Weierstraß-Institut
für Angewandte Analysis und Stochastik
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Linearized plasticity is the evolutionary Γ-limit of finite plasticity

Alexander Mielke[1,2], Ulisse Stefanelli[1,3]

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[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin-Adlershof
Germany

[3] Istituto di Matematica Applicata e Tecnologie Informatiche
Consiglio Nazionale delle Ricerche
via Ferrata 1
I-27100 Pavia
Italy
E-Mail: ulisse.stefanelli@imati.cnr.it

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. We provide a rigorous justification of the classical linearization approach in plasticity. By taking the small-deformations limit, we prove via Γ-convergence for rate-independent processes that energetic solutions of the quasi-static finite-strain elastoplasticity system converge to the unique strong solution of linearized elastoplasticity.

1. INTRODUCTION

This paper is devoted to the rigorous justification of the classical linearization approach in finite-strain elastoplasticity. When restricting to the small-deformation realm it is indeed customary to leave the nonlinear finite-strain frame and resort to linearized theories instead. This reduction is usually motivated by means of heuristic Taylor expansion arguments. Here, we aim at complement these formal motivations by providing a rigorous linearization proof by means of an *evolutionary* Γ*-convergence* analysis of rate-independent processes. In particular, we address the general time-dependent case, which e.g. allows for cyclic loading.

In the stationary framework, the pioneering contribution in this context goes back to DAL MASO, NEGRI, & PERCIVALE [DNP02] who devised a convergence proof of finite-strain elasticity to linearized elasticity. Later, the argument was extended to multi-well energies by SCHMIDT [Sch08] and to residually stressed materials by PARONI & TOMASSETTI [PT09, PT11]. The reader is also referred to [GN10, MN11, Neu10] for some related results in the direction of homogenization, to [AD11] for an application to the study of nematic elastomers, to [BSV07, Sch09] in the context of convergence of atomistic models, and to [SZ11] in relation with dislocation theory.

To our knowledge, this is the first result in the *evolutionary* case. With respect to the stationary case of [DNP02], the evolution situation is quite more involved. Indeed, the argument in [DNP02] relies on the Γ-convergence proof of the small-deformation energy functional to its linearization limit. Here, we are instead forced to cope with the occurrence of dissipative plastic evolution by means of a delicate recovery sequence construction relating energy and dissipation. We emphasize that finite-strain elastoplasticity is based on the multiplicative decomposition of the strain tensors. Moreover, the plastic tensor is to be considered as an element of a multiplicative matrix group. We have to control these noncommutative *multiplicative* structures in linear function spaces and to establish their convergence to the corresponding *linear* additive structures. In order to give some details in this direction we cannot avoid introducing some minimal notation.

Finite-strain elastoplasticity is usually based on the multiplicative decomposition $\nabla\varphi = F_{\rm el}F_{\rm pl}$ [Lee69]. Here $\varphi : \Omega \to \mathbb{R}^d$ is the deformation of the body with respect to the reference configuration $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ while $F_{\rm el}$ and $F_{\rm pl} \in$ SL(d) stand for the elastic and the plastic strain, respectively. Then, the stored energy in the body is written as

$$\int_\Omega W_{\rm el}(\nabla\varphi F_{\rm pl}^{-1})\,{\rm d}x + \int_\Omega W_{\rm h}(F_{\rm pl})\,{\rm d}x$$

where $W_{\rm el}$ is a frame-indifferent elastic stored-energy density and $W_{\rm h}$ describes hardening. The plastic flow rule is expressed by means of a suitably defined dissipation distance D : SL$(d) \times$ SL$(d) \to [0, \infty]$. In particular $D(F_{\rm pl}, \widehat{F}_{\rm pl})$ represents the minimal dissipated energy for an evolution from the plastic strain $F_{\rm pl}$ to $\widehat{F}_{\rm pl}$ and is given via a positively 1-homogeneous dissipation function R by

$$D(F_{\rm pl}, \widehat{F}_{\rm pl}) = D(I, \widehat{F}_{\rm pl}F_{\rm pl}^{-1}) = \inf \int_\Omega \int_0^1 R(\dot{P}P^{-1})\, {\rm d}t\, {\rm d}x,$$

the infimum being taken among all smooth trajectories $P : [0,1] \to \mathbb{R}^{d\times d}$ connecting $F_{\rm pl}$ to $\widehat{F}_{\rm pl}$. Staring from these functionals, by specifying loadings, boundary, and initial conditions, suitably weak solutions of the quasi-static finite-plasticity system (see Section 2) can be defined. We refer to [Mie03] for more information on the mathematical modeling of finite-strain elastoplasticity. There also models with additional hardening variables are given. Here we however refrain from maximal generality in order to emphasize the main features of the limiting process.

Let now the deformation and the plastic strain be small. In particular, for $\varepsilon > 0$ let $\varphi_\varepsilon = {\rm id}{+}\varepsilon u$ and $F_{{\rm pl},\varepsilon} = I{+}\varepsilon z$ where u is a *small* displacement and z is a *small* plastic strain. Correspondingly, we have that $F_{{\rm el},\varepsilon} = \nabla\varphi_\varepsilon F_{{\rm pl},\varepsilon}^{-1} = ({\rm id}{+}\varepsilon\nabla u)(I{+}\varepsilon z)^{-1}$ and we are lead to the consideration of the small-deformation finite-strain elastoplasticity energy and dissipation functionals

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_\Omega W_{\rm el}\big((I{+}\varepsilon\nabla u)(I{+}\varepsilon z)^{-1}\big)\,{\rm d}x + \frac{1}{\varepsilon^2}\int_\Omega W_{\rm h}(I{+}\varepsilon z)\,{\rm d}x,\\ &\frac{1}{\varepsilon}D((I{+}\varepsilon z),(I{+}\varepsilon\widehat{z})).\end{aligned}$$

Note that the rescalings above are such that, by assuming $W_{\rm el}$ and $W_{\rm h}$ to admit a quadratic expansion around identity, one can check that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega W_{\rm el}\big((I{+}\varepsilon\nabla u)(I{+}\varepsilon z)^{-1}\big)\,{\rm d}x \ &\to\ \frac12\int_\Omega(\nabla u{-}z){:}\mathbb{C}(\nabla u{-}z)\,{\rm d}x,\\ \frac{1}{\varepsilon^2}\int_\Omega W_{\rm h}(I{+}\varepsilon z)\,{\rm d}x \ &\to\ \frac12\int_\Omega z{:}\mathbb{H}z\,{\rm d}x\\ \frac{1}{\varepsilon}D((I{+}\varepsilon z),(I{+}\varepsilon\widehat{z})) \ &\to\ \int_\Omega R(\widehat{z}{-}z)\,{\rm d}x.\end{aligned}$$

This pointwise convergence is the classical justification of linearization in plasticity. On the other hand, it is not sufficient in itself for proving that finite-strain elastoplasticity trajectories actually convergence to a solution of the linearized-plasticity system.

Before going on let us mention that the solution concept which is here under consideration is that of *energetic solutions*. Starting from [MT04], this solution notion has been extensively applied in many different rate-independent contexts. We shall however record that one of the main motivations for introducing energetic solutions was exactly that of targeting existence theories for finite-strain elastoplasticity. In this respect, note that the only available existence result for finite-strain elastoplastic evolution has been recently obtained within the energetic solvability frame in [MM09] after adding the regularizing term $|\nabla F_{\rm pl}|^r$ for $r > 1$ (see also [MM06] for some preliminary result),

Our result consists in proving the convergence of energetic solutions of the finite-strain elastoplasticity system to linearized-plasticity solutions. In order to prove this convergence we follow the abstract evolutionary Γ-convergence theory for energetic solutions of rate-independent processes developed in [MRS08]. We shall mention that this evolutionary Γ-convergence method has recently attracted attention and has been successfully considered in connection with numerical approximations [KMR05, MR09, GP06a], damage [BRM09, TM10], fracture [GP06b], delamination [RSZ09], dimension reduction [FPZ10, LM11], homogenization [Tim09], and optimal control [Rin08, Rin09].

According to [MRS08], the convergence of the trajectories $(u_\varepsilon, z_\varepsilon)$ follows by proving two *separate* Γ–$\liminf$ inequalities for energy and dissipation and constructing of a *mutual recovery sequence* relating both. Note that separate Γ-convergence for energy and dissipation is not sufficient to pass to the limit within rate-independent processes. Apart from the additional technicalities due to the presence of the plastic strain and the dissipation functional, it is the delicate construction of the mutual recovery sequence that distinguishes our argument from all the already developed stationary analyses in the spirit of [DNP02].

2. Problem setup and results

Let the reference configuration $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary. Moreover, let $\Gamma \subset \partial\Omega$ be relatively open with $\mathcal{H}^{d-1}(\Gamma) > 0$. We define the state space as

$$\mathcal{Q} := \mathcal{U} \times \mathcal{Z} := \big\{ u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \mid u = 0 \text{ on } \Gamma \big\} \times \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}).$$

Note that the choice of the homogeneous Dirichlet condition on the displacement u is just motivated by the sake of simplicity. In particular, different boundary conditions may be considered as well.

For all given $A \in \mathbb{R}^{d\times d}$ we denote its symmetric and antisymmetric parts as $A^{\mathrm{sym}} := (A{+}A^\top)/2$ and $A^{\mathrm{anti}} = A{-}A^{\mathrm{sym}}$. We indicate by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\mathbb{R}^{d\times d}_{\mathrm{anti}}$ the subspaces of symmetric and antisymmetric tensors, respectively, whereas $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ stands for the subspace of symmetric and trace-free tensors, also called *deviatoric* tensors. The standard Euclidian tensor norm is denoted by $|\cdot|$ and, for all $A \in \mathbb{R}^{d\times d}$ and $\tau > 0$, $B_\tau(A)$ indicates the ball $B_\tau(A) := \{B \in \mathbb{R}^{d\times d} \mid |A-B| < \tau\}$. Moreover, the symbol $|\cdot|_{\mathbb{T}}$ stands for the seminorm

$$|A|^2_{\mathbb{T}} := \frac{1}{2} A{:}\mathbb{T}A$$

where the 4-tensor $\mathbb{T} \in \mathbb{R}^{d\times d\times d\times d}$ is symmetric ($\mathbb{T}_{ijk\ell} = \mathbb{T}_{k\ell ij}$) and positive semidefinite. For finite-strain elastoplasticity we use the classical notations

$$\begin{aligned}
\mathsf{SL}(d) &:= \{P \in \mathbb{R}^{d\times d} \mid \det P = 1\},\\
\mathsf{SO}(d) &:= \{R \in \mathbb{R}^{d\times d} \mid R^\top R = RR^\top = I\},\\
\mathsf{GL}_+(d) &:= \{Q \in \mathbb{R}^{d\times d} \mid \det Q > 0\}.
\end{aligned}$$

We assume that the elastic energy density functional W_{el} fulfills

(2.1a) $\quad W_{\mathrm{el}} \in \mathrm{C}^1(\mathsf{GL}_+(d)),$

(2.1b) $\quad \forall F \in \mathsf{GL}_+(d)\ \forall R \in \mathsf{SO}(d):\ W_{\mathrm{el}}(RF) = W_{\mathrm{el}}(F),$

(2.1c) $\quad \forall F \in \mathsf{GL}_+(d):\ W_{\mathrm{el}}(F) \geq c_1 \mathrm{dist}^2(F, \mathsf{SO}(d)),$

(2.1d) $\quad \forall F \in \mathsf{GL}_+(d):\ |F^\top \partial_F W_{\mathrm{el}}(F)| \leq c_2(W_{\mathrm{el}}(F) + c_3),$

(2.1e) $\quad \exists \mathbb{C} \geq 0\ \forall \delta > 0\ \exists c_{\mathrm{el}}(\delta) > 0\ \forall A \in B_{c_{\mathrm{el}}(\delta)}(0):\ \big|W_{\mathrm{el}}(I{+}A) - |A|^2_{\mathbb{C}}\big| \leq \delta |A|^2_{\mathbb{C}},$

for some positive c_1, c_2. Assumption (2.1b) is nothing but frame indifference and the nondegeneracy requirement (2.1c) is quite classical. Assumption (2.1d) entails the controllability of the *Mandel tensor* $F^\top \partial_F W_{\mathrm{el}}(F)$ by means of the energy. This is a crucial condition in finite-strain elastoplasticity (cf. [Bal84b, Bal02]) and was used in the context of rate-independent processes in [FM06, MM09]. Condition, (2.1e) encodes the local quadratic character of $W_{\rm el}$ around identity. More precisely, (2.1e) states that $|\cdot|_{\mathbb{C}}$ is the second order Taylor expansion of $W_{\rm el}$ at I, and may be reformulated by saying that $A \mapsto W_{\rm el}(I{+}A)$ is locally restrained between two multiples of $|\cdot|^2_{\mathbb{C}}$, namely,

$$\forall \delta > 0\, \forall A \in B_{c_{\rm el}(\delta)}(0): \quad (1{-}\delta)|A|^2_{\mathbb{C}} \leq W_{\rm el}(I{+}A) \leq (1{+}\delta)|A|^2_{\mathbb{C}}.$$

Moreover, (2.1e) entails

$$W_{\rm el}(I) = 0, \ \ \partial_F W_{\rm el}(I) = 0, \ \ \partial^2_F W_{\rm el}(I) = \mathbb{C}, \tag{2.2}$$

which, in particular, yields that the reference state is stress free. On the other hand, by assuming (2.2) and letting $W_{\rm el} \in {\rm C}^2$ in neighborhood of I, relation (2.1e) follows.

