 2, 3, 5, 16, 18], where fast FD and FE domain solvers have been developed. Recently fast numerical algorithms for second order equations based on nonoverlapping domain decomposition (DD) techniques and matrix compression for discrete harmonic Poincaré-Steklov operators (using truncation by frequency cutting) were developed in [13, 14]. Here we extend this approach to the case of the biharmonic equation.

The paper is organized as follows: In Sections 2 and 3 we state some results on boundary integral and Poincaré-Steklov operators for (1.1). It turns out that (1.1) is equivalent to the determination of $\Delta u|_\Gamma$, where u solves

$$\Delta^2 u = 0 \quad \text{in } \Omega \,, \quad u|_\Gamma = 0 \,,\ \partial_n u|_\Gamma = \partial_n \phi|_\Gamma - \partial_n w|_\Gamma \,, \tag{1.2}$$

and w is the harmonic function with the boundary value $w|_\Gamma = \phi|_\Gamma$. In Section 4 we see that for any biharmonic function $u \in H^2(\Omega) \cap H^1_0(\Omega)$ the mapping $S_{12}^{-1} : \partial_n u|_\Gamma \to \Delta u|_\Gamma$ (i.e. the restriction of the Poincaré-Steklov operator to certain subspace) is a symmetric and positive definite operator in appropriate trace spaces and that the problem (1.2) is equivalent to the operator equation

$$S_{12} \tau = \partial_n \phi|_\Gamma - \partial_n w|_\Gamma \,.$$

Using a mixed FE formulation with piecewise linear functions in Section 5 we construct discretizations $S_{12,h}$ of the operator S_{12}, which remain symmetric and positive definite. We show, that the approximate solutions converge with the order $h^{1/2}|\log h|$ to the exact one, where $h = O(N^{-1})$. In order to solve the discretized equations iteratively we construct efficient spectrally equivalent preconditioners for the discrete operators $S_{12,h}$. In Section 6 we adapt the idea of matrix compression for discrete harmonic Poincaré-Steklov operators to the efficient compression of the factorized stiffness matrix of $S_{12,h}$ related to rectangular boundaries. After that the extension to polygonal geometries is suggested based on successive inversion of harmonic interface operators. In this way the advanced tools of DD methods developed for second order elliptic equations may be directly adapted to the biharmonic problems as well. We conclude in Section 7 with some results of numerical experiments which confirm the theory.

2 Boundary integral equations for the bi-Laplacian

In this section we recall some results from [19] on boundary integral operators for the biharmonic Dirichlet problem on piecewise smooth boundaries.

Let Ω a bounded polygonal domain in the plane (x_1, x_2) with m corner points and the boundary $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, where Γ_i are straight lines, and by $C\Omega = \mathbb{R}^2 \setminus \overline{\Omega}$ we denote the exterior domain. The projections of the outward normal $\mathbf{n}$ onto the x_1– and x_2–axis are denoted by n_1 and n_2, respectively. The differentiation with respect to $\mathbf{n}$ is denoted by ∂_n. For the sequel functions on Γ are identified with periodic functions depending on arc length s, the derivative $\partial_s u$ is denoted by u'.

We introduce the trace space

$$V(\Gamma) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \;:\; u_1 \in H^1(\Gamma),\; n_1 u_2 - n_2 u_1' \in H^{1/2}(\Gamma),\; n_2 u_2 + n_1 u_1' \in H^{1/2}(\Gamma) \right\}$$

equipped with the canonical norm and define the generalized trace

$$\gamma u = \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix} := \begin{pmatrix} u|_\Gamma \\ \partial_n u|_\Gamma \end{pmatrix}.$$

Lemma 2.1 ([12]). *The linear mapping*

$$\gamma : H^2_{loc}(\mathbb{R}^2) \to V(\Gamma)$$

is continuous and has a continuous right inverse

$$\gamma^- : V(\Gamma) \to H^2_{loc}(\mathbb{R}^2).$$

The trace $\gamma u \in V(\Gamma)$ will be called the *Dirichlet datum* of $u \in H^2_{loc}(\mathbb{R}^2)$ on Γ. In the following we denote by $(V(\Gamma))'$ the dual space of $V(\Gamma)$ with respect to the duality form

$$\left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] := -\langle v_1, u_1 \rangle_\Gamma + \langle v_2, u_2 \rangle_\Gamma, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the extension of the usual L^2–scalar product on Γ. To define the *Neumann datum* of certain H^2-functions we introduce the space

$$H^2(\Omega,\Delta^2)=\{u\in H^2(\Omega):\Delta^2u\in L^2(\Omega)\}$$

equipped with the graph norm.

Lemma 2.2 *For any $u\in H^2(\Omega,\Delta^2)$ the functional*

$$\delta u:\psi\to[\delta u,\psi]:=\int_\Omega(\Delta u\,\Delta(\gamma^-\psi)-\gamma^-\psi\,\Delta^2u)dx \tag{2.2}$$

belongs to $(V(\Gamma))'$ and coincides, if u is sufficiently smooth, with the functional

$$\delta u=\begin{pmatrix}-\delta_3u\\ \delta_2u\end{pmatrix}:=\begin{pmatrix}\partial_n\Delta u|_\Gamma\\ \Delta u|_\Gamma\end{pmatrix}.$$

The linear operator $\delta:H^2(\Omega,\Delta^2)\to(V(\Gamma))'$ is continuous.

If we define the continuous bilinear form on $H^2(\Omega)\times H^2(\Omega)$

$$a_\Omega(u,v):=\int_\Omega\Delta u\,\Delta v dx\,,$$

then the *first Green formula* reads as

$$a_\Omega(u,v)-\int_\Omega v\,\Delta^2u dx=[\delta u,\gamma v]\;=\langle\delta_3u,\gamma_0v\rangle_\Gamma+\langle\delta_2u,\gamma_1v\rangle_\Gamma$$

for all $u\in H^2(\Omega,\Delta^2)$, $v\in H^2(\Omega)$.

Let us consider the variational form of the Dirichlet problem

$$\Delta^2u=0\quad\text{in }\Omega,\qquad\gamma u=\psi\in V(\Gamma)\,, \tag{2.3}$$

which can be written as

$$\begin{aligned}&\textit{Find }\;u\in H^2(\Omega)\;\textit{ with }\;\gamma u=\psi\;\textit{ such that}\\ &a_\Omega(u,z)=0\;\;\textit{for all }\;z\in H_0^2(\Omega):=\{u\in H^2(\Omega):\gamma u=0\}\,.\end{aligned} \tag{2.4}$$

Since $a_\Omega(u,v)$ is bounded and positive definite on $H_0^2(\Omega)$ (see [8]) Lemma 2.1 implies the unique solvability of (2.3) in the weak sense.

Lemma 2.3 *The Dirichlet problem* (2.3) *has for any $\psi\in V(\Gamma)$ a unique weak solution $u=T\psi\in H^2(\Omega)$. The solution operator $T:V(\Gamma)\to H^2(\Omega)$ is continuous.*

We now derive integral equations for the solution of the Dirichlet problem (2.3). The boundary integral operators for the bi–Laplacian Δ^2 are based on the fundamental solution

$$G(x,y):=\frac{1}{8\pi}|x-y|^2\log|x-y|\,,\quad x,y\in\mathbb{R}^2\,,$$

satisfying

$$\Delta_y^2G(x,y)=\Delta_x^2G(x,y)=\delta(x-y)\,.$$

For $x\in\mathbb{R}^2\backslash\Gamma$ the biharmonic single and double layer potentials are defined as

$$\begin{aligned}\mathcal{K}_0\chi(x)&:=[\chi,\gamma G(x,\cdot)]\,,\qquad &\chi\in(V(\Gamma))'\,,\\ \mathcal{K}_1\psi(x)&:=[\delta G(x,\cdot),\psi]\,,\qquad &\psi\in V(\Gamma)\,.\end{aligned}$$

Lemma 2.4 ([19]) *The mappings* $\mathcal{K}_0 : (V(\Gamma))' \to H^2(\Omega)$, $\mathcal{K}_1 : V(\Gamma) \to H^2(\Omega)$ *are continuous. If* $u \in H^2(\Omega)$ *solves the biharmonic equation* $\Delta^2 u = 0$ *then for all* $x \in \Omega$ *there holds*

$$u(x) = \mathcal{K}_0 \delta u(x) - \mathcal{K}_1 \gamma u(x) . \tag{2.5}$$

For $\chi \in (V(\Gamma))'$ and $\psi \in V(\Gamma)$ we introduce the boundary integral operators

$$\mathcal{A}\chi := \gamma \mathcal{K}_0 \chi \quad , \quad \mathcal{C}\psi := -\gamma(\mathcal{K}_1 \psi|_\Omega) .$$

Lemma 2.5 ([19]) *The mappings* $\mathcal{A} : (V(\Gamma))' \to V(\Gamma)$ *and* $\mathcal{C} : V(\Gamma) \to V(\Gamma)$ *are continuous,* $\mathcal{A}$ *is symmetric and strongly elliptic,* $\mathcal{C}^2 = \mathcal{C}$ *and* $\mathcal{CA} = \mathcal{AC}'$. *Furthermore, there exists a constant* $c > 0$ *such that for any* $\chi \in l(\Gamma)^\perp$ *the inequality*

$$[\chi, \mathcal{A}\chi] \geq c\|\chi\|^2_{(V(\Gamma))'}$$

holds, where $l(\Gamma)^\perp \subset (V(\Gamma))'$ *denotes the polar set of the space of traces of linear functions* $l(\Gamma) := \{\gamma(a_0 + a_1 x_1 + a_2 x_2) : a_0, a_1, a_2 \in \mathbb{R}\}$.

Note that the adjoint operators are taken with respect to the duality (2.1).

