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Chimera states are chaotic transients

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Abstract

Spatiotemporal chaos and turbulence are universal concepts for the explanation of irregular behavior in various physical systems. Recently, a remarkable new phenomenon, called "chimera states", has been described, where in a spatially homogeneous system regions of irregular incoherent motion coexist with regular synchronized motion, forming a self organized pattern in a population of nonlocally coupled oscillators. Whereas most of the previous studies of chimera states focused their attention to the case of large numbers of oscillators employing the thermodynamic limit of infinitely many oscillators, we investigate here the properties of chimera states in populations of finite size using concepts from deterministic chaos. Our calculations of the Lyapunov spectrum show that the incoherent motion, which is described in the thermodynamic limit as a stationary behavior, in finite size systems appears as weak spatially extensive chaos. Moreover, for sufficiently small populations the chimera states reveal their transient nature: after a certain time-span we observe a sudden collapse of the chimera pattern and a transition to the completely coherent state. Our results indicate that chimera states can be considered as chaotic transients, showing the same properties as type-II supertransients in coupled map lattices.

Since their first discovery by Kuramoto and his coauthors [1], chimera states have attracted considerable attention [2, 3, 4, 5, 6, 7, 8]. After the notion of chimera states was introduced by Abrams and Strogatz in [2] for spatially homogeneous systems of coupled oscillators in one space dimension, later similar spatio-temporal patterns were found in various heterogeneous systems [9, 10, 11, 12, 13, 14, 15] as well as two dimensional settings [16, 17]. The remarkable new phenomenon in all these systems is the coexistence of synchronized regions and regions with asynchronous motion, displaying together a self organized spatio-temporal pattern of coherent and incoherent motion. This new paradigm of dynamical behavior can serve as a prototype for various physical phenomena, e.g. coexistence of synchronous and asynchronous neural activity [18, 19] or turbulent-laminar flow patterns [20].

Starting from the pioneering work of Kuramoto [1] the thermodynamic limit $N \to \infty$, has become the most important tool for the study of chimera states. Following the approach of Pikovsky and Rosenblum [21], or alternatively Ott and Antonsen [22, 23], one can derive a limiting system of dynamical equations for macroscopic quantities, where chimera states appear as stable stationary patterns. However, in a similar way as Mirollo and Strogatz showed in [24] for partially locked states in the continuum limit of the classical Kuramoto model, it has been shown recently [25] that in the thermodynamic limit, chimera states are only neutrally stable, having continuous spectrum on the imaginary axis.

Based on the understanding of the thermodynamic limit $N \to \infty$, there appear the following natural questions about the finite size effects for chimera states in the finite dimensional setting: 1. How long do the chimera states persist when the number of oscillators N decreases, and in which way do they finally disappear? 2. How can their incoherent motion be understood in terms of classical deterministic chaos? At first glance, the finite size effects will manifest themselves only as noisy fluctuations with respect to the mean values given by the stationary macroscopic quantities obtained in the thermodynamic limit. However, due to the nonlinear nature of the system, these fluctuations may also induce qualitatively new phenomena. A first important feature of the finite dimensional chimera states that is not captured by the thermodynamic limit has been reported in [26]: the irregular motion of the coherent and incoherent region (see also Fig. 4). A second phenomenon resulting from finite size effects will be reported here: the collapse of the chimera. In this paper we demonstrate that after a long time-span a sudden collapse of the chimera pattern and a transition into the completely coherent state can be observed.

The observation of very long irregular transients dates back to the seminal paper of Grebogi, Ott, and Yorke [27] who discovered them in the vicinity of a bifurcation of a chaotic attractor in a low dimensional system. Later, so called supertransients have been found in spatially extended systems, where the length of the transients can grow exponentially with the system size (for recent surveys, see [28, 29]). While the first examples were based on coupled map lattices [30, 31], later also examples with both continuous time and space variable have been reported, see e.g. [32, 33, 34]. Possible applications of this general concept range here from fluid dynamics to chemical reaction kinetics and biological systems. Investigating the statistical properties of the collapse events for chimera states, we show that the average length of the transients grows exponentially with the system size, given by the number of oscillators. We complete this paper by starting with a study of the corresponding Lyapunov spectra, which turn out to be weakly chaotic and to remain stationary until the collapse. Based on these facts, we can conclude that chimera states are type-II supertransients in the sense of [30]. In this way, we can give an answer to the two questions raised above: We provide numerical evidence that finite size chimera states can be considered indeed as chaotic transients. For decreasing system size they disappear not in some kind of bifurcation, but are observed on shorter and shorter time scales.

