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Global existence result for phase transformations with heat transfer in shape memory alloys

Dedicated to 75th birthday of K. Gröger

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Abstract

We consider three-dimensional models for rate-independent processes describing materials undergoing phase transformations with heat transfer. The problem is formulated within the framework of generalized standard solids by the coupling of the momentum equilibrium equation and the flow rule with the heat transfer equation. Under appropriate regularity assumptions on the initial data, we prove the existence a global solution for this thermodynamically consistent system, by using a fixed-point argument combined with global energy estimates.

1 Introduction

Motivated by the study of shape memory alloys, we consider rate-independent processes describing materials undergoing phase transformations. In the framework of generalized standard solids due to Halphen and Nguyen (see [HaN75]), the unknowns are the displacement field u and an internal variable z and the problem is described by the momentum equilibrium equation combined with a flow rule for the evolution of the internal variable. A very powerful tool to study such problems is the so called energetic formulation, introduced in [MiT04, Mie05] and later on developed and intensively applied in [FrM06, MiR06, MiR07, Mie07, MiP07, MRS08]. Note that coupling rate-independent processes with rate-dependent processes makes, in general, the problems much more difficult; see for instance [EfM06, MPM08, BaR08, Rou09a, Rou09b].

In this paper, we are interested in coupling the rate-independent process with the thermal process, which is not rate-independent, and viscous damping. The model is based on the Helmholtz free energy $W(\mathrm{e}(u), z, \nabla z, \theta)$, depending on the *infinitesimal strain tensor* $\mathrm{e}(u) \overset{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ for the *displacement* u, where $(\cdot)^{\mathsf{T}}$ denotes the transpose of the tensor, the *internal variable* z and the *temperature* θ. For simplicity, we will omit any dependence on the material point $x \in \Omega$ and $t \in [0,T]$ with $T > 0$. We assume that W can be decomposed as follows

$$W(\mathrm{e}(u), z, \nabla z, \theta) \overset{\text{def}}{=} W_1(\mathrm{e}(u), z, \nabla z) - W_0(\theta) + \theta W_2(\mathrm{e}(u), z). \tag{1.1}$$

This partially linearized decomposition ensures that entropy separates the thermal and mechanical variables (see (2.8)). Let us emphasize that the last term $\theta W_2(\mathrm{e}(u), z)$ allows for coupling effects between the temperature and the internal variable, which is motivated by the phenomenological models for shape memory alloys presented in Section 2. Since coupling terms will appear both in the momentum equilibrium equation and in the inclusion describing the evolution of the internal variable, this setting is more general than the one presented in [Rou10]. We make the assumptions of small deformations. The problem is thus described by the following system

$$-\operatorname{div}(\sigma_{\text{el}} + \mathbb{L}\mathrm{e}(\dot u)) = \ell, \quad \sigma_{\text{el}} \overset{\text{def}}{=} \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta), \tag{1.2a}$$

$$\partial\Psi(\dot z) + \mathbb{M}\dot z + \sigma_{\text{in}} \ni 0, \quad \sigma_{\text{in}} \overset{\text{def}}{=} \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta) - \operatorname{div} \mathrm{D}_{\nabla z} W(\mathrm{e}(u), z, \nabla z, \theta), \tag{1.2b}$$

$$c(\theta)\dot\theta - \operatorname{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u) + \theta\partial_t W_2(\mathrm{e}(u), z) + \Psi(\dot z) + \mathbb{M}\dot z{:}\dot z. \tag{1.2c}$$

Here Ψ denotes the dissipation potential. As it is common in modeling hysteresis effect in mechanics, we assume that Ψ is positively homogeneous of degree 1, i.e., $\Psi(\gamma z) = \gamma\Psi(z)$ for all $\gamma \geq 0$. The viscosity tensors are denoted $\mathbb{L}$ and $\mathbb{M}$, $c(\theta)$ is the *heat capacity* and $\kappa(\mathrm{e}(u), z, \theta)$ is the *conductivity*. As usual, $(\dot{})$, D_z^i and ∂ denote the time derivative $\frac{\partial}{\partial t}$, the i-th derivative with respect to z and the subdifferential in the sense of convex analysis (for more details see [Bre73]), respectively. Observe that (1.2a), (1.2b) and (1.2c) are usually called the momentum equilibrium equation, the flow rule and the heat-transfer equation, respectively.

The paper is organized as follows. In Section 2, the thermodynamic consistency is justified and some illustrative examples are presented. Then the mathematical formulation of the problem in terms of displacement, internal variables and temperature is presented in Section 3. Our problem is reformulated in terms of enthalpy, which is a crucial ingredient to prove the existence result. Sections 4, 5 and 6 are devoted to the proof of a local existence result by a fixed point argument. More precisely, in Section 4, we consider first the system composed by the momentum equilibrium equation and flow rule for a given temperature θ and we prove existence and regularity results. Next in Section 5, we recall existence and regularity results for the enthalpy equation for any given right hand side. Then a local existence result follows in Section 6 by using a fixed-point argument. Finally a global energy estimate is established in Section 7 leading to a global existence result for the system (1.2).

2 Mechanical model

We justify here the thermodynamic consistency of the model (1.2). Starting from the Helmholtz free energy W, we introduce the specific *entropy* s via the Gibb's relation

$$s \stackrel{\text{def}}{=} -\mathrm{D}_\theta W(\mathrm{e}(u), z, \nabla z, \theta), \tag{2.1}$$

and the *internal energy*

$$W_{\text{in}}(\mathrm{e}(u), z, \nabla z, \theta) \stackrel{\text{def}}{=} W(\mathrm{e}(u), z, \nabla z, \theta) + \theta s. \tag{2.2}$$

Then the *entropy equation* is given by

$$\theta\dot{s} + \mathrm{div}(j) = \xi, \tag{2.3}$$

where j is the *heat flux* and ξ is the *dissipation rate*. We get

$$\xi = \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \mathbb{M}\dot{z}{:}\dot{z} + \Psi(\dot{z}) \geq 0,$$

and, assuming Fourier's law for the temperature, we have

$$j = -\kappa(\mathrm{e}(u), z, \theta)\nabla\theta.$$

We can check now that the second law of thermodynamics is satisfied if $\theta > 0$. Indeed, assuming that the system is thermally isolated, we may divide (2.3) by θ and Green's formula yields

$$\begin{aligned}\int_\Omega \dot{s}\,\mathrm{d}x &= \int_\Omega \frac{\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)}{\theta}\,\mathrm{d}x + \int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \mathbb{M}\dot{z}{:}\dot{z} + \Psi(\dot{z})}{\theta}\,\mathrm{d}x \\ &= \int_\Omega \frac{\kappa(\mathrm{e}(u),z,\theta)\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x + \int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \mathbb{M}\dot{z}{:}\dot{z} + \Psi(\dot{z})}{\theta}\,\mathrm{d}x \geq 0.\end{aligned}$$

We differentiate now $W_{\text{in}}(\mathrm{e}(u), z, \nabla z, \theta)$ with respect to time, we obtain by using the chain rule and (2.2) that

$$\begin{aligned}\dot{W}_{\text{in}}(\mathrm{e}(u), z, \nabla z, \theta) &= \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u), z, \nabla z, \theta){:}\mathrm{e}(\dot{u}) \\ &+ \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta){:}\dot{z} + \mathrm{D}_{\nabla z}W(\mathrm{e}(u), z, \nabla z, \theta)\cdot\nabla\dot{z} + \theta\dot{s}.\end{aligned} \tag{2.4}$$

We integrate (2.4) over Ω, thus we use the Green's formula and (2.3), we find

$$\begin{aligned}\int_\Omega \dot{W}_{\rm in}(\mathrm{e}(u),z,\nabla z,\theta)\,\mathrm{d}x &= \int_\Omega \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\nabla z,\theta){:}\mathrm{e}(\dot u)\,\mathrm{d}x \\ &+ \int_\Omega \mathrm{D}_z W(\mathrm{e}(u),z,\nabla z,\theta){:}\dot z\,\mathrm{d}x + \int_\Omega \mathrm{D}_{\nabla z}W(\mathrm{e}(u),z,\nabla z,\theta)\cdot\nabla\dot z\,\mathrm{d}x \\ &+ \int_\Omega (\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)+\mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u)+\mathbb{M}\dot z{:}\dot z+\Psi(\dot z))\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

On the one hand, we multiply (1.2a) by $\dot u$, and we integrate this expression over Ω to get

$$\int_\Omega \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\nabla z,\theta){:}\mathrm{e}(\dot u)\,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u)\,\mathrm{d}x = \int_\Omega \ell\cdot\dot u\,\mathrm{d}x. \tag{2.6}$$

On the other hand, the definition of the subdifferential $\partial\Psi(\dot z)$ leads to the variational equality associated to (1.2b)

$$\begin{aligned}&\int_\Omega \mathrm{D}_z W(\mathrm{e}(u),z,\nabla z,\theta){:}\dot z\,\mathrm{d}x + \int_\Omega \mathrm{D}_{\nabla z}W(\mathrm{e}(u),z,\nabla z,\theta)\cdot\nabla\dot z\,\mathrm{d}x \\ &+ \int_\Omega \mathbb{M}\dot z{:}\dot z\,\mathrm{d}x + \int_\Omega \Psi(\dot z)\,\mathrm{d}x = 0.\end{aligned} \tag{2.7}$$

We use (2.6) and (2.7) into (2.5), we obtain

$$\int_\Omega \dot{W}_{\rm in}(\mathrm{e}(u),z,\nabla z,\theta)\,\mathrm{d}x = \int_\Omega \ell\cdot\dot u\,\mathrm{d}x + \int_{\partial\Omega} \kappa(\mathrm{e}(u),z,\theta)\nabla\theta\cdot\eta\,\mathrm{d}x.$$

This means that the total energy balance can be expressed in terms of the internal energy, which is the sum of power of external load and heat.

Note that from (2.1), we get

$$s \stackrel{\text{def}}{=} \mathrm{D}_\theta W_0(\theta) - W_2(\mathrm{e}(u),z), \tag{2.8}$$

and

$$W_{\rm in}(\mathrm{e}(u),z,\nabla z,\theta) \stackrel{\text{def}}{=} W_1(\mathrm{e}(u),z,\nabla z,\theta) + \theta\mathrm{D}_\theta W_0(\theta) - W_0(\theta). \tag{2.9}$$

We use (2.9) into (2.3), we may deduce the heat-transfer equation (1.2c) with the heat capacity given by $c(\theta)=\theta\mathrm{D}^2_\theta W_0(\theta)$.

Motivated by the study of shape memory alloys, we will focus in the rest of the paper on the special case where W_1 is given by

$$W_1(\mathrm{e}(u),z,\nabla z) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbb{E}(\mathrm{e}(u){-}z){:}(\mathrm{e}(u){-}z) + \tfrac{\nu}{2}|\nabla z|^2 + H_1(z),$$

where the internal variable z is a deviatoric $d\times d$ tensor, H_1 is a hardening functional and the term $\frac{\nu}{2}|\nabla z|^2$, $\nu>0$, takes into account some nonlocal interaction effect for the internal variable. Moreover we will assume

$$W_2(\mathrm{e}(u),z) \stackrel{\text{def}}{=} \alpha\mathrm{tr}(\mathrm{e}(u)) + H_2(z).$$

Here $\alpha\mathrm{I}$, with $\alpha\ge 0$ and I the identity matrix, is the isothermal expansion tensor. Furthermore, the sum $H(z,\theta) \stackrel{\text{def}}{=} H_1(z)+\theta H_2(z)$ may be interpreted as a temperature-dependent hardening functional. Note that the system (1.2) is then rewritten as

$$\begin{aligned}&-\mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u)) = \ell, \\ &\partial\Psi(\dot z) + \mathbb{M}\dot z - \mathbb{E}(\mathrm{e}(u){-}z) + \mathrm{D}_z H_1(z) + \theta\mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \\ &c(\theta)\dot\theta - \mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot u)){+}\mathrm{D}_z H_2(z){:}\dot z) + \Psi(\dot z) + \mathbb{M}\dot z{:}\dot z.\end{aligned}$$

We may illustrate the presented model by a nontrivial example, namely, a three-dimensional macroscopic phenomenological model for shape-memory polycrystalline materials undergoing phase transformations. This model was introduced by Souza et al ([SMZ98]) and by Auricchio et al (see [AuP02]). The internal variable describes the inelastic part of the deformation due to the martensitic phase transformation and the hardening functional H takes the form

$$H(z, \theta) \stackrel{\text{def}}{=} c_1(\theta)|z| + c_2(\theta)|z|^2 + \chi(z).$$

Here $\chi : \mathbb{R}^{d\times d}_{\text{dev}} \to [0, +\infty]$ denotes the indicator function of the ball $\{z \in \mathbb{R}^{d\times d}_{\text{dev}} : \ |z| \le c_3(\theta)\}$ and the coefficients $c_i(\theta)$ are positive real numbers. Let us observe that $c_1(\theta) > 0$ is an activation threshold for initiation of martensitic phase transformations, $c_2(\theta)$ measures the occurrence of hardening with respect to the internal variable z and $c_3(\theta)$ represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants.

The dependence of the coefficients c_i, $i = 1, 2, 3$, with respect to θ is due to a strong thermo-mechanical constitutive coupling coming from the latent heat absorption or release, which is one of the main features of the behavior of shape-memory alloys (see [AuP04] for more details).

For mathematical purposes, we should regularize the hardening functional as in [MiP07]. Namely, we replace $H(z, \theta)$ by

$$H^\delta(z, \theta) \stackrel{\text{def}}{=} c_1(\theta)\sqrt{\delta^2{+}|z|^2} + c_2(\theta)|z|^2 + \tfrac{((|z|-c_3(\theta))_+)^4}{\delta(1+|z|^2)},$$

where $\delta > 0$ is a small parameter. Then, assuming that the mappings c_i, $i = 1, 2, 3$, are of class C^1 we can consider an affine approximation of $H^\delta(z, \theta)$ as $H^\delta_1(z) + \theta H^\delta_2(z)$ with

$$H^\delta_1(z) = \bar{c}_1\sqrt{\delta^2{+}|z|^2} + \bar{c}_2|z|^2 + \tfrac{((|z|-\bar{c}_3)_+)^4}{\delta(1+|z|^2)},$$

where $\bar{c}_i > 0$, $i = 1, 2, 3$.