Letting in (2.5) the point $x \in \Omega$ converge to the boundary Γ we obtain that any biharmonic function $u \in H^2(\Omega)$ satisfies the relation

$$\mathcal{A}\delta u + \mathcal{C}\gamma u = \gamma u .$$

Hence, if we consider the Dirichlet problem (2.3) then for given $\gamma u = \psi$ the unknown $\chi = \delta u$ has to solve the equation

$$\mathcal{A}\chi = (I - \mathcal{C})\psi . \tag{2.6}$$

Its unique solvability is subjected to the assumption

A1: The exterior homogeneous Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \complement\Omega , \qquad \gamma u = 0 ,$$

has no nontrivial solution satisfying the "radiation condition".

Here the "radiation condition" means that u can be represented in the form $u = \mathcal{K}_0\chi$ for suitable $\chi \in (V(\Gamma))'$, which is equivalent to certain asymptotic behaviour at infinity (see [7], [19]).

Recently Costabel and Dauge proved in [9] that the assumption **A1** is satisfied for the scaled curves

$$\rho\Gamma = \{\rho x \in \mathbb{R}^2, \ x \in \Gamma\} , \quad \rho > 0$$

where Γ is a arbitrary general curve, if and only $\rho \notin S_\Gamma$ and the set S_Γ of exceptional values has between 1 and 4 elements.

Theorem 2.1 *Suppose* **A1** *and let* $\psi \in V(\Gamma)$ *be given. Then the equation* (2.6) *has a unique solution* $\chi \in (V(\Gamma))'$ *and the variational solution* $u \in H^2(\Omega)$ *of the Dirichlet problem* (2.4) *can be obtained from the formula*

$$u(x) = \mathcal{K}_0\chi(x) - \mathcal{K}_1\psi(x) , \quad x \in \Omega .$$

3 Biharmonic Poincaré-Steklov operators

Since $\mathcal{A}$ is a 2×2 matrix of integral operators the standard Galerkin method applied to the equation (2.6) leads to a linear algebraic system with a rather large and in general dense matrix. Moreover, the integral operator with the kernel function $G(x,y)$ is of order 3 and thus the corresponding stiffness matrix possesses the condition number $\mathrm{O}(h^{-3})$. In absence of appropriate preconditioners we expect the complexity $\mathrm{O}(N^3)$ for BEM-Galerkin method provided by direct solvers. Therefore it is very reasonable to look for some alternative efficient discretization methods to solve (2.3) or equivalently (2.6).

To this end we start with the analysis of the biharmonic Poincaré-Steklov operator $\mathcal{T}$ which maps the Dirichlet data γu of a biharmonic function $u \in H^2(\Omega)$ to its Neumann data δu. Due to Lemma 2.3 this operator can be defined as

$$\mathcal{T}\psi := \delta(T\psi)\,, \quad \psi \in V(\Gamma)\,,$$

and Lemma 2.2 implies that $\mathcal{T} : V(\Gamma) \to (V(\Gamma))'$ is continuous. From Theorem 2.1 we see that under the assumption **A1** we have the equality

$$\mathcal{T} = \mathcal{A}^{-1}(I - \mathcal{C})\,. \tag{3.1}$$

The mapping properties of the operator $\mathcal{T}$ can be derived using an appropriate splitting of the trace space. Since

$$\Delta_y G(x,y) = \Delta_x G(x,y) = \frac{1}{2\pi} \log|x-y| + \frac{1}{2\pi}$$

for $\psi = \binom{v_1}{v_2} \in V(\Gamma)$ we get from (2.1) the representation

$$\mathcal{K}_1\psi(x) = -\frac{1}{2\pi}\int\limits_\Gamma v_1(y)\,\partial_{n_y}\log|x-y|\,ds_y + \frac{1}{2\pi}\int\limits_\Gamma v_2(y)\,(\log|x-y|+1)\,ds_y\,, \tag{3.2}$$

showing that $\mathcal{C}$ is the well–known Calderon projection for the Laplacian on Ω. Hence, the conjugate projection $I - \mathcal{C}$ maps onto the traces of functions u harmonic on $\complement\Omega$ which in view of (3.2) satisfy the condition

$$u(x) = a(\log|x|+1) + O(|x|^{-1}) \text{ for some } a \in \mathbb{R} \text{ as } |x| \to \infty\,. \tag{3.3}$$

This leads together with Lemma 2.5 to the following assertions.

Lemma 3.1 [19] *The trace space $V(\Gamma)$ is the direct sum $V(\Gamma) = V_1 \dot{+} V_2$ of the closed subspaces*

$$V_1 := \mathrm{im}\,(I-\mathcal{C}) = \{\gamma u \,:\, u \in H^2_{loc}(\complement\Omega)\,,\ \Delta u = 0\,,\ u \text{ satisfies } (3.3)\}\,,$$

$$V_2 := \mathrm{im}\,\mathcal{C} = \{\gamma u \,:\, u \in H^2(\Omega)\,,\ \Delta u = 0\}\,.$$

The mapping $\mathcal{A} : V_2^\perp \to V_1$ is bijective.

Thus the relation (3.1) holds even if the assumption **A1** is not valid.

Lemma 3.2 [19] *The biharmonic Poincaré-Steklov operator $\mathcal{T}$ is continuous, symmetric and* $\ker \mathcal{T} = V_2$. *The restriction to V_1 is positive definite*

$$[\mathcal{T}\psi, \psi] \geq c\,\|\psi\|^2_{V(\Gamma)}, \quad c > 0, \quad \forall \psi \in V_1 .$$

$\mathcal{T}$ has a symmetric and continuous pseudoinverse $\mathcal{S}$, i.e. $\mathcal{T}\mathcal{S}\mathcal{T} = \mathcal{T}$, which coincides with the restriction of $\mathcal{A}$ to $V_2{}^{\perp}$,

$$\mathcal{S} = \mathcal{A} : V_2{}^{\perp} \to V_1 \subset V(\Gamma) ,$$

and is positive definite on $V_2{}^{\perp}$,

$$[\chi, \mathcal{S}\chi] \geq c\,\|\chi\|^2_{(V(\Gamma))'}, \quad c > 0, \quad \forall \chi \in V_2{}^{\perp}.$$

Let us denote by $\mathcal{H}^2(\Omega)$ the closed subspace of $H^2(\Omega)$ containing the harmonic functions on Ω. Note that $V_2 = \gamma(\mathcal{H}^2(\Omega))$. From Lemma 3.1 we derive

Corollary 3.1 *An element $\chi \in (V(\Gamma))'$ coincides with the Neumann datum δu of a biharmonic function $u \in H^2(\Omega)$ if and only if $[\chi, \gamma v] = 0$ for any function $v \in \mathcal{H}^2(\Omega)$.*

This result confirms the well-known fact that the Neumann problem

$$\Delta^2 u = 0 \quad \text{in } \Omega , \qquad \delta u = \chi \in (V(\Gamma))'$$

is not solvable, in general. But it turns out that it makes sense to consider Neumann-type problems on certain subspaces of $H^2(\Omega)$. Let $W \subset H^2(\Omega)$ be some closed subspace and consider the problem:

$$\begin{aligned} &\textit{Given } \ \tau = \binom{\tau_3}{\tau_2} \in (V(\Gamma))' \ \textit{ find } \ u \in W \ \textit{ such that for all } \ z \in W \\ &a_\Omega(u, z) = [\tau, \gamma z] = \langle \tau_3, \gamma_0 z \rangle_\Gamma + \langle \tau_2, \gamma_1 z \rangle_\Gamma . \end{aligned} \tag{3.4}$$

The bilinear form $a_\Omega(u, v)$ is positive definite on $H^2_0(\Omega)$ for any domain Ω. In the following lemma we give some other subspaces of $H^2(\Omega)$ on which the bilinear form is positive definite if Ω is convex.

Lemma 3.3 *Suppose that the polygonal domain Ω is convex. Then the bilinear form $a_\Omega(u, v)$ is W_i - elliptic where*

$$W_0 := \{u \in H^2(\Omega)/\{1\} : \gamma_1 u = 0\} , \tag{3.5}$$

$$W_1 := H^2(\Omega) \cap H^1_0(\Omega) , \tag{3.6}$$

$$W_2 := H^2(\Omega)/\mathcal{H}^2(\Omega) . \tag{3.7}$$

Proof. Let $u \in W_1$. Then u solves the equation

$$\Delta u \; = \; f \qquad \text{in } \Omega , \qquad \gamma_0 u = 0 ,$$

with $f := \Delta u \in L^2(\Omega)$. The well-known regularity results for the solution of the Poisson equation in non-smooth convex domains imply

$$\|u\|_{H^2(\Omega)} \;\leq\; c\,\|f\|_{L^2(\Omega)} \;=\; c\,a_\Omega(u, u)^{1/2} , \tag{3.8}$$

which proves the assertion for (3.6). The same arguments apply also for (3.5).

Suppose $u \in W_2$ and determine $u_0 \in W_1$ as the solution of

$$\Delta u_0 = \Delta u \qquad \text{in } \Omega\,, \qquad \gamma_0 u_0 = 0\,.$$

Again we have

$$\|u_0\|_{H^2(\Omega)} \le c\|\Delta u\|_{L^2(\Omega)} = c\,a_\Omega(u,u)^{1/2}\,.$$

Since $u_1 = u_0 - u \in H^2(\Omega)$ is harmonic we obtain

$$\|u\|_{W_2} = \inf_{w\in\mathcal{H}^2(\Omega)} \|u+w\|_{H^2(\Omega)} \le \|u+u_1\|_{H^2(\Omega)} \le c\,a_\Omega(u,u)^{1/2}\,. \quad \blacksquare$$

Corollary 3.2 *For any* $\tau \in (V(\Gamma))'$ *and* $i = 0,1,2$ *the variational problem* (3.4) *has a unique solution* $u \in W_i$.

