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Generalized Prandtl–Ishlinskii operators arising from homogenization and dimension reduction

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Abstract

We consider rate-independent evolutionary systems over a physically domain Ω that are governed by simple hysteresis operators at each material point. For multiscale systems where ε denotes the ratio between the microscopic and the macroscopic length scale, we show that in the limit $\varepsilon \to 0$ we are led to systems where the hysteresis operators at each macroscopic point is a ***generalized Prandtl–Ishlinskii operator***.

1 Introduction

We are interested in the generation of complicated hysteresis in the process of taking multi-scale limits where the underlying problem on the small scale is described by simple hysteresis loops. For instance, we will show that homogenization of a problem with classical play operators, which do not have interior hysteresis loops, on the small scale will give rise to a homogenized macroscopic problem on the larger scale that has a complicated hysteresis operator of Prandtl-Ishlinskii type, which displays interior loops.

Our theory is based on the energetic formulation of rate-independent systems (RIS) $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ where the hysteresis is described by a differential inclusion for the state variable $q : [0, T] \to \mathcal{Q}$, namely

$$0 \in \partial\mathcal{R}(\dot{q}) + \mathrm{D}\mathcal{E}(t, q). \tag{1.1}$$

Here $\mathcal{E}$ is the energy potential, and the dissipation potential $\mathcal{R}$ is nonnegative, convex and homogeneous of degree 1, which leads to rate independency. The set $K^* := \partial\mathcal{R}(0) \subset \mathcal{Q}^*$ is called the play domain and its boundary is called the yields surface. In the case that the energy is quadratic in q, viz.

$$\mathcal{E}(t, q) = \tfrac{1}{2}\langle Aq, q\rangle - \langle \ell(t), q\rangle,$$

we call the solution operator for (1.1) the ***play operator*** (cf. [BrS96, Kre99]) associated to A and K^* and

$$q(t) = \mathfrak{P}_{A,K^*}[q(0), \ell]\,(t)$$

for the output $q \in \mathrm{W}^{1,1}([0, T]; \mathcal{Q})$, where $q(0)$ and $\ell \in \mathrm{W}^{1,1}([0, T]; \mathcal{Q}^*)$ are the inputs. Applications include elastoplasticity, isothermal shape-memory materials, piezo-electric materials, or micromagnetism. We refer to the surveys [Vis94, BrS96, Alb98, Mie05, Mie06] for further details on applications.

In Section 2 we recall the general theory of convergence of RIS $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ where ε is a small parameter tending to 0. As a special case of the abstract theory of Γ-convergence of energetic solutions derived in [MRS08] we present a fairly general convergence theory for play operators. Under the assumption that $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ converge in the sense of Mosco to $\mathcal{E}_0$ and $\mathcal{R}_0$, respectively, and that $\mathcal{R}_\varepsilon$ continuously converges to $\mathcal{R}_0$ we have the following statement (cf. [LiM11]): If

$$q^\varepsilon(0) \rightharpoonup q^0(0) \text{ and } \mathcal{E}_\varepsilon(0,q^\varepsilon(0)) \to \mathcal{E}_0(0,q^0(0))$$

then the solutions q^ε satisfy $q^\varepsilon(t) \to q^0(t)$ for all $t \in [0,T]$. The latter statement is the definition of the Γ-convergence of the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(\mathcal{Q},\mathcal{E}_0,\mathcal{R}_0)$. Application of this theory will be given in homogenization and in dimension reductions in Sections 4 and 5.

The Γ-convergence theory shows that the set of abstract play operators is closed under Γ-convergence for RIS. In this work we want to highlight that in such limit processes the class of simple hysteresis operators is not closed. In particular, we want to show that in limits for multiscale systems we can generate complex hysteresis operators in the large-scale system, when starting with simple hysteresis operators for the small-scale system. These hysteresis operators are obtained as ***symmetric B-contraction of a symmetric play operator***, namely

$$\mathcal{P}^B_{A,K^*}\big[q_0,\widetilde{\ell}\big] := B\mathfrak{P}_{A,K^*}\big[q_0,B^*\widetilde{\ell}\big].$$

We call these hysteresis operators ***generalized Prandtl-Ishlinskii operators*** (gPI operators). See Section 3 for further details and nontrivial examples.

In Section 4 we show how these operators appear in homogenization of elastoplastic materials, where the material properties are periodically modulated on the small scale with period ε. The mathematical tools is two-scale homogenization (cf. [Alb03, MiT07, Nes07, Vis08, Sch09, ScV10]), where the micro-cell problem defines the gPI operator. According to [MiT07, GiM10, Han11], the case of linearized elastoplasticity, where $q=(u,z)$ with the displacement $u:\Omega\to\mathbb{R}^d$ and the internal variable $z:\Omega\to\mathbb{R}^m$, one finds different macroscopic elastoplastic models depending on the strength of the gradient regularization $\varepsilon^{2\gamma}|\nabla z|^2$. In the case $\gamma<1$ one obtains classical models with homogenized elasticity and averaged yield strength. We refer also to [FrK00] for such a result in space dimension 1. However, for $\gamma \geq 1$ the macroscopic model can only be described in terms of a gPI operator. The occurrence of more complicated hysteresis operators for homogenized material models was also highlighted in [Vis08, Vis11].

In Section 5 we recall the rigorous derivation of an elastoplastic plate model from [Lie08, LiM11]. We show that it has a natural interpretation in terms of vector-valued gPI operators. While the case of pure bending was treated in [GKS08], we follow the general approach of [LiM11], where membrane and bending deformations are coupled via plastic effects.

2 Γ-convergence for rate-independent systems

Here we consider general families $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)_{\varepsilon\in]0,1]}$ of RIS and study the convergence of the associated solutions q^ε in the limit $\varepsilon\to 0$. The aim is to establish fairly general conditions on the convergences of $(\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(\mathcal{E}_0,\mathcal{R}_0)$ that guarantee that the solutions q_ε converge to the solution q of the limit system $(\mathcal{Q},\mathcal{E}_0,\mathcal{R}_0)$, which we then call the Γ-limit of the above family.

For rate-independent systems a general strategy for Γ-convergence was developed in [MRS08], which found numerous applications in, e.g., fracture [GiP06a], homogenization [MiT07], numerical approximation [KMR05, GiP06b, MiR09], and delamination [RSZ09, MRT10]. Here we specialize this theory to the case that $\mathcal{E}_\varepsilon(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ is a quadratic functional, as it is the case for play operators and in linearized elastoplasticity. Thus, the abstract theory is simplified in two respects. First, the systems under consideration have unique solutions and we do not need to consider subsequences. Second, the quadratic nature of the energy allows for a simpler construction of recovery sequences by using the quadratic trick introduced in [MiT07]. Thus, the strong compactness assumptions in [MRS08] can be avoided.