Let us emphasize that the existence proof presented in the next sections can be easily extended to more general models of shape memory alloys for which W_1 is given by

$$W_1(\mathrm{e}(u), z, \nabla z) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbb{E}(\mathrm{e}(u){-}E(z)){:}(\mathrm{e}(u){-}E(z)) + \tfrac{\nu}{2}|\nabla z|^2 + H_1(z),$$

where the internal variable z is a vector of $\mathbb{R}^{N-1}$, with $N \ge 2$, and E is an affine mapping from $\mathbb{R}^{N-1}$ to the set of $d \times d$ deviatoric tensors. In such models, N is the total number of phases, i.e., the austenite and all the variants of martensite, and the components $z_1, \dots, z_{N-1}$ of z and $z_N \stackrel{\text{def}}{=} 1 - \sum_{k=1}^{N-1} z_k$ are interpreted as phase fractions. Then $E(z)$ is the effective transformation strain of the mixture, given by

$$E(z) \stackrel{\text{def}}{=} \sum_{k=1}^{N-1} z_k E_k + \Big(1{-}\sum_{k=1}^{N-1} z_k\Big)E_N,$$

where E_k is the transformation strain of the phase k and the temperature dependent hardening functional $H(z, \theta)$ is the sum of a smooth part $w(z, \theta)$ and the indicator function of the set $[0, 1]^{N-1}$ (see [MiT99, Mie00, HaG02, GMH02, MTL02, GHH07]). The hardening functional can be regularized in a similar way as in the previous example by replacing $H(z, \theta)$ by

$$H^\delta(z, \theta) = w(z, \theta) + \sum_{k=1}^{N-1} \tfrac{((-z_k)_+)^4+((z_k-1)_+)^4}{\delta(1+|z_k|^2)},$$

and we may consider an affine approximation of $w(z, \theta)$ as $w_1(z) + \theta w_2(z)$. For more details on this example the reader is referred to [PaP11].

3 Mathematical formulation

We consider a reference configuration $\Omega \subset \mathbb{R}^3$. We assume that Ω is an bounded domain such that $\partial\Omega$ is of class $\mathrm{C}^{2+\rho}$. We will denote by $\mathbb{R}^{3\times 3}_{\text{sym}}$ the space of symmetric $3{\times}3$ tensors endowed with the natural scalar product $v{:}w \stackrel{\text{def}}{=} \mathrm{tr}(v^\mathsf{T} w)$ and the corresponding norm $|v|^2 \stackrel{\text{def}}{=} v{:}v$ for all $v, w \in \mathbb{R}^{3\times 3}_{\text{sym}}$. In particular, we assume that

$$
\begin{aligned}
&W_1(\mathrm{e}(u), z, \nabla z) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbb{E}(\mathrm{e}(u){-}z){:}(\mathrm{e}(u){-}z) + \tfrac{\nu}{2}|\nabla z|^2 + H_1(z),\\
&W_2(\mathrm{e}(u), z) \stackrel{\text{def}}{=} \alpha \mathrm{tr}(\mathrm{e}(u)) + H_2(z),
\end{aligned}
$$

where $\nu > 0$, $\alpha \geq 0$, is the isotropic thermal expansion coefficient, $\mathbb{E}$ denotes the *elastic tensor* and H_i, $i = 1, 2$, two *hardening functionals*. Given a function $\ell : \Omega{\times}(0,T) \to \mathbb{R}^3$, we look for a *displacement* $u : \Omega{\times}(0,T) \to \mathbb{R}^3$, a matrix of *internal variables* $z : \Omega{\times}(0,T) \to \mathbb{R}^{3\times 3}_{\text{dev}}$ and a *temperature* $\theta : \Omega{\times}(0,T) \to \mathbb{R}$ satisfying the following system:

$$
- \mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{3.1a}
$$

$$
\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{E}(\mathrm{e}(u){-}z) + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{3.1b}
$$

$$
\begin{aligned}
&c(\theta)\dot{\theta} - \mathrm{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta)\\
&= \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot{u})){+}\mathrm{D}_z H_2(z){:}\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}{:}\dot{z}.
\end{aligned} \tag{3.1c}
$$

We have naturally to prescribe initial conditions for the displacement, the internal variables, and the temperature, namely

$$
u(\cdot, 0) = u^0, \quad z(\cdot, 0) = z^0, \quad \theta(\cdot, 0) = \theta^0. \tag{3.2}
$$

The problem is to be completed with boundary conditions. More precisely, we suppose here that

$$
u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0, \quad \kappa\nabla\theta{\cdot}\eta_{|\partial\Omega} = 0, \tag{3.3}
$$

where η denotes the outward normal to the boundary $\partial\Omega$ of Ω. The original problem (3.1) can be rewritten in terms of enthalpy instead of temperature by employing the so-called enthalpy transformation

$$
g(\theta) = \vartheta \stackrel{\text{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s. \tag{3.4}
$$

Clearly, g is the unique primitive of the function c, which is supposed to be continuous, such that $g(0) = 0$. Furthermore, we will assume that for all $s \geq 0$, $c(s) \geq c^c > 0$ where c^c is a constant. Hence we deduce that g is a bijection from $[0, \infty)$ into $[0, \infty)$. We define

$$
\zeta(\vartheta) \stackrel{\text{def}}{=} \begin{cases} g^{-1}(\vartheta) & \text{if } \vartheta \geq 0,\\ 0 & \text{otherwise,} \end{cases} \tag{3.5a}
$$

$$
\kappa^c(\mathrm{e}(u), z, \vartheta) \stackrel{\text{def}}{=} \tfrac{\kappa(\mathrm{e}(u), z, \zeta(\vartheta))}{c(\zeta(\vartheta))}, \tag{3.5b}
$$

where g^{-1} is the inverse of g. For more details on the enthalpy transformation, the reader is referred to [Rou09b] and the references therein. Therefore the system (3.1) is transformed into the following form

$$
- \mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\zeta(\vartheta)\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{3.6a}
$$

$$
\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{E}(\mathrm{e}(u){-}z) + \mathrm{D}_z H_1(z) + \zeta(\vartheta)\mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{3.6b}
$$

$$
\begin{aligned}
&\dot{\vartheta} - \mathrm{div}(\kappa^c(\mathrm{e}(u), z, \vartheta)\nabla\vartheta)\\
&= \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \zeta(\vartheta)(\alpha\mathrm{tr}(\mathrm{e}(\dot{u})){+}\mathrm{D}_z H_2(z){:}\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}{:}\dot{z},
\end{aligned} \tag{3.6c}
$$

with boundary conditions

$$u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0, \quad \kappa^c \nabla\vartheta{\cdot}\eta_{|\partial\Omega} = 0, \tag{3.7}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \vartheta(\cdot,0) = \vartheta^0 = g(\theta^0). \tag{3.8}$$

The identity (3.6c) is called the enthalpy equation. As usual Korn's inequality will play a role in the mathematical analysis developed in the next sections. We have assumed that $\partial\Omega$ is of class $\mathrm{C}^{2+\rho}$, so that the Korn's inequality holds, i.e.

$$\exists C^{\mathsf{Korn}} > 0\, \forall u \in \mathrm{H}^1_0(\Omega):\ \|\mathrm{e}(u)\|^2_{\mathrm{L}^2(\Omega)} \geq C^{\mathsf{Korn}} \|u\|^2_{\mathrm{H}^1(\Omega)}, \tag{3.9}$$

(see [KoO88, DuL76]).

Let us introduce now the assumptions on the dissipation potential Ψ, on the hardening functions H_i, $i = 1, 2$, and on the data $\mathbb{E}$, $\mathbb{L}$, $\mathbb{M}$, ℓ, $c \overset{\mathsf{def}}{=} c(\theta)$ and $\kappa \overset{\mathsf{def}}{=} \kappa(\mathrm{e}(u), z, \theta)$, which will allow us to obtain some regularity properties that are needed to prove the existence result.

We assume that the dissipation potential Ψ is positively homogeneous of degree 1 and satisfies the triangle inequality, namely, we have

$$\forall \gamma \geq 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathsf{sym}}:\ \Psi(\gamma z) = \gamma \Psi(z), \tag{3.10a}$$

$$\exists C^\Psi > 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathsf{sym}}:\ 0 \leq \Psi(z) \leq C^\Psi |z|, \tag{3.10b}$$

$$\forall z_1, z_2 \in \mathbb{R}^{3\times3}_{\mathsf{sym}}:\ \Psi(z_1{+}z_2) \leq \Psi(z_1) + \Psi(z_2). \tag{3.10c}$$

It is clear that (3.10a), (3.10b) and (3.10c) imply that Ψ is convex and continuous. We impose that the hardening functionals H_i, $i = 1, 2$, belong to $\mathrm{C}^2(\mathbb{R}^{3\times3}_{\mathsf{dev}}; \mathbb{R})$ and satisfy the following inequalities

$$\exists c^{H_1}, \widetilde{c}^{H_1} > 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathsf{dev}}:\ H_1(z) \geq c^{H_1}|z|^2 - \widetilde{c}^{H_1}, \tag{3.11a}$$

$$\exists C^{H_i}_{zz} > 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathsf{dev}}:\ |\mathrm{D}^2_z H_i(z)| \leq C^{H_i}_{zz}. \tag{3.11b}$$

Note that (3.11b) leads to

$$\exists C^{H_i}_z > 0\, \forall z \in \mathbb{R}^{3\times3}_{\mathsf{dev}}:\ |\mathrm{D}_z H_i(z)| \leq C^{H_i}_z (1{+}|z|), \quad |H_i(z)| \leq C^{H_i}_z (1{+}|z|^2). \tag{3.12}$$

The *elastic tensor* $\mathbb{E} : \Omega \to \mathcal{L}(\mathbb{R}^{3\times3}_{\mathsf{sym}}, \mathbb{R}^{3\times3}_{\mathsf{sym}})$ is a symmetric positive definite operator such that

$$\exists c^{\mathbb{E}} > 0\, \forall z \in \mathrm{L}^2(\Omega; \mathbb{R}^{3\times3}_{\mathsf{sym}}):\ c^{\mathbb{E}} \|z\|^2_{\mathrm{L}^2(\Omega)} \leq \int_\Omega \mathbb{E}z{:}z\, \mathrm{d}x, \tag{3.13a}$$

$$\forall i, j, k = 1, 2, 3:\ \mathbb{E}(\cdot), \tfrac{\partial \mathbb{E}_{i,j}(\cdot)}{\partial x_k} \in \mathrm{L}^\infty(\Omega). \tag{3.13b}$$

We assume that $\mathbb{L}$ and $\mathbb{M}$ are symmetric positive definite tensors. This implies that

$$\exists c^{\mathbb{L}}, C^{\mathbb{L}} > 0\, \forall z \in \mathbb{R}^{3\times3}:\ c^{\mathbb{L}}|z|^2 \leq \mathbb{L}z{:}z \leq C^{\mathbb{L}}|z|^2, \tag{3.14a}$$

$$\exists c^{\mathbb{M}}, C^{\mathbb{M}} > 0\, \forall z \in \mathbb{R}^{3\times3}:\ c^{\mathbb{M}}|z|^2 \leq \mathbb{M}z{:}z \leq C^{\mathbb{M}}|z|^2. \tag{3.14b}$$

We assume furthermore that

$$\forall z \in \mathbb{R}^{3\times3}_{\mathsf{dev}}:\ \mathbb{M}z \in \mathbb{R}^{3\times3}_{\mathsf{dev}}. \tag{3.15}$$

We consider that ℓ is an external loading satisfying

$$\ell \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)). \tag{3.16}$$

Finally, for the heat capacity c and the conductivity κ^c we assume that

$$c : [0,\infty) \to [0,\infty) \text{ is continuous,} \tag{3.17a}$$

$$\exists \beta_1 \geq 2\ \exists c^c > 0\ \forall \theta \geq 0 :\ c^c(1{+}\theta)^{\beta_1 - 1} \leq c(\theta), \tag{3.17b}$$

$$\kappa^c : \mathbb{R}^{3\times 3}_{\mathrm{sym}} {\times} \mathbb{R}^{3\times 3}_{\mathrm{dev}} {\times} \mathbb{R} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}} \text{ is continuous, bounded and uniformy positive definite, i.e} \tag{3.17c}$$

$$\exists c^{\kappa^c} > 0\ \forall (e,z,\vartheta) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}} \times \mathbb{R}\ \forall v \in \mathbb{R}^3 :\ \kappa^c(e,z,\vartheta)v{\cdot}v \geq c^{\kappa^c}|v|^2, \tag{3.17d}$$

$$\exists C^{\kappa^c} > 0\ \forall (e,z,\vartheta) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}} \times \mathbb{R} :\ |\kappa^c(e,z,\vartheta)| \leq C^{\kappa^c}. \tag{3.17e}$$

Let us emphasize that the three dimensional model for shape memory alloys presented in Section 2 fulfills the previous assumptions so that we may apply the abstract result obtained in the next sections.

Let us end this section by some comments about the proof strategy. In order to obtain a local existence result for the coupled problem (3.6)–(3.8), a fixed point argument will be used. More precisely, for any given $\widetilde{\vartheta}$, we define $\theta \overset{\text{def}}{=} \zeta(\widetilde{\vartheta})$ and we solve first the system composed by the momentum equilibrium equation and the flow rule (3.1a)–(3.1b), then we solve the enthalpy equation (3.6c) with $\kappa^c \overset{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \zeta(\widetilde{\vartheta}))$. This allows us to define a mapping

$$\phi : \widetilde{\vartheta} \mapsto \vartheta,$$

and our aim is to prove that this mapping satisfies the assumptions of Schauder's fixed point theorem. Therefore, let us consider a given $\widetilde{\vartheta} \in \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ with $\bar p \geq 1$ and $\bar q \geq 1$. We define $\theta \overset{\text{def}}{=} \zeta(\widetilde{\vartheta})$. Since ζ is a Lipschitz continuous mapping from $\mathbb{R}$ to $\mathbb{R}$, we infer that the mapping

$$\begin{aligned} \phi_1 :\ \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega)) &\to \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega)) \\ \widetilde{\vartheta} &\mapsto \theta \end{aligned}$$

is also Lipschitz continuous. Furthermore (3.17b) implies that

$$\forall \theta \in [0,\infty) :\ \tfrac{c^c}{\beta_1}((1{+}\theta)^{\beta_1}{-}1) \overset{\text{def}}{=} g_1(\theta) \leq g(\theta).$$

Thus we have

$$\forall \vartheta \in [0,\infty) :\ 0 \leq \zeta(\vartheta) \leq \zeta_1(\vartheta) \overset{\text{def}}{=} g_1^{-1}(\vartheta),$$

and

$$\forall \vartheta \in \mathbb{R} :\ |\zeta(\vartheta)| \leq \big(\tfrac{\beta_1}{c^c}\vartheta^+{+}1\big)^{\frac{1}{\beta_1}} - 1,$$

with $\vartheta^+ \overset{\text{def}}{=} \max(\vartheta, 0)$ for all $\vartheta \in \mathbb{R}$. It follows that

$$\forall \beta \in [1,\beta_1]\ \forall \vartheta \in \mathbb{R} :\ |\zeta(\vartheta)| \leq \big(\tfrac{\beta_1}{c^c}\vartheta^+{+}1\big)^{\frac{1}{\beta}} - 1 \leq \big(\tfrac{\beta_1}{c^c}\vartheta^+\big)^{\frac{1}{\beta}}. \tag{3.18}$$

Hence, for all $\beta \in [1,\beta_1]$ and for all $\widetilde{\vartheta} \in \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$, we have $\theta = \zeta(\widetilde{\vartheta}) \in \mathrm{L}^{\beta\bar q}(0,T;\mathrm{L}^{\beta\bar p}(\Omega))$ with

$$\|\theta\|_{\mathrm{L}^{\beta\bar q}(0,T;\mathrm{L}^{\beta\bar p}(\Omega))} \leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta}} \|\widetilde{\vartheta}\|^{\frac{1}{\beta}}_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))}.$$

In the rest of the paper, we will assume that $\bar q > 4$ and $\bar p = 2$.