Let us introduce the spaces $Z_i := \gamma\, W_i$, $i = 0,1$. It is clear that the decomposition $V(\Gamma) = Z_1 \dot{+} Z_2$, which holds for a domain with smooth boundary Γ, is no longer true for polygonal domains. However, if the domain is convex then any $\psi \in V(\Gamma)$ can be represented in the form

$$\psi = \psi_0 + \psi_1 + \psi_2\,, \tag{3.9}$$

where $\psi_i \in Z_i$, $i = 0,1$, and $\psi_2 \in V_2 = \gamma(\mathcal{H}^2(\Omega))$.

For example, examining the proof of Lemma 3.3 we obtain the splitting

$$V(\Gamma) = Z_1 \dot{+} V_2 \tag{3.10}$$

given by $u = u_1 + w$, where

$$\begin{aligned} \Delta u_1 &= \Delta u && \text{in } \Omega\,, && \gamma_0 u_1 = 0\,,\\ \Delta w &= 0 && \text{in } \Omega\,, && \gamma_0 w = \gamma_0 u\,, \end{aligned}$$

such that for $\psi = \gamma u$ in (3.9) we have $\psi_0 = 0$, $\psi_1 = \gamma u_1$, $\psi_2 = \gamma w$.

Hence the Dirichlet problem (2.4) with $\psi = \binom{\varphi_0}{\varphi_1} \in V(\Gamma)$ can be reduced to the Dirichlet problems for the Laplace and the biharmonic equation in Ω

$$\begin{aligned} \Delta w &= 0 \quad , && \gamma_0 w = \varphi_0\,,\\ \Delta^2 u_1 &= 0 \quad , && \gamma_0 w = 0\,, \quad \gamma_1 u_1 = \varphi_1 - \partial_n w\,. \end{aligned} \tag{3.11}$$

Another splitting of $V(\Gamma)$ can be derived in the following way. Denote by U the solution of the Poisson equation

$$\Delta U = 1 \qquad \text{in } \Omega\,, \qquad \gamma_0 U = 0\,,$$

set $u = \gamma^- \psi$ and determine the constant c such that $\langle \gamma_1 u - c\gamma_1 U, 1\rangle_\Gamma = 0$. Then

$$\psi - c\gamma U = \psi_0 + \psi_2$$

with

$$\begin{aligned} &\psi_0 = \gamma u_0\,, \quad \text{where } \Delta u_0 = \Delta u - c\,, \quad \gamma_1 u_0 = 0\,,\\ &\psi_2 = \gamma w\,, \quad \text{where } \Delta w = 0\,, \qquad \gamma_1 w = \gamma_1 u - c\gamma_1 U\,. \end{aligned}$$

Hence for convex Ω we get also the splitting

$$V(\Gamma) = Z_0 \dotplus \operatorname{span}\{\gamma U\} \dotplus V_2 . \tag{3.12}$$

Thus the Dirichlet problem (2.4) can be reduced to a Dirichlet problem for the biharmonic equation with boundary data $v_0 \in Z_0$ and simpler Neumann and Dirichlet problems for the Laplace and Poisson equation on Ω.

At the end of this section we give a characterization of the spaces Z_i which follows from Lemma 2.1 and is valid for any polygonal domain. As usual $u|_{\Gamma_i} \in \widetilde{H}^{1/2}(\Gamma_i)$ means that the extension by zero of $u|_{\Gamma_i}$ to Γ belongs to $H^{1/2}(\Gamma)$.

Lemma 3.4

$$v \in Z_0 \quad \textit{if and only if} \quad \gamma_0 v \in Y_0 := \{u \in H^1(\Gamma)/\{1\} : u' \in \prod_{i=1}^{m} \widetilde{H}^{1/2}(\Gamma_i)\} ,$$
$$v \in Z_1 \quad \textit{if and only if} \quad \gamma_1 v \in Y_1 := \prod_{i=1}^{m} \widetilde{H}^{1/2}(\Gamma_i) .$$

Moreover, for $v \in Z_i$, $i = 0, 1$, the norms $\|v\|_{V(\Gamma)}$ and $\|\gamma_i v\|_{Y_i}$ are equivalent.

4 Poincaré-Steklov operators on subspaces

The approximation schemes corresponding to Dirichlet data from the subspaces Z_i, $i = 0, 1$, admit some special efficient solvers which will be derived in Sections 5 and 6. Here we consider the restrictions S_{03}^{-1} and S_{12}^{-1} of the operator $\mathcal{T}$ to the subspaces Z_0 and Z_1.

Let us consider the Dirichlet problem (2.4) with boundary data from Z_0 or Z_1. For $v_0 = \binom{\sigma_0}{0}$ and $v_1 = \binom{0}{\sigma_1}$ we introduce the mappings

$$S_{03}^{-1} \, : \, \sigma_0 \to \delta_3 T v_0 ,$$
$$S_{12}^{-1} \, : \, \sigma_1 \to \delta_2 T v_1 ,$$

which can be described variationally as follows. Let $\bar{v}_i$, $i = 0, 1$, be the weak solutions of (2.4) corresponding to the boundary values $v_i \in Z_i$. The first Green formula yields the equations

$$\langle S_{03}^{-1} \sigma_0, \gamma_0 z \rangle_\Gamma = a_\Omega(\bar{v}_0, z) \quad \text{for all } z \in W_0 \tag{4.1}$$

and

$$\langle S_{12}^{-1} \sigma_1, \gamma_1 z \rangle_\Gamma = a_\Omega(\bar{v}_1, z) \quad \text{for all } z \in W_1 . \tag{4.2}$$

¿From the Lemmas 2.1, 2.2, 3.3 and 3.4 we obtain the following assertion.

Theorem 4.1 *Let Ω be a convex domain. Then the operators*

$$S_{03}^{-1} : Y_0 \to Y_0' \quad \textit{and} \quad S_{12}^{-1} : Y_1 \to Y_1'$$

are both continuous, symmetric and positive definite (spd) *with respect to the duality form $\langle \cdot, \cdot \rangle_\Gamma$.*

Remark 4.1 *If* $u \in W_0$, $\Delta^2 u = 0$ *and* $\Delta u \in H^1(\Omega)$ *then*

$$S_{03}^{-1}\gamma_0 u := -\gamma_1(\Delta u)\,.$$

If $u \in W_1$, $\Delta^2 u = 0$ *and* $\Delta u \in H^{1/2+\varepsilon}(\Omega)$, $\varepsilon > 0$, *then*

$$S_{12}^{-1}\gamma_1 u \;:=\; \gamma_0(\Delta u)\,.$$

By $S_{03} : Y_0' \to Y_0$ and $S_{12} : Y_1' \to Y_1$ we denote the inverse of the operators S_{12}^{-1} and S_{03}^{-1}, respectively, which are both *spd.*

Proposition 4.1 *For any* $\tau_2 \in Y_1'$ *the function* $\sigma_1 = S_{12}\tau_2$ *is given by* $\sigma_1 = \gamma_1 u$, *where* $u \in W_1$ *solves the Neumann-type problem*

$$a_\Omega(u, z) = \langle \tau_2, \gamma_1 z\rangle_\Gamma\,, \quad \forall\, z \in W_1. \tag{4.3}$$

For any $\tau_3 \in Y_0'$ *the element* $\sigma_0 = S_{03}\tau_3 \in Y_0$ *is given by* $\sigma_0 = \gamma_0 u$, *where* $u \in W_0$ *solves the Neumann-type problem*

$$a_\Omega(u, z) = \langle \tau_3, \gamma_0 z\rangle_\Gamma\,, \quad \forall\, z \in W_0. \tag{4.4}$$

Note that in the case of a smooth boundary Γ the operators S_{03}^{-1}, S_{12}^{-1} and their inverses have been considered for example in [11], [17].

In the previous section we have seen that the Dirichlet problem (2.4) can be reduced to simple problems for the Laplace and Poisson equation and to one of the operator equations

$$S_{03}\tau_3 = \sigma_0\,, \qquad \tau_3 \in Y_0'\,, \tag{4.5}$$

$$S_{12}\tau_2 = \sigma_1\,, \qquad \tau_2 \in Y_1'\,, \tag{4.6}$$

where the actions of the operators S_{03} and S_{12} are carried out by solving the problems according to Proposition 4.1. In the sequel we choose the equation (4.6). In the next section we will construct an efficient mixed FE approximation of the operator S_{12}.

Let us deduce some consequences of Theorem 4.1, Proposition 4.1 and Corollary 3.2. We consider the problem:

$$\begin{aligned}&\textit{Given } \tau \in Y_1',\ \textit{find}\ \ u \in W_1 = H^2(\Omega) \cap H_0^1(\Omega)\ \ \textit{such that}\\ &a_\Omega(u, z) = \int_\Omega \Delta u\, \Delta z dx = \langle \tau, \gamma_1 z\rangle_\Gamma\,, \quad \forall\, z \in W_1\,.\end{aligned} \tag{4.7}$$

If we set $\phi := \Delta u \in L_2(\Omega)$ then the equation (4.7) can be written as

$$-\int_\Omega \nabla u \cdot \nabla z\, dx = \int_\Omega \phi\, z\, dx\,, \qquad \forall z \in H_0^1(\Omega)\,, \quad \text{and} \quad \gamma_0 u = 0\,, \tag{4.8}$$

$$\int_\Omega \phi\, \Delta z dx = \langle \tau, \gamma_1 z\rangle_\Gamma\,, \quad \forall\, z \in H^2(\Omega) \cap H_0^1(\Omega)\,. \tag{4.9}$$

Since Ω is convex the solution $u \in H_0^1(\Omega)$ of (4.8) belongs also to $H^2(\Omega)$, thus the system (4.8), (4.9) is equivalent to (4.7). On the other hand, equation (4.9) is a very week formulation of the Dirichlet problem for the Laplace equation (cf. [10]). If $\tau \in H^{1/2}(\Gamma)$ then the solution ϕ of (4.9) is the usual variational solution satisfying

$$\gamma_0\phi = \tau \quad \text{and} \quad \int_\Omega \nabla\phi \cdot \nabla z \, dx = 0 \,, \qquad \forall\, z \in H_0^1(\Omega) \,. \tag{4.10}$$

Therefore the solution operator of (4.9) defined by $\Lambda(\tau) = \phi$ is bounded from $H^{1/2}(\Gamma)$ into $H^1(\Omega)$. By applying Lemmas 2.1 and 3.4 and density arguments this operator can be extended to a bounded mapping $\Lambda : Y_1' \to L_2(\Omega)$.