The convergence result is formulated abstractly in terms of Γ-convergence of $\mathcal{E}_\varepsilon(t,\cdot)$ towards $\mathcal{E}_0(t,\cdot)$ and of $\mathcal{R}_\varepsilon$ to $\mathcal{R}_0$, where we use the weak and the strong topologies in the underlying separable Hilbert space $\mathcal{Q}$. It might be surprising that convergence of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ is enough to guarantee convergence of the solutions q^ε of the subdifferential inclusion (1.1), since from the equation it seems necessary to control the convergence of the (sub-) differentials. The relevance of the functionals is seen better if we use the equivalent ***energetic formulation*** for RIS. Here the equivalence holds, as $\mathcal{E}_\varepsilon(t,\cdot)$ is strictly convex, see [MiT04, Mie05]. A function q^ε is called ***energetic solution*** for the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ if for all $t\in[0,T]$ we have the stability (S) and the energy balance (E):

$$\text{(E)}\quad \mathcal{E}_\varepsilon(t,q^\varepsilon(t))\leq \mathcal{E}_\varepsilon(t,\tilde q)+\mathcal{R}_\varepsilon(\tilde q - q^\varepsilon(t)) \text{ for all } \tilde q\in\mathcal{Q};$$

$$\text{(S)}\quad \mathcal{E}_\varepsilon(t,q^\varepsilon(t))+\mathrm{Diss}_{\mathcal{R}_\varepsilon}(q^\varepsilon,[0,t]) = \mathcal{E}_\varepsilon(0,q^\varepsilon(0))+\textstyle\int_0^t \partial_s\mathcal{E}_\varepsilon(s,q^\varepsilon(s))\,\mathrm{d}s.$$

The dissipation $\mathrm{Diss}_{\mathcal{R}}(q,[r,s])$ is defined via

$$\mathrm{Diss}_{\mathcal{R}}(q,[r,s]) := \sup \textstyle\sum_{j=1}^N \mathcal{R}(q(t_j)-q(t_{j-1}))$$

where the supremum is taken over all $N\in\mathbb{N}$ and all partitions $r<t_0<t_1<\cdots<t_N<s$. Note that the dissipation is defined along any curve $q:[0,T]\to\mathcal{Q}$ without any assumptions on continuity or differentiability. For absolutely continuous functions we have

$$\mathrm{Diss}_{\mathcal{R}}(q,[r,s]) = \textstyle\int_r^s \mathcal{R}(\dot q(t))\,\mathrm{d}t.$$

We recall that the energetic formulation via (S) and (E) is totally equivalent to the subdifferential inclusion for play operators, where $\mathcal{E}_\varepsilon(t,\cdot)$ is uniformly convex. Its importance is that it is totally derivative free. We neither need derivatives of the solution $q^\varepsilon:[0,T]\to\mathcal{Q}$ nor of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$. Thus, it is ideally suited for limiting processes in the variational sense, where the convergence of functionals is studied, see [Att84, Dal93]. We use the notions of ***Mosco convergence*** and ***continuous convergence*** for functionals $\mathcal{I}_n$. The first is written $\mathcal{I}_n \xrightarrow{\mathrm{M}} \mathcal{I}$ and defined via (i) and (ii):

(i) Liminf estimate:

$$q_n \rightharpoonup q \quad\Longrightarrow\quad \mathcal{I}(q)\leq \liminf_{n\to\infty}\mathcal{I}_n(q_n),$$

(ii) Limsup estimate ($\hat{=}\ \exists$ recovery sequences)

$$\forall\,\hat q\in\mathcal{Q}\ \exists\,(\hat q_n)_n:\quad \hat q_n\to\hat q \text{ and } \mathcal{I}(\hat q)\geq \limsup_{n\to\infty}\mathcal{I}_n(\hat q_n).$$

The continuous convergence (with respect to the norm topology) is written as $\mathcal{I}_n \stackrel{\mathrm{c}}{\rightsquigarrow} \mathcal{I}$ and defined via

$$\mathcal{I}_n \stackrel{\mathrm{c}}{\rightsquigarrow} \mathcal{I} \;\Leftrightarrow\; \big(q_n \to q \;\Rightarrow\; \mathcal{I}_n(q_n) \to \mathcal{I}(q) \big).$$

Our precise assumptions on the family $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon\in[0,1]}$ are the following. Note that often the limit functionals $\mathcal{E}_0$ and $\mathcal{R}_0$ are included in the assumptions via $\varepsilon = 0$. The assumptions (2.1a)–(2.1c) provide some uniform a priori estimates, while (2.1d) and (2.1e) are the main convergence assumptions.

$$\begin{aligned}
&\mathcal{E}_\varepsilon(t,q) = \mathcal{B}_\varepsilon(q) - \langle \ell_\varepsilon(t), q\rangle \text{ where } \mathcal{B}_\varepsilon \\
&\quad \text{is quadratic, wlsc and } \ell_\varepsilon \in \mathrm{C}^1([0,T];\mathcal{Q}^*); && \text{(2.1a)}\\
&\mathcal{R}_\varepsilon : \mathcal{Q} \to [0,\infty] \text{ is 1-homogeneous,} \\
&\quad \text{wlsc, and convex;} && \text{(2.1b)}\\
&\exists\, \beta, C > 0\; \forall\, (t,q) \in [0,T]\times\mathcal{Q}\; \forall\, \varepsilon\in[0,1]: \\
&\quad \mathcal{B}_\varepsilon(q) \geq \tfrac{\beta}{2}\|q\|^2,\;\; \|\ell_\varepsilon(t)\|_{\mathcal{Q}^*} + \|\dot\ell_\varepsilon(t)\|_{\mathcal{Q}^*} \leq C; && \text{(2.1c)}\\
&\mathcal{B}_\varepsilon \stackrel{\mathrm{M}}{\longrightarrow} \mathcal{B}_0 \;\text{ and }\; \forall\, t:\; \ell_\varepsilon(t) \to \ell_0(t) \text{ in } \mathcal{Q}^*; && \text{(2.1d)}\\
&\mathcal{R}_\varepsilon \stackrel{\mathrm{c}}{\rightsquigarrow} \mathcal{R}_0 \quad\text{and}\quad \mathcal{R}_\varepsilon \stackrel{\mathrm{M}}{\longrightarrow} \mathcal{R}_0. && \text{(2.1e)}
\end{aligned}$$

In the last condition "$\stackrel{\mathrm{c}}{\rightsquigarrow}$" implies that every strongly converging sequence is a recovery sequence. The additional condition "$\stackrel{\mathrm{M}}{\longrightarrow}$" is needed to guarantee $\mathcal{R}_0(q_0) \leq \liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(q_\varepsilon)$ whenever $q_\varepsilon \rightharpoonup q_0$. Note that we only ask for continuous convergence in the norm topology, which is in contrast to [KMR05, MRS08, MiR09], where the more restrictive continuous convergence in the weak topology is used. Thus, one can follow [MiT07] and exploit the quadratic structure (2.1a) of $\mathcal{E}_\varepsilon$ for the construction of ***mutual recovery sequences***.

The following convergence result for the solutions q_ε of the RIS $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ is established in [LiM11, Thm. 3.1]. It shows strong convergence of the solutions q^ε towards solutions q^0 of the limiting problem $(\mathcal{Q}, \mathcal{E}_0, \mathcal{R}_0)$. Moreover, the solutions $q^\varepsilon(t)$ are recovery sequences for $q_0(t)$, see (2.3b).