When there is not any confusion, we will use simply the notation $\mathrm{X}(\Omega)$ instead of $\mathrm{X}(\Omega;\mathrm{Y})$ where X is a functional space and Y is a vectorial space.

4 Existence and regularity results for the system composed by the momentum equilibrium equation and the flow rule

This section is devoted to the proof of existence and uniqueness results for the system composed by the momentum equilibrium equation and the flow rule (3.1a)–(3.1b) under the consideration that $\theta = \zeta(\widetilde{\vartheta})$ is given in a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ with $q = \beta_1\bar{q}$ and $p \in \big[4, \min(\beta_1\bar{p}, 6)\big]$. More precisely, we look for a solution of the problem (P_{uz}):

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-z)+\alpha\theta\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{4.1a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{E}(\mathrm{e}(u)-z) + \mathrm{D}_z H_1(z) + \theta\mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{4.1b}$$

with initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \tag{4.2}$$

and boundary conditions

$$u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0. \tag{4.3}$$

Furthermore we will establish some a priori estimates and some regularity results for the solution of (P_{uz}).

As a first step, we use classical results for Partial Differential Equations (PDE) and Ordinary Differential Equations (ODE) to obtain an existence result. Let $\mathcal{A} : \mathrm{H}^1(\Omega) \to (\mathrm{H}^1(\Omega))'$ be the linear continuous mapping defined by

$$\forall (u,v) \in (\mathrm{H}^1(\Omega))^2 : \ \langle \mathcal{A}u, v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Omega \nu\mathbb{M}^{-1}\nabla u{:}\nabla v \,\mathrm{d}x.$$

Classical results about elliptic operators implies that $\mathcal{A}$ generates an analytic semigroup on $\mathrm{L}^2(\Omega)$, which extends to a C^0-semigroup of contractions on $\mathrm{L}^p(\Omega)$. We denote by $\mathcal{A}_p$ (resp. $\mathcal{A}_{\frac{p}{2}}$) the realization of its generator in $\mathrm{L}^p(\Omega)$ (resp. $\mathrm{L}^{p/2}(\Omega)$) and by $\mathrm{X}_{q,p}(\Omega)$ the intersection of interpolation spaces

$$\mathrm{X}_{q,p}(\Omega) \stackrel{\text{def}}{=} (\mathrm{L}^p(\Omega), \mathcal{D}(\mathcal{A}_p))_{1-\frac{2}{q},\frac{q}{2}} \cap (\mathrm{L}^{p/2}(\Omega), \mathcal{D}(\mathcal{A}_{\frac{p}{2}}))_{1-\frac{1}{q},q},$$

(see for instance [HiR08, PrS01] and the references therein). Here $\mathcal{D}(\mathcal{A}_p)$ (resp. $\mathcal{D}(\mathcal{A}_{\frac{p}{2}})$) denotes the domain of $\mathcal{A}_p$ (resp. $\mathcal{A}_{\frac{p}{2}}$).

In the sequel, the notations for the constants introduced in the proofs are valid only in the proof and we also use the set $\mathcal{Q}_\tau \stackrel{\text{def}}{=} \Omega \times (0,\tau)$ with $\tau \in [0,T]$.

Theorem 4.1 (Existence for $(\mathbf{P}_{uz})$) *Let θ be given in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$. Assume that $u^0 \in \mathrm{H}^1(\Omega)$, $z^0 \in \mathrm{H}^1(\Omega)$ and that* (3.10)*,* (3.11)*,* (3.14) *and* (3.16) *hold. Then the problem* (4.1)–(4.3) *admits a solution $(u,z) \in \mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega){\times}\mathrm{H}^1(\Omega))$. Furthermore if $z^0 \in \mathrm{X}_{q,p}(\Omega)$ then $z \in \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{C}^0([0,T],\mathrm{H}^1(\Omega))$ and $\dot{z} \in \mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$.*

Proof. Observe first that for all $f \in \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega);\mathbb{R}^3)')$ and for all $u^* \in \mathrm{H}^1_0(\Omega;\mathbb{R}^3)$ the following problem

$$-\operatorname{div}(\mathbb{E}\mathrm{e}(u)+\mathbb{L}\mathrm{e}(\dot{u})) = f,$$

with initial conditions

$$u(\cdot,0) = u^* \in \mathrm{H}^1_0(\Omega;\mathbb{R}^3),$$

and boundary conditions

$$u_{|\partial\Omega} = 0,$$

possesses a unique solution $u \overset{\text{def}}{=} \mathcal{L}(u^*, f) \in \mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)) \cap \mathrm{C}^0([0,T];\mathrm{H}^1_0(\Omega))$. Note that

$$\forall f_1, f_2 \in \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega);\mathbb{R}^3)')\ \forall u^* \in \mathrm{H}^1_0(\Omega;\mathbb{R}^3) :\ \mathcal{L}(u^*, f_1{+}f_2) = \mathcal{L}(u^*, f_1) + \mathcal{L}(0, f_2),$$

and the mapping

$$\begin{aligned}\mathcal{L}_0(\cdot) \overset{\text{def}}{=} \mathcal{L}(0,\cdot) :\ \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega)')) &\to \mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)) \cap \mathrm{C}^0([0,T];\mathrm{H}^1_0(\Omega)),\\ f &\mapsto u,\end{aligned}$$

is linear and continuous. Moreover, the classical energy estimate gives

$$\begin{aligned}\forall t \in [0,T] : c^{\mathbb{E}}\|\mathrm{e}(u(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbb{L}}C^{\mathsf{Korn}}\int_0^t \|\dot{u}(\cdot,s)\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}s\\ \leq \tfrac{1}{c^{\mathbb{L}}C^{\mathsf{Korn}}}\int_0^t \|f(\cdot,s)\|^2_{(\mathrm{H}^1_0(\Omega))'}\,\mathrm{d}s,\end{aligned} \tag{4.4}$$

for all $f \in \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega)'))$ and $u = \mathcal{L}_0(f)$. It follows that (4.1a) and (4.1b) can be rewritten as

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} + \mathbb{E}z + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) - \nu\Delta z + g_1(\theta) + g_2(z) \ni 0, \tag{4.5}$$

with initial conditions

$$z(\cdot,0) = z^0 \in \mathrm{H}^1(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}}), \tag{4.6}$$

and boundary conditions

$$\nabla z{\cdot}\eta_{|\partial\Omega} = 0, \tag{4.7}$$

where $g_1(\theta) \overset{\text{def}}{=} -\mathbb{E}\mathrm{e}(\mathcal{L}(u^0,\ell)) - \mathbb{E}\mathrm{e}(\mathcal{L}_0(\mathrm{div}(\alpha\theta\mathrm{I}))) \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{sym}}))$ and

$$\begin{aligned}g_2 :\ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}})) &\to \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{sym}})),\\ z &\mapsto \mathbb{E}\mathrm{e}(\mathcal{L}_0(\mathrm{div}(\mathbb{E}z))).\end{aligned}$$

Let $\varphi : \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}}) \to (-\infty,+\infty]$ defined by

$$z \mapsto \begin{cases}\frac{\nu}{2}\|\nabla z\|^2_{\mathrm{L}^2(\Omega)} & \text{if } z \in \mathrm{H}^1(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}}),\\ +\infty & \text{otherwise},\end{cases}$$

is a proper, convex lower semicontinuous function. Clearly, $\partial\varphi$ is a maximal monotone operator, for more details on the maximal monotone operators and their properties, the reader is referred to [Bre73]. The resolvant of the subdifferential $\partial\varphi$ is defined by

$$\forall\epsilon > 0 :\ \mathcal{J}_\epsilon \overset{\text{def}}{=} (\mathrm{I}{+}\epsilon\partial\varphi)^{-1}.$$

Note that the resolvant $\mathcal{J}_\epsilon$ is a contraction defined on all $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}})$. Let us introduce also

$$\forall\epsilon > 0\ \forall z \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}}) :\ \varphi_\epsilon(z) \overset{\text{def}}{=} \min_{\bar{z}\in\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}})}\big\{\tfrac{1}{2\varepsilon}\|z{-}\bar{z}\|^2_{\mathrm{L}^2(\Omega)}{+}\varphi(\bar{z})\big\},$$

which, is a convex and Fréchet differentiable mapping from $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathsf{dev}})$ to $\mathbb{R}$. Using [Bre73, Prop. 2.11], we know that the Yosida approximation of $\partial\varphi$ coincide with the Fréchet differential of φ_ϵ, i.e., we have

$$\forall\epsilon > 0 :\ \partial\varphi_\epsilon \overset{\text{def}}{=} \tfrac{1}{\epsilon}(\mathrm{I}{-}\mathcal{J}_\epsilon), \tag{4.8}$$

and $\partial\varphi_\epsilon$ is a monotone and $\frac{1}{\epsilon}$-Lipschitz continuous mapping on $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}})$.

We approximate now the problem (4.5)–(4.7) by

$$\partial\Psi(\dot{z}_\epsilon) + \mathbb{M}\dot{z}_\epsilon + \partial\varphi_\epsilon(z_\epsilon) + \mathbb{E}z_\epsilon + \mathrm{D}_z H_1(z_\epsilon) \ni -(g_1(\theta_\epsilon)+g_2(\mathcal{J}_\epsilon z_\epsilon)+\theta_\epsilon \mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon)), \tag{4.9}$$

with the initial condition

$$z_\epsilon(0,\cdot) = z^0. \tag{4.10}$$

Here $\theta_\epsilon \in \mathrm{C}^\infty_0(0,T)\otimes \mathrm{C}^\infty_0(\Omega)$ and $\partial\Psi(\dot{z}_\epsilon)$ is taken in the sense of the $\mathrm{L}^2(\Omega)$-extension of the subdifferential of the convex function Ψ (see examples 2.1.3 and 2.3.3 in [Bre73]), i.e., (4.9) is equivalent to

$$\begin{aligned}
&-\mathbb{M}\dot{z}_\epsilon(t,x) - \partial\varphi_\epsilon(z_\epsilon(t,x)) - \mathbb{E}z_\epsilon(t,x) - \mathrm{D}_z H_1(z_\epsilon(t,x)) - g_1(\theta_\epsilon(t,x))\\
&- g_2(\mathcal{J}_\epsilon z_\epsilon(t,x)) - \theta_\epsilon(t,x)\mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon(t,x))) \in \partial\Psi(\dot{z}_\epsilon(t,x)).
\end{aligned}$$

for all $t\in[0,T]$ and almost every $x\in\Omega$. Since we are looking for a solution with values in $\mathbb{R}^{3\times 3}_{\text{dev}}$, the test-functions should be taken in $\mathbb{R}^{3\times 3}_{\text{dev}}$. More precisely, we have

$$\begin{aligned}
&\Psi(\widetilde{z}) - \Psi(\dot{z}_\epsilon(t,x)) + (\mathbb{M}\dot{z}_\epsilon(t,x)+\partial\varphi_\epsilon(z_\epsilon(t,x))+\mathbb{E}z_\epsilon(t,x)+\mathrm{D}_z H_1(z_\epsilon(t,x))\\
&+g_1(\theta_\epsilon(t,x))+g_2(\mathcal{J}_\epsilon z_\epsilon(t,x))+\theta_\epsilon(t,x)\mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon(t,x))):(\widetilde{z}-z_\epsilon(t,x)) \geq 0,
\end{aligned} \tag{4.11}$$

for all $\widetilde{z}\in\mathbb{R}^{3\times 3}_{\text{dev}}$ and $t\in[0,T]$ and almost every $x\in\Omega$. Observe that (4.11) is equivalent to

$$\begin{aligned}
&\Psi_{\text{dev}}(\widetilde{z}) - \Psi_{\text{dev}}(\dot{z}_\epsilon(t,x)) + \mathrm{Proj}_{\mathbb{R}^{3\times 3}_{\text{dev}}}(\mathbb{M}\dot{z}_\epsilon(t,x)+\partial\varphi_\epsilon(z_\epsilon(t,x))+\mathbb{E}z_\epsilon(t,x)+\mathrm{D}_z H_1(z_\epsilon(t,x))\\
&+g_1(\theta_\epsilon(t,x))+g_2(\mathcal{J}_\epsilon z_\epsilon)(t,x)+\theta_\epsilon(t,x)\mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon(t,x))):(\widetilde{z}-z_\epsilon(t,x)) \geq 0,
\end{aligned}$$

for all $\widetilde{z}\in\mathbb{R}^{3\times 3}_{\text{dev}}$ and $t\in[0,T]$ and almost every $x\in\Omega$. Here Ψ_{dev} is the restriction of Ψ to $\mathbb{R}^{3\times 3}_{\text{dev}}$ and $\mathrm{Proj}_{\mathbb{R}^{3\times 3}_{\text{dev}}}$ denotes the projection on $\mathbb{R}^{3\times 3}_{\text{dev}}$ relatively to the inner product of $\mathbb{R}^{3\times 3}_{\text{sym}}$. Using the definition of the subdifferential $\partial\Psi_{\text{dev}}(\dot{z}_\epsilon(t,x))$ leads to the following differential inclusion

$$\begin{aligned}
&-\mathrm{Proj}_{\mathbb{R}^{3\times 3}_{\text{dev}}}(\partial\varphi_\epsilon(z_\epsilon(t,x))+\mathbb{E}z_\epsilon(t,x)+\mathrm{D}_z H_1(z_\epsilon(t,x)) + g_1(\theta_\epsilon(t,x))\\
&+g_2(\mathcal{J}_\epsilon z_\epsilon(t,x))+\theta_\epsilon(t,x)\mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon(t,x))) \in \partial\Psi_{\text{dev}}(\dot{z}_\epsilon(t,x)) + \mathbb{M}\dot{z}_\epsilon(t,x),
\end{aligned}$$

for all $t\in[0,T]$ and almost every $x\in\Omega$. Note that the operator $\partial\Psi_{\text{dev}}+\mathbb{M}$ is strongly monotone on $\mathbb{R}^{3\times 3}_{\text{dev}}$ as well as its $\mathrm{L}^2(\Omega)$-extension, still denoted $\partial\Psi_{\text{dev}}+\mathbb{M}$, on $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}})$. This operator is invertible and its inverse is $\frac{1}{c^{\mathbb{M}}}$-Lipschitz continuous. Thus (4.9) can be rewritten as

$$\dot{z}_\epsilon = \Phi(\theta_\epsilon, z_\epsilon), \tag{4.12}$$

with

$$\begin{aligned}
\Phi(\theta_\epsilon, z) \stackrel{\text{def}}{=} (\partial\Psi_{\text{dev}}+\mathbb{M})^{-1}\big(&\mathrm{Proj}_{\mathbb{R}^{3\times 3}_{\text{dev}}}(-\partial\varphi_\epsilon(z)-\mathbb{E}z-\mathrm{D}_z H_1(z)\\
&-g_1(\theta_\epsilon)-g_2(\mathcal{J}_\epsilon z)-\theta_\epsilon \mathrm{D}_z H_2(\mathcal{J}_\epsilon z))\big),
\end{aligned}$$

for all $z\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}}))$ and we have to solve this differential equation for the unknown function z_ϵ. We may observe that we are not in the classical framework of ODE in $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}})$ (see [Car90]), since $g_2(\mathcal{J}_\epsilon z)$ can not be defined for $z\in\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}})$ but only for $z\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}}))$. However we can solve (4.12) with the Picard's iterations technique. Indeed, $\Phi(\theta_\epsilon,z)\in\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$ for all $z\in\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$ and $z_\epsilon\in\mathrm{C}^1([0,T];\mathrm{L}^2(\Omega))$ is a solution of (4.9)–(4.10) if and only if z_ϵ is a fixed point of the mapping

$$\begin{aligned}
\Lambda_\epsilon:\ &\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \to \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega))\\
&\qquad z\mapsto \Lambda_\epsilon(z) : t\mapsto z^0 + \int_0^t \Phi(\theta_\epsilon(\cdot,s), z(\cdot,s))\,\mathrm{d}s.
\end{aligned}$$