Using Proposition 4.1 and (4.9) we see that for τ, $\omega \in Y_1'$ there holds

$$\langle S_{12}\tau, \omega\rangle_\Gamma = \int_\Omega \Lambda(\tau)\, \Lambda(\omega)\, dx \,, \tag{4.11}$$

whereas in the case $\omega \in H^{1/2}(\Gamma)$ Green's formula yields the representation

$$\langle S_{12}\tau, \omega\rangle_\Gamma = \int_\Omega \phi w \, dx + \int_\Omega \nabla u \cdot \nabla w \, dx \,, \quad \text{where } w \in H^1(\Omega) \text{ with } \gamma_0 w = \omega \,. \tag{4.12}$$

In particular, from (4.11) and Theorem 4.1 we obtain that for any convex polygonal domain and any Dirichlet data $\tau \in Y_1'$ there holds the equivalence of norms

$$c_1 \, \|\tau\|_{Y_1'} \le \|\Lambda(\tau)\|_{L_2(\Omega)} \le c_2 \, \|\tau\|_{Y_1'} \,. \tag{4.13}$$

5 Mixed FE approximations of S_{12}

We now propose some alternative fast algorithms for solving the discrete counterparts of equation (4.6) corresponding to a mixed FE approximation of the underlying operator. Consider the boundary reduction of the Dirichlet problem for $v_1 \in Z_1$

$$\Delta^2 u = 0 \quad \text{on} \quad \Omega \,, \qquad \gamma_0 u = 0 \,, \quad \gamma_1 u = \sigma \,, \tag{5.1}$$

Ω is a convex polygonal domain, which is equivalent to the solution of the boundary equation

$$S_{12}\tau = \sigma \,, \qquad \tau = \delta_2 u \in Y_1' = \prod_{i=1}^{m} H^{-1/2}(\Gamma_i) \,. \tag{5.2}$$

In view of the decomposition (3.11) it is quite reasonable to make the following assumptions. Let the Dirichlet datum $\psi = \binom{\varphi_0}{\varphi_1} \in V(\Gamma)$ in (2.4) be the generalized trace of a sufficiently smooth function. Accordingly to (3.11) we have to solve equation (5.2) with $\sigma = \varphi_1 - \partial_n w$, where the harmonic in Ω function w has the trace $\gamma_0 w = \varphi_0$. Due to the convexity of Ω there holds $w \in H^{2+\varepsilon}(\Omega)$ with $\varepsilon = (\pi - \alpha)/\alpha > 0$, α denotes the largest interior angle at the corner points of Ω, such that in general $\sigma \in \prod_{i=1}^{m} \widetilde{H}^{1/2+\varepsilon}(\Gamma_i)$ and the solution of (5.1) $u \in H^{2+\varepsilon}(\Omega)$. But on the other hand, since the solution of (2.4) belongs to $H^{3+\mu}(\Omega)$ for some $\mu > 0$ (see [6]) we obtain $\phi = \Delta u \in H^{1+\mu}(\Omega)$ and therefore the solution $\tau = \delta_2 u = \phi|_\Gamma$ of (5.2) satisfies $\tau \in H^{1/2+\mu}(\Gamma)$. Note that for a rectangular domain Ω the boundary condition $\gamma_0 u = 0$ implies even $\tau \in \prod_{i=1}^{m} \widetilde{H}^{1/2+\mu}(\Gamma_i)$.

Let us consider the formal approximate solution of equation (5.2) by a Galerkin boundary element method. This means we have finite dimensional spaces $S_h \subset Y_1'$ of functions given on Γ and determine $\tau_h \in S_h$ such that

$$\langle S_{12}\tau_h, \omega_h\rangle_\Gamma = \langle \sigma, \omega_h\rangle_\Gamma \ , \quad \forall\, \omega_h \in S_h \ . \tag{5.3}$$

Due to Theorem 4.1 the discrete systems (5.3) are uniquely solvable for any h and the quasioptimal estimate

$$\|\tau - \tau_h\|_{Y_1'} \le c \min_{\omega_h \in S_h} \|\tau - \omega_h\|_{Y_1'} \tag{5.4}$$

holds. It is clear that this approximation method is only of theoretical interest. To derive a more practical method we use the fact that for the given σ the solution of (5.1) satisfies $\phi = \Delta u \in H^{1+\mu}(\Omega)$. Therefore (u, ϕ) is a solution of the mixed formulation: find $u \in H_0^1(\Omega)$, $\phi \in H^1(\Omega)$ satisfying the equations

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla z \, dx + \int_\Omega \phi\, z\, dx &= \int_\Gamma \sigma\, \gamma_0 z\, ds \ , && \forall\, z \in H^1(\Omega) \ , \\ \int_\Omega \nabla \phi \cdot \nabla z\, dx &= 0 \ , && \forall\, z \in H_0^1(\Omega) \ . \end{aligned} \tag{5.5}$$

Since any solution of this system satisfies

$$\int_\Omega \phi^2\, dx = \int_\Gamma \sigma\, \gamma_0 \phi\, ds \quad , \quad \int_\Omega |\nabla u|^2\, dx + \int_\Omega \phi\, u\, dx = 0$$

we get from (4.13) the inequalities

$$\|u\|_{H^1(\Omega)} \le c_1\, \|\phi\|_{L_2(\Omega)} \le c_2\, \|\sigma\|_{Y_1} \ ,$$

hence $(u, \Delta u)$ is the unique solution of (5.5).

Now we introduce the simplest FE solution of (5.5). Let be given a family of regular triangulations Υ_h of Ω of size h and spaces $X_h \subset H^1(\Omega)$ of piecewise linear functions related to Υ_h and denote $X_{h,\Gamma} := X_h|_\Gamma$ and $X_{0h} := X_h \cap H_0^1(\Omega)$. These spaces satisfy the following approximation and inverse properties:
For $u \in H^s(\Omega)$, $1 \le s \le 2$, there holds

$$\inf_{u_h \in X_h} \Big(\|u - u_h\|_{L_2(\Omega)} + h \|u - u_h\|_{H^1(\Omega)} \Big) \le c\, h^s\, \|u\|_{H^s(\Omega)} \ . \tag{5.6}$$

There exists $c > 0$, independent of h and $\tau_h \in X_{h,\Gamma}$, such that

$$\|\tau_h\|_{H^s(\Gamma)} \le c\, h^{-s-1/2}\, \|\tau_h\|_{Y_1'} \ , \quad -1/2 < s < 3/2 \ . \tag{5.7}$$

The mixed finite element approximation $(u_h, \phi_h) \in X_{0h} \times X_h$ of (5.1) is given as the unique solution of the system

$$\begin{aligned} \int_\Omega \nabla u_h \cdot \nabla z_h \, dx + \int_\Omega \phi_h\, z_h\, dx &= \int_\Gamma \sigma\, \gamma_0 z_h\, ds \ , && \forall\, z_h \in X_h \ , \\ \int_\Omega \nabla \phi_h \cdot \nabla z_h\, dx &= 0 \ , && \forall\, z_h \in X_{0h} \ . \end{aligned} \tag{5.8}$$

Let us denote by $\lambda(\tau_h) \in X_h$ the discrete harmonic extension of $\tau_h \in X_{h,\Gamma}$ to Ω, i.e.

$$\gamma_0 \lambda(\tau_h) = \tau_h \quad \text{and} \quad \int_\Omega \nabla\lambda(\tau_h) \cdot \nabla z_h \, dx = 0 \, , \qquad \forall \, z_h \in X_{0h} \, , \tag{5.9}$$

and define the operator $S_{12,h}$ by

$$\langle S_{12,h}\tau_h, \omega_h\rangle_\Gamma = \int_\Omega \lambda(\tau_h) \, w_h \, dx + \int_\Omega \nabla u_h \cdot \nabla w_h \, dx \, , \tag{5.10}$$

where τ_h , $\omega_h \in X_{h,\Gamma}$, $w_h \in X_h$ with $\gamma_0 w_h = \omega_h$, and $u_h \in X_{0h}$ solves the equation

$$-\int_\Omega \nabla u_h \cdot \nabla z_h \, dx = \int_\Omega \lambda(\tau_h) \, z_h \, dx \, , \qquad \forall \, z_h \in X_{0h} \, . \tag{5.11}$$

Then obviously the system (5.8) can be written as the equation

$$\langle S_{12,h}\tau_h, \omega_h\rangle_\Gamma = \langle \sigma, \omega_h\rangle_\Gamma \, , \quad \forall \, \omega_h \in X_{h,\Gamma} \, , \tag{5.12}$$

providing the solution $\tau_h = \gamma_0 \phi_h \in X_{h,\Gamma}$.