Our model is an array of N identical non-locally coupled phase oscillators with phases $\vec{\Psi} = (\Psi_1, \ldots, \Psi_N)$ evolving according to

$$\dot{\Psi}_k(t) = \omega - \frac{1}{2R} \sum_{j=k-R}^{k+R} \sin(\Psi_k(t) - \Psi_j(t) + \alpha). \tag{1}$$

The indices have to be considered modulo N, inducing a ring structure on the array. With ω, we denote the natural frequency of the oscillators that can be set to zero, and $\alpha \in (0, \pi/2)$ is Sakaguchi's phase lag parameter [35]. The coupling range R should satisfy $R > 1$, excluding the trivial case of local (next-neighbor) coupling, and $R < (N-1)/2$, excluding also the case of global coupling. A typical chimera solution for model (1) with $N = 40$ and $R = 14$ is shown in Figure 1. Even for this rather small number of oscillators, we can clearly distinguish between oscillators with coherent and incoherent motion. Taking the time averages of the phase velocities, see Fig. 1(b), we still obtain a rather continuous inhomogeneous profile, similar as in the thermodynamic limit (cf. [25]).

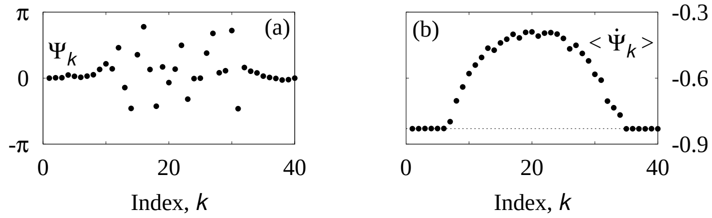

Figure 1: (a) Phase snapshot of a chimera state observed in system (1); (b) time-averaged frequencies. Parameters: $N = 40$, $R = 14$, and $\alpha = 1.46$.

1 The chaotic nature of the finite size chimera

In this section we show that the incoherent motion of a finite size chimera carries the characteristics of weak spatially extended deterministic chaos. We present our calculations of the Lyapunov spectra of chimera states focusing our attention here on the spectra of chimera states in systems with a comparatively small number of oscillators N.

A detailed investigation of the behavior of the Lyapunov spectra for large N can be found in [25]. In particular the limit $N \to \infty$ has been studied there. It has been shown that within the incoherent region the chaos has a spatially extensive nature and that the corresponding exponents tend to zero for $N \to \infty$; the Lyapunov dimension is given asymptotically by the number of incoherent oscillators. Corresponding to the coherent region there is a stable part of the spectrum that has a negative limit; moreover, both parts of the limiting spectrum can be calculated explicitly as the continuous spectrum of the linearized evolution operator for the thermodynamic limit system.

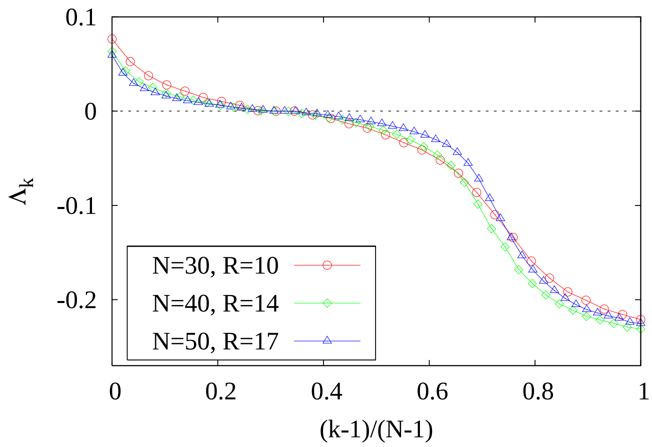

Figure 2: Lyapunov spectra computed for chimera trajectories of (1).

For our numerical computations we used the common fourth-order Runge-Kutta scheme with fixed time step $dt = 0.01$ to integrate system (1) together with the standard algorithm for Lyapunov exponents using continuous Gram-Schmidt orthonormalization [36]. In Figure 2, we show the complete Lyapunov spectra for chimera states with three different values of N. Note, that we have rescaled here the exponent index by the system size N in order to demonstrate the extensive nature of the chaos. Moreover, it can be seen that the positive exponents decay for increasing N. Note that with changing N, we adapted also the coupling range R in order to obtain an approximately fixed ratio between these quantities. Figure 3 indicates that the positive exponents stabilize nicely after a computation over 15000 time units. However, it was not possible to extend the time span of our calculations of the Lyapunov spectra arbitrarily, since for these values of N we could not find chimera trajectories that persist over an arbitrarily long time.