Theorem 2.1 *Let the assumptions* (2.1) *hold. Moreover, choose a family* $(q^0_\varepsilon)_{\varepsilon\in[0,1]}$ *of initial data such that the following conditions hold:*

$$\begin{aligned}
&\forall\, \varepsilon\in[0,1]\; \forall\, \tilde q \in \mathcal{Q}: \\
&\qquad \mathcal{E}_\varepsilon(0,q^0_\varepsilon) \leq \mathcal{E}_\varepsilon(0,\tilde q) + \mathcal{R}_\varepsilon(\tilde q - q^0_\varepsilon), && \text{(2.2a)}\\
&q^0_\varepsilon \rightharpoonup q^0_0 \quad \textit{and} \quad \mathcal{E}_\varepsilon(0,q^0_\varepsilon) \to \mathcal{E}_0(0,q^0_0). && \text{(2.2b)}
\end{aligned}$$

Then, the unique solutions $q_\varepsilon : [0,T] \to \mathcal{Q}$ *for the RIS* $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *with* $q_\varepsilon(0) = q^0_\varepsilon$ *satisfy, for all* $t \in [0,T]$*, the convergences*

$$\begin{aligned}
&q_\varepsilon(t) \to q_0(t), && \text{(2.3a)}\\
&\mathcal{E}_\varepsilon(t,q_\varepsilon(t)) \to \mathcal{E}_0(t,q_0(t)), && \text{(2.3b)}\\
&\mathrm{Diss}_{\mathcal{R}_\varepsilon}(q_\varepsilon,[0,t]) \to \mathrm{Diss}_{\mathcal{R}_0}(q_0,[0,t]). && \text{(2.3c)}
\end{aligned}$$

The assumption (2.2b) on the initial conditions should be seen as the initialization of (2.3a) and (2.3b). Similarly, (2.2a) is necessary as energetic solutions are stable for all times, also $t = 0$.

We finish this section by mentioning that the assumptions of Theorem 2.1 are sufficient but not necessary for the Γ-convergence of the RIS, even when restricting to the case of play operators. Another set of assumptions would arises if we weaken the convergence of the energies $\mathcal{E}_\varepsilon$ and strengthen the convergence of the dissipation potentials $\mathcal{R}_\varepsilon$. For instance, we may impose the Γ-convergence of $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ (where the recovery sequences $\hat{q}^\varepsilon$ need to converge only in the weak sense) and ask continuous convergence in the weak topology of $\mathcal{R}_\varepsilon$ to $\mathcal{R}_0$. However, in general one needs to analyze the mutual convergences of the pairs $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ to $(\mathcal{E}_0, \mathcal{R}_0)$ in the sense of mutual recovery sequence, see [MRS08].

3 Generalized Prandtl-Ishlinskii operators

Using the energetic framework for RIS (Y, E, R), a ***symmetric play operator*** is defined on a Hilbert space Y with a quadratic energy functional $E(t, y) = \frac{1}{2}\langle Ay, y\rangle - \langle \ell(t), y\rangle$, where $A : Y \to Y^*$ is a symmetric and positive definite and bounded linear operator. The dissipation potential $R : Y \to [0, \infty]$ is lower semicontinuous, convex, and positively homogeneous of degree 1. Hence, the play domain $K^* = \partial R(0) \subset Y^*$ is closed, convex, and contains 0. The play operator $\mathfrak{P}_{A,K^*}$ associates to every $\ell \in \mathrm{W}^{1,1}([0, T]; Y^*)$ and every initial datum $y_0 \in Y$ with $\ell(0) - Ay_0 \in K^*$ the unique solution $y(t) = \mathfrak{P}_{A,K^*}[y_0, \ell](t)$ of

$$0 \in \partial R(\dot{y}(t)) + Ay(t) - \ell(t), \quad y(0) = y_0.$$

Definition 3.1 *A* ***symmetric*** B***-contraction*** $\mathcal{P}^B_{A,K^*}$ *of the symmetric play operator* $\mathfrak{P}_{A,K^*}$ *is defined using a Hilbert space* $\hat{Y}$ *and a bounded linear and surjective operator* $B : Y \to \hat{Y}$ *via*

$$\mathcal{P}^B_{A,K^*}[y_0, \hat{\ell}](t) := B\, \mathfrak{P}_{A,K^*}[y_0, B^* \hat{\ell}](t) \in \hat{Y}$$

for $\hat{\ell} \in \mathrm{W}^{1,1}([0, T]; \hat{Y}^*)$*. We say shortly that* $\mathcal{P}^B_{A,K^*}$ *is a* B*-contraction of* $\mathfrak{P}_{A,K^*}$*.*

A hysteresis operator is called ***generalized Prandtl-Ishlinskii operator*** *(gPI operator) if it is a* B*-contraction of a symmetric play operator* $\mathfrak{P}_{A,K^*}$ *for suitable* $A,\ K^*,\ B$ *and Hilbert spaces* Y *and* $\hat{Y}$*.*

By definition $\mathcal{P}^B_{A,K^*}[y_0, \cdot]$ maps $\mathrm{W}^{1,1}([0, T], \hat{Y}^*)$ into $\mathrm{W}^{1,1}([0, T], \hat{Y})$. Note that we cannot project the initial datum y_0 from Y into $\hat{Y}$, as the internal memory remains to be defined via the variable y in the bigger space Y.

We now give a few examples of gPI operators.

Example 3.2 (Classical Prandtl-Ishlinskii operator) *For classical Prandtl-Ishlinskii operators we use a measure space* $(\Sigma; \mu)$ *and let* $Y = \mathrm{L}^2(\Sigma)$ *using the measure* μ*. For functions* $a \in \mathrm{L}^\infty(\Sigma)$ *and* $\rho \in \mathrm{L}^2(\Sigma)$ *with* $a(\sigma) \geq \delta > 0$ *and* $\rho(\sigma) \geq 0$ μ*-a.e. we define*

$$E(t, y) = \int_\Sigma \tfrac{1}{2} a(\sigma) y(\sigma)^2 \,\mathrm{d}\mu - \langle \ell(t), y\rangle$$
$$\text{and} \quad R(v) = \int_\Sigma \rho(\sigma) |v(\sigma)| \,\mathrm{d}\mu,$$

giving $K^* = \{\,\xi\,|\,\xi(\sigma) \in k(\sigma)^* := B_{\rho(\sigma)}(0)\ \mu$*-a.e.*$\,\} \subset \mathrm{L}^2(\Sigma)$*. The total play operator* $\mathfrak{P}_{A,K^*}$ *consists of independent scalar plays* $P_{a(\sigma),k(\sigma)^*}$ *for each* $\sigma \in \Sigma$*, viz.*

$$\mathfrak{P}_{A,K^*}[y_0,\ell](t,\sigma) = P_{a(\sigma),k(\sigma)^*}[y_0(\sigma),\ell(\cdot,\sigma)](t).$$

The classical Prandtl-Ishlinskii operator is obtained by loading the individual plays $P_{a(\sigma),k(\sigma)^*}$ *with* $b(\sigma)\hat\ell$ *with* $\hat\ell(t) \in \hat Y^* = \mathbb{R}$ *independent of* σ*. The output is then obtained by averaging via* b*. In fact, we set* $\hat Y = \mathbb{R}$*, choose* $b \in \mathrm{L}^2(\Sigma)$*, and define* $B : Y \to \hat Y$ *with*

$$By = \textstyle\int_\Sigma b(\sigma)y(\sigma)\,\mathrm{d}\mu, \text{ giving } B^*\hat\eta = \hat\eta\, b.$$

Following [Vis94, BrS96, Kre99] the PI operator reads

$$\mathcal{P}^B_{A,K^*}[y_0,\hat\ell](t) = \textstyle\int_\Sigma b(\sigma)P_{a(\sigma),k(\sigma)^*}[y_0(\sigma),b(\sigma)\hat\ell\,](t)\,\mathrm{d}\mu,$$

The next example starts with vector-valued play operators giving again vector-valued gPI operators.