Note that the symmetry of the elastic tensor $\mathbb{C}$ (implicitly assumed in the notation $|\cdot|_{\mathbb{C}}$) may be directly obtained from the last of (2.2) by assuming additional smoothness on $W_{\rm el}$. Moreover, letting $A \in \mathbb{R}^{d\times d}$ be given, as we have that $\exp(A^{\rm anti}) \in \mathsf{SO}(d)$, the frame indifference (2.1b) entails that the function $t \mapsto \partial_F W_{\rm el}(\exp(tA^{\rm anti}))$ is constantly equal to $\partial_F W_{\rm el}(I) = 0$. Hence, by taking its derivative with respect to t and evaluating it at $t = 0$ we get $\mathbb{C}A^{\rm anti} = 0$. Namely, $\mathbb{C}$ necessarily fulfills also the so called *minor symmetries* $\mathbb{C}_{ijk\ell} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}$ and we have

$$\forall A \in \mathbb{R}^{d\times d}: \ \mathbb{C}A = \mathbb{C}A^{\rm sym}. \tag{2.3}$$

On the other hand, as effect of the nondegeneracy (2.1c) and assumption (2.1e) we have that $\mathbb{C}$ is positive definite on $\mathbb{R}^{d\times d}_{\rm sym}$. Indeed, by linearizing $d(\cdot, \mathsf{SO}(d))$ around identity we have [FJM02, (3.21)]

$$\forall B \in \mathbb{R}^{d\times d}: \quad d(B, \mathsf{SO}(d)) = |B^{\rm sym}{-}I| + O(|B{-}I|^2). \tag{2.4}$$

Hence, given $A \in \mathbb{R}^{d\times d}$ and $\eta,\ \delta > 0$, by choosing $B = I{+}\eta A$ in the latter we have

$$\begin{aligned} c_1|A^{\rm sym}|^2 &\overset{(2.4)}{=} \lim_{\eta\to 0} \frac{c_1}{\eta^2} d^2(I{+}\eta A, \mathsf{SO}(d)) \\ &\overset{(2.1c)}{\leq} \lim_{\eta\to 0} \frac{1}{\eta^2} W_{\rm el}(I{+}\eta A) \overset{(2.1e)}{\leq} (1{+}\delta)|A|^2_{\mathbb{C}} \end{aligned}$$

so that, by taking $\delta \to 0$, we have

$$\forall A \in \mathbb{R}^{d\times d}: \quad c_1|A^{\rm sym}|^2 \leq |A|^2_{\mathbb{C}} = |A^{\rm sym}|^2_{\mathbb{C}}. \tag{2.5}$$

Note that all assumptions (2.1a)-(2.1e) are consistent with the usual polyconvexity framework

$$\begin{aligned} &F \mapsto W_{\rm el}(F) \ \text{ polyconvex}, \\ &W_{\rm el}(F) \to \infty \ \text{ for } \ \det F \to 0. \end{aligned}$$

Our assumptions on the hardening functional $W_\mathrm{h} : \mathbb{R}^{d\times d} \to [0,\infty]$ read

(2.6a) $$W_\mathrm{h}(P) := \begin{cases} \widetilde{W}_\mathrm{h}(P) & \text{if } P \in K, \\ \infty & \text{if } P \in \mathbb{R}^{d\times d} \setminus K, \end{cases}$$

(2.6b) where K is compact in $\mathsf{SL}(d)$ and contains a neighborhood of I,

(2.6c) $\widetilde{W}_\mathrm{h} : \mathbb{R}^{d\times d} \to \mathbb{R}$ is locally Lipschitz continuous and

(2.6d) $\exists \mathbb{H} \geq 0\ \forall \delta > 0\ \exists c_\mathrm{h}(\delta) > 0\ \forall A \in B_{c_\mathrm{h}(\delta)}(0) : \ \big|\widetilde{W}_\mathrm{h}(I{+}A) - |A|^2_\mathbb{H}\big| \leq \delta |A|^2_\mathbb{H}$,

(2.6e) $\exists c_3 > 0\ \forall A \in \mathbb{R}^{d\times d} : \ W_\mathrm{h}(I{+}A) \geq c_3 |A|^2$.

Note that by assumption (2.6b) we can find a constant $c_K > 0$ such that

(2.7) $$P \in K \ \ \Rightarrow \ \ |P| + |P^{-1}| \leq c_K,$$

(2.8) $$P \in \mathsf{SL}(d) \setminus K \ \ \Rightarrow \ |P - I| \geq \frac{1}{c_K}.$$

The rather strong technical assumption on W_h that its effective domain $K = \{P \in \mathsf{SL}(d) \mid W_\mathrm{h}(P) < \infty\}$ fulfills (2.7) is crucial as it will provide L^∞-bounds that are essential in order to control the multiplicative terms $(I{+}\varepsilon\nabla u)(I{+}\varepsilon z)^{-1}$. Moreover, by combining (2.6d) and (2.6e) we check that

(2.9) $$\forall A \in \mathbb{R}^{d\times d} : \quad c_3 |A|^2 \leq |A|^2_\mathbb{H}.$$

As for the dissipation we assume that

(2.10a) $R^\mathrm{dev} : \mathbb{R}^{d\times d}_\mathrm{dev} \to [0,\infty]$ convex and positively 1-homogeneous,

(2.10b) $\forall P \in \mathbb{R}^{d\times d}_\mathrm{dev} : \quad c_4 |P| \leq R^\mathrm{dev}(P) \leq c_5 |P|$,

(2.10c) $$R : \mathbb{R}^{d\times d} \to [0,\infty]; \ \ R(z) := \begin{cases} R^\mathrm{dev}(z) & \text{if } z \in \mathbb{R}^{d\times d}_\mathrm{dev}, \\ \infty & \text{else,} \end{cases}$$

for positive $c_4,\ c_5$. Moreover, we define

$$D : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty], \ \text{ with } \ D(P,\widehat{P}) = D(I, \widehat{P}P^{-1}) \ \text{ given by}$$

(2.11) $$D(I,\widehat{P}) := \inf \left\{ \int_0^1 R(\dot{P}P^{-1})\,\mathrm{d}t \ \middle| \ P \in \mathrm{C}^1(0,1;\mathbb{R}^{d\times d}),\ P(0) = I,\ P(1) = \widehat{P} \right\}.$$

Note in particular that $D(I,P) < \infty$ implies $\det P = 1$. Moreover, one can prove that there exists $c_6 > 0$ such that

(2.12) $$\forall P,\, Q \in \mathsf{SL}(d) : \ D(P,Q) \leq c_6\big(1{+}|P|{+}|Q|\big), \quad D(I,P) \leq c_6 |P{-}I|.$$

The quasistatic evolution of the finite-strain and linearized elastoplasticity systems are driven by the energy functionals $\mathcal{W}_\varepsilon,\ \mathcal{W}_0 : \mathcal{Q} \to (-\infty,\infty]$ given by

$$\mathcal{W}_\varepsilon(u,z) := \frac{1}{\varepsilon^2} \int_\Omega W_\mathrm{el}\big((I{+}\varepsilon\nabla u)(I{+}\varepsilon z)^{-1}\big)\,\mathrm{d}x + \frac{1}{\varepsilon^2} \int_\Omega W_\mathrm{h}(I{+}\varepsilon z)\,\mathrm{d}x,$$
$$\mathcal{W}_0(u,z) := \int_\Omega |\nabla u^\mathrm{sym}{-}z^\mathrm{sym}|^2_\mathbb{C}\,\mathrm{d}x + \int_\Omega |z|^2_\mathbb{H}\,\mathrm{d}x.$$

We prescribe the generalized loading as

$$\ell \in W^{1,1}(0,T;\mathcal{U}') \tag{2.13}$$

and, by letting $\ell_\varepsilon := \varepsilon\ell$, we introduce some notation for the total energy functionals $\mathcal{E}_\varepsilon,\ \mathcal{E}_0 : [0,T]\times\mathcal{Q}\to(-\infty,\infty]$ as

$$\mathcal{E}_\varepsilon(t,u,z) := \mathcal{W}_\varepsilon(u,z) - \frac{1}{\varepsilon}\langle \ell_\varepsilon(t),u\rangle = \mathcal{W}_\varepsilon(u,z) - \langle \ell(t),u\rangle$$
$$\mathcal{E}_0(t,u,z) := \mathcal{W}_0(u,z) - \langle \ell(t),u\rangle,$$

Eventually, the dissipative character of the evolution is encoded into the dissipation functions $D_\varepsilon, D_0 : \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\to[0,\infty]$ and functionals $\mathcal{D}_\varepsilon,\mathcal{D}_0 : (L^1(\Omega;\mathbb{R}^{d\times d}))^2\to[0,\infty]$ given by

$$D_\varepsilon(z_1,z_2) := \frac{1}{\varepsilon}D(I{+}\varepsilon z_1, I{+}\varepsilon z_2), \quad D_0(z_1,z_2) := R(z_2{-}z_1),$$
$$\mathcal{D}_\varepsilon(z_1,z_2) := \int_\Omega D_\varepsilon(z_1,z_2)\,\mathrm{d}x, \quad \mathcal{D}_0(z_1,z_2) := \int_\Omega D_0(z_1,z_2)\,\mathrm{d}x.$$

The total dissipation of the process over the time interval $[0,t]\subset[0,T]$ will be given by

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z;[0,t]) := \sup\left\{\sum_{i=1}^N \mathcal{D}_\varepsilon(z(t^i),z(t^{i-1})) \mid \{0=t^0<\cdots<t^N=t\}\right\}$$

where the sup is taken over all partitions of $[0,t]$.

From here on, we term *Rate-Independent System* (RIS) the triple $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ given by the choice of the state space $\mathcal{Q}$ and the energy and dissipation functionals $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$. The term *evolutionary* Γ*-convergence* refers to a suitable notion of convergence for rate-independent systems in the spirit of [MRS08] which in particular entails the convergence of the respective energetic solutions.

A crucial structure in the energetic formulation of RIS is the set $\mathcal{S}_\varepsilon(t)$ of *stable states* at time $t\in[0,T]$, which is defined via

$$\mathcal{S}_\varepsilon(t) := \Big\{(u,z)\in\mathcal{Q} \mid \mathcal{E}_\varepsilon(t,u,z)<\infty \ \text{ and }$$
$$\mathcal{E}_\varepsilon(t,u,z)\le\mathcal{E}_\varepsilon(t,\widehat{u},\widehat{z})+\mathcal{D}_\varepsilon(z(t),\hat{z})\Big\}.$$

Our assumption on the initial data reads

$$\mathcal{S}_\varepsilon(0)\ni(u_\varepsilon^0,z_\varepsilon^0)\to(u_0^0,z_0^0) \ \text{ weakly in } \ \mathcal{Q},\ z_0^0\in\mathbb{R}^{d\times d}_{\mathrm{dev}},$$
$$\mathcal{E}_\varepsilon(0,u_\varepsilon^0,z_\varepsilon^0)\to\mathcal{E}_0(0,u_0^0,z_0^0). \tag{2.14}$$

Note that the latter assumption is not empty as it is fulfilled at least by the natural choice $(u_0,z_0)=(0,0)$ if $\ell(0)=0$.