Let us assume that $z_1, z_2 \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$. By using (3.11b) and (3.13b), we find that

$$\|\Lambda_\epsilon(z_1(\cdot,t))-\Lambda_\epsilon(z_2(\cdot,t))\|_{\mathrm{L}^2(\Omega)} \leq \tfrac{1}{c^{\mathbb{M}}}\int_0^t \big(\|\partial\varphi_\epsilon(z_1(\cdot,s))-\partial\varphi_\epsilon(z_2(\cdot,s))\|_{\mathrm{L}^2(\Omega)}$$
$$+(\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}+C_{zz}^{H_1}+C_{zz}^{H_2}\|\theta_\epsilon\|_{\mathrm{L}^\infty(\mathcal{Q}_T)})\|z_1(\cdot,s)-z_2(\cdot,s)\|_{\mathrm{L}^2(\Omega)}$$
$$+\|g_2(\mathcal{J}_\epsilon z_1(\cdot,s))-g_2(\mathcal{J}_\epsilon z_2(\cdot,s))\|_{\mathrm{L}^2(\Omega)}\big)\,\mathrm{d}s,$$

for all $t\in[0,T]$. Since $\partial\varphi_\epsilon$ is $\frac{1}{\epsilon}$-Lipschitz continuous on $\mathrm{L}^2(\Omega)$ and g_2 is linear, it follows that

$$\begin{aligned}\forall t\in[0,T]: \|\Lambda_\epsilon(z_1(\cdot,t))-\Lambda_\epsilon(z_2(\cdot,t))\|_{\mathrm{L}^2(\Omega)} &\leq C^\epsilon\int_0^t\|z_1(\cdot,s)-z_2(\cdot,s)\|_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s\\ &+\tfrac{1}{c^{\mathbb{M}}}\int_0^t\|g_2(\mathcal{J}_\epsilon z_1(\cdot,s)-\mathcal{J}_\epsilon z_2(\cdot,s))\|_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s,\end{aligned}\tag{4.13}$$

where $C^\epsilon \overset{\text{def}}{=} \frac{1}{c^{\mathbb{M}}}\big(\frac{1}{\epsilon}+\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}+C_{zz}^{H_1}+C_{zz}^{H_2}\|\theta_\epsilon\|_{\mathrm{L}^\infty(\mathcal{Q}_T)}\big)$. Thus the energy estimate (4.4) allows us to infer that

$$\begin{aligned}\|g_2(z(\cdot,s))\|_{\mathrm{L}^2(\Omega)} &\leq \tfrac{\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}}{\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\Big(\int_0^s\|\mathrm{div}(\mathbb{E}z(\cdot,\sigma))\|^2_{(\mathrm{H}_0^1(\Omega))'}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\\ &\leq \tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}}{\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\Big(\int_0^s\|z(\cdot,\sigma)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}},\end{aligned}$$

for all $s\in[0,T]$ and all $z\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. This leads to

$$\begin{aligned}&\int_0^t\|g_2(\mathcal{J}_\epsilon z_1(\cdot,s)-\mathcal{J}_\epsilon z_2(\cdot,s))\|_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s\\ &\leq \tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}}{\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\int_0^t\Big(\int_0^s\|\mathcal{J}_\epsilon z_1(\cdot,\sigma)-\mathcal{J}_\epsilon z_2(\cdot,\sigma)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\mathrm{d}s,\end{aligned}$$

for all $t\in[0,T]$. Since $\mathcal{J}_\epsilon$ is a contraction on $\mathrm{L}^2(\Omega)$ we infer

$$\begin{aligned}\int_0^t\|g_2(\mathcal{J}_\epsilon z_1(\cdot,s)-\mathcal{J}_\epsilon z_2(\cdot,s))\|_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s &\leq \tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}}{\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\int_0^t\Big(\int_0^s\|z_1(\cdot,\sigma)-z_2(\cdot,\sigma)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}\mathrm{d}s\\ &\leq \tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}\sqrt{T}}{\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}t\|z_1-z_2\|_{\mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))},\end{aligned}$$

for all $t\in[0,T]$. Finally, we find

$$\forall t\in[0,T]:\ \|\Lambda_\epsilon(z_1(\cdot,t))-\Lambda_\epsilon(z_2(\cdot,t))\|_{\mathrm{L}^2(\Omega)} \leq \big(C^\epsilon+\tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}\sqrt{T}}{c^{\mathbb{M}}\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\big)t\|z_1-z_2\|_{\mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))}.$$

We iterate the previous computation to get

$$\|\Lambda_\epsilon^m(z_1(\cdot,t))-\Lambda_\epsilon^m(z_2(\cdot,t))\|_{\mathrm{L}^2(\Omega)} \leq \big(C^\epsilon+\tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}\sqrt{T}}{c^{\mathbb{M}}\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\big)^m\tfrac{t^m}{m!}\|z_1-z_2\|_{\mathrm{C}^0(0,T;\mathrm{L}^2(\Omega))}.$$

for all $t\in[0,T]$ and $m\geq 1$. It follows that there exists $m_0\geq 1$ such that

$$\big(C^\epsilon+\tfrac{\|\mathbb{E}\|^2_{\mathrm{L}^\infty(\Omega)}\sqrt{T}}{c^{\mathbb{M}}\sqrt{c^{\mathbb{E}}c^{\mathbb{L}}C^{\text{Korn}}}}\big)^{m_0}\tfrac{T^{m_0}}{m_0!}<1,$$

and $\Lambda_\epsilon^{m_0}$ possesses a unique fixed point in $\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$, which is also the unique fixed point of Λ_ϵ. We may conclude that there exists a unique solution $z_\epsilon\in\mathrm{C}^1([0,T];\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}))$ to the problem (4.9)–(4.10).

Let us choose now a sequence $(\theta_\epsilon)_{\epsilon>0}$ such that

$$\theta_\epsilon \to \theta \text{ in } \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega)).$$

Since $p, q \geq 2$, it follows that

$$\theta_\epsilon \to \theta \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)).$$

We will establish that there exists a subsequence of $(z_\epsilon)_{\epsilon>0}$, still denoted by $(z_\epsilon)_{\epsilon>0}$, such that

$$\begin{aligned} &z_\epsilon \rightharpoonup z \text{ in } \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}})) \text{ weak,}\\ &z_\epsilon \rightharpoonup z \text{ in } \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega;\mathbb{R}^{3\times3}_{\text{dev}})) \text{ weak } *, \end{aligned}$$

where z is a solution of the problem (4.5)–(4.7). To do so, we notice that

$$\begin{aligned} \forall t \in [0,T]:\ \dot z_\epsilon(\cdot,t) =& (\partial\Psi_{\text{dev}}+\mathbb{M})^{-1}\big(\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))\\ &-\partial\varphi_\epsilon(z_\epsilon(\cdot,t))-\mathbb{E}z_\epsilon(\cdot,t)-\mathrm{D}_z H_1(z_\epsilon(\cdot,t)))\big), \end{aligned}$$

where $g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t)) \overset{\text{def}}{=} -(g_1(\theta_\epsilon(\cdot,t))+g_2(\mathcal{J}_\epsilon z_\epsilon(\cdot,t))+\theta_\epsilon(\cdot,t)\mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon(\cdot,t)))$. Define

$$\forall t \in [0,T]:\ w_\epsilon(\cdot,t) \overset{\text{def}}{=} g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t)) - \partial\varphi_\epsilon(z_\epsilon(\cdot,t)) - \mathbb{E}z_\epsilon(\cdot,t) - \mathrm{D}_z H_1(z_\epsilon(\cdot,t)).$$

We have $w_\epsilon \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}}))$ and

$$\forall t \in [0,T]:\ w_\epsilon(\cdot,t) + \partial\varphi_\epsilon(z_\epsilon(\cdot,t)) + \mathbb{E}z_\epsilon(\cdot,t) + \mathrm{D}_z H_1(z_\epsilon(\cdot,t)) = g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t)), \tag{4.14a}$$

$$\forall t \in [0,T]:\ \dot z_\epsilon(\cdot,t) = (\partial\Psi_{\text{dev}}+\mathbb{M})^{-1}\big(\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w_\epsilon(\cdot,t))\big). \tag{4.14b}$$

In order to obtain a priori estimates, we multiply (4.14a) by $\dot z_\epsilon$, and we integrate this expression over $\mathcal{Q}_\tau$ to get

$$\begin{aligned} &\int_{\mathcal{Q}_\tau} w_\epsilon{:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_\tau} \partial\varphi_\epsilon(z_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_\tau} \mathbb{E}z_\epsilon{:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\mathcal{Q}_\tau} \mathrm{D}_z H_1(z_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_\tau} g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

We observe that $\int_{\mathcal{Q}_\tau} \partial\varphi_\epsilon(z_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t = \varphi_\epsilon(z_\epsilon(\cdot,\tau)) - \varphi_\epsilon(z^0)$ and $\int_{\mathcal{Q}_\tau} \mathrm{D}_z H_1(z_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega H_1(z_\epsilon(\cdot,\tau))\,\mathrm{d}x - \int_\Omega H_1(z^0)\,\mathrm{d}x$. Moreover, recalling that $\dot z_\epsilon \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}))$, we have

$$\int_{\mathcal{Q}_\tau} w_\epsilon{:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_\tau} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w_\epsilon){:}\dot z_\epsilon \,\mathrm{d}x\,\mathrm{d}t \geq c^{\mathbb{M}} \int_0^\tau \|\dot z_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t.$$

Since $z^0 \in \mathrm{H}^1(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}) = \mathcal{D}(\varphi)$, we have $\varphi_\epsilon(z^0) \leq \varphi(z^0)$ (for technical details, the reader is referred to [Bre73]) and

$$\begin{aligned} &\tfrac{c^{\mathbb{M}}}{2}\int_0^\tau \|\dot z_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \varphi_\epsilon(z_\epsilon(\cdot,\tau)) + \int_\Omega H_1(z_\epsilon(\cdot,\tau))\,\mathrm{d}x + \tfrac12\int_\Omega \mathbb{E}z_\epsilon(\cdot,\tau){:}z_\epsilon(\cdot,\tau)\,\mathrm{d}x\\ &\leq \varphi(z^0) + \int_\Omega H_1(z^0)\,\mathrm{d}x + \tfrac12\int_\Omega \mathbb{E}z^0{:}z^0\,\mathrm{d}x + \tfrac{1}{2c^{\mathbb{M}}}\int_0^\tau \|g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t. \end{aligned}$$

Furthermore

$$\forall \epsilon > 0\ \forall \bar z \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}):\ \varphi_\epsilon(\bar z) = \tfrac{\epsilon}{2}\|\partial\varphi_\epsilon(\bar z)\|^2_{\mathrm{L}^2(\Omega)} + \varphi(\mathcal{J}_\epsilon \bar z) = \tfrac{1}{2\epsilon}\|\bar z - \mathcal{J}_\epsilon \bar z\|^2_{\mathrm{L}^2(\Omega)} + \varphi(\mathcal{J}_\epsilon \bar z).$$

Thus $\mathcal{J}_\epsilon \bar{z} \in \mathrm{C}^0([0,T];\mathrm{H}^1(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}}))$ for all $\bar{z} \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}}))$ and

$$\begin{aligned}&\tfrac{c^{\mathbb{M}}}{2}\int_0^\tau \|\dot{z}_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{1}{2\epsilon}\|z_\epsilon(\cdot,\tau){-}\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)\|^2_{\mathrm{L}^2(\Omega)} + \varphi(\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)) + \int_\Omega H_1(z_\epsilon(\cdot,\tau))\,\mathrm{d}x\\ &+ \tfrac12\int_\Omega \mathbb{E}z_\epsilon(\cdot,\tau){:}z_\epsilon(\cdot,\tau)\,\mathrm{d}x \le \varphi(z^0) + \int_\Omega H_1(z^0)\,\mathrm{d}x\\ &+ \tfrac12\int_\Omega \mathbb{E}z^0{:}z^0\,\mathrm{d}x + \tfrac{1}{2c^{\mathbb{M}}}\int_0^\tau \|g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t.\end{aligned}$$

By using (3.11a), (3.12) and (3.13a), we infer that

$$\begin{aligned}&\tfrac{c^{\mathbb{M}}}{2}\int_0^\tau \|\dot{z}_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{1}{2\epsilon}\|z_\epsilon(\cdot,\tau){-}\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)\|^2_{\mathrm{L}^2(\Omega)} + C_1\|z_\epsilon(\cdot,\tau)\|^2_{\mathrm{L}^2(\Omega)}\\ &+ C_1\|\nabla(\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau))\|^2_{\mathrm{L}^2(\Omega)} \le C_0 + \tfrac{1}{2c^{\mathbb{M}}}\int_0^\tau \|g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t,\end{aligned} \tag{4.15}$$

where $C_0 \overset{\text{def}}{=} \varphi(z^0) + \int_\Omega H_1(z^0)\,\mathrm{d}x + \widetilde{c}^{H_1}|\Omega| + \frac12\int_\Omega \mathbb{E}z^0{:}z^0\,\mathrm{d}x$ and $C_1 \overset{\text{def}}{=} \min\big(c^{H_1}{+}\frac{c^{\mathbb{E}}}{2}, \frac{\nu}{2}\big)$. Since $\mathcal{J}_\epsilon$ is a contraction and $\mathcal{J}_\epsilon 0 = 0$, we have

$$\forall \bar{z} \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\text{dev}}):\ \|\mathcal{J}_\epsilon \bar{z}\|_{\mathrm{L}^2(\Omega)} = \|\mathcal{J}_\epsilon \bar{z}{-}\mathcal{J}_\epsilon 0\|_{\mathrm{L}^2(\Omega)} \le \|\bar{z}\|_{\mathrm{L}^2(\Omega)}.$$