Note that the definition of $S_{12,h}$ admits two specific forms important for further developments. Choosing in (5.10) $w_h = \lambda(\omega_h)$ we have the equality

$$\langle S_{12,h}\tau_h, \omega_h\rangle_\Gamma = \int_\Omega \lambda(\tau_h) \, \lambda(\omega_h) \, dx \, . \tag{5.13}$$

On the other hand, let us choose in (5.10) $w_h \in X_h$ in such a way that $w_h = 0$ for any interior node of the triangulation Υ_h. Then we obtain

$$\langle S_{12,h}\tau_h, \omega_h\rangle_\Gamma = \int_{\Gamma_h} \lambda(\tau_h) \, w_h \, dx + \int_{\Gamma_h} \nabla u_h \cdot \nabla w_h \, dx \, , \tag{5.14}$$

where $\Gamma_h \subset \Omega$ denotes the union of all elements of the triangulation Υ_h bordering on Γ. We shall use (5.13) to study the mapping properties of the operator $S_{12,h}$, while (5.14) is important for constructing a fast matrix times vector multiplication algorithm for the corresponding stiffness matrix.

Let us denote by $X'_{h,\Gamma}$ the dual space of $X_{h,\Gamma}$ equipped with the norm $\|\cdot\|_{Y_1}$.

Theorem 5.1 *The operator $S_{12,h} : X_{h,\Gamma} \to X'_{h,\Gamma}$ is* (spd) *and for any $\tau_h \in X_{h,\Gamma}$ there holds $\langle S_{12,h}\tau_h, \tau_h\rangle_\Gamma \asymp \|\tau_h\|^2_{Y'_1}$, i.e.*

$$c_1 \|\tau_h\|^2_{Y'_1} \leq \langle S_{12,h}\tau_h, \tau_h\rangle_\Gamma \leq c_2 \|\tau_h\|^2_{Y'_1} \, , \tag{5.15}$$

with constants not depending on h.

The proof follows immediately from (5.13) and

Lemma 5.1 *If the polygonal domain Ω is convex then there exist constants not depending on h such that for any $\tau_h \in X_{h,\Gamma}$ the estimate*

$$c_1 \|\tau_h\|_{Y'_1} \leq \|\lambda(\tau_h)\|_{L_2(\Omega)} \leq c_2 \|\tau_h\|_{Y'_1}$$

is valid.

Proof. Since $\lambda(\tau_h)$ is the FE–approximation of $\Lambda(\tau_h)$ after applying usual error estimates and the inverse property (5.7) we get

$$\|\lambda(\tau_h)\|_{L_2(\Omega)} \leq \|\Lambda(\tau_h)\|_{L_2(\Omega)} + \|\Lambda(\tau_h) - \lambda(\tau_h)\|_{L_2(\Omega)} \leq \|\tau_h\|_{Y_1'} + c\,h\,\|\Lambda(\tau_h)\|_{H^1(\Omega)}$$
$$\leq \|\tau_h\|_{Y_1'} + c\,h\,\|\tau_h\|_{H^{1/2}(\Gamma)} \leq c_2\,\|\tau_h\|_{Y_1'}\,.$$

To prove the other direction we show that

$$\Big|\int_\Gamma \tau_h\,\varphi\,ds\Big| \leq c\,\|\lambda(\tau_h)\|_{L_2(\Omega)}\,\|\varphi\|_{Y_1}\,, \quad \forall\,\varphi \in Y_1\,. \tag{5.16}$$

For given $\varphi \in Y_1$ we solve the Dirichlet problem for the biharmonic equation

$$\Delta^2 v = 0 \quad \text{in } \Omega\,, \quad \gamma_0 v = 0\,, \quad \gamma_1 v = \varphi\,.$$

Due to the Lemmas 2.3 and 3.4 we have $\|v\|_{H^2(\Omega)} \leq c\,\|\varphi\|_{Y_1}$ and Green's formula yields

$$\int_\Gamma \tau_h\,\varphi\,ds = \int_\Omega \lambda(\tau_h)\,\Delta v\,dx + \int_\Omega \nabla\lambda(\tau_h)\cdot\nabla v\,dx\,.$$

Now (5.16) follows immediately from

$$\Big|\int_\Gamma \tau_h\,\varphi\,ds\Big| \leq \Big|\int_\Omega \lambda(\tau_h)\,\Delta v\,dx\Big| + \inf_{\omega_h\in X_{0h}}\Big|\int_\Omega \nabla\lambda(\tau_h)\cdot\nabla(v-\omega_h)\,dx\Big|$$
$$\leq \|\lambda(\tau_h)\|_{L_2(\Omega)}\|\Delta v\|_{L_2(\Omega)} + \|\lambda(\tau_h)\|_{H^1(\Omega)} \inf_{\omega_h\in X_{0h}}\|v-\omega_h\|_{H^1(\Omega)}\,. \quad \blacksquare$$

In contrast to the Galerkin method (5.4) the solutions τ_h of (5.12) corresponding to the mixed formulation do not converge quasioptimal to the exact solution. However, using results of Scholz [20] and the assumptions mentioned above we get the following estimates.

Theorem 5.2 *If the given σ in* (5.1) *is such that the solution satisfies $\Delta u \in H^{1+\mu}(\Omega)$, $\mu > 0$, then the mixed finite element approximation provides the estimate*

$$\|u-u_h\|_{L_2(\Omega)} + h^{1/2}|\log h|\,\|\Delta u - \phi_h\|_{L_2(\Omega)} \leq c\,h|\log h|^2\,\|\Delta u\|_{H^{1+\mu}(\Omega)}\,, \tag{5.17}$$

where $(u_h,\phi_h) \in X_{0h}\times X_h$ is the solution of (5.8). *Moreover, for $\tau_h = \gamma_0\phi_h \in X_{h,\Gamma}$ there holds*

$$\|\gamma_0\Delta u - \tau_h\|_{Y_1'} \leq c\,h^{1/2}|\log h|\,\|\Delta u\|_{H^{1+\mu}(\Omega)}, \tag{5.18}$$

with some constant not depending on h.

Proof. Since (5.17) can be proven quite similar to Theorem 1 in [20] we consider only the estimate (5.18). Let us denote by $\rho_h \in X_{h,\Gamma}$ the Galerkin approximation of τ, i.e.

$$\langle S_{12}\rho_h,\omega_h\rangle_\Gamma = \langle\sigma,\omega_h\rangle_\Gamma\,, \quad \forall\,\omega_h \in X_{h,\Gamma}\,,$$

providing the estimate (see (5.4))

$$\|\tau-\rho_h\|_{Y_1'} \leq c\,h^{1+\mu}\|\tau\|_{H^{1/2+\mu}(\Gamma)}\,. \tag{5.19}$$

¿From (5.5) and (5.8) we know that

$$\int_\Omega \nabla(u_h - v_h)\cdot\nabla z_h\,dx + \int_\Omega (\phi_h - \Lambda(\rho_h))\,z_h\,dx = 0\,, \qquad \forall\, z_h \in X_h\,, \tag{5.20}$$

where $v_h \in H^2(\Omega)\cap H_0^1(\Omega)$ solves the equation $\Delta v_h = \Lambda(\rho_h)$. Now we use the Ritz projection $P_h v_h \in X_{0h}$, defined by

$$\int_\Omega \nabla(v_h - P_h v_h)\cdot\nabla z_h\,dx = 0\,, \qquad \forall\, z_h \in X_{0h}\,,$$

to estimate

$$\|\lambda(\tau_h) - \lambda(\rho_h)\|^2_{L_2(\Omega)} = \int_\Omega (\lambda(\tau_h) - \lambda(\rho_h))^2\,dx + \int_\Omega \nabla(u_h - P_h v_h)\cdot\nabla(\lambda(\tau_h) - \lambda(\rho_h))\,dx\,.$$

Here the second integral vanishes due to the definition of the discrete harmonic extension λ. Noting that $\phi_h = \lambda(\tau_h)$ we obtain from (5.20) that

$$\begin{aligned}\|\lambda(\tau_h) - \lambda(\rho_h)\|^2_{L_2(\Omega)} &= \int_\Omega (\Lambda(\rho_h) - \lambda(\rho_h))(\lambda(\tau_h) - \lambda(\rho_h))\,dx \\ &\quad + \int_\Omega \nabla(v_h - P_h v_h)\cdot\nabla(\lambda(\tau_h) - \lambda(\rho_h))\,dx\,.\end{aligned}$$

Now we use that for any $z_h \in X_h$

$$\Big|\int_\Omega \nabla(v_h - P_h v_h)\cdot\nabla z_h\,dx\Big| \le c\,h^{1/2}|\log h|\,\|\Delta v_h\|_{L_\infty(\Omega)}\,\|z_h\|_{L_2(\Omega)}\,,$$

as proved in [20]. Thus we get

$$\|\lambda(\tau_h) - \lambda(\rho_h)\|_{L_2(\Omega)} \le \|\Lambda(\rho_h) - \lambda(\rho_h)\|_{L_2(\Omega)} + c\,h^{1/2}|\log h|\,\|\Delta v_h\|_{L_\infty(\Omega)}\,.$$

Since (5.19) and the properties of piecewise linear functions imply that

$$\|\rho_h\|_{H^{1/2+\mu}(\Gamma)} \le c\,\|\tau\|_{H^{1/2+\mu}(\Gamma)}$$

we have

$$\|\Lambda(\rho_h)\|_{H^{1+\mu}(\Omega)} \le c\,\|\tau\|_{H^{1/2+\mu}(\Gamma)}$$

uniformly in h. Hence

$$\|\lambda(\tau_h) - \lambda(\rho_h)\|_{L_2(\Omega)} \le c\,h^{1/2}|\log h|\,\|\tau\|_{H^{1/2+\mu}(\Gamma)}\,,$$

proving (5.18) by applying Lemma 5.1. ∎

¿From Theorem 5.1 it is clear that equation (5.12) is suited for iterative solution methods. As main ingredients of an efficient solver we underline the following issues:

(i) easily invertible spectrally close preconditioners for the operator $S_{12,h}$;

(ii) a fast matrix-vector multiplication procedure for the stiffness matrix of the operator $S_{12,h}$ on a polygonal boundary.