Example 3.3 (Vector-valued PI operators) *The above example can easily be extended by letting* $Y = \mathrm{L}^2(\Sigma;\mathbb{R}^n)$*,* $a \in \mathrm{L}^2(\Sigma;\mathbb{R}^{n\times n})$ *with* $\xi\cdot a(\sigma)\xi \geq \delta|\xi|^2$ *and considering a family* $k^*(\sigma) \subset \mathbb{R}^n$ *of closed convex sets containing* 0*. Then, with* $(Ay)(\sigma) = a(\sigma)y(\sigma)$ *and* $K^* = \{\,\eta \in \mathrm{L}^2(\Sigma)\mid \eta(\sigma) \in k^*(\sigma)\ \mu$*-a.e.*$\,\}$ *we obtain the Hilbert-space valued play operator* $\mathfrak{P}_{A,K^*}[y_0,\cdot]$ *mapping* $\mathrm{W}^{1,1}([0,T];\mathrm{L}^2(\Sigma;\mathbb{R}^n))$ *into itself. For the contraction we take* $\hat Y = \mathbb{R}^m$ *and the mapping* $By = \int_\Sigma by\,\mathrm{d}\mu$ *with* $b \in \mathrm{L}^2(\Sigma;\mathbb{R}^{n\times m})$*. Thus, the gPI operator* $\mathcal{P}^B_{A,K^*}$ *reads*

$$\mathcal{P}^B_{A,K^*}[y_0,\hat\ell](t) = \textstyle\int_\Sigma b(\sigma)\mathfrak{P}_{a(\sigma),k^*(\sigma)}[y_0,b(\sigma)^*\hat\ell\,](t)\,\mathrm{d}\mu.$$

Hence, $\mathcal{P}^B_{A,K^*}[y_0,\cdot]$ *maps* $\mathrm{W}^{1,1}([0,T];\mathbb{R}^m)$ *into itself.*

Finally we provide an example that does not have a representation by an integral over a measure space (Σ,μ). This example is the main reason for introducing our notion of gPI operators.

Example 3.4 (A genuine gPI operator) *We now consider a bounded and smooth, physical domain* $\Omega \subset \mathbb{R}^d$ *and the differential inclusion*

$$\begin{aligned} &0 \in \rho\,\mathrm{Sign}(\dot y) - \Delta y + \alpha y - \ell(t,x) \ \text{ on }]0,T[\times\Omega,\\ &y(t,x) = 0 \ \text{ in }]0,T[\times\partial\Omega, \end{aligned} \tag{3.1}$$

where $\alpha,\ \rho > 0$*. This subdifferential inclusion describes the play associated with* (Y,E,R) *where*

$$Y = \mathrm{H}^1_0(\Omega), \qquad R(v) = \textstyle\int_\Omega \rho|v|\,\mathrm{d}x,$$
$$E(t,y) = \textstyle\int_\Omega \frac12|\nabla y|^2 + \frac{\alpha}{2}y^2 - \ell(t)y\,\mathrm{d}x.$$

With $K^* = \{\,\eta \in \mathrm{H}^{-1}(\Omega)\mid \|\eta\|_{\mathrm{L}^\infty} \leq \rho\,\}$ *we find the play operator* $\mathfrak{P}_{A,K^*}[y_0,\cdot]$*, which maps* $\mathrm{W}^{1,1}([0,T];\mathrm{H}^{-1}(\Omega))$ *into* $\mathrm{W}^{1,1}([0,T];\mathrm{H}^1_0(\Omega))$*. For a contraction to* $\hat Y = \mathbb{R}$ *we choose* $\beta \in \mathrm{H}^{-1}(\Omega)$ *giving* $By = \langle\beta,y\rangle$*. The gPI operator* $\mathcal{P}^B_{A,K^*}$ *then takes the form* $\mathcal{P}^B_{A,K^*}[y_0,\hat\ell\,](t) := \Big\langle \beta, \mathfrak{P}_{A,K^*}[y_0,\hat\ell(\cdot)\beta](t)\Big\rangle$ *and maps* $\mathrm{W}^{1,1}([0,T])$ *into itself.*

The general class of gPI operators is not yet investigated. It may still have inherited good features from the original play operators. Thus, it remains to find suitable characterization of gPI operators and analyze their structural properties. As a first observation we mention that play operators do not describe inner loops, while Prandtl-Ishlinskii operators do. The gPI operators provide an even wider class of operators describing hysteresis with inner loops.

As an example of a potential property of general gPI operators, we may consider the case $\hat{Y} = \mathbb{R}$, which means that the input and the output are scalar. It is then natural to ask whether all such gPI operators have a return-point memory like the classical Prandtl-Ishlinskii operators. We conjecture that $\mathcal{P}^{B}_{A,K^*}$ constructed in Example 3.4 has a return-point memory. If this is the case, then there must be a representation as a Preisach operator, see [BrS96]. However, it is not clear how the corresponding inner and outer loops can directly be described through properties of Ω, α, and ρ.

4 Homogenization in elastoplasticity

In the case of homogenization we assume that the constitutive relations have coefficients that vary on a microscale with a small period $\varepsilon > 0$. We will encounter hysteresis occurring in the microscopic level as well as operators on the macroscopic level.

To be more precise, we consider an open bounded domain Ω in the physical space $\mathbb{R}^d$. Periodicity is expressed by a periodic lattice $\Lambda \subset \mathbb{R}^d$. The corresponding unit cell Ξ is defined as the torus $\mathbb{R}^d/_{\Lambda}$. For notational simplicity we assume vol$(\Xi) = 1$. On Ω we consider the elastic displacement $u : \Omega \to \mathbb{R}^d$ as well as an internal variable $z : \Omega \to \mathbb{R}^m$. Depending on the applications at hand, the latter may denote the plastic tensor in $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ and hardening variables, the magnetization in $z \in \mathbb{R}^d$, or a remanent polarization, see [Vis94, BrS96, MiT06, Mie06]. We especially refer to [Nes07, MiT07, Han11, Sch09, GiM10, ScV10] for the case of linearized elastoplasticity.