Then we find

$$\tfrac{c^{\mathbb{M}}}{2}\int_0^\tau \|\dot{z}_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + C_1\|\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)\|^2_{\mathrm{H}^1(\Omega)} \le C_0 + \tfrac{1}{2c^{\mathbb{M}}}\int_0^\tau \|g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t. \tag{4.16}$$

For the bound on $g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))$, we use (3.12) to get

$$\begin{aligned}&\int_0^\tau \|g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \le 3\int_0^\tau \|g_1(\theta_\epsilon)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + 3\int_0^\tau \|g_2(\mathcal{J}_\epsilon z_\epsilon)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\\ &+ 6(C_z^{H_2})^2\int_0^\tau \|\theta_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + 6(C_z^{H_2})^2\int_{\mathcal{Q}_\tau} |\theta_\epsilon|^2|\mathcal{J}_\epsilon z_\epsilon|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.17}$$

By using the definition of mappings g_1 and g_2, the first two terms on the right hand side of (4.17) can be estimated. More precisely, it is plain that there exists $C_2 > 0$ such that

$$\begin{aligned}&\int_0^\tau \|g_1(\theta_\epsilon)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \int_0^\tau \|g_2(\mathcal{J}_\epsilon z_\epsilon)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\\ &\le \tfrac{C_2}{3}\Big(\|u^0\|^2_{\mathrm{H}^1(\Omega)}{+}\int_0^\tau \|\ell\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t{+}\int_0^\tau \|\theta_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t{+}\int_0^\tau \|\mathcal{J}_\epsilon z_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\Big).\end{aligned} \tag{4.18}$$

The last term on the right hand side of (4.17) is estimated by using Hölder's inequality and the continuous embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$. Namely, it follows that there exists $C_3 > 0$ such that

$$\int_{\mathcal{Q}_\tau} |\theta_\epsilon|^2|\mathcal{J}_\epsilon z_\epsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le C_3\int_0^\tau \|\theta_\epsilon\|^2_{\mathrm{L}^4(\Omega)}\|\mathcal{J}_\epsilon z_\epsilon\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}t. \tag{4.19}$$

We insert (4.18) and (4.19) into (4.16), we obtain

$$\begin{aligned}&\tfrac{c^{\mathbb{M}}}{2}\int_0^\tau \|\dot{z}_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + C_1\|\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)\|^2_{\mathrm{H}^1(\Omega)} \le C_0 + \tfrac{C_2}{2c^{\mathbb{M}}}\Big(\|u^0\|^2_{\mathrm{H}^1(\Omega)}{+}\int_0^\tau \|\ell\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\Big)\\ &+ \tfrac{C_2+6(C_z^{H_2})^2}{2c^{\mathbb{M}}}\int_0^\tau \|\theta_\epsilon\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \int_0^\tau \tfrac{C_2+6(C_z^{H_2})^2C_3\|\theta_\epsilon\|^2_{\mathrm{L}^4(\Omega)}}{2c^{\mathbb{M}}}\|\mathcal{J}_\epsilon z_\epsilon\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{4.20}$$

Since θ_ϵ is bounded in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$, with $q \geq 2$ and $p \geq 4$, we may deduce from Grönwall's lemma that $\mathcal{J}_\epsilon z_\epsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$, $g_\epsilon(\cdot,\mathcal{J}_\epsilon z_\epsilon)$ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and z_ϵ is bounded in $\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$. Therefore (3.12) implies that $D_z H_1(z_\epsilon)$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$. Furthermore the definition of the subdifferential $\partial\Psi_{\text{dev}}(\dot{z}_\epsilon)$ enables us to infer from (4.14b) that

$$-\int_{\mathcal{Q}_\tau} \mathbb{M}\dot{z}_\epsilon{:}w_\epsilon^p \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \|w_\epsilon^p\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \leq \int_{\mathcal{Q}_\tau} (\Psi(w_\epsilon^p{+}\dot{z}_\epsilon){-}\Psi(\dot{z}_\epsilon))\,\mathrm{d}x\,\mathrm{d}t \leq \int_{\mathcal{Q}_\tau} \Psi(w_\epsilon^p)\,\mathrm{d}x\,\mathrm{d}t,$$

where $w_\epsilon^p = \mathrm{Proj}_{\mathbb{R}^{3\times 3}_{\text{dev}}}(w_\epsilon)$, and using (3.10b), we get

$$\|w_\epsilon^p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \leq C^\Psi\sqrt{T|\Omega|} + C^{\mathbb{M}}\|\dot{z}_\epsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}. \tag{4.21}$$

Observe that $\dot{z}_\epsilon$ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$, allows us to conclude that w_ϵ^p is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$. On the other hand, we multiply (4.14a) by $\partial\varphi_\epsilon(z_\epsilon)$ and we integrate this result over $\mathcal{Q}_\tau$. Recalling that $\partial\varphi_\epsilon(z_\epsilon)$ takes its values in $L^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}})$, the Cauchy-Schwarz's inequality, (3.12) and (4.21) give

$$\begin{aligned}\|\partial\varphi_\epsilon(z_\epsilon)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} &\leq (C_z^{H_1}{+}C^\Psi)\sqrt{T|\Omega|} + (C_z^{H_1}{+}\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)})\|z_\epsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}\\ &+ C^{\mathbb{M}}\|\dot{z}_\epsilon\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \|g_\epsilon(\cdot,\mathcal{J}_\epsilon z_\epsilon)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}.\end{aligned}$$

Thus $\partial\varphi_\epsilon(z_\epsilon)$ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$ and finally w_ϵ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon > 0$. Furthermore, there exists $z \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, $\widetilde{z} \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$, $w, v \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ such that it is possible to extract subsequences, still denoted by z_ϵ, $\mathcal{J}_\epsilon z_\epsilon$, w_ϵ and $\partial\varphi_\epsilon(z_\epsilon)$ satisfying the following convergences

$$\begin{aligned} z_\epsilon &\rightharpoonup z \ \text{ in } \ \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak},\\ z_\epsilon &\rightharpoonup z \ \text{ in } \ \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak } *,\\ \mathcal{J}_\epsilon z_\epsilon &\rightharpoonup \widetilde{z} \ \text{ in } \ \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \ \text{ weak } *,\\ w_\epsilon &\rightharpoonup w \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak},\\ \partial\varphi_\epsilon(z_\epsilon) &\rightharpoonup v \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak}.\end{aligned}$$

Moreover, using (4.15) there exists $C_4 > 0$ such that

$$\forall\tau \in [0,T]: \ \|z_\epsilon(\cdot,\tau){-}\mathcal{J}_\epsilon z_\epsilon(\cdot,\tau)\|^2_{\mathrm{L}^2(\Omega)} \leq C_4\epsilon\big(1{+}\|g_\epsilon(\cdot,\mathcal{J}_\epsilon z_\epsilon)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}\big), \tag{4.22}$$

which, allows us to infer that $z = \widetilde{z}$. Since $\mathcal{J}_\epsilon$ is a contraction on $\mathrm{L}^2(\Omega)$, it follows also that, possibly extracting another subsequence, still denoted by $\mathcal{J}_\epsilon z_\epsilon$, we have

$$\forall r \in [2,6): \ \mathcal{J}_\epsilon z_\epsilon \to z \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^r(\Omega)).$$

We may deduce from (4.22) that

$$z_\epsilon \to z \ \text{ in } \ \mathrm{C}^0(0,T;\mathrm{L}^2(\Omega)).$$

Our aim now consists to pass to the limit in (4.14). To do so, we observe that the mapping $\mathcal{L}_0$ is linear and continuous, the mappings g_1 and g_2 are also continuous, which, gives

$$\begin{aligned} g_1(\theta_\epsilon) &\to g_1(\theta) \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)),\\ g_2(\mathcal{J}_\epsilon z_\epsilon) &\to g_2(z) \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)).\end{aligned}$$

Since $\mathrm{D}_z H_i$, $i = 1, 2$, is Lipschitz continuous, it follows that

$$\mathrm{D}_z H_1(z_\epsilon) + \mathbb{E}z_\epsilon \to \mathrm{D}_z H_1(z) + \mathbb{E}z \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)),$$
$$\forall r \in [2,6) : \ \mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon) \to \mathrm{D}_z H_2(z) \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^r(\Omega)).$$

Hence we consider r such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ to get

$$\theta_\epsilon \mathrm{D}_z H_2(\mathcal{J}_\epsilon z_\epsilon) \to \theta \mathrm{D}_z H_2(z) \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)).$$

We can pass to the limit in all terms of (4.14a), we obtain

$$w + v + \mathrm{D}_z H_1(z) + \mathbb{E}z = -(g_1(\theta) + g_2(z) + \theta \mathrm{D}_z H_2(z)). \tag{4.23}$$

It remains to pass to the limit in (4.14b) and to prove that z solves (4.5)–(4.7). Using (4.10) and the strong convergence of $(z_\epsilon)_{\epsilon>0}$ to z in $\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$, we infer that $z(0,\cdot) = z^0$. Then we prove that $v(t,\cdot) \in \partial\varphi(z(t,\cdot))$ for almost every $t \in [0,T]$, i.e, $v \in \partial\varphi(z)$ where $\partial\varphi$ is identified to its $\mathrm{L}^2(0,T)$-extension. To do so, we use the classical results for maximal monotone operators (see [Bre73]). More precisely, since $\partial\varphi_\epsilon(z_\epsilon) \in \partial\varphi(\mathcal{J}_\epsilon z_\epsilon)$, it is sufficient to prove that

$$\limsup_{\epsilon \to 0} \int_{\mathcal{Q}_T} (\partial\varphi_\epsilon(z_\epsilon)){:}(\mathcal{J}_\epsilon z_\epsilon) \,\mathrm{d}x \,\mathrm{d}t \le \int_{\mathcal{Q}_T} v{:}z \,\mathrm{d}x \,\mathrm{d}t, \tag{4.24}$$

which is an immediate consequence of the convergence results for the subsequences $(\partial\varphi_\epsilon(z_\epsilon))_{\epsilon>0}$ and $(\mathcal{J}_\epsilon z_\epsilon)_{\epsilon>0}$. Hence, using the definitions of w and φ, we may conclude that z is a solution of (4.5)–(4.7) if we can prove that $\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w(t,\cdot)) - \mathbb{M}\dot{z}(t,\cdot) \in \partial\Psi_{\text{dev}}(\dot{z}(t,\cdot))$ for almost every $t \in [0,T]$. So we may use the same trick as previously and we only need to check that

$$\limsup_{\epsilon \to 0} \int_{\mathcal{Q}_T} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w_\epsilon){:}\dot{z}_\epsilon \,\mathrm{d}x \,\mathrm{d}t \le \int_{\mathcal{Q}_T} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w){:}\dot{z} \,\mathrm{d}x \,\mathrm{d}t. \tag{4.25}$$

But $\dot{z}_\epsilon$ takes its values in $\mathbb{R}^{3\times3}_{\text{dev}}$ and thus $\dot{z}$ takes also its values in $\mathbb{R}^{3\times3}_{\text{dev}}$. It follows that (4.25) is equivalent to

$$\limsup_{\epsilon \to 0} \int_{\mathcal{Q}_T} w_\epsilon{:}\dot{z}_\epsilon \,\mathrm{d}x \,\mathrm{d}t \le \int_{\mathcal{Q}_T} w{:}\dot{z} \,\mathrm{d}x \,\mathrm{d}t.$$

We compute $\int_{\mathcal{Q}_T} w_\epsilon{:}\dot{z}_\epsilon \,\mathrm{d}x \,\mathrm{d}t$ and $\int_{\mathcal{Q}_T} w{:}\dot{z} \,\mathrm{d}x \,\mathrm{d}t$ from (4.14a) and (4.23), then the convergence results obtained above imply that (4.25) holds if and only if

$$\liminf_{\epsilon \to 0} \int_{\mathcal{Q}_T} \partial\varphi_\epsilon(z_\epsilon){:}\dot{z}_\epsilon \,\mathrm{d}x \,\mathrm{d}t \ge \int_{\mathcal{Q}_T} \partial\varphi(z){:}\dot{z} \,\mathrm{d}x \,\mathrm{d}t.$$

We observe that $\int_{\mathcal{Q}_T} \partial\varphi_\epsilon(z_\epsilon){:}\dot{z}_\epsilon \ \mathrm{d}x \ \mathrm{d}t = \varphi_\epsilon(z_\epsilon(T)) - \varphi_\epsilon(z^0) \ge \varphi(\mathcal{J}_\epsilon z_\epsilon(T)) - \varphi_\epsilon(z^0)$ and $\int_{\mathcal{Q}_T} \partial\varphi(z){:}\dot{z} \,\mathrm{d}x \,\mathrm{d}t = \varphi(z(T)) - \varphi(z^0)$. Recalling that $z^0 \in \mathcal{D}(\varphi)$, we get $\lim_{\epsilon \to 0} \varphi_\epsilon(z^0) = \varphi(z^0)$ and the lower semicontinuity of φ, allows us to conclude. This proves the existence result.