The item (ii) will be considered in Section 6. The item (i) is important since in view of Theorem 5.1 the operator $S_{12,h}$ possesses a condition number estimate

$$\kappa(S_{12,h}) = O(h^{-1})$$

requiring an efficient preconditioning in order to construct asymptotically optimal iterative solvers for equation (5.12). We notice that due to (5.15) any easily invertible *spd* operator $B_h : X_{h,\Gamma} \to X'_{h,\Gamma}$ providing the equivalence

$$\langle B_h u, u\rangle_\Gamma \asymp \|u\|^2_{Y_1'} , \quad \forall u \in X_{h,\Gamma} \tag{5.21}$$

gives an appropriate preconditioner. Let us construct the stiffness matrix of such an operator. We first split the space $X_{h,\Gamma}$ in the form

$$X_{h,\Gamma} = X_{h,cor} \dot{+} \widetilde{X}_{h,\Gamma} , \quad \text{where} \quad \widetilde{X}_{h,\Gamma} = X_{h,\Gamma} \cap Y_1 \tag{5.22}$$

and the m–dimensional space $X_{h,cor}$ is spanned by the hat functions corresponding to the corner points t_i of Ω, i.e. the piecewise linear functions φ_i, $i = 1,..,m$, with minimal support such that $\varphi_i(t_i) \neq 0$. It can be easily seen that the norm equivalence

$$\|u_c + \tilde{u}_h\|_{Y_1'} \asymp \|u_c\|_{Y_1'} + \|\tilde{u}_h\|_{Y_1'} , \quad \forall\, u_c \in X_{h,cor},\ \tilde{u}_h \in \widetilde{X}_{h,\Gamma}$$

holds. Indeed, we can choose a regular triangulation of Ω with the property that at most two triangles meet at any corner point and the piecewise linear functions supported on these triangles are discrete harmonic. By using (5.13) and Theorem 5.1 the inequality follows. Hence we obtain for $u_c = \sum_{i=1}^{m} a_i\varphi_i \in X_{h,cor}$

$$\|u_c\|^2_{Y_1'} \asymp h^2 \sum_{i=1}^{m} a_i^2 ,$$

and moreover

$$\langle S_{12,h}(u_c + \tilde{u}_h), u_c + \tilde{u}_h\rangle_\Gamma \asymp \|u_c\|^2_{Y_1'} + \|\tilde{u}_h\|^2_{Y_1'} , \quad \forall\, u_c \in X_{h,cor},\ \tilde{u}_h \in \widetilde{X}_{h,\Gamma} .$$

Thus we choose the operator B_h from (5.21) in the form $B_h = B_c \dot{+} \tilde{B}_h$ with respect to the splitting (5.22), where B_c is a m-dimensional scalar operator defined by $B_c u_c = h\, u_c$. To define $\tilde{B}_h$ we note that the operators

$$\Big(-\frac{d^2}{ds^2}\Big)^{1/2} : \widetilde{H}^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$$

are symmetric and positive definite. Hence if we define the discrete mapping

$$\langle D_h u_h, v_h\rangle_\Gamma := \langle u_h', v_h'\rangle_\Gamma , \quad \forall\, u_h, v_h \in \widetilde{X}_{h,\Gamma} , \tag{5.23}$$

then $D_h^{1/2}$ possesses similar mapping properties. Thus the matrix representation $\mathcal{B}_h$ for a preconditioner $\tilde{B}_h$ realizing the norm equivalence

$$\langle \tilde{B}_h \tilde{u}_h, \tilde{u}_h\rangle_\Gamma \asymp \|\tilde{u}_h\|^2_{Y_1'} , \quad \forall\, \tilde{u}_h \in \widetilde{X}_{h,\Gamma} ,$$

can be given by $\mathcal{B}_h = \mathcal{M}_h \mathcal{D}_h^{-1/2} \mathcal{M}_h$, where $\mathcal{M}_h$ is the mass matrix corresponding to the nodal basis $\{\omega_k\} \subset \widetilde{X}_{h,\Gamma}$ and $\mathcal{D}_h$ is the stiffness matrix of the operator D_h from (5.23)

$$(\mathcal{M}_h)_{j,k} = \langle \omega_j, \omega_k\rangle_\Gamma , \qquad (\mathcal{D}_h)_{j,k} = \langle D_h\omega_j, \omega_k\rangle_\Gamma .$$

Theorem 5.3 *The relation of spectral equivalence*

$$\langle S_{12,h}u_h, u_h\rangle_\Gamma \asymp \langle B_c u_c, u_c\rangle_\Gamma + (\mathcal{M}_h \mathcal{D}_h^{-1/2} \mathcal{M}_h U_h, U_h)$$

holds uniformly in $h > 0$ for any $u_h = u_c + \tilde{u}_h$ with $u_c \in X_{h,cor}$, $\tilde{u}_h \in \widetilde{X}_{h,\Gamma}$, where U_h is the vector representation of $\tilde{u}_h$ in the nodal basis and $B_c u_c = h\, u_c$.

Note that the matrix-vector multiplication for $\mathcal{B}_h^{-1} = \mathcal{M}_h^{-1}\mathcal{D}_h^{1/2}\mathcal{M}_h^{-1}$ which appears in corresponding PCG iterations for solving the equation (5.12) costs $\sum_i O(n_i \log n_i)$ operations since the stiffness matrix of the operator D_h from (5.23) has a tridiagonal form providing the Fourier eigenbasis.

6 Matrix compression and fast iterative boundary solvers

In this section we propose and analyze an efficient matrix times vector multiplication algorithm for the operator $S_{12,h}$ with complexity of the order $O(N_\Gamma \log^2 N_\Gamma)$ in the case of a rectangle, where N_Γ is the number of the degrees of freedom on the underlying boundary $\Gamma = \cup_{i=1}^4 \Gamma_i$. After that we extend the proposed technique to the case of right triangles and to convex polygons applying the well-developed iterative substructuring interface solvers for the Laplace equation.

Let Ω be the rectangular domain which we assume, to simplify the exposition, to be the unit square. Consider the approximate solution of the problem

$$\Delta^2 u = 0, \quad \gamma_0 u = 0, \quad \Delta u|_{\Gamma_1} = \varphi \in \widetilde{H}^{1/2}(\Gamma_1), \quad \Delta u|_{\Gamma \setminus \Gamma_1} = 0 \tag{6.1}$$

related to the mixed FE scheme. Let φ have the Fourier expansion

$$\varphi(x) = \sum_{k=1}^{n-1} c_k \sin k\pi x\,, \quad x \in [0,1]\,.$$

The idea of our method is based on the discrete counterpart of the representation

$$u(x,y) = \sum_{k=1}^{n-1} \frac{c_k}{2\pi k} E(k,y) \sin k\pi x \tag{6.2}$$

for the corresponding solution of (6.1), where

$$E(k,y) = y \cdot \frac{e^{\pi k y} + e^{-\pi k y}}{e^{\pi k} - e^{-\pi k}} - \frac{e^{\pi k} + e^{-\pi k}}{e^{\pi k} - e^{-\pi k}} \cdot \frac{e^{\pi k y} - e^{-\pi k y}}{e^{\pi k} - e^{-\pi k}}\,.$$

Since $E(k,y)$ behaves like $e^{-k\pi(1-y)}$ as $y \to 0$ the expansion (6.2) exhibits fast exponential decay in $y \to 0$ for the high frequency components of the solution u, see Fig. 1 and 2.

The point is that this property can be extended to the mixed FE discretization of (6.1) related to uniform meshes. Let X_h be the space of piecewise linear finite elements defined for the uniform triangulation Υ_h of Ω with mesh size $h = n^{-1} = 2^{-p}$, $p \in \mathbb{N}$. To apply the representation (5.14) with

$$\tau_h = \begin{cases} \varphi(x)\,, & x \in \Gamma_1\,, \\ 0\,, & x \in \Gamma \setminus \Gamma_1\,, \end{cases}$$

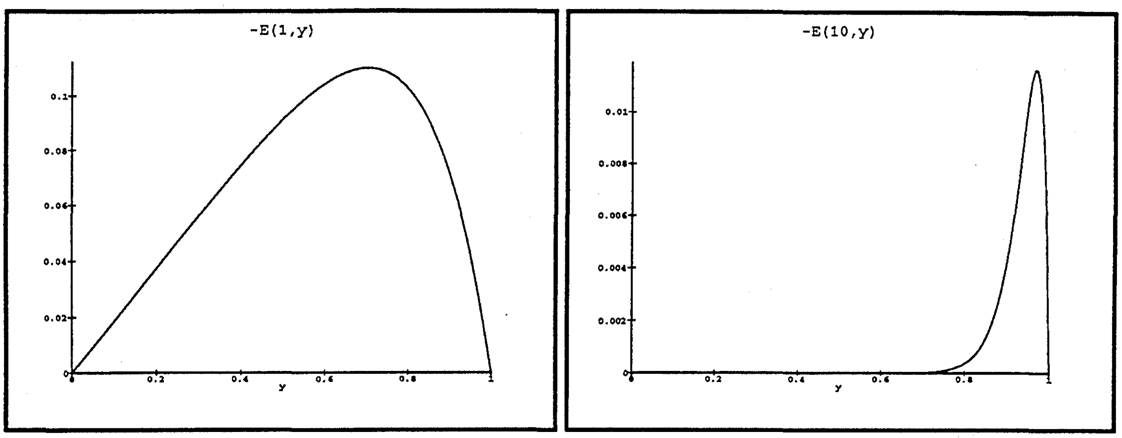

Figure 1: Graphs of $E(k,y)$ vs. y for $k=1$ and $k=10$.

we have to compute the solution u_h on the near boundary grid layer only.