Denoting by $q = (u, z) \in \mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ the state of the system, the rate-independent system (RIS) is the triple $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ leading to the differential inclusion

$$0 = \mathrm{D}_u\mathcal{E}_\varepsilon(t, u, z), \quad 0 \in \partial\mathcal{R}_\varepsilon(\dot{z}) + \mathrm{D}_z\mathcal{E}_\varepsilon(t, u, z).$$

The state spaces are $\mathcal{Z} = \mathrm{H}^1(\Omega; \mathbb{R}^m)$ or $\mathcal{Z} = \mathrm{L}^2(\Omega; \mathbb{R}^m)$ and $\mathcal{U} = \mathrm{H}^1_{\mathsf{Dir}}(\Omega; \mathbb{R}^d)$, where "Dir" means the Dirichlet boundary $\Gamma_{\mathsf{Dir}} \subset \partial\Omega$ with $\int_{\Gamma_{\mathsf{Dir}}} 1 \,\mathrm{d}a > 0$ and $u|_{\Gamma_{\mathsf{Dir}}} = 0$. The energy functional $\mathcal{E}_\varepsilon : [0, T] \times \mathcal{Q} \to \mathbb{R}$ and the dissipation potential $\mathcal{R}_\varepsilon : \mathcal{Z} \to [0, \infty]$ are given by

$$\begin{aligned}\mathcal{E}_\varepsilon(t, z) = \textstyle\int_\Omega W(x, \tfrac{1}{\varepsilon}x, \mathbf{e}(u), z(x)) \\ + \frac{\varepsilon^{2\gamma}}{2} a(x, \tfrac{1}{\varepsilon}x)|\nabla z(x)|^2 - \ell(t, x)u(x)\mathrm{d}x,\\ \mathcal{R}_\varepsilon(\dot{z}) = \textstyle\int_\Omega R(x, \tfrac{1}{\varepsilon}x, \dot{z}(x))\,\mathrm{d}x,\end{aligned} \tag{4.1}$$

where $\mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$. Here $\xi = \frac{x}{\varepsilon}$ denotes the microscopic spatial scale on which the constitutive functions W, a, and R are assumed to be periodic, viz.

$$\begin{aligned}&W(x, \xi+\lambda, e, z) = W(x, \xi, e, z),\\ &a(x, \xi+\lambda) = a(x, \xi), \quad R(x, \xi+\lambda, z) = R(x, \xi, z),\end{aligned}$$

for all $(\lambda, x, \xi, \mathbf{e}, z)$. The stored energy density $W(x,\xi,\cdot,\cdot)$ is assumed to be a quadratic form that is uniformly bounded and coercive on $\Omega\times\Xi$.

We study convergence of q_ε for $\varepsilon\to 0$ in the sense of ***two-scale convergence*** $z_\varepsilon \overset{2}{\rightharpoonup} Z$ defined via

$$\int_\Omega z_\varepsilon(x)\cdot\Phi(x,\tfrac{1}{\varepsilon}x)\,\mathrm{d}x \to \int_\Omega\int_\Xi Z(x,\xi)\cdot\Phi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$
$$\text{for all } \Phi\in \mathrm{C}_c(\Omega\times\Xi;\mathbb{R}^m),$$

where (z_ε) is a bounded sequence in $\mathrm{L}^2(\Omega;\mathbb{R}^m)$ while the two-scale limit Z lies in $\mathrm{L}^2(\Omega\times\Xi;\mathbb{R}^m)$.

The results developed in [MiT07, Tim09, Han11] (for elastoplasticity and piezoelectricity) show that the solutions $q_\varepsilon=(u_\varepsilon,z_\varepsilon)$ of the RIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ have a two-scale limit that is again a solution of a RIS $(\mathcal{Q}_0,\mathcal{E}_0,\mathcal{R}_0)$. However, the form of the limit system strongly depends on the strength of the gradient term $|\nabla z|^2$ in the energy $\mathcal{E}_\varepsilon$, i.e. in the value of $\gamma\geq 0$. We will denote the different limit RIS by $(\mathcal{Q}^\gamma,\mathcal{E}_0^\gamma,\mathcal{R}_0^\gamma)$.

Case $\gamma=0$. In this case we obtain a classical macroscopic system, since z_ε cannot develop microstructure due to the gradient term $|\nabla z|^2$, which prevents oscillations. We obtain $q_\varepsilon(t)\rightharpoonup q_0(t)$ in $\mathrm{H}^1(\Omega;\mathbb{R}^{d\times d}_{\mathsf{sym}}\times\mathbb{R}^m)$, where q_0 solves the RIS $(\mathcal{Q}^0,\mathcal{E}_0^0,\mathcal{R}_0^0)$ with $\mathcal{Q}^0=\mathcal{U}\times\mathcal{Z}$,

$$\mathcal{E}_0^0(t,u,z)=\int_\Omega W_0(x,\mathbf{e}(u),z)+\tfrac12\nabla z\cdot a_0(x)\nabla z-\ell(t)\cdot u\,\mathrm{d}x$$
$$\mathcal{R}_0^0(\dot z)=\int_\Omega R_0(x,\dot z(x))\,\mathrm{d}x,\quad R_0(x,\hat z)=\int_\Xi R(x,\xi,\hat z)\,\mathrm{d}\xi.$$

Here we have used the notation $W_0(x,\hat{\mathbf{e}},z)=\min\int_\Xi W(x,\xi,\hat{\mathbf{e}}+\mathbf{e}(U),z)\mathrm{d}\xi$ and $p\cdot a_0(x)p=\min\int_\Xi a(x,\xi)|p+\nabla\zeta|^2\,\mathrm{d}\xi$, where the minima are taken over all microscopic fluctuations $U\in\mathrm{H}^1(\Xi;\mathbb{R}^d)$ and $\zeta\in\mathrm{H}^1(\Xi;\mathbb{R}^m)$, respectively.

Thus, the system $0\in\partial\mathcal{R}_0^0(\dot z)+\mathrm{D}_z\mathcal{E}_0^0(t,u,z),\ 0=\mathrm{D}_u\mathcal{E}_0^0(t,u,z)$ provides a macroscopic material model where the elastic tensor and the gradient regularization term are classically homogenized, while the dissipation potential is simply averaged over $\xi\in\Xi$.

Case $\gamma\in\,]0,1[$. For $\gamma>0$ the gradient regularization in $\mathcal{E}_\varepsilon$ is too weak to appear on the macroscopic scale, and for $\gamma<1$ it is still strong enough to prevent oscillations on the microscale. Hence we have $\mathcal{Q}^\gamma=\mathcal{U}\times\mathrm{L}^2(\Omega;\mathbb{R}^m)$, $\mathcal{R}_0^\gamma=\mathcal{R}_0^0$, and

$$\mathcal{E}_0^0(t,u,z)=\int_\Omega W_0(x,\mathbf{e}(u),z)-\ell(t,\cdot)u\ \mathrm{d}x.$$

Again the macroscopic model has homogenized elasticity and an averaged yield strength. The difference to $\gamma=0$ is just the missing gradient ∇z.