Definition 2.1 (Energetic solutions)**.** *Let* $\varepsilon\ge0$*. We say that a trajectory* $q_\varepsilon : [0,T]\to(u_\varepsilon,z_\varepsilon)\in\mathcal{Q}$ *is an* energetic solution (related to the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$) *if* $(u_\varepsilon(0),z_\varepsilon(0)) =$

$(u^0_\varepsilon, z^0_\varepsilon)$*, the map* $t \mapsto \langle \dot\ell, u_\varepsilon\rangle$ *is integrable, and, for all* $t \in [0,T]$*,*

(2.15) $$(u_\varepsilon(t), z_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t),$$

(2.16) $$\mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) = \mathcal{E}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) - \int_0^t \langle \dot\ell, u_\varepsilon\rangle \,\mathrm{d}s.$$

An energetic solution will be called a finite-plasticity solution *if* $\varepsilon > 0$ *and a* linearized-plasticity solution *for* $\varepsilon = 0$*.*

Note that linearized-plasticity solutions (u_0, z_0) are unique as effect of the quadratic and uniformly convex character of $\mathcal{W}_0$. Moreover, from assumption (2.13) we get that $(u_0, z_0) \in \mathrm{W}^{1,1}(0,T;\mathcal{Q})$ and

$$\forall t \in [0,T]:\ \mathrm{Diss}_{\mathcal{D}_0}(z_0; [0,t]) = \int_0^t R(\dot z_0)\,\mathrm{d}s.$$

The reader is referred to [Hil50, Lub90, Mar75] for some general introduction to plasticity and to [HR99, Joh76, Suq81] for the classical well-posedness theory for linearized elastoplasticity.

Our main result reads as follows and will be proved in Section 3 as a special instance of the general theory of [MRS08].

Theorem 2.2 (Finite plasticity Γ-converges to linearized plasticity)**.** *Assume* (2.1)-(2.6)*,* (2.10)*, and* (2.13)-(2.14)*. Let* $(u_\varepsilon, z_\varepsilon)$ *be a finite-plasticity solution. Then,* $(u_\varepsilon(t), z_\varepsilon(t)) \to (u_0(t), z_0(t))$ *weakly in* $\mathcal{Q}$ *for all* $t \in [0,T]$ *where* (u_0, z_0) *is the unique linearized-plasticity solution.*

Theorem 2.2 is exclusively a convergence result. In particular, we *assume* that finite-plasticity solutions exist. Note however that the existence of finite-plasticity solutions is presently not known within our minimal assumption frame. A possibility here would be that of considering directly some more regular situations including extra compactifying terms like $|\nabla F_{\rm pl}|^r$ $(r > 1)$ such that finite-plasticity solutions exist [MM09]. We shall not follow this line here but rather present a second result based on *approximate minimizers* of the related incremental problems. Indeed, given the time partitions $\{0 = t^i_\varepsilon < \cdots < t^{N_\varepsilon}_\varepsilon = T\}$ with diameters $\tau_\varepsilon := \max_{i=1,\dots,N_\varepsilon}(t^i_\varepsilon - t^{i-1}_\varepsilon) \to 0$ as $\varepsilon \to 0$, the (iterative) incremental problem

$$(u^i_\varepsilon, z^i_\varepsilon) \in \operatorname*{Arg\,min}_{(u,v)\in\mathcal{Q}} \big(\mathcal{E}_\varepsilon(t^i_\varepsilon, u, z) + \mathcal{D}_\varepsilon(z^{i-1}_\varepsilon, z)\big) \quad \text{for } i = 1, \dots, N_\varepsilon$$

may not be solvable (cf. [CHM02], still see [Mie04, MM06] for some additional discussion). Hence, following [MRS08, Sec. 4] we fix a sequence $0 < \alpha_\varepsilon \to 0$ in order to control the tolerances for the minimizations and consider the following *approximate incremental problem*

(2.17) $$\begin{aligned} &\text{Find iteratively } (u^i_\varepsilon, z^i_\varepsilon) \in \mathcal{Q} \text{ such that}\\ &\mathcal{E}_\varepsilon(t^i_\varepsilon, u^i_\varepsilon, z^i_\varepsilon) + \mathcal{D}_\varepsilon(z^{i-1}_\varepsilon, z^i_\varepsilon)\\ &\le (t^i_\varepsilon - t^{i-1}_\varepsilon)\alpha_\varepsilon + \inf_{(u,v)\in\mathcal{Q}} \big(\mathcal{E}_\varepsilon(t^i_\varepsilon, u, z) + \mathcal{D}_\varepsilon(z^{i-1}_\varepsilon, z)\big).\end{aligned}$$

By the definition of infimum the latter always admits solutions and we will show the following convergence result.

Theorem 2.3 (Convergence of approximate incremental minimizers)**.** *Under the assumptions of Theorem* 2.2 *let* $(u^i_\varepsilon, z^i_\varepsilon)$ *be approximate incremental minimizers and* $(\overline{u}_\varepsilon, \overline{z}_\varepsilon)$ *be the corresponding right-continuous, piecewise-constant interpolants on the time partitions. Then,* $(\overline{u}_\varepsilon, \overline{z}_\varepsilon) \to (u_0, z_0)$ *pointwise weakly in* $\mathcal{Q}$ *where* (u_0, z_0) *is the unique linearized-plasticity solution.*

In the finite-elasticity case (stationary), using ideas from [DNP02] the convergence of approximate minimizers has been considered in [PT09].

3. Proofs

The argument basically follows the lines of the abstract analysis of [MRS08]. Still, our setting cannot be completely recovered from the application of the above-mentioned abstract theory as extra care is needed for the treatment of the multiplicative nonlinearities. We hence resort in providing here an independent proof. After establishing the coercivity of the energy in Subsection 3.1, the proof strategy relies in providing *two separate* Γ–$\liminf$ inequalities for $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ and a *mutual recovery sequence* argument relating both. This is done in Subsections 3.2 and 3.3 below. Eventually, the proofs of Theorems 2.2 and 2.3 are outlined in Subsections 3.4 and 3.5, respectively.

A caveat on notation: henceforth the symbol c stands for any positive constant independent of ε and δ but possibly depending on the fixed data. In particular, note that c may change from line to line. Moreover, in the following we use the short-hand notation, for all $A \in \mathbb{R}^{d\times d}$,

$$W_{\rm el}^\varepsilon(A) := \frac{1}{\varepsilon^2} W_{\rm el}(I{+}\varepsilon A), \quad W_{\rm h}^\varepsilon(A) := \frac{1}{\varepsilon^2} W_{\rm h}(I{+}\varepsilon A), \quad \widetilde{W}_{\rm h}^\varepsilon(A) := \frac{1}{\varepsilon^2} \widetilde{W}_{\rm h}(I{+}\varepsilon A).$$

3.1. **Energy coercivity.**

We start by providing a uniform coercivity result for the energy. It follows the ideas in [DNP02] and relies on the Rigidity Lemma [FJM02, Thm. 3.1].

Lemma 3.1 (Coercivity)**.** *There exists $c>0$ such that, for all $(u,z)\in\mathcal{Q}$*

$$\|\nabla u\|^2_{\rm L^2} + \|z\|^2_{\rm L^2} + \|\varepsilon z\|^2_{\rm L^\infty} \leq c\big(1{+}\mathcal{W}_\varepsilon(u,z)\big). \tag{3.1}$$

Proof. Let us assume with no loss of generality that $\mathcal{W}_\varepsilon(u,z) < \infty$. Hence, $|I{+}\varepsilon z| \leq c_K$ almost everywhere from property (2.7). Thus, we have that $\|\varepsilon z\|_{\rm L^\infty} \leq c$. Moreover, one readily checks from the coercivity (2.6e) that

$$c_3 \|z\|^2_{\rm L^2} \leq \int_\Omega W_{\rm h}^\varepsilon(z)\,{\rm d}x \leq \mathcal{W}_\varepsilon(u,z). \tag{3.2}$$

For the displacement u we follow ideas from [DNP02]. Given any $Q \in \mathsf{SO}(d)$ by letting $\varphi = {\rm id}{+}\varepsilon u$ and $F_{\rm el} = \nabla\varphi(I{+}\varepsilon z)^{-1}$ we have

$$\begin{aligned}|\nabla\varphi{-}Q|^2 &= |\nabla\varphi - Q(I{+}\varepsilon z) + \varepsilon Q z|^2 = |(F_{\rm el}{-}Q)(I{+}\varepsilon z) + \varepsilon Q z|^2 \\ &\leq c\big(|F_{\rm el}{-}Q|^2|I{+}\varepsilon z|^2 + \varepsilon^2|z|^2\big) \leq c\big(|F_{\rm el}{-}Q|^2 + \varepsilon^2|z|^2\big).\end{aligned}$$

In particular, by passing to the infimum for $Q\in\mathsf{SO}(d)$ we have checked that

$${\rm dist}^2(\nabla\varphi, \mathsf{SO}(d)) \leq c\big({\rm dist}^2(F_{\rm el},\mathsf{SO}(d)) + \varepsilon^2|z|^2\big).$$

By taking the integral in space and using the nondegeneracy condition (2.1c) we obtain that

$$\begin{aligned}\int_\Omega {\rm dist}^2(\nabla\varphi,\mathsf{SO}(d))\,{\rm d}x &\leq c\int_\Omega {\rm dist}^2(F_{\rm el},\mathsf{SO}(d))\,{\rm d}x + c\varepsilon^2\int_\Omega |z|^2\,{\rm d}x \\ &\overset{(3.2)}{\leq} \varepsilon^2 c\big(1{+}\mathcal{W}_\varepsilon(u,z)\big).\end{aligned}$$

Hence, the Rigidity Lemma [FJM02, Thm. 3.1] ensures that

$$\|\nabla\varphi{-}\widehat{Q}\|^2_{\rm L^2} \leq \varepsilon^2 c(1{+}\mathcal{W}_\varepsilon(u,z))$$

for some constant rotation $\widehat{Q} \in \mathrm{SO}(d)$. Finally, using [DNP02, Prop. 3.4] and $\varphi|_\Gamma = \mathrm{id}$ as $u \in \mathcal{U}$, we conclude $|\widehat{Q}-I|^2 \le \varepsilon^2 c(1+\mathcal{W}_\varepsilon(u,z))$. Then, we have

$$\|\nabla u\|^2_{\mathrm{L}^2} = \frac{1}{\varepsilon^2}\|\nabla\varphi - I\|^2_{\mathrm{L}^2} \le \frac{2}{\varepsilon^2}\|\nabla\varphi-\widehat{Q}\|^2_{\mathrm{L}^2} + \frac{2}{\varepsilon^2}\|\widehat{Q}-I\|^2_{\mathrm{L}^2} \le c(1+\mathcal{W}_\varepsilon(u,z))$$

and the bound (3.1) follows. □

3.2. **Γ–$\liminf$ inequalities.** Next, we turn our attention to the proof of the separate Γ–$\liminf$ inequalities for energy and dissipation. Let us start with a statement concerning the energy densities.

Lemma 3.2. *Under assumptions* (2.1e) *and* (2.6d)*, we have*

$$W^\varepsilon_{\mathrm{el}} \to |\cdot|^2_{\mathbb{C}} \text{ and } \widetilde{W}^\varepsilon_{\mathrm{h}} \to |\cdot|^2_{\mathbb{H}} \text{ locally uniformly.} \tag{3.3}$$

Moreover, we have

$$|z|^2_{\mathbb{H}} \le \inf\Big\{ \liminf_{\varepsilon\to 0} W^\varepsilon_{\mathrm{h}}(z_\varepsilon) \;\Big|\; z_\varepsilon \to z\Big\}. \tag{3.4}$$

Proof. Let $K_0 \Subset \mathbb{R}^{d\times d}$, fix $\delta > 0$ and find the corresponding $c_{\mathrm{el}}(\delta) > 0$ from condition (2.1e). As $\varepsilon K_0 \subset B_{c_{\mathrm{el}}(\delta)}(0)$ for ε sufficiently small we have that

$$\limsup_{\varepsilon\to 0}\, \sup_{K_0} \big|W^\varepsilon_{\mathrm{el}} - |\cdot|^2_{\mathbb{C}}\big| \le \delta \sup_{K_0} |\cdot|^2 \le \delta c$$

and local uniform convergence follows from $\delta > 0$ being arbitrary. The same argument applies to $\widetilde{W}^\varepsilon_{\mathrm{h}}$.