Finally we observe that

$$-\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w) = \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(g_1(\theta) + g_2(z) + \mathbb{E}z) + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) + v,$$

and, using the definition of the mappings g_1, g_2 and φ

$$-\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w) = \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(-\mathbb{E}(\mathrm{e}(u) - z)) + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) - \nu\Delta z.$$

So we may rewrite (4.1b) as follows

$$\dot{z} - \nu\mathbb{M}^{-1}\Delta z = \mathbb{M}^{-1}f^z. \tag{4.26}$$

with $f^z \stackrel{\text{def}}{=} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(\mathbb{E}(\mathrm{e}(u)-z)) - \mathrm{D}_z H_1(z) - \theta \mathrm{D}_z H_2(z) - \psi$ and $\psi = \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}}(w) - \mathbb{M}\dot{z} \in \partial\Psi_{\text{dev}}(\dot{z})$. With (3.10) we infer that

$$\forall \psi \in \partial\Psi_{\text{dev}}(\dot{z}) : \ |\psi| \leq C^{\Psi} \ \text{ a.e. } (x,t) \in \Omega \times (0,T).$$

Since $z \in \mathrm{L}^{\infty}(0,T;\mathrm{H}^1(\Omega))$, we infer by using (3.12) that $\mathrm{D}_z H_i(z)$ belongs to $\mathrm{L}^{\infty}(0,T;\mathrm{L}^p(\Omega))$ for $i = 1,2$. Then it follows that $\mathbb{M}^{-1}f^z$ belongs to $\mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$. Since $z^0 \in \mathrm{X}_{q,p}(\Omega)$, we may deduce from the maximal regularity result for parabolic systems that $z \in \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{C}^0([0,T];\mathrm{H}^1(\Omega))$ and $\dot{z} \in \mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$. We refer to [Dor93, HiR08, PrS01] and the references therein for more details on the maximal regularity result for parabolic systems and its consequences. □

Let us observe that here neither $\partial\Psi_{\text{dev}} + \mathbb{M}$ nor $\partial\varphi + \mathrm{D}_z H_1 + \theta\mathrm{D}_z H_2 + \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\text{dev}}} \circ (\mathbb{E}+g_1+g_2)$ are linear and self-adjoint operators of $\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}})$, so we can not use the ideas proposed in [CoV90] to prove uniqueness. The uniqueness result proved below relies on the boundedness assumption (3.11b) for the hardening functional H_1 combined with Grönwall's lemma. More precisely let $h_1 \in \mathrm{C}^2(\mathbb{R}^{3\times3}_{\text{dev}};\mathbb{R})$ be defined by

$$\forall z \in \mathbb{R}^{3\times3}_{\text{dev}} : \ h_1(z) \stackrel{\text{def}}{=} H_1(z) - C^{H_1}|z|^2, \tag{4.27}$$

with a real number $C^{H_1} > 0$. Assumption (3.11b) implies that there exists $C^{h_1} > 0$ such that

$$\forall z_1, z_2 \in \mathbb{R}^{3\times3}_{\text{dev}} : \ |\mathrm{D}_z h_1(z_1) - \mathrm{D}_z h_1(z_2)| \leq C^{h_1}|z_1 - z_2|. \tag{4.28}$$

Proposition 4.2 (Uniqueness for $(\mathbf{P}_{uz})$) *Assume that θ is given in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$,* (3.10), (3.11), (3.14) *and* (3.16) *hold and $u^0 \in \mathrm{H}^1(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$. Then the problem* (4.1)–(4.3) *admits a unique solution.*

Proof. Let $\xi_1 \stackrel{\text{def}}{=} (u_1, z_1)$ and $\xi_2 \stackrel{\text{def}}{=} (u_2, z_2)$ be two solutions of (4.1a)–(4.1b) satisfying (4.2) and (4.3). With the results of the previous theorem we already know that $(u_i, z_i) \in \mathrm{C}^0([0,T];\mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega))$ and $\Delta z_i \in \mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$, $i = 1,2$. Define

$$\begin{aligned} \gamma(t) &\stackrel{\text{def}}{=} \tfrac{1}{2}\int_{\Omega} \mathbb{E}((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2)):((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2))\,\mathrm{d}x \\ &+ \tfrac{\nu}{2}\int_{\Omega} \nabla(z_1-z_2):\nabla(z_1-z_2)\,\mathrm{d}x + C^{H_1}\int_{\Omega}|z_1-z_2|^2\,\mathrm{d}x \end{aligned} \tag{4.29}$$

for all $t \in [0,T]$. Since

$$\begin{aligned} \gamma(t) &= \tfrac{1}{2}\int_{\Omega} \mathbb{E}((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2)):((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2))\,\mathrm{d}x \\ &- \tfrac{\nu}{2}\int_{\Omega} \Delta(z_1-z_2)\cdot(z_1-z_2)\,\mathrm{d}x + C^{H_1}\int_{\Omega}|z_1-z_2|^2\,\mathrm{d}x, \end{aligned}$$

the mapping γ is continuous on $[0,T]$ and its derivative in the sense of distributions belongs to $\mathrm{L}^1(0,T)$. Thus γ is absolutely continuous on $[0,T]$ and with assumptions (3.12) and (3.13a) combined with Korn's inequality, we infer that there exists a real number $\kappa > 0$ such that

$$\forall t \in [0,T] : \ \gamma(t) \geq \kappa\big(\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)} + \|z_1(\cdot,t)-z_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)}\big). \tag{4.30}$$

On the one hand, recalling that $\partial\Psi_{\text{dev}}(\cdot)$ is a monotone operator, the Green's formula and (4.3) enable us to deduce from (4.1b) that

$$\begin{aligned}\forall \widetilde{z}_i \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}):\ &\int_\Omega (\mathbb{M}\dot{z}_i - \mathbb{E}(\mathrm{e}(u_i)-z_i)+\mathrm{D}_z H_1(z_i)+\theta\mathrm{D}_z H_2(z_i)):(\dot{z}_i-\widetilde{z}_i)\,\mathrm{d}x\\ &-\nu\int_\Omega \Delta z_i\cdot(\dot{z}_i-\widetilde{z}_i)\,\mathrm{d}x \le \int_\Omega(\Psi_{\text{dev}}(\widetilde{z}_i)-\Psi_{\text{dev}}(\dot{z}_i))\,\mathrm{d}x.\end{aligned} \tag{4.31}$$

On the other hand, we multiply (4.1a) by $\dot{u}_i - \widetilde{u}_i$, we integrate this expression over Ω, and with the help of the Green's formula together with (4.3), we obtain

$$\forall \widetilde{u}_i \in \mathrm{H}^1_0(\Omega):\ \int_\Omega (\mathbb{E}(\mathrm{e}(u_i)-z_i)+\alpha\theta\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u}_i)):(\mathrm{e}(\dot{u}_i)-\mathrm{e}(\widetilde{u}_i))\,\mathrm{d}x = \int_\Omega \ell\cdot(\dot{u}_i-\widetilde{u}_i)\,\mathrm{d}x. \tag{4.32}$$

Thus we add (4.31) and (4.32), we get

$$\begin{aligned}&\int_\Omega (\mathbb{M}\dot{z}_i - \mathbb{E}(\mathrm{e}(u_i)-z_i)+\mathrm{D}_z H_1(z_i)+\theta\mathrm{D}_z H_2(z_i)):(\dot{z}_i-\widetilde{z}_i)\,\mathrm{d}x - \nu\int_\Omega \Delta z_i\cdot(\dot{z}_i-\widetilde{z}_i)\,\mathrm{d}x\\ &+\int_\Omega (\mathbb{E}(\mathrm{e}(u_i)-z_i)+\alpha\theta\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u}_i)):(\mathrm{e}(\dot{u}_i)-\mathrm{e}(\widetilde{u}_i))\,\mathrm{d}x\\ &\le \int_\Omega(\ell\cdot(\dot{u}_i-\widetilde{u}_i)+\Psi_{\text{dev}}(\widetilde{z}_i)-\Psi_{\text{dev}}(\dot{z}_i))\,\mathrm{d}x.\end{aligned} \tag{4.33}$$

We choose now $(\widetilde{u}_i,\widetilde{z}_i) = (\dot{u}_{3-i},\dot{z}_{3-i})$ for $i=1,2$ in (4.33), we add these two inequalities, and with the help of (3.14), we find

$$\begin{aligned}&\dot{\gamma}(t) + c^{\mathbb{M}}\int_\Omega |\dot{z}_1-\dot{z}_2|^2\,\mathrm{d}x + c^{\mathbb{L}}\int_\Omega |\mathrm{e}(\dot{u}_1)-\mathrm{e}(\dot{u}_2)|^2\,\mathrm{d}x\\ &\le -\int_\Omega (\mathrm{D}_z h_1(z_1)-\mathrm{D}_z h_1(z_2)):(\dot{z}_1-\dot{z}_2)\,\mathrm{d}x\\ &-\int_\Omega \theta(\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)):(\dot{z}_1-\dot{z}_2)\,\mathrm{d}x \ \text{ a.e. } t\in[0,T].\end{aligned}$$

We estimate the terms of the right hand side of the previous inequality as follows:

$$\begin{aligned}&\Big|\int_\Omega (\mathrm{D}_z h_1(z_1)-\mathrm{D}_z h_1(z_2)):(\dot{z}_1-\dot{z}_2)\,\mathrm{d}x\Big| \le C^{h_1}\int_\Omega |z_1-z_2||\dot{z}_1-\dot{z}_2|\,\mathrm{d}x\\ &\le \tfrac{c^{\mathbb{M}}}{4}\int_\Omega |\dot{z}_1-\dot{z}_2|^2\,\mathrm{d}x + \tfrac{(C^{h_1})^2}{c^{\mathbb{M}}}\int_\Omega |z_1-z_2|^2\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_\Omega \theta(\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)):(\dot{z}_1-\dot{z}_2)\,\mathrm{d}x\Big| \le C^{H_2}_{zz}\int_\Omega |\theta||z_1-z_2||\dot{z}_1-\dot{z}_2|\,\mathrm{d}x\\ &\le \tfrac{c^{\mathbb{M}}}{4}\int_\Omega |\dot{z}_1-\dot{z}_2|^2\,\mathrm{d}x + \tfrac{(C^{H_2}_{zz})^2}{c^{\mathbb{M}}}\int_\Omega |\theta|^2|z_1-z_2|^2\,\mathrm{d}x\\ &\le \tfrac{c^{\mathbb{M}}}{4}\int_\Omega |\dot{z}_1-\dot{z}_2|^2\,\mathrm{d}x + \tfrac{(C^{H_2}_{zz})^2}{c^{\mathbb{M}}}\|\theta\|^2_{\mathrm{L}^4(\Omega)}\|z_1-z_2\|^2_{\mathrm{L}^4(\Omega)}.\end{aligned}$$

Since $\mathrm{H}^1(\Omega)\hookrightarrow\mathrm{L}^4(\Omega)$ with continuous embedding, we infer that there exists $C>0$ such that

$$\dot{\gamma}(t) \le C\big(1+\|\theta(\cdot,t)\|^2_{\mathrm{L}^4(\Omega)}\big)\|z_1(\cdot,t)-z_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)} \le \tfrac{C}{\kappa}\big(1+\|\theta(\cdot,t)\|^2_{\mathrm{L}^4(\Omega)}\big)\gamma(t) \ \text{ a.e. } t\in[0,T].$$

But $\theta \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ with $q \geq 4$ and $p \geq 4$, thus we get

$$\forall t \in [0,T] : \ \gamma(t) \leq \gamma(0)\mathrm{e}^{\frac{C}{\kappa}\int_0^t (1+\|\theta(\cdot,s)\|^2_{\mathrm{L}^4(\Omega)})\,\mathrm{d}s},$$

which, allows us to conclude. □

We provide that $\widetilde{\vartheta} \mapsto (u,z)$ is a continuous mapping from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ into $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1(\Omega))$ where (u,z) is the unique solution of (P_{uz}) when $\theta = \zeta(\widetilde{\vartheta})$.

Lemma 4.3 *Assume that* (3.10)*,* (3.11)*,* (3.14) *and* (3.16) *hold and that* $u^0 \in \mathrm{H}^1(\Omega)$*,* $z^0 \in \mathrm{X}_{q,p}(\Omega)$*. Then* $\vartheta \mapsto (u,z)$ *is continuous mapping from* $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ *into* $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega))$*.*

Proof. We consider $\vartheta_i \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ and for $i = 1,2$, let $\theta_i \stackrel{\text{def}}{=} \zeta(\vartheta_i) \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ and (u_i,z_i) the solution of the following system:

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u_i){-}z_i){+}\alpha\theta_i\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u}_i)) = \ell, \tag{4.34a}$$

$$\partial\Psi(\dot{z}_i) + \mathbb{M}\dot{z}_i - \mathbb{E}(\mathrm{e}(u_i){-}z_i) + \mathrm{D}_z H_1(z_i) + \theta\mathrm{D}_z H_2(z_i) - \nu\Delta z_i \ni 0, \tag{4.34b}$$

together with initial conditions

$$u_i(0,\cdot) = u^0, \quad z_i(0,\cdot) = z^0, \tag{4.35}$$

and boundary conditions

$$u_{i|\partial\Omega} = 0, \quad \nabla z_i{\cdot}\eta_{|\partial\Omega} = 0. \tag{4.36}$$

Since the mapping $\phi_1 : \vartheta \mapsto \theta = \zeta(\vartheta)$ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, it is enough to check that the mapping $\theta = \zeta(\vartheta) \to (u,z)$ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$.

Once again the key tool is the structural decomposition (4.27) combined with Grönwall's lemma. We reproduce the same kind of computations as in Proposition 4.2. More precisely, the total energy inequality associated to (4.34)–(4.36) is obtained by multiplying (4.34a) and (4.34b) by $\dot{u}_{3-i}{-}\dot{u}_i$ and $\dot{z}_{3-i}{-}\dot{z}_i$ respectively, and integrating over Ω. Then, we add these two inequalities, we find

$$\begin{aligned}
&\int_\Omega \mathbb{E}(\mathrm{e}(u_i){-}z_i){:}((\mathrm{e}(\dot{u}_{3-i}){-}\dot{z}_{3-i}){-}(\mathrm{e}(\dot{u}_i){-}\dot{z}_i))\,\mathrm{d}x + \int_\Omega \mathbb{M}\dot{z}_i{:}(\dot{z}_{3-i}{-}\dot{z}_i)\,\mathrm{d}x\\
&+ \int_\Omega \alpha\theta_i\mathrm{tr}(\mathrm{e}(\dot{u}_{3-i}){-}\mathrm{e}(\dot{u}_i))\,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot{u}_i){:}(\mathrm{e}(\dot{u}_{3-i}){-}\mathrm{e}(\dot{u}_i))\,\mathrm{d}x\\
&- \int_\Omega \nu\Delta z_i{\cdot}(\dot{z}_{3-i}{-}\dot{z}_i)\,\mathrm{d}x + \int_\Omega \mathrm{D}_z H_1(z_i){:}(\dot{z}_{3-i}{-}\dot{z}_i)\,\mathrm{d}x\\
&+ \int_\Omega \theta_i\mathrm{D}_z H_2(z_i){:}(\dot{z}_{3-i}{-}\dot{z}_i)\,\mathrm{d}x - \int_\Omega \ell{\cdot}(\dot{u}_{3-i}{-}\dot{u}_i)\,\mathrm{d}x\\
&+ \Psi_{\text{dev}}(\dot{z}_{3-i}) - \Psi_{\text{dev}}(\dot{z}_i) \geq 0 \ \text{ a.e. } \ t \in [0,T].
\end{aligned} \tag{4.37}$$