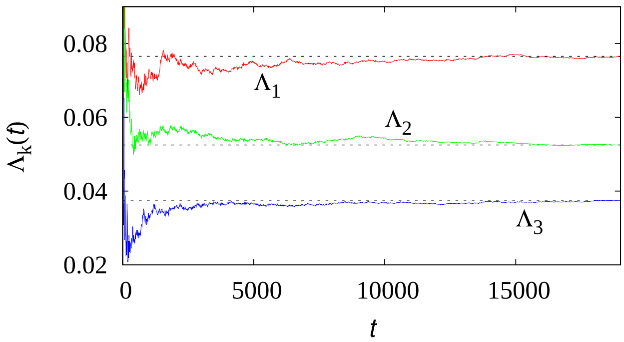

Figure 3: Stabilization of the leading three finite-time Lyapunov exponents calculated along chimera trajectories of increasing lengths. Parameters: $N = 30$, $R = 10$, and $\alpha = 1.46$.

2 The chimera's collapse

In our numerical simulations of Eq. (1) with $N \lesssim 40$, we discovered a surprising phenomenon: The collapse of the chimera. After an apparently stable existence for a quite long time span, the chimera state disappears suddenly and the system changes over to completely coherent motion (see Fig. 4). Note that for such small values of N the irregular motion of the coherent region described in [26] is also very pronounced. The moment τ of the collapse shows a sensitive dependence on the initial data. We used simulations with slightly varying initial conditions to investigate the statistical properties of the collapse events. In Fig. 5, we show a histogram of collapse times τ that we obtained from 2000 trajectories with initial data obtained by small random perturbations (with the amplitude 10^{-3}) of the reference solution in Fig. 4. The collapse event can be easily detected from the global mean field

$$Z(t) := \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\Psi_j(t)} \right|,$$

that for $t > \tau$ suddenly stabilizes at $Z(t) = 1$ (cf. Fig 4). We clearly see that the distribution of the collapse times $\rho(\tau)$ follows an exponential law

$$\rho(\tau) = \lambda e^{-\lambda\tau},$$

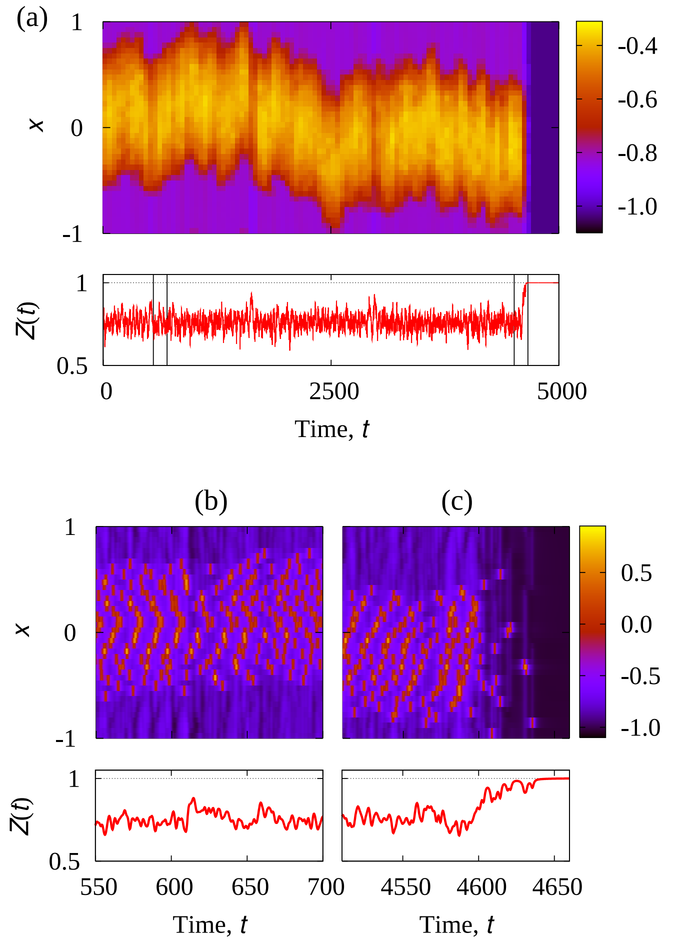

Figure 4: The chimera's collapse: (a) space time plot of averaged phase velocities and global mean field $Z(t)$ for a chimera trajectory that collapses after approx. 4600 time units to the completely coherent state (dark region: slow coherent motion). Panels (b) and (c) show a magnification of segments well before the collapse and directly at the collapse. Parameters: $N = 40$, $R = 14$, and $\alpha = 1.46$.