Case $\gamma=1$. Now the gradient regularization appears on the microscopic scale giving true two-scale limits. Using the microscopic fluctuation strain tensor $\mathbf{e}_\xi(U)=\frac12(\nabla_\xi U+\nabla_\xi U^{\mathsf{T}})$ we find

$$u_\varepsilon(t)\rightharpoonup u_0(t) \text{ in } \mathcal{U},\quad z_\varepsilon(t)\overset{2}{\rightharpoonup} Z(t),$$
$$\mathbf{e}(u_\varepsilon(t))\overset{2}{\rightharpoonup}\mathbf{e}_x(u_0(t))+\mathbf{e}_\xi(U(t)),$$

such that $(u_0, u, Z) \in \mathcal{Q}^1 = \mathcal{U} \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathsf{av}}(\Xi; \mathbb{R}^d)) \times \mathrm{L}^2(\Omega; \mathrm{H}^1(\Xi))$. Here $U(t, x, \cdot) \in \mathrm{H}^1_{\mathsf{av}}(\Xi; \mathbb{R}^d)$ is called microfluctuation, and the subscript "av" means $\int_\Xi U(t, x, \xi)\,\mathrm{d}\xi = 0$. The limit functionals are

$$
\begin{aligned}
&\mathcal{E}^1_0(t, u, U, Z) = \\
&\int_\Omega \int_\Xi W(x, \xi, \mathbf{e}(u){+}\mathbf{e}_y(U), Z) {+} \tfrac{a(x,\xi)}{2} |\nabla_\xi Z|^2 \,\mathrm{d}\xi {-} \ell(t){\cdot} u \,\mathrm{d}x, \\
&\mathcal{R}^1_0(\hat{Z}) = \int_\Omega \int_\Xi R(x, \xi, \hat{Z}(x, \xi)) \,\mathrm{d}\xi \,\mathrm{d}x.
\end{aligned}
\tag{4.2}
$$

To write down a macroscopic model for this system, we have to be more careful as we need to generate gPI operators. In particular, we have to eliminate the microfluctuations U and $Z(x, \cdot)$ jointly. For this we write out the equations more explicitly, but suppress the dependence on x and ξ:

$$
0 = -\operatorname{div}_x \Big(\int_\Xi \partial_{\mathbf{e}} W(\mathbf{e}_x(u){+}\mathbf{e}_y(U), Z) \,\mathrm{d}\xi \Big) - \ell(t, x), \tag{4.3a}
$$

$$
0 = -\operatorname{div}_\xi \partial_{\mathbf{e}} W(\mathbf{e}_x(u){+}\mathbf{e}_\xi(U), Z), \tag{4.3b}
$$

$$
\begin{aligned}
0 \in \partial R(\dot{Z}) &+ \partial_Z W(\mathbf{e}_x(u){+}\mathbf{e}_\xi(U), Z) \\
&- \operatorname{div}_\xi \big(a \nabla_\xi Z\big).
\end{aligned}
\tag{4.3c}
$$

Whereas (4.3a) is posed on Ω and hence is a macroscopic equation (after doing the indicated integration), (4.3b) and (4.3c) are posed on $\Omega \times \Xi$.

The main observation is that equations (4.3b) and (4.3c) can be solved on $\{x\} \times \Xi$ as soon as we know the macroscopic "loading" $\mathbf{e}(u)(\cdot, x) \in \mathrm{W}^{1,1}([0, T]; \mathbb{R}^{d \times d}_{\mathsf{sym}})$.

Defining $Y = \mathrm{H}^1_{\mathsf{av}}(\Xi; \mathbb{R}^d) \times \mathrm{H}^1(\Xi; \mathbb{R}^m)$, $y = (U, Z)$,

$$
\begin{aligned}
\boldsymbol{E}_x(t, U, Z) &= \int_\Xi W(x, \xi, \mathbf{e}_y(U), Z) + \tfrac{a(x,\xi)}{2} |\nabla_\xi Z|^2 \\
&\qquad + U{\cdot}\hat{\ell}_U(t) + Z{\cdot}\hat{\ell}_Z(t) \,\mathrm{d}\xi, \\
\text{and} \quad \boldsymbol{R}_x(\dot{U}, \dot{Z}) &= \int_\Xi R(x, \xi, \dot{Z}) \,\mathrm{d}\xi
\end{aligned}
$$

we obtain a play operator for each $x \in \Omega$:

$$
(U(t), Z(t)) = \boldsymbol{P}_x[(U_0, Z_0), (\hat{\ell}_U, \hat{\ell}_Z)](t).
$$

Since the loading for this play operator on the microscopic level only occurs via the macroscopic quantity $\mathbf{e}(u(t))(x)$ we define a suitable contraction leading to a gPI operator. Using that the second derivatives $\partial^2_{\mathbf{e}} W$ and $\partial_{\mathbf{e}} \partial_z W$ are independent of $\mathbf{e}$ and z, we define

$$
\begin{aligned}
&B_x : Y \to \mathbb{R}^{d \times d}_{\mathsf{sym}} \text{ with} \\
&B_x(U, Z) = \int_\Xi \partial^2_{\mathbf{e}} W(x, \xi) \mathbf{e}_\xi(U) {+} \partial_{\mathbf{e}} \partial_z W(x, \xi) Z(\xi) \,\mathrm{d}\xi
\end{aligned}
$$

and find that the unique solutions (U, Z) of (4.3b) and (4.3c) on $\{x\} \times \Xi$ are obtained by $\boldsymbol{P}_x[(U_0, Z_0), B^*_x \mathbf{e}(u(\cdot, x))]$. However, by symmetry of the second derivatives of W, we see that B exactly provides the coupling of (U, Z) into the macroscopic equation (4.3a). Hence inserting the formula for (U, Z) and using the abbreviation $A_{\mathbf{e}}(x) = \int_\Xi \partial^2_{\mathbf{e}} W(x,\xi)\,\mathrm{d}\xi$ we obtain the purely macroscopic material model

$$0 = -\operatorname{div}_x \Big(\int_\Xi A_{\mathbf{e}} \mathbf{e}_x(u) + \mathcal{P}_x[(U_0, Z_0), \mathbf{e}_x(u)(\cdot)] \Big) - \ell(t)$$

on Ω, where $\mathcal{P}_x$ is the B-contraction of by $\boldsymbol{P}_x$, viz.

$$\mathcal{P}_x[(U_0, Z_0), \mathbf{e}(\cdot)] := B_x \boldsymbol{P}_x[(U_0, Z_0), B_x^* \mathbf{e}(\cdot)].$$

We emphasize that the gPI operator $\mathcal{P}_x$ does not only contain the plastic response, but also include the elastic corrector for the homogenization procedure.

Case $\gamma > 1$. In this case the gradient regularization is so small that it does not even matter on the microscopic scale. Hence we obtain a similar two-scale limit, but now $\mathcal{Q}_0^\gamma = \mathcal{U} \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathsf{av}}(\Xi; \mathbb{R}^d)) \times \mathrm{L}^2(\Omega; \mathrm{L}^2(\Xi))$. The limit functionals $\mathcal{E}^\gamma$ has the same form as $\mathcal{E}^1$, see (4.2), except that the term involving $\nabla_\xi Z$ is missing. As in the case $\gamma = 1$ we obtain, for each macroscopic point $x \in \Omega$, a play operator $\boldsymbol{P}_x$, but now defined on $Y = \mathrm{H}^1_{\mathsf{av}}(\Xi; \mathbb{R}^d) \times \mathrm{L}^2(\Xi)$. The contraction B to a gPI operator is exactly the same as for $\gamma = 1$. Since the term $\mathbf{e}_\xi(U)$ provides coupling on Ξ, the B-contraction is a generalized and not a classical Prandtl-Ishlinskii operator.