As for the Γ–$\liminf$ inequality (3.4), let $z_\varepsilon \to z$ and assume with no loss of generality that $\sup_\varepsilon W^\varepsilon_{\mathrm{h}}(z_\varepsilon) < \infty$. Hence, $W^\varepsilon_{\mathrm{h}}(z_\varepsilon) = \widetilde{W}^\varepsilon_{\mathrm{h}}(z_\varepsilon)$ and the inequality follows from the above proved uniform convergence. □

We are now in the position of proving the Γ–$\liminf$ estimate for the energy. It follows indeed from (3.3) and the lower-semicontinuity result of Lemma 4.2.

Lemma 3.3 (Γ–$\liminf$ for the energy)**.** *For all* $(u,z) \in \mathcal{Q}$ *we have*

$$\mathcal{W}_0(u,z) \le \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon,z_\varepsilon) \;\Big|\; (u_\varepsilon,z_\varepsilon) \to (u_0,z_0) \text{ weakly in } \mathcal{Q}\Big\}.$$

Proof. Let $(u_\varepsilon,z_\varepsilon) \to (u,z)$ weakly in $\mathcal{Q}$. We can assume with no loss of generality that $\sup_\varepsilon \mathcal{W}_\varepsilon(u_\varepsilon,z_\varepsilon) < \infty$. Owing to the Γ–$\liminf$ inequality (3.4) and the lower semicontinuity Lemma 4.2 we readily conclude that

$$\int_\Omega |z|^2_{\mathbb{H}} \le \liminf_{\varepsilon\to 0} \int_\Omega W^\varepsilon_{\mathrm{h}}(z_\varepsilon)\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\,\mathrm{d}x. \tag{3.5}$$

Moreover, $\mathcal{W}_\varepsilon(u_\varepsilon,z_\varepsilon) < \infty$ implies $\varepsilon z_\varepsilon \in K - I$ almost everywhere. In particular, $\varepsilon z_\varepsilon$ are bounded in L^∞. The same holds for $(I+\varepsilon z_\varepsilon)^{-1}$ as

$$(I+\varepsilon z_\varepsilon)^{-1} = \det(I+\varepsilon z_\varepsilon)\,\mathrm{cof}(I+\varepsilon z_\varepsilon) = \mathrm{cof}(I+\varepsilon z_\varepsilon).$$

We define the auxiliary tensors

$$w_\varepsilon := \frac{1}{\varepsilon}\Big((I{+}\varepsilon z_\varepsilon)^{-1} - I + \varepsilon z_\varepsilon\Big) = \varepsilon(I{+}\varepsilon z_\varepsilon)^{-1} z_\varepsilon^2, \tag{3.6}$$

so that $(I{+}\varepsilon z_\varepsilon)^{-1} = I{-}\varepsilon z_\varepsilon{+}\varepsilon w_\varepsilon$. By the first equality in (3.6) we have $\|\varepsilon w_\varepsilon\|_{\mathrm{L}^\infty} \le c$, while the second gives

$$\|w_\varepsilon\|_{\mathrm{L}^1} = \varepsilon\|(I{+}\varepsilon z_\varepsilon)^{-1} z_\varepsilon^2\|_{\mathrm{L}^1} \le c\varepsilon\|z_\varepsilon\|_{\mathrm{L}^2}^2 \le c\varepsilon$$

where we have also used the boundedness in L^2 of z_ε from (3.1). Thus, by interpolation, w_ε is bounded in L^2 as well, so that $w_\varepsilon \to 0$ weakly in L^2.

Given $A_\varepsilon := (F_{\mathrm{el},\varepsilon}{-}I)/\varepsilon$ we want to show the weak L^2 convergence $A_\varepsilon \to \nabla u{-}z$. From

$$A_\varepsilon = \frac{1}{\varepsilon}\left((I{+}\varepsilon\nabla u_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1} - I\right) \tag{3.7}$$

we find $I{+}\varepsilon A_\varepsilon = (I{+}\varepsilon\nabla u_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1}$ and compute that

$$A_\varepsilon = \frac{1}{\varepsilon}\Big((I{+}\varepsilon\nabla u_\varepsilon)(I{-}\varepsilon z_\varepsilon{+}\varepsilon w_\varepsilon) - I\Big) = \nabla u_\varepsilon - z_\varepsilon + w_\varepsilon - \varepsilon\big(\nabla u_\varepsilon z_\varepsilon{-}\nabla u_\varepsilon w_\varepsilon\big).$$

Hence, as we have that $\nabla u_\varepsilon{-}z_\varepsilon \to \nabla u{-}z$ and $w_\varepsilon \to 0$ weakly in L^2, we have to show $v_\varepsilon := \nabla u_\varepsilon(\varepsilon z_\varepsilon{-}\varepsilon w_\varepsilon) \to 0$ weakly in L^2 as well. Indeed, the boundedness in L^2 of v_ε follows from $\|\nabla u_\varepsilon\|_{\mathrm{L}^2} \le c$ (see (3.1)) and the L^∞-boundedness of $\varepsilon z_\varepsilon$ and $\varepsilon w_\varepsilon$. Moreover, since z_ε and w_ε are bounded in L^2 we have $\|v_\varepsilon\|_{\mathrm{L}^1} \le c\varepsilon$ and conclude $v_\varepsilon \to 0$ weakly in L^2.

Eventually, owing to Lemma 3.2, we are in the position of exploiting the lower semicontinuity Lemma 4.2 in order to obtain that

$$\begin{aligned}\int_\Omega |\nabla u{-}z|_{\mathbb{C}}^2 &\le \liminf_{\varepsilon\to 0}\int_\Omega W_{\mathrm{el}}^\varepsilon(A_\varepsilon)\,\mathrm{d}x\\ &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{el}}\big((I{+}\varepsilon\nabla u_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1}\big)\,\mathrm{d}x.\end{aligned}$$

Finally, by recalling relation (2.3) and the already established (3.5) the assertion follows. ☐

Before moving to the Γ–$\liminf$ inequality for the dissipation functionals $\mathcal{D}_\varepsilon$, we prepare here a preliminary result on the functions D_ε.

Lemma 3.4 (Γ-convergence of D_ε)**.** *$D_\varepsilon \to D_0$ in the sense of Γ-convergence.*

Proof. Γ–$\liminf$ *inequality.* Let $(z_\varepsilon, \widehat{z}_\varepsilon) \to (z, \widehat{z})$ and assume with no loss of generality that $\sup_\varepsilon D_\varepsilon(z_\varepsilon, \widehat{z}_\varepsilon) < \infty$. In particular, we have that $(I{+}\varepsilon\widehat{z}_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1} \in \mathrm{SL}(d)$. By defining

$$\zeta_\varepsilon := \frac{1}{\varepsilon}\Big((I{+}\varepsilon\widehat{z}_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1} - I\Big) = \widehat{z}_\varepsilon - z_\varepsilon + w_\varepsilon - \varepsilon\widehat{z}_\varepsilon z_\varepsilon + \widehat{z}_\varepsilon w_\varepsilon$$

where w_ε is given in (3.6), we readily check that $I{+}\varepsilon\zeta \in \mathrm{SL}(d)$ and $\zeta_\varepsilon \to \widehat{z} - z$.

Let now $t \mapsto P_\varepsilon(t) \in \mathrm{C}^1(0,1;\mathbb{R}^{d\times d})$ be such that $P_\varepsilon(0) = I$, $P_\varepsilon(1) = I{+}\varepsilon\zeta_\varepsilon$, and

$$D(I, I{+}\varepsilon\zeta_\varepsilon) \ge (1{-}\varepsilon)\int_0^1 R(\dot{P}_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t.$$

Such function P_ε exists by the very definition of D. By possibly reparametrizing P_ε and using assumption (2.10b) and the bound (2.12) we can assume that

$$c_4|\dot{P}_\varepsilon(t)P_\varepsilon^{-1}(t)| \overset{\text{(2.10b)}}{\leq} R(\dot{P}_\varepsilon(t)P_\varepsilon^{-1}(t)) \leq 2D(I, I{+}\varepsilon\zeta_\varepsilon) \overset{\text{(2.12)}}{\leq} c\varepsilon. \tag{3.8}$$

Hence, $P_\varepsilon \to I$ uniformly as

$$|P_\varepsilon(t){-}I| \leq \int_0^t |\dot{P}_\varepsilon P_\varepsilon^{-1}|\,|P_\varepsilon|\,\mathrm{d}s \leq c\varepsilon \int_0^t |P_\varepsilon|\,\mathrm{d}s \leq c\varepsilon\left(1 + \int_0^t |P_\varepsilon{-}I|\,\mathrm{d}s\right).$$

By defining $\widehat{P}_\varepsilon(t) = I + (P_\varepsilon(t){-}I)/\varepsilon$ one has that $\widehat{P}_\varepsilon(0) = I$ and $\widehat{P}_\varepsilon(1) = I{+}\zeta_\varepsilon$. Moreover, as $\varepsilon\dot{\widehat{P}}_\varepsilon = \dot{P}_\varepsilon$ and R is positively 1-homogeneous (2.10a), we have that

$$\frac{1}{\varepsilon}D(I, I{+}\varepsilon\zeta_\varepsilon) \geq (1{-}\varepsilon)\int_0^1 R(\dot{\widehat{P}}_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t.$$

Owing now to bound (3.8), by possibly extracting not relabeled subsequences, we have that $\dot{\widehat{P}}_\varepsilon \to Q$ weakly-star in $\mathrm{L}^\infty(0,1;\mathbb{R}^{d\times d})$ and

$$\begin{aligned}\liminf_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \widehat{z}_\varepsilon) &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}D(I, I{+}\varepsilon\zeta_\varepsilon)\\ \geq \liminf_{\varepsilon\to 0}\int_0^1 R(\dot{\widehat{P}}_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t &\geq \int_0^1 R(Q)\,\mathrm{d}t \geq R(\tilde{Q})\end{aligned}$$

where we have exploited the lower semicontinuity tool of Lemma 4.2 and used Jensen's inequality with $\tilde{Q} = \int_0^1 Q\,\mathrm{d}t$.

Finally, by integrating we have that

$$\tilde{Q} = \int_0^1 Q\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^1 \dot{\widehat{P}}_\varepsilon\,\mathrm{d}t = \lim_{\varepsilon\to 0}\zeta_\varepsilon = \widehat{z} - z$$

so that we have checked

$$\liminf_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \widehat{z}_\varepsilon) \geq R(\widehat{z}{-}z).$$

Recovery sequence. Given $\zeta \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ we have that $\exp(\zeta) \in \mathsf{SL}(d)$ and, by taking $P(t) := \exp(t\zeta)$ into the definition of D, we readily check that $D(I, \exp(\zeta)) \leq R(\zeta)$.

Let now $z,\ \widehat{z} \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ be given and define

$$\widehat{z}_\varepsilon = \frac{1}{\varepsilon}\big(\exp(\varepsilon(\widehat{z}{-}z))(I{+}\varepsilon z) - I\big).$$

As $(I{+}\varepsilon\widehat{z}_\varepsilon)(I{+}\varepsilon z)^{-1} = \exp(\varepsilon(\widehat{z}{-}z))$, we have that

$$\limsup_{\varepsilon\to 0} D_\varepsilon(z, \widehat{z}_\varepsilon) = \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}D(I, \exp(\varepsilon(\widehat{z}{-}z))) \leq R(\widehat{z}{-}z) = D_0(z, \widehat{z})$$

so that $(z, \widehat{z}_\varepsilon)$ is a recovery sequence. $\square$

Owing to Lemma 3.4, it suffices now to apply the lower semicontinuity result in Lemma 4.2 in order to establish the Γ–$\liminf$ inequality for the dissipation functionals. More precisely, we have following.

Lemma 3.5 (Γ–$\liminf$ for the dissipation)**.**

$$\mathcal{D}_0(z,\widehat{z}) \le \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{D}_\varepsilon(z_\varepsilon,\widehat{z}_\varepsilon) \ \Big| \ (z_\varepsilon,\widehat{z}_\varepsilon) \to (z,\widehat{z}) \text{ weakly in } (\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}))^2 \Big\}. \tag{3.9}$$

3.3. **Mutual recovery sequence.** We now come to the construction of a *mutual recovery sequence*. Let us recall from [MRS08] that indeed *two separate* Γ–$\limsup$ inequalities for energy and dissipation generally do not suffice for passing to the limit in RIS. In particular, the construction of recovery sequences for energy and dissipation has to be *mutually* coordinated.