Let the vector representation $\overline{\varphi} = \{\varphi_i\}_{i=1}^{n-1}$ of the given function $\varphi \in \widetilde{X}_{h,\Gamma_1}$, where $\widetilde{X}_{h,\Gamma_j} := \widetilde{X}_{h,\Gamma}|_{\Gamma_j}$, have the Fourier expansion

$$\varphi_i = \sum_{k=1}^{n-1} c_k \sin k\pi\frac{i}{n}, \qquad i = 1,\dots,n-1.$$

Then the solution ϕ_h of (5.8) has the vector representation

$$(\phi_h)_{ij} = \sum_{k=1}^{n-1} c_k \left(\frac{\lambda_k^j - \lambda_k^{-j}}{\lambda_k^n - \lambda_k^{-n}}\right) \sin k\pi\frac{i}{n}, \quad i,j \in [0,n], \tag{6.3}$$

with

$$\lambda_k = 1 + 2\sin^2\frac{k\pi}{2n} + 2\sin\frac{k\pi}{2n}\sqrt{1+\sin^2\frac{k\pi}{2n}}.$$

Substituting (6.3) in (5.11) one obtains

$$u_{ij} := (u_h)_{ij} = \sum_{k=1}^{n-1} c_k \frac{\lambda_k^2 - 1}{n\,\lambda_k}\left(\frac{j}{n}\sigma_1(k,j) - \sigma_1(k,n)\,\sigma_2(k,j)\right) \sin k\pi\frac{i}{n}, \tag{6.4}$$

where we denote

$$\sigma_1(k,j) = \frac{\lambda_k^j + \lambda_k^{-j}}{\lambda_k^n - \lambda_k^{-n}}, \quad \sigma_2(k,j) = \frac{\lambda_k^j - \lambda_k^{-j}}{\lambda_k^n - \lambda_k^{-n}}.$$

Computations with (6.4) on the grid-lines $j=1$ and $j=n-1$ may be performed by FFT with the complexity $O(n\log n)$. Furthermore, for the grid-line $i=1$ (the same for $i=n-1$) we get

$$u_j = u_{1j} = \sum_{k=1}^{n-1} c_k\,\alpha_k \left(\frac{j}{n}\sigma_1(k,j) - \sigma_1(k,n)\,\sigma_2(k,j)\right), \quad \alpha_k = \frac{\lambda_k^2-1}{n\,\lambda_k}\sin\frac{k\pi}{n}. \tag{6.5}$$

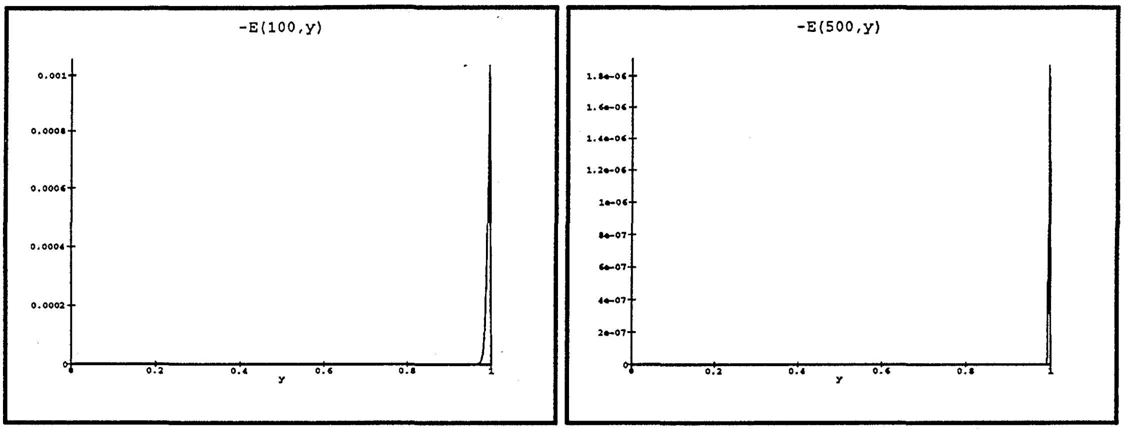

Figure 2: Graphs of $E(k,y)$ vs. y for $k=100$ and $k=500$.

Thus the vector $\overline{u}_1 = (u_j)_{j=1}^{n-1}$ admits the representation

$$\overline{u}_1 = \mathcal{Z}\,\mathcal{D}_0\,\mathcal{F}_n\,\overline{\varphi},$$

where $\mathcal{D}_0$ and $\mathcal{F}_n$ are diagonal and FFT matrices, respectively. The dense matrix $\mathcal{Z} = \{z_{jk}\}_{j,k=1}^{n-1}$ with

$$z_{jk} = \frac{j}{n}\,\sigma_1(k,j) - \sigma_1(k,n)\,\sigma_2(k,j)$$

may be approximated with given accuracy $O(n^{-\alpha})$, $\alpha > 0$, by the sparse matrix

$$\mathcal{Z}^l = \{z^l_{jk}\}_{j,k=1}^{n-1}, \qquad z^l_{jk} = \begin{cases} z_{jk}\ , & k \le l(j) \\ 0\ , & k > l(j) \end{cases} \tag{6.6}$$

with indices $l(j)$, $j = 1,\ldots,n-1$, satisfying the estimate

$$\sum_{j=1}^{n-1} l(j) = O(n\log^2 n)\ .$$

Theorem 6.1 *With given $\varepsilon = O(n^{-\alpha})$, $\alpha > 0$, let $l(j) \ge \xi_j \log(b\,\xi_j)$, where*

$$\xi_j = \frac{n}{a(n+1-j)}\ , \quad j = 1,\ldots n-1\ ,$$

$a = \log(3+2\sqrt{2})$ and $b = \|\varphi\|_{L^2(\Gamma_1)} \cdot O(\varepsilon)$. Then the estimate

$$\max_j |u_j - (\mathcal{Z}^l\,\mathcal{D}_0\,\mathcal{F}_n\,\overline{\varphi})_j| \le \varepsilon$$

holds, where the number of nonzero entries in $\mathcal{Z}^l$ is $\sum_{j=1}^{n-1} l(j) = O(n\log^2 n)$.

Proof. The proof is based on the estimate [4]

$$\lambda_k \geq \exp(a \cdot k \cdot n^{-1})$$

applied to both terms in the right-hand side of (6.5) with summation over $k \geq l(j)+1$ only. ∎

Since for a rectangle the exact solution $\tau \in \prod_{i=1}^{4} \widetilde{H}^{1/2+\mu}(\Gamma_i)$ it suffices to choose $\widetilde{X}_{h,\Gamma} \subset X_{h,\Gamma}$. as the space of test and trial functions. The compressed operator $S^l_{12,h}$ (related to $S_{12,h}$) being defined on the subspace $\prod_{i=1}^{4} \widetilde{X}_{h,\Gamma_i}$ now admits the factorized matrix form

$$S^l_{12,h} \cong \begin{pmatrix} \mathcal{F}_n\mathcal{D}_1 & \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_2 & \mathcal{Z}^l\mathcal{D}_3 \\ \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_1 & \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_2 \\ \mathcal{F}_n\mathcal{D}_2 & \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_1 & \mathcal{Z}^l\mathcal{D}_3 \\ \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_2 & \mathcal{Z}^l\mathcal{D}_3 & \mathcal{F}_n\mathcal{D}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{F}_n & & & O \\ & \mathcal{F}_n & & \\ & & \mathcal{F}_n & \\ O & & & \mathcal{F}_n \end{pmatrix} \tag{6.7}$$

where $\mathcal{D}_i$, $i = 1, 2, 3$, are diagonal matrices and $\mathcal{Z}^l$ is the truncated matrix defined by (6.6). Here "$\cong$" means the representation up to some permutation transform.

Now one can solve the equation

$$\langle S^l_{12,h}\tau_h, v_h\rangle_\Gamma = \langle \sigma, v_h\rangle_\Gamma \ , \quad \forall v_h \in X_{h,\Gamma} \ ,$$

approximating (5.12) for the unit square Ω by the iterative PCG method using the spectrally equivalent preconditioner $\mathcal{B}_h = \mathcal{M}_h\mathcal{D}_h^{-1/2}\mathcal{M}_h$ for the related stiffness matrix $S^l_{12,h}$ (see Theorem 5.3). In the case of a rectangle the proposed algorithm has the complexity $O(n \log^2 n) \cdot \log \varepsilon^{-1}$. Here $\varepsilon > 0$ is the reduction factor in the stopping criteria of the iterative process.

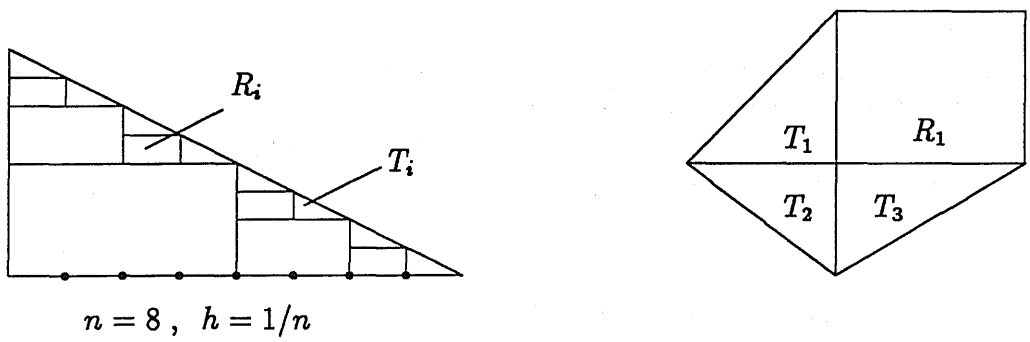

Figure 3: Decompositions of right triangular and polygonal domains.