5 Dimension reduction in elastoplasticity

Dimension reduction for plate-like domains $\Omega_\varepsilon = \omega \times \,]-\varepsilon, \varepsilon[$ is treated for small-strain elastoplasticity with hardening. In [LiM11] a plate model was rigorously derived that includes bending, in-plane stretch and compression, and a vector-valued Prandtl-Ishlinskii operator for the plastic effects. This model includes the elastoplastic Kirchhoff-plate model derived in [GKS08] for the more general case including inertial terms. A corresponding beam models was studied in [KrS08].

The plate model starts with the same rate-independent elastoplastic material model $(\mathcal{Q}, \hat{\mathcal{E}}_\varepsilon, \hat{\mathcal{R}}_\varepsilon)$ as given in (4.1), but now without the gradient ∇z (i.e. $a \equiv 0$). The internal variable is the plastic strain $\hat{z} \in \mathbb{R}^{3\times3}_{\mathsf{dev}} = \{\, A \in \mathbb{R}^{3\times3} \mid A = A^\mathsf{T},\ \operatorname{tr} A = 0 \,\}$, and the typical stored-energy density, to which we restrict for simplicity, reads

$$W(\hat{\mathbf{e}}, \hat{z}) = \tfrac{\lambda}{2}(\operatorname{tr}\hat{\mathbf{e}})^2 + \mu|\hat{\mathbf{e}} - \hat{z}|^2 + \tfrac{h}{2}|\hat{z}|^2. \tag{5.1}$$

The multiscale feature arises from the thin plate domain $\Omega_\varepsilon = \omega \times \,]-\varepsilon, \varepsilon[$. We blow up the domain Ω_ε to the ε-independent domain Ω_1 via $S_\varepsilon : \Omega_1 \to \Omega_\varepsilon; x \mapsto \hat{x} = (x_1, x_2, \varepsilon x_3)$, i.e. $S_\varepsilon = \operatorname{diag}(1, 1, \varepsilon)$. For the displacements and plastic variables we denote by $\hat{u} : \Omega_\varepsilon \to \mathbb{R}^d$ and $\hat{z} : \Omega_\varepsilon \to \mathbb{R}^{3\times3}_{\mathsf{dev}}$ the unscaled variables on the thin domain and introduce the scaled variables

$$u(x) = S_\varepsilon \hat{u}(S_\varepsilon x) \quad \text{and} \quad z(x) = \hat{z}(S_\varepsilon x).$$

Note that the true strains $\hat{\mathbf{e}} = \frac{1}{2}(\nabla_{\hat{x}}\hat{u} + \nabla_{\hat{x}}\hat{u}^\mathsf{T})$ takes the form $\hat{\mathbf{e}} = S_\varepsilon^{-1}\mathbf{e}S_\varepsilon^{-1}$, where $\mathbf{e} = \mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\mathsf{T})$ is the strain calculated in the new scaled variables.

This leads to the RIS $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ given by

$$\begin{aligned}
&\mathcal{Q} = \mathrm{H}^1_{\mathsf{Dir}}(\Omega_1; \mathbb{R}^d) \times \mathrm{L}^2(\Omega_1; \mathbb{R}^{3\times 3}_{\mathsf{dev}}),\\
&\mathcal{R}_\varepsilon(v) = \mathcal{R}(v) = \textstyle\int_{\Omega_1} R(v(x))\,\mathrm{d}x,\\
&\mathcal{E}_\varepsilon(t,u,z) = \textstyle\int_{\Omega_1} W_\varepsilon(\mathbf{e}(u), z) - \ell(t,x)\cdot u(t)\,\mathrm{d}x,\\
&\qquad\qquad \text{where } W_\varepsilon(\mathbf{e}, z) = W(S_\varepsilon^{-1}\mathbf{e}S_\varepsilon^{-1}, z).
\end{aligned}$$

The scaling of the strains $\mathbf{e}$ in W_ε via $\hat{\mathbf{e}} = S_\varepsilon \mathbf{e} S_\varepsilon$ leads to a singular limit, for which the transversal strains $\mathbf{e}_{j3}$ for $j = 1, 2, 3$ must be 0. The rigorous Γ-limit for $\varepsilon \to 0$ was analyzed in [LiM11] following the ideas of [BC*92] for the purely elastic case. It leads to the so-called Kirchhoff-Love displacements

$$\begin{aligned}
\mathcal{U}_{\mathrm{KL}} = \Big\{ \mathcal{K}U \;\Big|\; & U = (U_1, U_2, U_3)^{\mathsf{T}} \text{ and}\\
& U_1, U_2 \in \mathrm{H}^1(\omega),\ U_3 \in \mathrm{H}^2(\omega) \Big\},\\
\text{where } \mathcal{K}U(x) = & \begin{pmatrix} U_1(x_1,x_2) - x_3\partial_{x_1}U_3(x_1,x_2) \\ U_2(x_1,x_2) - x_3\partial_{x_2}U_3(x_1,x_2) \\ U_3(x_1,x_2) \end{pmatrix}.
\end{aligned}$$

Note that the functions U_j are defined on the midplane ω, while $\mathcal{K}U$ defines functions on Ω_1.

The limiting RIS $(\mathcal{Q}, \mathcal{E}_0, \mathcal{R}_0)$ is such that $\mathcal{E}_0(t,u,z) = \infty$ if $u \notin \mathcal{U}_{\mathrm{KL}}$. Hence, the limit problem can be stated in terms of the state $q = (U, z)$ in the subdifferential form $0 \in \partial\mathcal{R}_0(\dot{q}) + \mathrm{D}\mathcal{E}_0(t,q)$, namely

$$0 = -\operatorname{div}_{\mathrm{ip}}\Big(\Sigma_0\big(2\mathbf{e}_{\mathrm{ip}}(U) - [z_{\mathrm{ip}}]_0\big)\Big) - g_{\mathsf{memb}}(t), \tag{5.2a}$$

$$0 = \operatorname{div}_{\mathrm{ip}}\operatorname{div}_{\mathrm{ip}}\big(\Sigma_0\big(\tfrac{2}{3}\mathrm{D}^2U_3 + [z_{\mathrm{ip}}]_1\big)\big) - g_{\mathsf{bend}}(t), \tag{5.2b}$$

$$\begin{aligned}
0 \in{}& \partial R(\dot{z}) + hz\\
&+ \operatorname{dev}\big(\big[\!\big[\, \Sigma_0(z_{\mathrm{ip}} - \mathbf{e}_{\mathrm{ip}}(U) + x_3\mathrm{D}^2U_3) \,\big\|\, 0 \,\big]\!\big]\big),
\end{aligned} \tag{5.2c}$$

where the subscript "ip" stands for "in-plane". Equations (5.2a) and (5.2b) are stated on the midplane ω only, while the plastic flow-rule (5.2c) is stated on the three-dimensional domain $\Omega_1 = \omega \times {]{-1},1[}$. Equation (5.2a) is the membrane equation for (U_1, U_2), which is coupled to the averaged plastic strain $[z_{\mathrm{ip}}]_0$. Equation (5.2b) is a generalization of Kirchhoff's plate equation (of order four) for U_3, which is coupled to plasticity via the weighted average $[z_{\mathrm{ip}}]_1$ (see below for the definitions).