Lemma 3.6 (Mutual recovery sequence)**.** *Let $t\in[0,T]$, $(u_\varepsilon,z_\varepsilon)\to(u_0,z_0)$ weakly in $\mathcal{Q}$, and*

$$\sup_\varepsilon \mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon) < \infty. \tag{3.10}$$

Moreover, let $(\widehat{u}_0,\widehat{z}_0) := (u_0,z_0)+(\tilde{u},\tilde{z})$ with $(\tilde{u},\tilde{z})$ smooth and compactly supported in Ω and $\tilde{z}\in\mathbb{R}^{d\times d}_{\mathrm{dev}}$ everywhere. Then, there exist $(\widehat{u}_\varepsilon,\widehat{z}_\varepsilon)\in\mathcal{Q}$ such that

$$\begin{aligned} &(\widehat{u}_\varepsilon,\widehat{z}_\varepsilon)\to(\widehat{u}_0,\widehat{z}_0) \ \text{ weakly in } \mathcal{Q} \text{ and}\\ &\limsup_{\varepsilon\to0}\Big(\mathcal{E}_\varepsilon(t,\widehat{u}_\varepsilon,\widehat{z}_\varepsilon)-\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon)+\mathcal{D}_\varepsilon(z_\varepsilon,\widehat{z}_\varepsilon)\Big)\\ &\le \Big(\mathcal{E}_0(t,\widehat{u}_0,\widehat{z}_0)-\mathcal{E}_0(t,u_0,z_0)+\mathcal{D}_0(z_0,\widehat{z}_0)\Big). \end{aligned}\tag{3.11}$$

Proof. For the sake of clarity, we decompose this argument into subsequent steps. The general strategy of the proof is to choose $(\widehat{u}_\varepsilon,\widehat{z}_\varepsilon)$ and show convergence to $(\widehat{u}_0,\widehat{z}_0)$,

$$\limsup_{\varepsilon\to0}\mathcal{D}_\varepsilon(z_\varepsilon,\widehat{z}_\varepsilon)\le\mathcal{D}_0(z_0,\widehat{z}_0)=R(\tilde{z}),$$

and

$$\limsup_{\varepsilon\to0}\Big(\mathcal{E}_\varepsilon(t,\widehat{u}_\varepsilon,\widehat{z}_\varepsilon)-\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon)\Big)\le\mathcal{E}_0(t,\widehat{u}_0,\widehat{z}_0)-\mathcal{E}_0(t,u_0,z_0).$$

Note that in order to establish the latter we cannot argue on individual terms but rather aim at exploiting certain cancellations. This resembles the situation of the so-called *quadratic trick* (see, e.g., [MT05]) and crucially uses (2.1d) as well as the smoothness of $(\tilde{u},\tilde{z})$. In particular, note that within this proof the constant c may depend on $\tilde{u}$ and $\tilde{z}$ as well.

Step 1: Choice of the mutual recovery sequence. By defining the functions $\psi_\varepsilon := \mathrm{id}+\varepsilon\tilde{u}$ and $\varphi_\varepsilon := \mathrm{id}+\varepsilon u_\varepsilon$ and the set

$$K_\varepsilon := \big\{x\in\Omega \ \big| \ \exp(\varepsilon\tilde{z}(x))(I+\varepsilon z_\varepsilon(x))\in K\big\},$$

the proof of the lemma follows by checking that the choices

$$\begin{aligned} \widehat{u}_\varepsilon &:= \frac{1}{\varepsilon}\big(\psi_\varepsilon\circ\varphi_\varepsilon-\mathrm{id}\big),\\ \widehat{z}_\varepsilon &:= \begin{cases} \dfrac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I+\varepsilon z_\varepsilon)-I\big) & \text{on } K_\varepsilon\\ z_\varepsilon & \text{else,}\end{cases} \end{aligned}$$

fulfill (3.11). The construction of $\widehat{u}_\varepsilon$ via a composition and of $\widehat{z}_\varepsilon$ via matrix exponential and multiplication is necessary in order to deal with the multiplicative nature of finite-strain elastoplasticity.

From the bound (3.10) we readily have that $I{+}\varepsilon z_\varepsilon \in \mathsf{SL}(d)$ almost everywhere. Hence, upon noting that

$$I{+}\varepsilon\widehat{z}_\varepsilon = \begin{cases} \exp(\varepsilon\tilde{z})(I{+}\varepsilon z_\varepsilon) & \text{on } K_\varepsilon \\ I{+}\varepsilon z_\varepsilon & \text{else,} \end{cases}$$

we immediately check that $(I{+}\varepsilon\widehat{z}_\varepsilon) \in K \subset \mathsf{SL}(d)$ almost everywhere and is bounded in L^∞. Using the fact that $\operatorname{tr}\tilde{z} = 0$ we have $\det\exp(\varepsilon\tilde{z}) = \exp(\operatorname{tr}\tilde{z}) = 1$ and hence $\exp(\varepsilon\tilde{z})(I{+}\varepsilon z_\varepsilon) \in \mathsf{SL}(d)$ almost everywhere.

Next, note that the measure of the complement of K_ε can be controlled by means of a Chebyshev estimate. Indeed, relation (2.8) gives

$$\begin{aligned} |\Omega\setminus K_\varepsilon| &= \int_{\Omega\setminus K_\varepsilon} 1\,\mathrm{d}x \le c_K^2 \int_\Omega \big|\exp(\varepsilon\tilde{z})(I{+}\varepsilon z_\varepsilon) - I\big|^2\,\mathrm{d}x \\ &= c_K^2 \int_\Omega \big|\exp(\varepsilon\tilde{z}) - I + \varepsilon\exp(\varepsilon\tilde{z})z_\varepsilon\big|^2\,\mathrm{d}x \le c\varepsilon^2\left(1{+}\int_\Omega z_\varepsilon^2\,\mathrm{d}x\right) \le c\varepsilon^2. \end{aligned}$$

Now, one has that

$$\begin{aligned} &\widehat{z}_\varepsilon - z_\varepsilon = \frac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I{+}\varepsilon z_\varepsilon) - I\big) - z_\varepsilon = \frac{1}{\varepsilon}(\exp(\varepsilon\tilde{z}){-}I)(I{+}\varepsilon z_\varepsilon) \quad \text{on } K_\varepsilon, \\ &\widehat{z}_\varepsilon - z_\varepsilon = 0 \quad \text{on } \Omega\setminus K_\varepsilon, \end{aligned}$$

the convergence $|\Omega\setminus K_\varepsilon| \to 0$, and that $\widehat{z}_\varepsilon$ and z_ε are bounded in L^2. Hence, we readily check that

$$\widehat{z}_\varepsilon + z_\varepsilon \to \widehat{z}_0 + z_0 \ \text{ weakly in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}), \tag{3.12}$$

$$\widehat{z}_\varepsilon - z_\varepsilon \to \tilde{z} \ \text{ strongly in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}). \tag{3.13}$$

From the energy bound (3.10) and the coercivity Lemma 3.1 we have that u_ε is bounded in H^1 and $\varepsilon u_\varepsilon \to 0$ strongly in L^2. Hence, one has that $\|\varphi_\varepsilon{-}\mathrm{id}\|_{\mathrm{L}^2} = \varepsilon\|u_\varepsilon\|_{\mathrm{L}^2} \le c\varepsilon$ and, by the Lipschitz continuity of $\nabla\tilde{u}$, we conclude that

$$\|\nabla\tilde{u}(\varphi_\varepsilon) - \nabla\tilde{u}\|_{\mathrm{L}^2} \le c\|\varphi_\varepsilon{-}\mathrm{id}\|_{\mathrm{L}^2} = c\varepsilon\|u_\varepsilon\|_{\mathrm{L}^2} \le c\varepsilon. \tag{3.14}$$

Moreover, by computing

$$\begin{aligned} \nabla\widehat{u}_\varepsilon &= \frac{1}{\varepsilon}\big(\nabla\psi_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon{-}I\big) = \frac{1}{\varepsilon}\big((I{+}\varepsilon\nabla\tilde{u})(\varphi_\varepsilon)\nabla\varphi_\varepsilon{-}I\big) \\ &= \frac{1}{\varepsilon}\big(\nabla\varphi_\varepsilon{+}\varepsilon\nabla\tilde{u}(\varphi_\varepsilon)\nabla\varphi_\varepsilon{-}I\big) = \nabla u_\varepsilon + \nabla\tilde{u}(\varphi_\varepsilon) + \varepsilon\nabla\tilde{u}(\varphi_\varepsilon)\nabla u_\varepsilon \end{aligned}$$

we obtain that

$$\begin{aligned} \|(\nabla\widehat{u}_\varepsilon{-}\nabla u_\varepsilon) - \nabla\tilde{u}\|_{\mathrm{L}^2} &\le \|\nabla\tilde{u}(\varphi_\varepsilon){-}\nabla\tilde{u}\|_{\mathrm{L}^2} + \|\varepsilon\nabla\tilde{u}(\varphi_\varepsilon)\nabla u_\varepsilon\|_{\mathrm{L}^2} \\ &\overset{(3.14)}{\le} c\varepsilon + c\varepsilon\|\nabla u_\varepsilon\|_{\mathrm{L}^2} \le c\varepsilon. \end{aligned} \tag{3.15}$$

The tensors $A_\varepsilon = (F_{\mathrm{el},\varepsilon}{-}I)/\varepsilon$ and $\widehat{A}_\varepsilon = (\widehat{F}_{\mathrm{el},\varepsilon}{-}I)/\varepsilon$ fulfill

$$A_\varepsilon = \frac{1}{\varepsilon}\big((I{+}\varepsilon\nabla u_\varepsilon)(I{+}\varepsilon z_\varepsilon)^{-1} - I\big), \quad \widehat{A}_\varepsilon = \frac{1}{\varepsilon}\big((I{+}\varepsilon\nabla\widehat{u}_\varepsilon)(I{+}\varepsilon\widehat{z}_\varepsilon)^{-1} - I\big)$$

and are hence both bounded in L^2.

Fix now δ and let $c_{\rm el}(\delta)$ and $c_{\rm h}(\delta)$ be given by conditions (2.1e) and (2.6d), respectively. For all $\varepsilon > 0$ we define the sets

$$U_\varepsilon^\delta := \Big\{x \in \Omega \;\Big|\; |\varepsilon A_\varepsilon(x)| + |\varepsilon \widehat{A}_\varepsilon(x)| \le c_{\rm el}(\delta)\Big\},$$
$$Z_\varepsilon^\delta := \Big\{x \in \Omega \;\Big|\; |\varepsilon z_\varepsilon(x)| + |\varepsilon \widehat{z}_\varepsilon(x)| \le c_{\rm h}(\delta)\Big\},$$

We refer to the latter as *good sets* as strains are there under control and we can replace the nonlinear densities $W_{\rm el}$ and $W_{\rm h}$ by their quadratic expansions via (2.1e) and (2.6d). In particular, on the good sets the quadratic character of the expansions will entail the control of the difference of the energy contributions by means of a suitable cancellation (*quadratic trick*). On the other hand, we term *bad sets* the corresponding complements $\Omega\setminus U_\varepsilon^\delta$ and $\Omega\setminus Z_\varepsilon^\delta$ where the quadratic expansions are a priori not available. Using some nontrivial cancellations, we will show that the difference of the energy contributions on the bad sets is infinitesimal. Note preliminarily that the integrands on the bad sets blow up while the bad sets have small measure. Indeed,

$$|\Omega \setminus U_\varepsilon^\delta| = \int_{\Omega\setminus U_\varepsilon^\delta} 1\,{\rm d}x \le \frac{1}{c_{\rm el}^2(\delta)} \int_\Omega (|A_\varepsilon| + |\widehat{A}_\varepsilon|)^2\,{\rm d}x \le \frac{c\varepsilon^2}{c_{\rm el}^2(\delta)}, \tag{3.16}$$

$$|\Omega \setminus Z_\varepsilon^\delta| = \int_{\Omega\setminus Z_\varepsilon^\delta} 1\,{\rm d}x \le \frac{1}{c_{\rm h}^2(\delta)} \int_\Omega (|z_\varepsilon| + |\widehat{z}_\varepsilon|)^2\,{\rm d}x \le \frac{c\varepsilon^2}{c_{\rm h}^2(\delta)}. \tag{3.17}$$