It is convenient to introduce the notations: $\bar{u} \stackrel{\text{def}}{=} u_1{-}u_2$, $\bar{z} \stackrel{\text{def}}{=} z_1{-}z_2$ and $\bar{\theta} \stackrel{\text{def}}{=} \theta_1{-}\theta_2$. Therefore, we take $i = 1,2$ in (4.37) and we add these two inequalities, we get

$$\begin{aligned}
&\int_\Omega \mathbb{E}(\mathrm{e}(\bar{u}){-}\bar{z}){:}(\mathrm{e}(\dot{\bar{u}}){-}\dot{\bar{z}})\,\mathrm{d}x + \int_\Omega \mathbb{M}\dot{\bar{z}}{:}\dot{\bar{z}}\,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot{\bar{u}}){:}\mathrm{e}(\dot{\bar{u}})\,\mathrm{d}x\\
&- \int_\Omega \nu\Delta\bar{z}{\cdot}\dot{\bar{z}}\,\mathrm{d}x + \int_\Omega (\mathrm{D}_z H_1(z_1){-}\mathrm{D}_z H_1(z_2)){:}\dot{\bar{z}}\,\mathrm{d}x\\
&\leq -\int_\Omega \alpha\bar{\theta}\mathrm{tr}(\mathrm{e}(\dot{\bar{u}}))\,\mathrm{d}x - \int_\Omega (\theta_1\mathrm{D}_z H_2(z_1){-}\theta_2\mathrm{D}_z H_2(z_2)){:}\dot{\bar{z}}\,\mathrm{d}x.
\end{aligned}$$

Define

$$\forall t \in [0,T]: \gamma(t) \overset{\text{def}}{=} \frac{1}{2}\int_\Omega \mathbb{E}((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2)):((\mathrm{e}(u_1)-z_1)-(\mathrm{e}(u_2)-z_2))\,\mathrm{d}x$$
$$-\frac{\nu}{2}\int_\Omega \nabla(z_1-z_2):\nabla(z_1-z_2)\,\mathrm{d}x + C^{H_1}\int_\Omega |z_1-z_2|^2\,\mathrm{d}x.$$

As in Proposition 4.2, we can check that γ is absolutely continuous on $[0,T]$ and (3.14) and (4.27) imply that

$$\dot\gamma(t) + c^{\mathbb{M}}\|\dot{\bar z}\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbb{L}}\|\mathrm{e}(\dot{\bar u})\|^2_{\mathrm{L}^2(\Omega)} \le -\int_\Omega (\mathrm{D}_z h_1(z_1)-\mathrm{D}_z h_1(z_2)):\dot{\bar z}\,\mathrm{d}x$$
$$-\int_\Omega \alpha\bar\theta \mathrm{tr}(\mathrm{e}(\dot{\bar u}))\,\mathrm{d}x - \int_\Omega (\theta_1 \mathrm{D}_z H_2(z_1)-\theta_2\mathrm{D}_z H_2(z_2)):\dot{\bar z}\,\mathrm{d}x,$$

for almost every $t\in[0,T]$. Clearly, it follows from (4.28) that

$$\dot\gamma(t) + c^{\mathbb{M}}\|\dot{\bar z}\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbb{L}}\|\mathrm{e}(\dot{\bar u})\|^2_{\mathrm{L}^2(\Omega)} \le -\int_\Omega \alpha\bar\theta \mathrm{tr}(\mathrm{e}(\dot{\bar u}))\,\mathrm{d}x$$
$$-\int_\Omega (\theta_1 \mathrm{D}_z H_2(z_1)-\theta_2\mathrm{D}_z H_2(z_2)):\dot{\bar z}\,\mathrm{d}x + C^{h_1}\int_\Omega |\bar z||\dot{\bar z}|\,\mathrm{d}x.$$

We estimate the first and third term on the right hand side with the help of Cauchy-Schwarz's inequality, while for the second term, we use the following decomposition

$$(\theta_1\mathrm{D}_z H_2(z_1)-\theta_2\mathrm{D}_z H_2(z_2)):\dot{\bar z} = (\bar\theta\mathrm{D}_z H_2(z_1)+\theta_2(\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2))):\dot{\bar z}.$$

Hence, we obtain

$$\dot\gamma(t) + \frac{3c^{\mathbb{M}}}{4}\|\dot{\bar z}\|^2_{\mathrm{L}^2(\Omega)} + \frac{c^{\mathbb{L}}}{2}\|\mathrm{e}(\dot{\bar u})\|^2_{\mathrm{L}^2(\Omega)} \le \frac{(C^{h_1})^2}{c^{\mathbb{M}}}\|\bar z\|^2_{\mathrm{L}^2(\Omega)} + \frac{3\alpha^2}{2c^{\mathbb{L}}}\|\bar\theta\|^2_{\mathrm{L}^2(\Omega)}$$
$$+\int_\Omega \big(|\bar\theta||\mathrm{D}_z H_2(z_1)||\dot{\bar z}|+|\theta_2||\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)||\dot{\bar z}|\big)\,\mathrm{d}x. \tag{4.38}$$

It remains to estimate the last term on the right hand side of (4.38). We use (3.11b) and (3.12) to get

$$\int_\Omega \big(|\bar\theta||\mathrm{D}_z H_2(z_1)||\dot{\bar z}|+|\theta_2||\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)||\dot{\bar z}|\big)\,\mathrm{d}x$$
$$\le C_z^{H_2}\int_\Omega \big(1+|z_1|\big)|\bar\theta||\dot{\bar z}|\,\mathrm{d}x + C_{zz}^{H_2}\int_\Omega |\theta_2||\bar z||\dot{\bar z}|\,\mathrm{d}x.$$

The Young's inequality implies that there exists $\gamma_i>0$, $i=1,2,3$, such that

$$\int_\Omega \big(|\bar\theta||\mathrm{D}_z H_2(z_1)||\dot{\bar z}|+|\theta_2||\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)||\dot{\bar z}|\big)\,\mathrm{d}x \le \frac{C_z^{H_2}}{2\gamma_1}\|\bar\theta\|^2_{\mathrm{L}^2(\Omega)}$$
$$+\frac{C_z^{H_2}}{2\gamma_2}\int_\Omega |\bar\theta|^2|z_1|^2\,\mathrm{d}x + \frac{C_{zz}^{H_2}}{2\gamma_3}\int_\Omega |\theta_2|^2|\bar z|^2\,\mathrm{d}x + \frac{C_z^{H_2}(\gamma_1+\gamma_2)+C_{zz}^{H_2}\gamma_3}{2}\|\dot{\bar z}\|^2_{\mathrm{L}^2(\Omega)}.$$

Since $z_1\in\mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$ and $\mathrm{H}^2(\Omega)\hookrightarrow\mathrm{L}^\infty(\Omega)$ with continuous embedding, we get

$$\int_\Omega \big(|\bar\theta||\mathrm{D}_z H_2(z_1)||\dot{\bar z}|+|\theta_2||\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)||\dot{\bar z}|\big)\,\mathrm{d}x \le \frac{C_z^{H_2}}{2\gamma_1}\|\bar\theta\|^2_{\mathrm{L}^2(\Omega)}$$
$$+\frac{C_z^{H_2}}{2\gamma_2}\|z_1\|^2_{\mathrm{L}^\infty(\Omega)}\|\bar\theta\|^2_{\mathrm{L}^2(\Omega)} + \frac{C_{zz}^{H_2}}{2\gamma_3}\|\theta_2\|^2_{\mathrm{L}^4(\Omega)}\|\bar z\|^2_{\mathrm{L}^4(\Omega)} + \frac{C_z^{H_2}(\gamma_1+\gamma_2)+C_{zz}^{H_2}\gamma_3}{2}\|\dot{\bar z}\|^2_{\mathrm{L}^2(\Omega)}. \tag{4.39}$$

We insert (4.39) in (4.38) and we choose $\gamma_1 = \gamma_2 = \frac{c^{\mathbb{M}}}{4C_z^{H_2}}$ and $\gamma_3 = \frac{c^{\mathbb{M}}}{2C_{zz}^{H_2}}$. Therefore the continuous embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$ and (4.30) give

$$\begin{aligned} \dot{\gamma}(t) + \tfrac{c^{\mathbb{M}}}{4}\|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbb{L}}}{2}\|\mathrm{e}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(\Omega)} &\le C(\bar{\theta}, z_1) + C(\theta_2)\|\bar{z}\|^2_{\mathrm{H}^1(\Omega)} \\ &\le C(\bar{\theta}, z_1) + \tfrac{C(\theta_2)}{\kappa}\gamma(t), \end{aligned} \tag{4.40}$$

for a.e. $t \in [0,T]$. Here $C(\bar{\theta}, z_1) \overset{\text{def}}{=} \big(\frac{(3\alpha)^2}{2c^{\mathbb{L}}} + \frac{2(C_z^{H_2})^2}{c^{\mathbb{M}}}\big)\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \frac{2(C_z^{H_2})^2}{c^{\mathbb{M}}}\|z_1\|^2_{\mathrm{L}^\infty(\Omega)}\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)}$ and $C(\theta_2) \overset{\text{def}}{=} \frac{(C^{h_1})^2}{c^{\mathbb{M}}} + \frac{C^2(C_{zz}^{H_2})^2}{c^{\mathbb{M}}}\|\theta_2\|^2_{\mathrm{L}^4(\Omega)}$ where $C > 0$ is the generic constant involved in the continuous embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$. Since $\bar{q} \ge 3$ we can check that $C(\bar{\theta}, z_1) \in \mathrm{L}^1(0,T)$ and

$$\begin{aligned} \int_0^T C(\bar{\theta}, z_1)\,\mathrm{d}t &\le \big(\tfrac{(3\alpha)^2}{2c^{\mathbb{L}}} + \tfrac{2(C_z^{H_2})^2}{c^{\mathbb{M}}}\big)\|\bar{\theta}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \\ &+ \tfrac{2(C_z^{H_2})^2}{c^{\mathbb{M}}}\|z_1\|^2_{\mathrm{L}^{\frac{2\bar{q}}{\bar{q}-2}}(0,T;\mathrm{L}^\infty(\Omega))}\|\bar{\theta}\|^2_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^2(\Omega))}. \end{aligned}$$

Note that $C(\theta_2) \in \mathrm{L}^1(0,T)$, which, thanks to Grönwall's lemma, gives

$$\forall t \in [0,T]: \ \gamma(t) \le \int_0^t C(\bar{\theta}(\cdot,s), z_1(\cdot,s))\mathrm{e}^{\frac{1}{\kappa}\int_s^t C(\theta_1(\cdot,\sigma) - \bar{\theta}(\cdot,\sigma))\,\mathrm{d}\sigma}\,\mathrm{d}s.$$

Recalling that the mapping $\widetilde{\vartheta} \mapsto \theta$ is Lipschitz continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ and maps any bounded subset of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ into a bounded subset of $\mathrm{L}^{q}(0,T;\mathrm{L}^{p}(\Omega))$ with $\bar{p} = 2$ and $\bar{q} > 4$, $p \ge 4$, $q = \beta_1\bar{q} > 8$, the last estimate allows us to conclude. $\square$

Let us observe furthermore that the image of a bounded set of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ by the mapping $\widetilde{\vartheta} \mapsto \zeta(\widetilde{\vartheta}) = \theta \mapsto (u,z)$ is a bounded subset of $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1(\Omega))$.

Let us introduce now some new notations. For any $r > 1$, let

$$\mathrm{V}^r(\Omega;\mathbb{R}^3) \overset{\text{def}}{=} \big\{u \in \mathrm{L}^2(\Omega;\mathbb{R}^3): \ \nabla u \in \mathrm{L}^r(\Omega;\mathbb{R}^{3\times3})\big\},$$

and for any $r \ge 2$, let

$$\mathrm{V}^r_0(\Omega;\mathbb{R}^3) \overset{\text{def}}{=} \big\{u \in \mathrm{V}^r(\Omega;\mathbb{R}^3): \ u_{|\partial\Omega} = 0\big\}.$$

We endowed $\mathrm{V}^r(\Omega;\mathbb{R}^3)$ with the following norm

$$\forall u \in \mathrm{V}^r(\Omega;\mathbb{R}^3): \ \|u\|_{\mathrm{V}^r(\Omega)} \overset{\text{def}}{=} \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^r(\Omega)}.$$

The aim of the next two lemmas is to prove further regularity results for the solutions (u,z) of the system composed by the momentum equilibrium equation and the flow rule. More precisely, assuming that θ remains in a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$, we will prove that $\mathrm{e}(\dot{u})$, $\dot{z}$ and z remain in a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$, $\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)) \cap \mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))$ and $\mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$, respectively.

Lemma 4.4 *Assume that* (3.10), (3.11), (3.13b), (3.14) *and* (3.16) *hold. Assume moreover that* $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *and* $u^0 \in \mathrm{V}^p_0(\Omega;\mathbb{R}^3)$. *Then* $\mathrm{e}(u)$ *belongs to* $\mathrm{W}^{1,q}(0,T;\mathrm{L}^p(\Omega))$ *and* $\theta \mapsto \mathrm{e}(u)$ *maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $\mathrm{W}^{1,q}(0,T;\mathrm{L}^p(\Omega))$.

Proof. The idea of the proof is to interpret (4.1a) as an ODE for u in an appropriate Banach space. More precisely, let $\mathcal{F}_p \overset{\text{def}}{=} \mathrm{L}^2(\Omega;\mathbb{R}^3)\times\mathrm{L}^p(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})$ endowed with the norm

$$\forall\varphi=(\varphi_1,\varphi_2)\in\mathcal{F}_p:\ \|\varphi\|_{\mathcal{F}_p}\overset{\text{def}}{=}\|(\varphi_1,\varphi_2)\|_{\mathcal{F}_p}=\|\varphi_1\|_{\mathrm{L}^2(\Omega)}+\|\varphi_2\|_{\mathrm{L}^p(\Omega)}.$$

It follows that $\mathcal{F}_p$ is a Banach space. Let us introduce now the mapping $\mathcal{A}_{\mathbb{E}}$

$$\begin{aligned}\mathcal{A}_{\mathbb{E}}:\ &\mathrm{V}^p_0(\Omega;\mathbb{R}^3)\to\mathcal{F}_p,\\ &\qquad\qquad u\mapsto\varphi\overset{\text{def}}{=}(0,\mathbb{E}\mathrm{e}(u)).\end{aligned}$$

Since $\mathbb{E}\in\mathrm{L}^\infty(\Omega)$, we infer that $\mathcal{A}_{\mathbb{E}}$ is a linear continuous mapping from $\mathrm{V}^p_0(\Omega)$ to $\mathcal{F}_p$. Besides since $\mathbb{L}$ is a symmetric, positive definite tensor, classical results about PDE in Banach spaces imply that, for all $\varphi=(\varphi_1,\varphi_2)\in\mathcal{F}_p$, there exists a unique $u\in\mathrm{V}^p_0(\Omega;\mathbb{R}^3)$, denoted $u=\Lambda_p(\varphi)$, such that

$$\forall v\in\mathrm{V}^{p^*}(\Omega;\mathbb{R}^3):\ \int_\Omega\mathbb{L}\mathrm{e}(u){:}\mathrm{e}(v)\,\mathrm{d}x=\int_\Omega\varphi_1{\cdot}v\,\mathrm{d}x+\int_\Omega\varphi_2{:}\mathrm{e}(v)\,\mathrm{d}x,$$

where p^* is the conjugate of p, i.e $\frac{1}{p^*}+\frac{1}{p}=1$. Furthermore there exists a real number $C>0$, independent of φ, such that

$$\|u\|_{\mathrm{V}^p(\Omega;\mathbb{R}^3)}\le C\big(\|\varphi_1\|_{\mathrm{L}^2(\Omega)}+\|\varphi_2\|_{\mathrm{L}^p(\Omega)}\big)=C\|\varphi\|_{\mathcal{F}_p},$$

and Λ_p is linear continuous from $\mathcal{F}_p$ to $\mathrm{V}^p_0(\Omega;\mathbb{R}^3)$ (for more details, the reader is referred to [Val88]). It follows that (4.1a) can be rewritten as

$$\dot u=\mathcal{G}_p(\varphi_{z\theta},u),\tag{4.41}$$

with $\varphi_{z\theta}\overset{\text{def}}{=}(\ell,\mathbb{E}z-\alpha\theta\mathrm{I})$ and $\mathcal{G}_p(\varphi_{z\theta},u)\overset{\text{def}}{=}\Lambda_p(\varphi_{z\theta}-\mathcal{A}_{\mathbb{E}}u)=\Lambda_p(\varphi_{z\theta})-\Lambda_p(\mathcal{A}_{\mathbb{E}}u)$.