Here, $\mathbf{e}_{\mathrm{ip}}(U) = \frac{1}{2}(\nabla_{\mathrm{ip}}U_{\mathrm{ip}} + \nabla_{\mathrm{ip}}U_{\mathrm{ip}}^{\mathsf{T}}) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is the in-plane strain tensor, and $\mathrm{D}^2U_3 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is the bending strain tensor. For the specific choice (5.1) for W, the 2-dimensional stress-strain relation is $\Sigma_0(\mathbf{e}_{\mathrm{ip}}) = \frac{2\lambda\mu}{\lambda+2\mu}\operatorname{tr}(\mathbf{e}_{\mathrm{ip}})I_2 + 2\mu\mathbf{e}_{\mathrm{ip}}$. The forcings $g_{\mathsf{memb}}(t)$ and $g_{\mathsf{bend}}(t)$ are obtained from suitable averages of $\ell(t)$.

The plastic variable $z \in \mathbb{R}^{3\times 3}_{\mathsf{dev}}$ as well as its in-plane part $z_{\mathrm{ip}} = (z_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times 2}_{\mathsf{sym}}$ still depend on x_3. The notation $[\![\, A_{\mathrm{ip}} \,\|\, 0 \,]\!] \in \mathbb{R}^{3\times 3}_{\mathsf{sym}}$ in (5.2c) means that $A_{\mathrm{ip}} \in \mathbb{R}^{2\times 2}_{\mathsf{sym}}$ is enlarged by adding a row and a column of 0. Moreover, (5.2c) defines a play operator $\mathfrak{P}_{\mathbb{A},\mathbb{K}^*}$ on $\mathrm{L}^2(\Omega_1; \mathbb{R}^{3\times 3}_{\mathsf{dev}})$, if we replace $\operatorname{dev}\big(\llbracket\,\Sigma_0(-\mathbf{e}_{\rm ip}(U){+}x_3\mathrm{D}^2U_3)\;\|\;0\,\rrbracket\big)$ by a general loading $\mathbb{L}(t)\in \mathrm{L}^2(\Omega_1;\mathbb{R}^{3\times3}_{\mathsf{dev}})$. Of course, this play operator acts independently for each $x\in\Omega_1$. This is important to construct the macroscopic problem on ω.

The coupling of $\mathfrak{P}_{\mathbb{A},\mathbb{K}^*}$ to the equations for U occurs via suitable averages over $x_3\in\,]{-1},1[$, namely $[z_{\rm ip}]_j=\int_{-1}^1 x_3^j z_{\rm ip}(x_3)\,\mathrm{d}x_3\in\mathbb{R}^{2\times2}_{\mathsf{sym}}$. Using the spaces

$$Y:=\mathrm{L}^2(]{-1},1[\,;\mathbb{R}^{3\times3}_{\mathsf{dev}})\quad\text{and}\quad \hat Y:=\mathbb{R}^{2\times2}_{\mathsf{sym}}\times\mathbb{R}^{2\times2}_{\mathsf{sym}}$$

we define the contraction operator

$$B:\begin{cases} Y & \to \qquad \hat Y,\\ z & \mapsto\ (B_{\mathsf{memb}}z,B_{\mathsf{bend}}z),\end{cases}$$
$$B_{\mathsf{memb}}z=-\Sigma_0([z_{\rm ip}]_0),\ \ B_{\mathsf{bend}}z=\Sigma_0([z_{\rm ip}]_1).$$

With these notations (5.2) takes the form

$$0=-\operatorname{div}_{\rm ip}\big(2\Sigma_0(\mathbf{e}_{\rm ip}(U)){+}B_{\mathsf{memb}}z\big)-g_{\mathsf{memb}}(t), \tag{5.3a}$$
$$0=\operatorname{div}_{\rm ip}\operatorname{div}_{\rm ip}\big(\tfrac23\Sigma_0(\mathrm{D}^2U_3){+}B_{\mathsf{bend}}z\big)-g_{\mathsf{bend}}(t), \tag{5.3b}$$
$$0\in\partial R(\dot z)+hz+\Sigma_{00}(z)+B^*(\mathbf{e}_{\rm ip}(U),\mathrm{D}^2U_3). \tag{5.3c}$$

Thus, we are led to define the contraction

$$\mathcal{P}^B_{\mathbb{A},\mathbb{K}^*}[z^0,\tilde\ell]:=B\mathfrak{P}_{\mathbb{A},\mathbb{K}^*}[z^0,B^*\tilde\ell],$$
$$\mathcal{P}^B_{\mathbb{A},\mathbb{K}^*}[z^0,\tilde\ell]=(\mathcal{P}_{\mathsf{memb}}[z^0,\tilde\ell],\mathcal{P}_{\mathsf{bend}}[z^0,\tilde\ell]),$$

where the latter splitting into two parts associates to the two components of B, but does not give rise to a block structure via $B^*\tilde\ell$. The whole operator $\mathcal{P}^B_{\mathbb{A},\mathbb{K}^*}$ maps $\mathrm{W}^{1,1}([0,T];\hat Y)$ into itself and is defined via a (classical) vector-valued Prandtl-Ishlinskii operator for each midplane point $(x_1,x_2)\in\omega$. The classical averaging occurs via the averaging in the membrane part and via the weighted average for the bending part. In the pure bending case, the averaging with the weight x_3 leads to a vector-valued family of play operators that are the same up to scaling, cf. [GKS08]. For the plastic plate this means that plasticity will always start on the outer part of the plate, i.e. $|x_3|\approx1$ and will then develop towards the midplane for large bendings.

Following [GKS08] and our Section 3 the system can be reduced to a PDE system on ω via the gPI operator $\mathcal{P}^B_{\mathbb{A},\mathbb{K}^*}$ with components $\mathcal{P}_{\mathsf{memb}}$ and $\mathcal{P}_{\mathsf{bend}}$, namely

$$-\operatorname{div}_{\rm ip}\Big(\Sigma_0\big(\mathbf{e}_{\rm ip}(U)\big)+\mathcal{P}_{\mathsf{memb}}[z^0,(\mathbf{e}_{\rm ip}(U),\mathrm{D}^2U_3)](t)\Big)$$
$$=g_{\mathsf{memb}}(t)\text{ in }\omega, \tag{5.4a}$$
$$\operatorname{div}_{\rm ip}\operatorname{div}_{\rm ip}\Big(\Sigma_0\big(\mathrm{D}^2U_3\big)+\mathcal{P}_{\mathsf{bend}}\big[z^0,(\mathbf{e}_{\rm ip}(U),\mathrm{D}^2U_3)\big](t)\Big)$$
$$=g_{\mathsf{bend}}(t)\text{ in }\omega. \tag{5.4b}$$

The membrane displacement (U_1,U_2) is coupled to the bending displacement U_3 via the hysteresis operator $\mathcal{P}^B_{\mathbb{A},\mathbb{K}^*}=(\mathcal{P}_{\mathsf{memb}},\mathcal{P}_{\mathsf{bend}})$ only. Applying first a stretching and then a bending gives a different result as repeating the same loadings in opposite order, see [Lie08].

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