with a constant collapse rate λ and the average lifetime

$$T_c := \langle \tau \rangle = \lambda^{-1}.$$

In this way, for a given set of parameters the collapse rate and the average lifetime of the chimera state can be obtained by a straightforward fitting procedure.

Varying now the number of oscillators N and extracting the average lifetime $T_c(N)$ in the way described above, we observe an exponential growth $T_c(N) \sim e^{\kappa N}$, see Figure 6. In our example with $\alpha = 1.46$ and $R/N \approx 0.35$, we observed an exponential rate $\kappa = 0.23$. Due to the exponential growth, a numerical evaluation of the collapse statistics for 2000 collapse events was only possible for a system size up to $N = 45$. For $N > 60$ it is already very unlikely to observe even a single collapse event within a time span that is amenable to numerical simulation.

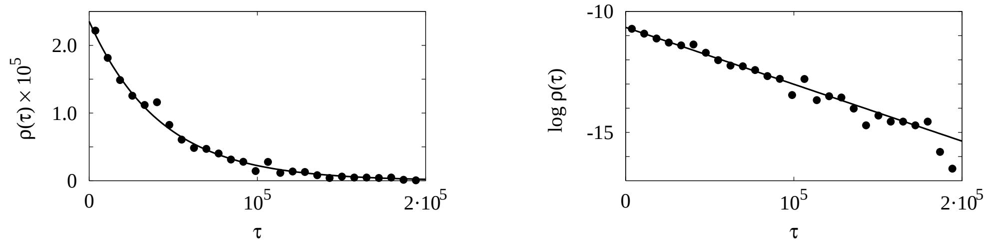

Figure 5: Histogram of collapse times (circles) in natural (left panel) and logarithmic (right panel) scale with fitted exponential distribution (solid line). Parameters as in Fig. 4.

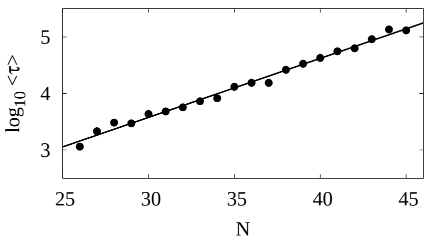

Figure 6: Average lifetimes of chimera states for increasing N from numerical simulations (circles); fitted exponential growth (solid line). Parameters: $R/N \approx 0.35$, and $\alpha = 1.46$.

Regardless, we can conclude that for all values of N the chimera states will eventually collapse to the completely coherent state, and hence have to be considered chaotic transients.

3 Conclusions

The observed exponential growth of the transient time with the system size together with the chaotic Lyapunov spectrum is typical for type-II supertransients in spatially extended systems. In contrast to all earlier examples, the collapsing spatiotemporal chaos appears here together with a regular pattern in space. A further striking difference is the completely trivial dynamics of the single elements, being identical phase oscillators. A key role for the both the appearance of the incoherent motion and the spatial pattern is played by the non-local coupling structure that seems to be essential for the chimera phenomenon. Their transient nature shows that they constitute a large chaotic saddle, introducing a fractal structure at the basin boundary of the completely coherent state.

Our conclusions are based on system (1), which we have chosen as the simplest equation where chimera states can be observed. However, our results seem not to depend on our specific choices, such as the piecewise constant coupling function, the identical natural frequencies, or the coupling of Kuramoto–Sakaguchi type. Instead, we have some indications, that our main findings — the collapse and the weakly chaotic Lyapunov spectra — can be observed similarly in other systems that exhibit the chimera phenomenon.

Based on the thermodynamic limit $N = \infty$, several authors have already calculated stability boundaries, e.g. saddle-node bifurcations, for chimera states. It seems to be an open question, how these results should now be interpreted for finite N chimeras, keeping in mind their transient nature. In particular, the behavior of the average lifetime when approaching the stability boundary is an interesting open problem and will be addressed in a forthcoming paper.

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