Step 2: Treatment of the dissipation term. As $\widehat{z}_\varepsilon = z_\varepsilon$ on $\Omega \setminus K_\varepsilon$ one has that

$$\mathcal{D}_\varepsilon(z_\varepsilon, \widehat{z}_\varepsilon) = \int_{K_\varepsilon} D_\varepsilon(I, \exp(\varepsilon\tilde{z}))\,{\rm d}x \le \int_\Omega D_\varepsilon(I, \exp(\varepsilon\tilde{z}))\,{\rm d}x. \tag{3.18}$$

By recalling the construction of the recovery sequence in the proof of Lemma 3.4 we conclude that

$$D_\varepsilon(I, \exp(\varepsilon\tilde{z})) \to R(\tilde{z}) \ \text{ in } \ C^0(\overline{\Omega}). \tag{3.19}$$

Eventually, by taking the $\limsup$ in relation (3.18) and using (3.19) we have proved that

$$\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \widehat{z}_\varepsilon) \le \lim_{\varepsilon\to 0} \int_\Omega D_\varepsilon(I, \exp(\varepsilon\tilde{z}))\,{\rm d}x = \int_\Omega R(\tilde{z})\,{\rm d}x = \mathcal{D}_0(z_0, \widehat{z}_0). \tag{3.20}$$

Step 3: Limsup for the differences of the elastic energy terms. Let us start by rewriting the tensors A_ε as

$$A_\varepsilon = \nabla u_\varepsilon - z_\varepsilon + w_\varepsilon - \varepsilon\nabla u_\varepsilon z_\varepsilon + \varepsilon \nabla u_\varepsilon w_\varepsilon.$$

On the other hand, as regards the tensors $\widehat{A}_\varepsilon$ we have that

$$\begin{aligned}
\widehat{A}_\varepsilon &= \frac{1}{\varepsilon}\big((I + \varepsilon\nabla\widehat{u}_\varepsilon)(I - \varepsilon z_\varepsilon + w_\varepsilon)\exp(-\varepsilon\tilde{z}) - I\big)\\
&= \big(\nabla\widehat{u}_\varepsilon - z_\varepsilon + w_\varepsilon - \varepsilon\nabla\widehat{u}_\varepsilon z_\varepsilon + \varepsilon\nabla\widehat{u}_\varepsilon w_\varepsilon\big)\exp(-\varepsilon\tilde{z})\\
&\quad + \frac{1}{\varepsilon}\big(\exp(-\varepsilon\tilde{z}) - I\big) \quad \text{on } K_\varepsilon\\
\widehat{A}_\varepsilon &= \frac{1}{\varepsilon}\big((I + \varepsilon\nabla\widehat{u}_\varepsilon)(I - \varepsilon z_\varepsilon + w_\varepsilon) - I\big)\\
&= \nabla\widehat{u}_\varepsilon - z_\varepsilon + w_\varepsilon - \varepsilon\nabla\widehat{u}_\varepsilon z_\varepsilon + \varepsilon\nabla\widehat{u}_\varepsilon w_\varepsilon \quad \text{on } \Omega\setminus K_\varepsilon.
\end{aligned}$$

Hence, one can compute that

$$
\begin{aligned}
\widehat{A}_\varepsilon - A_\varepsilon &= (\nabla\widehat{u}_\varepsilon{-}\nabla u_\varepsilon)(I{-}\varepsilon z_\varepsilon{+}\varepsilon w_\varepsilon) + \frac{1}{\varepsilon}\big(\exp(-\varepsilon\tilde{z}){-}I\big)\\
&\qquad + (\nabla\widehat{u}_\varepsilon{-}z_\varepsilon{+}w_\varepsilon{-}\varepsilon\nabla\widehat{u}_\varepsilon z_\varepsilon{+}\varepsilon\nabla\widehat{u}_\varepsilon w_\varepsilon)(\exp(-\varepsilon\tilde{z}){-}I) \quad \text{on } K_\varepsilon\\
\widehat{A}_\varepsilon - A_\varepsilon &= (\nabla\widehat{u}_\varepsilon{-}\nabla u_\varepsilon)(I{-}\varepsilon z_\varepsilon{+}\varepsilon w_\varepsilon) \quad \text{on } \Omega\setminus K_\varepsilon.
\end{aligned}
$$

In particular, owing to convergence (3.15) we have that $(\nabla\widehat{u}_\varepsilon{-}\nabla u_\varepsilon)(I{-}\varepsilon z_\varepsilon{+}\varepsilon w_\varepsilon)$ converges to $\nabla\tilde{u}$ strongly in L^2. Thus, it is a standard matter to check that

$$
\widehat{A}_\varepsilon + A_\varepsilon \to (\nabla\widehat{u}_0{-}\widehat{z}_0) + (\nabla u_0{-}z_0) \text{ weakly in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}), \tag{3.21}
$$

$$
\widehat{A}_\varepsilon - A_\varepsilon \to \nabla\tilde{u} - \tilde{z} \text{ strongly in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}). \tag{3.22}
$$

On the good set U_ε^δ we will use the assumption (2.1e) in order to have that

$$
\begin{aligned}
W_{\mathrm{el}}^\varepsilon(\widehat{A}_\varepsilon) - W_{\mathrm{el}}^\varepsilon(A_\varepsilon) &\le |\widehat{A}_\varepsilon|_{\mathbb{C}}^2 - |A_\varepsilon|_{\mathbb{C}}^2 + 2c\delta c_{\mathrm{el}}^2(\delta)\\
&= \frac{1}{2}(\widehat{A}_\varepsilon{-}A_\varepsilon){:}\mathbb{C}(\widehat{A}_\varepsilon{+}A_\varepsilon) + 2c\delta c_{\mathrm{el}}^2(\delta).
\end{aligned} \tag{3.23}
$$

Let us now argue on the bad set $\Omega\setminus U_\varepsilon^\delta$ by defining

$$
G_{1,\varepsilon} := (I{+}\varepsilon\nabla\widehat{u}_\varepsilon)(I{+}\varepsilon\nabla u_\varepsilon)^{-1}, \quad G_{2,\varepsilon} := (I{+}\varepsilon z_\varepsilon)(I{+}\varepsilon\widehat{z}_\varepsilon)^{-1}.
$$

Note that $G_{1,\varepsilon}$ and $G_{2,\varepsilon}$ are chosen in such a way that $\widehat{F}_{\mathrm{el},\varepsilon} = G_{1,\varepsilon}F_{\mathrm{el},\varepsilon}G_{2,\varepsilon}$. We readily compute that

$$
G_{1,\varepsilon} - I = \nabla\psi_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon(I{+}\varepsilon\nabla u_\varepsilon)^{-1} - I = \nabla\psi_\varepsilon(\varphi_\varepsilon) - I = \varepsilon\nabla\tilde{u}(\varphi_\varepsilon)
$$

so that $\|G_{1,\varepsilon}{-}I\|_{\mathrm{L}^\infty(\Omega\setminus U_\varepsilon^\delta;\mathbb{R}^{d\times d})} = \varepsilon\|\nabla\tilde{u}(\varphi_\varepsilon)\|_{\mathrm{L}^\infty(\Omega\setminus U_\varepsilon^\delta;\mathbb{R}^{d\times d})} \le c\varepsilon$. Moreover, one has that

$$
G_{2,\varepsilon} = \begin{cases} \exp(-\varepsilon\tilde{z}) & \text{on } (\Omega\setminus U_\varepsilon^\delta)\cap K_\varepsilon,\\ I & \text{elsewhere in } \Omega\setminus U_\varepsilon^\delta. \end{cases}
$$

Hence, $\|G_{2,\varepsilon} - I\|_{\mathrm{L}^\infty(\Omega\setminus U_\varepsilon^\delta;\mathbb{R}^{d\times d}_{\mathrm{dev}})} \le c\varepsilon$ as well. Next, estimate (4.1) and bound (3.10) allow us to control the elastic part of the energy on the bad set $\Omega\setminus U_\varepsilon^\delta$ (where ∇u_ε and z_ε are not under control) by cancellation. For this we employ the multiplicative estimate (2.1d) provided in (4.1):

$$
\begin{aligned}
&\int_{\Omega\setminus U_\varepsilon^\delta}\big(W_{\mathrm{el}}^\varepsilon(\widehat{A}_\varepsilon) - W_{\mathrm{el}}^\varepsilon(A_\varepsilon)\big)\,\mathrm{d}x = \frac{1}{\varepsilon^2}\int_{\Omega\setminus U_\varepsilon^\delta}\big(W_{\mathrm{el}}(\widehat{F}_{\mathrm{el},\varepsilon}) - W_{\mathrm{el}}(F_{\mathrm{el},\varepsilon})\big)\,\mathrm{d}x\\
&= \frac{1}{\varepsilon^2}\int_{\Omega\setminus U_\varepsilon^\delta}|W_{\mathrm{el}}(G_{1,\varepsilon}F_{\mathrm{el},\varepsilon}G_{2,\varepsilon}) - W_{\mathrm{el}}(F_{\mathrm{el},\varepsilon})|\,\mathrm{d}x\\
&\overset{(4.1)}{\le} \frac{c_7}{\varepsilon^2}\int_{\Omega\setminus U_\varepsilon^\delta}\big(W_{\mathrm{el}}(F_{\mathrm{el},\varepsilon}) + c_8\big)\big(|G_{1,\varepsilon}{-}I| + |G_{2,\varepsilon}{-}I|\big)\,\mathrm{d}x\\
&\le c_7\left(\frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{el}}(F_{\mathrm{el},\varepsilon})\,\mathrm{d}x + \frac{c_8}{\varepsilon^2}|\Omega\setminus U_\varepsilon^\delta|\right)\big(\|G_{1,\varepsilon}{-}I\|_{\mathrm{L}^\infty} + \|G_{2,\varepsilon}{-}I\|_{\mathrm{L}^\infty}\big)\\
&\overset{(3.16)}{\le} c\varepsilon.
\end{aligned} \tag{3.24}
$$

Thus, we have controlled the difference of the energy contributions in the bad set $\Omega\setminus U_\varepsilon^\delta$ where the gradients are big.

Finally, by using convergences (3.21)-(3.22), equation (3.23) on the good set U^δ_ε, and relation (3.24) on the bad set $\Omega \setminus U^\delta_\varepsilon$, we conclude that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left(\int_\Omega W^\varepsilon_{\rm el}(\widehat{A}_\varepsilon)\,{\rm d}x - \int_\Omega W^\varepsilon_{\rm el}(A_\varepsilon)\,{\rm d}x\right)\\
&\overset{(3.23)}{\leq} \limsup_{\varepsilon\to 0}\left(\frac{1}{2\varepsilon^2}\int_{U^\delta_\varepsilon}\big(\widehat{A}_\varepsilon - A_\varepsilon\big):\mathbb{C}\big(\widehat{A}_\varepsilon + A_\varepsilon\big)\,{\rm d}x + 2c|\Omega|\delta c^2_{\rm el}(\delta)\right.\\
&\qquad\qquad\left. + \int_{\Omega\setminus U^\delta_\varepsilon}\big(W^\varepsilon_{\rm el}(\widehat{A}_\varepsilon) - W^\varepsilon_{\rm el}(A_\varepsilon)\big)\,{\rm d}x\right)\\
&\overset{(3.24)}{=} \limsup_{\varepsilon\to \varepsilon}\left(\frac{1}{2\varepsilon^2}\int_{U^\delta_\varepsilon}\big(\widehat{A}_\varepsilon - A_\varepsilon\big):\mathbb{C}\big(\widehat{A}_\varepsilon + A_\varepsilon\big)\,{\rm d}x + 2c|\Omega|\delta c^2_{\rm el}(\delta) + c\varepsilon\right)\\
&= \frac{1}{2}\int_\Omega\big(\nabla\tilde{u} - \tilde{z}\big):\mathbb{C}\big(\nabla(\widehat{u}_0 + u_0) - (\widehat{z}_0 + z_0)\big)\,{\rm d}x + 2c|\Omega|\delta c^2_{\rm el}(\delta)\\
&= \int_\Omega |\nabla\widehat{u}^{\rm sym}_0 - \widehat{z}^{\rm sym}_0|^2_{\mathbb{C}}\,{\rm d}x - \int_\Omega |\nabla u^{\rm sym}_0 - z^{\rm sym}_0|^2_{\mathbb{C}}\,{\rm d}x + 2c|\Omega|\delta c^2_{\rm el}(\delta)
\end{aligned}\tag{3.25}$$

where we have made use of relation (2.3).