Now we consider a convex polygonal domain Ω composed of M matching rectangles R_i and K regular right triangles T_i, such that $\bar{\Omega} = (\cup_{i=1}^{M}\bar{R}_i) \cup (\cup_{i=M+1}^{M+K}\bar{T}_i)$ (see Fig.3). We do not discuss here the geometrical problem of producing such a decomposition for an arbitrary convex polygon. Let Υ_h be the piecewise uniform triangulation of Ω aligned with the skeleton $\Gamma_{sk} = \cup_i\Gamma_i$, $\Gamma_i = \partial R_i$ or $\Gamma_i = \partial T_i$, including $\Gamma = \partial\Omega$. Assume the triangulation of any subdomain R_i or T_i to be uniform. Starting with the above coarse decomposition of Ω we introduce a more refined one assuming that any triangle T_i is composed of $n_i - 1$ rectangles and n_i triangles of size $O(h)$, as proposed in [13], see Fig. 3. This produces the so-called refined skeleton $\Gamma'_{sk} \supset \Gamma_{sk}$ which defines the resultant decomposition of Ω into M' rectangles R_i and K' triangles T_i (we use just the same notations for corresponding subdomains as in the case of coarse decomposition).

The strategy for the construction of a compressed operator approximating $S_{12,h}$ is based on the fast computation of the biharmonic Poincaré-Steklov operators $S_{12,i}$ on each subdomain R_i and T_i producing Γ'_{sk} and on the asymptotically optimal interface solvers [13, 15] of the complexity

$$\log \varepsilon^{-1}\Big(\sum_{i=1}^{M} O(n_i \log^2 n_i) + \sum_{i=M+1}^{M+K} O(n_i \log^3 n_i)\Big)$$

to invert the discrete harmonic interface operator $S_{\Gamma'_{sk}}$ defined on the skeleton Γ'_{sk} by

$$\langle S_{\Gamma'_{sk}} u, v\rangle_{\Gamma'_{sk}} = \sum_{i=1}^{M'+K'} \langle S_{\Gamma_i}^{-1} u_i, v_i\rangle_{\Gamma_i}\ , \quad \forall\, u, v \in Y_h := X_h|_{\Gamma'_{sk}}\ .$$

Here $S_{\Gamma_i}^{-1}$ denote the FE discretization of the local harmonic Poincaré-Steklov operators related to R_i or to T_i, while $u_i = u|_{\Gamma_i}$, $v_i = v|_{\Gamma_i}$.

First we note that for the smallest subdomains Ω_i, which produce the refined skeleton Γ'_{sk}, i.e. all triangles T_i and the rectangles containing two triangles of Υ_h, the computation of $S_{12,i}\tau$, $\tau \in X_h|_{\Gamma_i}$, follows simply from formula (5.13), since $\lambda(\tau)|_{\Omega_i}$ coincides with the only element of $X_h|_{\Omega_i}$ having τ as boundary value on Γ_i. Since the number of these smallest subdomains Ω_i is proportional to N, the number of degrees of freedom on Γ, the matrix times vector multiplication for these operators $S_{12,i}$ has the complexity of the order $O(N)$.

In the case of all other rectangular subdomains R_i we apply the compression technique described at the beginning of this section, where we denote by $S^l_{12,i}$ the compressed biharmonic Poincaré-Steklov operator related R_i and to the matrices $\mathcal{Z}^l$, chosen in accordance with Theorem 6.1.

We propose the following algorithm for the computation of $\sigma_h = S^l_{12,h}\tau_h$ on Γ, where now $S^l_{12,h}$ denotes the 'truncated' operator of $S_{12,h}$ corresponding to the choice (6.7) of truncated matrices $S^l_{12,i}$. We introduce $Y_{0h} = X_{0h}|_{\Gamma'_{sk}}$ and denote by $S^l_{12,i}$ also the biharmonic Poincaré-Steklov operator related to the smallest subdomains Ω_i, described above.

Algorithm 1. *Given* $\tau_h \in X_{h,\Gamma}$,

1) Solve for $\phi_h \in Y_h$ *the Dirichlet interface problem on the skeleton* Γ'_{sk}

$$\phi_h|_\Gamma = \tau_h \quad \textit{and} \quad \langle S_{\Gamma'_{sk}}\phi_h,\, v\rangle_\Gamma = 0\ , \quad \forall\, v \in Y_{0h}\ ;$$

2) For all $i \le M' + K'$ *compute the elements*

$$\sigma_i = S^l_{12,i}\phi_i\ , \quad \textit{where} \quad \phi_i = (\phi_h)|_{\Gamma_i}\ ;$$

3) Solve for $\overline{u}_h \in Y_{0h}$ *the Dirichlet interface problem on the skeleton* Γ'_{sk}

$$\langle S_{\Gamma'_{sk}}\overline{u}_h,\, v\rangle_\Gamma = \sum_{i=1}^{M'+K'} \langle \sigma_i, v_i\rangle_{\Gamma_i}\ , \quad \forall\, v \in Y_{0h}\ ;$$

4) Compute for any index $i \leq M' + K'$ such that $\Gamma_i \cap \Gamma \neq \emptyset$

$$\sigma_h|_{\Gamma_i\cap\Gamma} = (\sigma_i + S_{\Gamma_i}^{-1}\overline{u}_h)|_{\Gamma_i\cap\Gamma_i} .$$

Lemma 6.1 *Algorithm* 1 *has the complexity*

$$\log \varepsilon^{-1} \cdot \Big(\sum_{i=1}^{M} O(n_i \log^2 n_i) \; + \sum_{i=M+1}^{M+K} O(n_i \log^3 n_i)\Big) \; .$$

Here $\varepsilon > 0$ is the reduction factor in the stopping criteria of the iterative process for performing steps 1) *and* 3) *of Algorithm* 1.

Similar to the case of the unit square one can solve the equation

$$\langle S^l_{12,h}\tau_h, v_h\rangle_\Gamma = \langle \sigma, v_h\rangle_\Gamma \; , \quad \forall v_h \in X_{h,\Gamma} \; ,$$

approximating (5.12) for the polygonal domain Ω by the iterative PCG method using the spectrally equivalent preconditioner described in Theorem 5.3.

Remind that the case of general Dirichlet data with $\gamma_0 u \neq 0$ is reduced to the just considered case $\gamma_0 u = 0$ by computing beforehand the discrete harmonic Dirichlet–Neumann mapping on Γ providing the complexity $O(N \log^q N)$ with $q = 2$ or $q = 3$ in the case of rectangular or polygonal boundaries, respectively.

Finally we emphasize that the proposed approach can be carried out also for equation (4.5) involving the Poincaré-Steklov operator S_{03}. An important issue on this way is again an efficient preconditioning for the corresponding mixed FE discretization of S_{03}.

7 Numerical example

Here we provide the results of numerical experiments confirming the asymptotic almost optimal performance of the proposed algorithms in the case of rectangular boundary. Note that the iterative substructuring solvers for the interface reductions of the second order elliptic problems used in Algorithm 1 are rather standard tools. So we further restrict ourselves to examine the matrix generation and the matrix times vector multiplication procedures for the operator $S^l_{12,h}$, which actually present the principal part of our algorithm. The results of numerical experiments with fast solvers of the complexity $O(n \log^2 n)$ for the second order elliptic interface problems may be found in [14, 13].

Consider the problem (6.1) on $\Omega = [0,1] \times [0,1]$ with the exact solution u^* corresponding to the given function $\varphi = \sin k\pi x$, $k \in \mathbb{N}$. Table 1 gives the numerical results for $k = 1$ on a sequence of five grids with $n_i = 2^{i+5}$, $i = 1, \dots, 5$. The number of degrees of freedom on Γ for the finest grid is $4n_5 = 4096$. Here $\varepsilon_{rel} = \|\gamma_1 u^* - S^l_{12,h}\tau\|_{L^\infty(\Gamma)} / \|\gamma_1 u^*\|_{L^\infty(\Gamma)}$ is the relative L^∞-error on Γ and τ is the extension of φ by zero to Γ. N_c is the number of nonzero entries in the sparse off-diagonal blocks of the compressed matrix $S^l_{12,h}$ of the form (6.7), while N_{full} is the dimension of the corresponding dense nontruncated blocks. T_{matr} and T_{mult} are the

n_i	ε_{rel}	N_c	N_{full}	T_{matr}	T_{mult}
64	$2.2 \cdot 10^{-4}$	286	4096	0.061	0.049
128	$3.5 \cdot 10^{-5}$	759	16384	0.16	0.11
256	$1.4 \cdot 10^{-5}$	1963	65536	0.5	0.27
512	$3.4 \cdot 10^{-6}$	4930	262144	1.37	0.6
1024	$8.6 \cdot 10^{-7}$	12034	1048576	4.23	1.32

Table 1: Implementation of $S^l_{12,h}$ for rectangular boundary.

times (in sec) exhibited for computations of nonzero matrix entries and for matrix-vector multiplication, respectively, related to $S^l_{12,h}$. We use the cutting parameter $\varepsilon = O(h^2)$, see Theorem 6.1, estimating from below the actual approximation error.

The presented runs were performed on IBM PC 486/66. The actual compression rate achieved on the finest grid is about 1.2%. These results indicate almost optimal, i.e. linear, growth of computing time (up to logarithmic terms) with respect to the number of degrees of freedom on the boundary Γ as in the case of second order elliptic equations. It is expected that the algorithm shows the same performance for general convex polygons.

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