Step 4: Upper bound on the hardening energy term. Let us now turn our attention to the hardening part of the energy. On the good set Z^δ_ε we have that

$$\begin{aligned}
W^\varepsilon_{\rm h}(\widehat{z}_\varepsilon) - W^\varepsilon_{\rm h}(z_\varepsilon) &\leq |\widehat{z}_\varepsilon|^2_{\mathbb{H}} - |z_\varepsilon|^2_{\mathbb{H}} + 2c\delta c^2_{\rm h}(\delta)\\
&= \frac{1}{2}(\widehat{z}_\varepsilon - z_\varepsilon):\mathbb{H}(\widehat{z}_\varepsilon + z_\varepsilon) + 2c\delta c^2_{\rm h}(\delta).
\end{aligned}\tag{3.26}$$

As regards the bad set $\Omega\setminus Z^\delta_\varepsilon$ one has that

$$W^\varepsilon_{\rm h}(\widehat{z}_\varepsilon) - W^\varepsilon_{\rm h}(z_\varepsilon) = \begin{cases} \dfrac{1}{\varepsilon^2}\widetilde{W}_{\rm h}(\exp(\varepsilon\tilde{z})(I+\varepsilon z_\varepsilon)) - \dfrac{1}{\varepsilon^2}\widetilde{W}_{\rm h}(I+\varepsilon z_\varepsilon) & \text{on } (\Omega\setminus Z^\delta_\varepsilon)\cap K_\varepsilon\\ 0 & \text{on } \Omega\setminus(Z^\delta_\varepsilon\cup K_\varepsilon). \end{cases}$$

Hence, by exploiting the local Lipschitz continuity of $\widetilde{W}_{\rm h}$ we have that

$$\begin{aligned}
&\int_{\Omega\setminus Z^\delta_\varepsilon}\big(W^\varepsilon_{\rm h}(\widehat{z}_\varepsilon) - W^\varepsilon_{\rm h}(z_\varepsilon)\big)\,{\rm d}x \leq \frac{c}{\varepsilon^2}\int_{\Omega\setminus Z^\delta_\varepsilon}|\exp(\varepsilon\tilde{z}) - I|\,|I+\varepsilon z_\varepsilon|\,{\rm d}x\\
&\leq \frac{c}{\varepsilon^2}|\Omega\setminus Z^\delta_\varepsilon|\,c\varepsilon \overset{(3.17)}{\leq} c\varepsilon.
\end{aligned}\tag{3.27}$$

Eventually, owing to convergences (3.12)-(3.13) we compute that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\left(\int_\Omega W_{\rm h}^\varepsilon(\widehat{z}_\varepsilon)\,{\rm d}x-\int_\Omega W_{\rm h}^\varepsilon(z_\varepsilon)\,{\rm d}x\right)\\
&\overset{(3.26)}{\le}\limsup_{\varepsilon\to 0}\Bigg(\int_{Z_\varepsilon^\delta}\frac{1}{2}(\widehat{z}_\varepsilon{-}z_\varepsilon){:}\mathbb{H}(\widehat{z}_\varepsilon{+}z_\varepsilon)\,{\rm d}x+2c|\Omega|\delta c_{\rm h}^2(\delta)\\
&\qquad\qquad+\int_{\Omega\setminus Z_\varepsilon^\delta}\big(W_{\rm h}^\varepsilon(\widehat{z}_\varepsilon)-W_{\rm h}^\varepsilon(z_\varepsilon)\big)\,{\rm d}x\Bigg)\\
&\overset{(3.27)}{=}\limsup_{\varepsilon\to 0}\left(\int_{Z_\varepsilon^\delta}\frac{1}{2}(\widehat{z}_\varepsilon{-}z_\varepsilon){:}\mathbb{H}(\widehat{z}_\varepsilon{+}z_\varepsilon)\,{\rm d}x+2c|\Omega|\delta c_{\rm h}^2(\delta)+c\varepsilon\right)\\
&=\int_\Omega\frac{1}{2}\tilde{z}{:}\mathbb{H}(\widehat{z}_0{+}z_0)\,{\rm d}x+2c|\Omega|\delta c_{\rm h}^2(\delta)\\
&=\int_\Omega|\widehat{z}_0|_{\mathbb{H}}^2\,{\rm d}x-\int_\Omega|z_0|_{\mathbb{H}}^2\,{\rm d}x+2c|\Omega|\delta c_{\rm h}^2(\delta).
\end{aligned}
\tag{3.28}
$$

Step 5: Conclusion of the proof. By collecting relations (3.25) and (3.28), and recalling that $\langle\ell(t),u_\varepsilon-\widehat{u}_\varepsilon\rangle\to\langle\ell(t),u_0-\widehat{u}_0\rangle$ we have proved that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\Big(\mathcal{E}_\varepsilon(t,\widehat{u}_\varepsilon,\widehat{z}_\varepsilon){-}\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon)\Big)\\
&\le\Big(\mathcal{E}_0(t,\widehat{u}_0,\widehat{z}_0){-}\mathcal{E}_0(t,u_0,z_0)\Big)+c\delta(c_{\rm el}^2(\delta){+}c_{\rm h}^2(\delta)).
\end{aligned}
$$

Finally, the assertion (3.11) follows by taking $\delta\to 0$ and employing (3.20). □

Remark 3.7. *Note that the construction of the mutual recovery sequence is compatible with the standard constraint* $\det(I{+}\varepsilon\nabla\widehat{u}_\varepsilon)>0$. *Indeed, by letting* ε *be small enough we have that* $I{+}\varepsilon\nabla\tilde{u}$ *is everywhere positive definite, hence* $\det(I{+}\varepsilon\nabla\tilde{u})>0$. *In particular, as* $\det(I{+}\varepsilon\nabla u_\varepsilon)>0$ *almost everywhere, we have that*

$$
\det(I{+}\varepsilon\nabla\widehat{u}_\varepsilon)=\det(\nabla\psi_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon)=\det(I{+}\varepsilon\nabla\tilde{u}(\varphi_\varepsilon))\det(I{+}\varepsilon\nabla u_\varepsilon)>0
$$

almost everywhere as well. That is, $I{+}\varepsilon\nabla\widehat{u}_\varepsilon\in GL_+(d)$ *almost everywhere.*

3.4. **Proof of Theorem 2.2.** Owing to the the above-obtained Γ–$\liminf$ and mutual-recovery-sequence results, the proof of Theorem 2.2 now follows along the lines of the general theory of [MRS08]. We limit ourselves in sketching the main points of the argument and refer the reader to [MRS08] for the details.

Let $(u_\varepsilon,z_\varepsilon)$ be a sequence of finite-plasticity solutions. The coercivity of the energy (3.1) entails an a priori bound on $(u_\varepsilon,z_\varepsilon)$. In particular, we have the following.

Corollary 3.8 (A priori bound)**.** *There exists* $c>0$ *such that all finite-plasticity solutions* $(u_\varepsilon,z_\varepsilon)$ *fulfill*

$$
\forall t\in[0,T]:\quad \|u_\varepsilon(t)\|_{{\rm H}^1}+\|z_\varepsilon(t)\|_{{\rm L}^2}+\|\varepsilon z_\varepsilon(t)\|_{{\rm L}^\infty}+{\rm Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])\le c.
\tag{3.29}
$$

Proof. We exploit the energy balance (2.16) and the bound (3.1) in order to get that, for all $t \in [0,T]$,

$$
\begin{aligned}
&\|\nabla u_\varepsilon(t)\|^2_{\mathrm{L}^2} + \|z_\varepsilon(t)\|^2_{\mathrm{L}^2} + \|\varepsilon z_\varepsilon(t)\|^2_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \\
&\overset{(3.1)}{\le} c\big(1+\mathcal{W}_\varepsilon(u_\varepsilon(t), z_\varepsilon(t))\big) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \\
&\le c\big(1 + \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \langle \ell(t), u(t)\rangle + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])\big) \\
&\overset{(2.16)}{=} c\left(1 + \mathcal{E}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) + \langle \ell(t), u(t)\rangle - \int_0^t \langle \dot\ell, u\rangle\,\mathrm{d}s\right) \\
&\le c\left(1 + \mathcal{E}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) + \|\ell(t)\|_{\mathrm{H}^{-1}}\|u(t)\|_{\mathrm{H}^1} + \int_0^t \|\dot\ell\|_{\mathrm{H}^{-1}}\|u\|_{\mathrm{H}^1}\,\mathrm{d}s\right)
\end{aligned}
$$

so that the assertion follows by Gronwall's Lemma. ☐

Owing to the a priori bound (3.29), we may now exploit the generalized version of Helly's Selection Principle in [MRS08, Thm. A.1] (consider also the comments thereafter) and deduce that, at least for some nonrelabeled subsequence, and all $s,\ t \in [0,T]$ with $s < t$,

$$
\begin{aligned}
&\delta_0(t) := \lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]), \\
&z_\varepsilon(t) \to z_0(t) \quad \text{weakly in } \mathcal{Z}, \\
&\mathrm{Diss}_{\mathcal{D}_0}(z_0;[s,t]) \le \delta_0(t) - \delta_0(s),
\end{aligned}
$$

Moreover, by letting $t \in [0,T]$ be fixed we may extract a further subsequence (still not relabeled, possibly depending on t) such that $u_\varepsilon(t) \to u_*$ weakly in $\mathcal{U}$. We now check that indeed $(u_*, z_0(t)) \in \mathcal{S}_0(t)$. To this aim, by density it suffices to consider competitors $(\widehat{u}_0, \widehat{z}_0) = (u_*, z_0(t)) + (\tilde u, \tilde z)$ with $(\tilde u, \tilde z)$ smooth and compactly supported. By applying Lemma 3.6 we find a mutual recovery sequence $(\widehat{u}_\varepsilon, \widehat{z}_\varepsilon)$ such that

$$
\begin{aligned}
&\mathcal{E}_0(t, \widehat{u}_0, \widehat{z}_0) - \mathcal{E}_0(t, u_*, z_0(t)) + \mathcal{D}_0(z_0(t), \widehat{z}_0) \\
&\ge \limsup_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(t, \widehat{u}_\varepsilon, \widehat{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathcal{D}_\varepsilon(z_\varepsilon(t), \widehat{z}_\varepsilon)\right) \ge 0
\end{aligned}
\tag{3.30}
$$

where the last inequality follows from the stability (2.15) of $(u_\varepsilon(t), z_\varepsilon(t))$. Hence, we have proved that $(u_*, z_0(t)) \in \mathcal{S}_0(t)$. Note that, given $z_0(t) \in \mathcal{Z}$, as the functional $u \in \mathcal{U} \mapsto \mathcal{E}_0(t, u, z_0(t))$ is uniformly convex there exists a unique $u_0(t) \in \mathcal{U}$ such that $(u_0(t), z_0(t)) \in \mathcal{S}_0(t)$. From the fact that $(u_*, z_0(t)) \in \mathcal{S}_0(t)$ we conclude that $u_* \equiv u_0(t)$. In particular $u_\varepsilon(t) \to u_0(t)$ weakly in $\mathcal{U}$ for all $t \in [0,T]$ and the whole sequence converges.

Let now be given a partition $\{0 = t_0 < t_1 < \cdots < t_N = t\}$. By passing to the $\liminf$ in the energy balance (2.16) and using Lemmas 3.3 and 3.5 we get that

$$
\begin{aligned}
&\mathcal{E}_0(t, u_0(t), z_0(t)) + \sum_{i=1}^N \mathcal{D}_0(z_0(t_i), z_0(t_{i-1})) \\
&\le \liminf_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \sum_{i=1}^N \mathcal{D}_\varepsilon(z_\varepsilon(t_i), z_\varepsilon(t_{i-1}))\right) \\
&\le \liminf_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) - \int_0^t \langle \dot\ell, u_\varepsilon\rangle\,\mathrm{d}s\right) = \mathcal{E}_0(0, u^0_0, z^0_0) - \int_0^t \langle \dot\ell, u_0\rangle\,\mathrm{d}s
\end{aligned}
$$

where for the last equality we have used (2.14) and the convergence of u_ε. Hence, the upper energy estimate follows by taking the $\sup$ among all partitions of the interval $[0,t]$. The lower energy estimate can classically recovered from stability as in [Mie05, Prop. 2.7]. This proves that (u_0, z_0) is a linearized-plasticity solution. In particular, as linearized-plasticity solutions are unique, the whole sequence $(u_\varepsilon, z_\varepsilon)$ converges and no extraction of subsequences is actually needed.

Along the lines of the proof of Theorem 2.2 (see also [MRS08, Thm. 3.1]) we also obtain the following convergences.

Corollary 3.9 (Improved convergences)**.** *Under the assumptions of Theorem* 2.2 *we have that, for all* $t \in [0,T]$*,*

$$\int_\Omega \big(W^\varepsilon_{\rm el}(A_\varepsilon) + W^\varepsilon_{\rm h}(z_\varepsilon)\big)\,{\rm d}x \to \int_\Omega \big(|\nabla u_0 - z_0|^2_{\mathbb{C}} + |z_0|^2_{\mathbb{H}}\big)\,{\rm d}x, \tag{3.31}$$

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \to \int_0^t R(\dot z)\,{\rm d}s. \tag{3.32}$$

In particular, owing to the energy convergence (3.31) we are in the position of deducing some strong convergence of finite-plasticity solutions to linearized-plasticity solutions.

Corollary 3.10 (Strong convergence)**.** *Under the assumptions of Theorem* 2.2 *for all* $t \in [0,T]$ *we have that* $(u_\varepsilon(t), z_\varepsilon(t)) \to (u_0(t), z_0(t))$ *strongly in* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \times \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ *for all* $p \in [1,2)$*.*

Proof. Let ν denote the Young measure generated by the sequence $(A_\varepsilon, z_\varepsilon)$ and define the measure $\nu^{\rm sym}(A^s, Z) := \nu(A^s \oplus \mathbb{R}^{d\times d}_{\rm anti}, Z)$ for all Borel sets $(A^s, Z) \subset \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}^{d\times d}$. Note that $\nu^{\rm sym}$ is indeed the Young measure generated by $(A^{\rm sym}_\varepsilon, z_\varepsilon)$. By using the lower semicontinuity Lemma 4.2 and the energy convergence (3.31) we deduce that

$$\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^{d\times d}_{\rm sym}\times\mathbb{R}^{d\times d}} \big(|A^{\rm sym}|^2_{\mathbb{C}} + |z|^2_{\mathbb{H}}\big){\rm d}\nu^{\rm sym}_x(A^{\rm sym}, z)\right){\rm d}x \\
&= \int_\Omega \left(\int_{\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}} \big(|A|^2_{\mathbb{C}} + |z|^2_{\mathbb{H}}\big){\rm d}\nu_x(A, z)\right){\rm d}x \\
&\le \liminf_{\varepsilon\to0} \int_\Omega \big(W^\varepsilon_{\rm el}(A_\varepsilon) + W^\varepsilon_{\rm h}(z_\varepsilon)\big)\,{\rm d}x \overset{(3.31)}{=} \int_\Omega \big(|\nabla u_0 - z_0|^2_{\mathbb{C}} + |z_0|^2_{\mathbb{H}}\big)\,{\rm d}x.
\end{aligned} \tag{3.33}$$

Recall from (3.7) that

$$A^{\rm sym}_\varepsilon = \nabla u^{\rm sym}_\varepsilon - z^{\rm sym}_\varepsilon - \varepsilon(\nabla u_\varepsilon z_\varepsilon - \nabla u_\varepsilon w_\varepsilon)^{\rm sym}$$

where the remainder term $\varepsilon(\nabla u_\varepsilon z_\varepsilon - \nabla u_\varepsilon w_\varepsilon)^{\rm sym}$ converges strongly to 0 in L^p for all $p \in [1,2)$. Hence, the barycenter of $\nu^{\rm sym}$ is clearly $(\nabla u^{\rm sym}_0 - z^{\rm sym}_0, z_0)$.

We readily check that the measure $\nu^{\rm sym}$ is concentrated in its barycenter. Indeed, if this was not the case, by uniform convexity we would have that

$$\begin{aligned}
&\int_\Omega \big(|\nabla u^{\rm sym}_0 - z^{\rm sym}_0|^2_{\mathbb{C}} + |z_0|^2_{\mathbb{H}}\big)\,{\rm d}x \\
&< \int_\Omega \left(\int_{\mathbb{R}^{d\times d}_{\rm sym}\times\mathbb{R}^{d\times d}} \big(|A^{\rm sym}|^2_{\mathbb{C}} + |z|^2_{\mathbb{H}}\big){\rm d}\nu^{\rm sym}_x(A^{\rm sym}, z)\right){\rm d}x
\end{aligned}$$

contradicting relation (3.33). Here we have used positive definiteness from (2.5) and (2.9). As ν^{sym} is concentrated, we exploit [AGS08, Thm. 5.4.4.iii, p. 127] and deduce that

$$\int_\Omega f(x, A_\varepsilon^{\mathrm{sym}}(x), z_\varepsilon(x))\,\mathrm{d}x \to \int_\Omega \left(\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}} f(x, A^{\mathrm{sym}}, z)\mathrm{d}\nu_x^{\mathrm{sym}}(A^{\mathrm{sym}}, z) \right) \mathrm{d}x$$

along with the choice

$$f(x, A^{\mathrm{sym}}, z) := \left|(\nabla u_0^{\mathrm{sym}}(x) - z_0^{\mathrm{sym}}(x), z_0^{\mathrm{sym}}(x)) - (A^{\mathrm{sym}} - z^{\mathrm{sym}}, z)\right|^p.$$

Hence, we have that $(A_\varepsilon^{\mathrm{sym}}, z_\varepsilon) \to (\nabla u_0^{\mathrm{sym}} - z_0^{\mathrm{sym}}, z_0)$ strongly in $\mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ for all $p \in [1,2)$. In particular,

$$\nabla u_\varepsilon^{\mathrm{sym}} = A_\varepsilon^{\mathrm{sym}} + z_\varepsilon^{\mathrm{sym}} + \varepsilon(\nabla u_\varepsilon z_\varepsilon - \nabla u_\varepsilon w_\varepsilon)^{\mathrm{sym}} \to \nabla u_0^{\mathrm{sym}} \text{ strongly in } \mathrm{L}^p$$

for all $p \in [1,2)$ and the assertion follows by Korn's inequality. ☐

3.5. **Sketch of the proof of Theorem 2.3.** The argument for Theorem 2.2 can be adapted to prove Theorem 2.3 as well. The only notable difference is that one has to cope with the fact that the piecewise constant interpolants $(\overline{u}_\varepsilon, \overline{z}_\varepsilon)$ of the approximate incremental minimizers need not be stable but rather just *approximately stable*. More precisely, from (2.17) and the triangle inequality we have that

$$\forall (\widehat{u}, \widehat{z}) \in \mathcal{Q} : \mathcal{E}_\varepsilon(t, \widehat{u}, \widehat{z}) - \mathcal{E}_\varepsilon(t, \overline{u}_\varepsilon(t), \overline{z}_\varepsilon(t)) + \mathcal{D}_\varepsilon(\overline{z}_\varepsilon(t), \widehat{z}) \geq -\tau_\varepsilon\alpha_\varepsilon.$$

By coordinating to the sequence $(\overline{u}_\varepsilon(t), \overline{z}_\varepsilon(t))$ a mutual recovery sequence $(\widehat{u}_\varepsilon, \widehat{z}_\varepsilon)$ via Lemma 3.6 (with $(\overline{u}_\varepsilon(t), \overline{z}_\varepsilon(t))$ instead of $(u_\varepsilon(t), z_\varepsilon(t))$) the lower bound (3.30) still follows as $\tau_\varepsilon\alpha_\varepsilon \to 0$. Hence, the stability of the limit can be recovered. Finally, improved and strong convergences in the spirit of Corollaries 3.9-3.10 can be established as well.

4. Appendix

4.1. **Estimate on left and right multiplication.** In the proof of Theorems 2.2-2.3 we have made use of the following estimate combining left and right multiplication.

Lemma 4.1. *Assume* (2.1a) *and* (2.1d). *Then,*

$$\exists c_7,\, c_8, \gamma > 0\ \forall G_1,\, G_2 \in B_\gamma(I)\ \forall F \in \mathit{GL}_+(d) :$$
$$|W_{\mathrm{el}}(G_1 F G_2) - W_{\mathrm{el}}(F)| \leq c_7(W(F) + c_8)\big(|G_1 - I| + |G_2 - I|\big). \tag{4.1}$$

Proof. Following [Bal02, Lemma 2.5], we find positive constants $c_0,\ \widehat{c}_0,\ \gamma$ such that, for all $G \in B_\gamma(I)$ and all $F \in \mathrm{GL}_+(d)$, one has that

$$W_{\mathrm{el}}(GF) \leq \widehat{c}_0 W_{\mathrm{el}}(F) + c_0, \quad W_{\mathrm{el}}(FG) \leq \widehat{c}_0 W_{\mathrm{el}}(F) + c_0, \tag{4.2}$$
$$|\partial_F W(GF) F^\top| \leq \widehat{c}_0 W_{\mathrm{el}}(F) + c_0, \tag{4.3}$$
$$|F^\top \partial_F W(FG)| \leq \widehat{c}_0 W_{\mathrm{el}}(F) + c_0. \tag{4.4}$$

For $s \in [0,1]$, let now $H_j(s) := (1-s)I + sG_j$ for $j = 1, 2$, and note that $H_j \in B_\gamma(I)$. As $H_j' = G_j - I$ is constant we can compute that

$$
\begin{aligned}
&W_{\mathrm{el}}(G_1FG_2) - W_{\mathrm{el}}(F) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} W_{\mathrm{el}}(H_1(s)FH_2(s))\,\mathrm{d}s \\
&= \int_0^1 \partial_F W_{\mathrm{el}}(H_1FH_2)(H_1'FH_2 + H_1FH_2')\,\mathrm{d}s \\
&= \int_0^1 \partial_F W_{\mathrm{el}}(H_1FH_2)(FH_2)^\top\,\mathrm{d}s : H_1' + \int_0^1 (H_1F)^\top W_{\mathrm{el}}(H_1FH_2)\,\mathrm{d}s : H_2'.
\end{aligned}
$$

We control the above right-hand side as

$$
\begin{aligned}
\left|\int_0^1 \partial_F W_{\mathrm{el}}(H_1FH_2)(FH_2)^\top\,\mathrm{d}s : H_1'\right| &\overset{(4.3)}{\leq} \left(\int_0^1 \big(\widehat{c}_0 W_{\mathrm{el}}(FH_2) + c_0\big)\,\mathrm{d}s\right)|G_1 - I| \\
&\overset{(4.2)}{\leq} \big(\widehat{c}_0^2 W_{\mathrm{el}}(F) + c_0\widehat{c}_0 + c_0\big)|G_1 - I|, \\
\left|\int_0^1 (H_1F)^\top W_{\mathrm{el}}(H_1FH_2)\,\mathrm{d}s : H_2'\right| &\overset{(4.4)}{\leq} \left(\int_0^1 \big(\widehat{c}_0 W_{\mathrm{el}}(H_1F) + c_0\big)\,\mathrm{d}s\right)|G_2 - I| \\
&\overset{(4.2)}{\leq} \big(\widehat{c}_0^2 W_{\mathrm{el}}(F) + c_0\widehat{c}_0 + c_0\big)|G_2 - I|,
\end{aligned}
$$

whence the assertion follows. □

4.2. **Lower semicontinuity tool.** In Section 3 the following lower-semicontinuity lemma is used.

Lemma 4.2 (Lower-semicontinuity). *Let $f_0,\ f_\varepsilon : \mathbb{R}^n \to [0,\infty]$ be lower semicontinuous,*

$$
\forall v_0 \in \mathbb{R}^n : \quad f_0(v_0) \leq \inf\big\{ \liminf_{\varepsilon\to 0} f_\varepsilon(v_\varepsilon) \mid v_\varepsilon \to v_0 \big\},
$$

and $w_n \to w_0$ weakly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$. Denoting by ν the Young measure generated by w_n we have that

$$
\int_\Omega \left(\int_{\mathbb{R}^n} f_0(w)\mathrm{d}\nu_x(w)\right)\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(w_\varepsilon)\,\mathrm{d}x.
$$

In particular, if f_0 is convex we have

$$
\int_\Omega f_0(w_0)\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(w_\varepsilon)\,\mathrm{d}x.
$$

This lemma is in the same spirit of the results by BALDER [Bal84a, Thm. 1] and IOFFE [Iof77] and can be proved via augmenting the variables by including the parameter ε. The reader is referred to [Ste08, Thm 4.3, Cor. 4.4] or [MRS09, Lemma 3.1] for a proof in the case $d = 1$. In case of local uniform convergence, a proof can be found in [Li96].

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