With the assumption (3.16) and the previous results, we already know that $\varphi_{z\theta}\in\mathrm{L}^q(0,T;\mathcal{F}_p)$ and we can apply classical results for ODE in Banach spaces to conclude that $u\in\mathrm{W}^{1,q}(0,T;\mathrm{V}^p_0(\Omega;\mathbb{R}^3))$, the reader is referred to [Car90] for more details.

We can also obtain estimates for u and $\dot u$ in $\mathrm{V}^p_0(\Omega;\mathbb{R}^3)$. To this aim, we introduce the following notations: $C_{\Lambda_p}\overset{\text{def}}{=}\|\Lambda_p\|_{\mathcal{L}(\mathcal{F}_p,\mathrm{V}^p_0(\Omega))}$ and $C_{\mathcal{A}_{\mathbb{E}}}\overset{\text{def}}{=}\|\mathcal{A}_{\mathbb{E}}\|_{\mathcal{L}(\mathrm{V}^p_0(\Omega),\mathcal{F}_p)}$. Then, we observe that (4.41) gives

$$\|\dot u(\cdot,t)\|_{\mathrm{V}^p(\Omega)}\le C_{\Lambda_p}\big(\|\varphi_{z\theta}(\cdot,t)-\mathcal{A}_{\mathbb{E}}u^0\|_{\mathcal{F}_p}+C_{\mathcal{A}_{\mathbb{E}}}\|u(\cdot,t)-u^0\|_{\mathrm{V}^p(\Omega)}\big)\ \text{a.e. } t\in[0,T].\tag{4.42}$$

Let us turn now to the term $\|u(\cdot,t)-u^0\|_{\mathrm{V}^p(\Omega)}$. It is clear that

$$\begin{aligned}&\|u(\cdot,t)-u^0\|_{\mathrm{V}^p(\Omega)}\le\int_0^t\|\mathcal{G}_p(\varphi_{z\theta}(\cdot,s),u(\cdot,s))\|_{\mathrm{V}^p(\Omega)}\,\mathrm{d}s\\ &\le C_{\Lambda_p}\int_0^t\|\varphi_{z\theta}(\cdot,s)-\mathcal{A}_{\mathbb{E}}u^0\|_{\mathcal{F}_p}\,\mathrm{d}s+C_{\Lambda_p}C_{\mathcal{A}_{\mathbb{E}}}\int_0^t\|u(\cdot,s)-u^0\|_{\mathrm{V}^p(\Omega)}\,\mathrm{d}s.\end{aligned}$$

Therefore, we may infer from Grönwall's lemma and the continuous embedding $\mathrm{H}^1(\Omega)\hookrightarrow\mathrm{L}^p(\Omega)$ that there exists a generic constant $C_1>0$ such that

$$\begin{aligned}&\|u(\cdot,t)-u^0\|_{\mathrm{V}^p(\Omega)}\le\mathrm{e}^{C_{\Lambda_p}C_{\mathcal{A}_{\mathbb{E}}}t}C_{\Lambda_p}\int_0^t\|\varphi_{z\theta}(s)-\mathcal{A}_{\mathbb{E}}u^0\|_{\mathcal{F}_p}\,\mathrm{d}s\\ &\le\mathrm{e}^{C_{\Lambda_p}C_{\mathcal{A}_{\mathbb{E}}}t}C_{\Lambda_p}C_1\int_0^t\big(\|\ell(s)\|_{\mathrm{L}^2(\Omega)}+\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}\|z(\cdot,s)\|_{\mathrm{H}^1(\Omega)}\\ &+\alpha\|\theta(\cdot,s)\|_{\mathrm{L}^p(\Omega)}+C_{\mathcal{A}_{\mathbb{E}}}\|u^0\|_{\mathrm{V}^p(\Omega)}\big)\,\mathrm{d}s.\end{aligned}\tag{4.43}$$

We insert (4.43) in (4.42), we find that there exists $C_2 > 0$ such that

$$
\begin{aligned}
&\|\dot u\|_{\mathrm{L}^q(0,T;\mathrm{V}^p(\Omega))}\\
&\le C_2\big(\|\ell\|_{\mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))}+\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}\|z\|_{\mathrm{L}^q(0,T;\mathrm{H}^1(\Omega))}+\alpha\|\theta\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))}+\|u^0\|_{\mathrm{V}^p(\Omega)}\big).
\end{aligned}
$$

Recalling Lemma 4.3 we infer that the image of any bounded set of $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ by the mappings $\widetilde\vartheta \mapsto \zeta(\widetilde\vartheta) = \theta \mapsto \mathrm{e}(u)$ and $\widetilde\vartheta \mapsto \zeta(\widetilde\vartheta) = \theta \mapsto \mathrm{e}(\dot u)$ are still bounded sets in $\mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega))$ and $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$, respectively. ☐

Let us conclude this section with some regularity results for z and $\dot z$.

Lemma 4.5 *Assume that* (3.10)*,* (3.11)*,* (3.13b)*,* (3.14) *and* (3.16) *hold. Assume moreover that* $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *and* $u^0 \in \mathrm{V}^p_0(\Omega;\mathbb{R}^3)$*. Then* $\dot z$ *and* Δz *belong to* $\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))\cap\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega))$ *and* $z \in \mathrm{C}^0([0,T],\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$*. Moreover* $\theta \mapsto (\dot z, \Delta z, z)$ *maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $(\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)))^2 \times (\mathrm{C}^0([0,T];\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)))$*.*

Proof. Let us rewrite again (4.1b) as

$$
\dot z - \nu\mathbb{M}^{-1}\Delta z = \mathbb{M}^{-1} f^z. \tag{4.44}
$$

with $f^z \overset{\text{def}}{=} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\mathrm{dev}}}(\mathbb{E}(\mathrm{e}(u){-}z)) - \mathrm{D}_z H_1(z) - \theta\mathrm{D}_z H_2(z) - \psi$ and $\psi \overset{\text{def}}{=} \mathrm{Proj}_{\mathbb{R}^{3\times3}_{\mathrm{dev}}}(w) - \mathbb{M}\dot z \in \partial\Psi_{\mathrm{dev}}(\dot z)$ where w has been defined in the proof of Theorem 4.1. With the assumption (3.10b), we have

$$
\forall \psi \in \partial\Psi_{\mathrm{dev}}(\dot z): \ |\psi| \le C^\Psi \ \text{ a.e. } (x,t) \in \Omega \times (0,T).
$$

Then Lemma 4.4 enables us to infer that $\mathrm{Proj}_{\mathbb{R}^{3\times3}_{\mathrm{dev}}}(\mathbb{E}\mathrm{e}(u)) - \psi$ remains bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega))$. Furthermore, we know with Lemma 4.3 that z is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$, so, using (3.12), we infer that $\mathrm{D}_z H_i(z)$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega))$ for $i = 1,2$. Then it follows that $\mathbb{M}^{-1}f^z$ is bounded in $\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega))$ if θ belongs to a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$. We may deduce from the maximal regularity result for parabolic systems that z is bounded in $\mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$ and $\dot z$ is bounded in $\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega))$ (see [Dor93, HiR08, PrS01]).

Since $\mathrm{H}^2(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ with continuous embedding, z is bounded in $\mathrm{L}^q(0,T;\mathrm{L}^\infty(\Omega))$ and thus $\theta\mathrm{D}_z H_2(z)$ is bounded in $\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))$. We may deduce that f^z is bounded in $\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))$ and, the maximal regularity result for parabolic systems allows us to infer that $\dot z$ and Δz belong to a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)) \cap \mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega))$ and z belongs to a bounded subset of $\mathrm{C}^0([0,T];\mathrm{X}_{q,p}(\Omega))$ whenever θ belongs to a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$. ☐

5 Existence and regularity results for the enthalpy equation

In this section existence and uniqueness results for the enthalpy equation are recalled and some regularity results are obtained. More precisely, let us consider the enthalpy equation (P_ϑ):

$$
\dot\vartheta - \mathrm{div}(\widetilde\kappa^c\nabla\vartheta) = f \tag{5.1}
$$

together with initial conditions

$$
\vartheta(0) = \vartheta^0, \tag{5.2}
$$

and boundary conditions

$$\widetilde{\kappa}^c \nabla\vartheta\cdot\eta_{|\partial\Omega} = 0. \tag{5.3}$$

We assume that the initial enthalpy ϑ^0 belongs to $\mathrm{L}^2(\Omega)$ and f belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Furthermore, we assume that $\widetilde{\kappa}^c \in \mathrm{L}^\infty(\mathcal{Q}_T;\mathbb{R}^{3\times3}_{\mathsf{sym}})$ and satisfies

$$\exists c^{\kappa^c} > 0\ \forall v \in \mathbb{R}^3:\ \widetilde{\kappa}^c(x,t)v\cdot v \geq c^{\kappa^c}|v|^2 \text{ a.e. } (x,t)\in\mathcal{Q}_T, \tag{5.4a}$$

$$\exists C^{\kappa^c} > 0:\ |\widetilde{\kappa}^c(x,t)| \leq C^{\kappa^c} \text{ a.e. } (x,t)\in\mathcal{Q}_T. \tag{5.4b}$$

The weak formulation of the problem is given by

$$\begin{cases} \text{Find } \vartheta:[0,T]\to\mathrm{H}^1(\Omega) \text{ such that } \vartheta(0)=\vartheta^0 \text{ and for all } \xi\in\mathrm{H}^1(\Omega), \\ \displaystyle\int_\Omega \dot\vartheta\xi\,\mathrm{d}x + \int_\Omega \widetilde{\kappa}^c\nabla\vartheta\cdot\nabla\xi\,\mathrm{d}x = \int_\Omega f\xi\,\mathrm{d}x \text{ in the sense of distributions.}\end{cases} \tag{5.5}$$

Theorem 5.1 (Existence and uniqueness for $(\mathbf{P}_\vartheta)$) *Under the previous assumptions,* (5.1)–(5.3) *possesses a unique solution* $\vartheta \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ *with* $\dot\vartheta \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))$. *Moreover we have*

$$\forall\tau\in[0,T]:\ \|\vartheta(\tau)\|^2_{\mathrm{L}^2(\Omega)} + 2c^{\kappa^c}\int_0^\tau \|\nabla\vartheta(t)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \leq \mathrm{e}^\tau\big(\|\vartheta^0\|^2_{\mathrm{L}^2(\Omega)} + \|f\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}\big).$$

Proof. The proof of existence and uniqueness of a solution is quite classical and can be found in [Bre83, RaT83]. The estimate is straightforward and its verification is left to the reader. ☐

Let us introduce the following functional space

$$\mathcal{W} \overset{\mathsf{def}}{=} \big\{\vartheta\in\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)):\ \dot\vartheta\in\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)')\big\},$$

endowed with the norm

$$\forall\vartheta\in\mathcal{W}:\ \|\vartheta\|_{\mathcal{W}} \overset{\mathsf{def}}{=} \|\vartheta\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\vartheta\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|\dot\vartheta\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)')}.$$

Due to [Sim87], we know that $\mathcal{W}$ is compactly embedded in $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$. Note that the previous estimate implies that there exists a generic constant $C>0$ such that the solution of problem (P_ϑ) satisfies

$$\|\vartheta\|_{\mathcal{W}} \leq C\big(\|\vartheta^0\|_{\mathrm{L}^2(\Omega)} + \|f\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}\big).$$

6 Local existence result

This section is dedicated to the proof of a local existence result for (3.6)–(3.8) by using a fixed-point argument. To this aim, for any given $\widetilde{\vartheta} \in \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$, we consider $\widetilde{\kappa}^c \overset{\mathsf{def}}{=} \kappa^c(\mathrm{e}(u),z,\theta)$ and $f = f^{\widetilde{\vartheta}} \overset{\mathsf{def}}{=} \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot u)) + \mathrm{D}_z H_2(z){:}\dot z) + \Psi(\dot z) + \mathbb{M}\dot z{:}\dot z$ in (P_ϑ), where (u,z) are the solutions of (P_{uz}) with $\theta = \zeta(\widetilde{\vartheta})$. With the results obtained in the Section 4, we already know that $f^{\widetilde{\vartheta}}$ belongs to $\mathrm{L}^{q/4}(0,T;\mathrm{L}^{p/2}(\Omega))$. Since $p\geq4$ and $q>8$, we infer that $f^{\widetilde{\vartheta}}$ belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and we can define $\vartheta\in\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))\cap\mathcal{W}$ as the unique solution of (P_ϑ). Thus we can introduce the fixed point mapping $\phi:\widetilde{\vartheta}\mapsto\vartheta$ from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$.

Proposition 6.1 *The mapping ϕ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$.*

Proof. Let $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ be a converging sequence of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ and let $\widetilde{\vartheta}_*$ be its limit. We denote by $\vartheta_n \overset{\text{def}}{=} \phi(\widetilde{\vartheta}_n)$ for all $n \geq 0$ and $\vartheta_* \overset{\text{def}}{=} \phi(\widetilde{\vartheta}_*)$. Since $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ is a bounded family of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we infer from the previous results that $(\vartheta_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathcal{W}$. We may deduce that $(\vartheta_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ (see [Sim87]) and there exists a subsequence, still denoted $(\vartheta_n)_{n\in\mathbb{N}}$, such that

$$\begin{aligned} &\vartheta_n \rightharpoonup \vartheta \text{ in } \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \text{ weak,} \\ &\vartheta_n \to \vartheta \text{ in } \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega)). \end{aligned}$$

Let us define $\mathcal{V}_T \overset{\text{def}}{=} \big\{ w \in \mathrm{C}^\infty([0,T]) : \ w(T) = 0 \big\}$. Hence we observe that for all $n \geq 0$, we have

$$\begin{aligned} &\forall \xi \in \mathrm{H}^1(\Omega)\ \forall w \in \mathcal{V}_T : \\ &-\int_{\mathcal{Q}_T} \vartheta_n(x,t)\xi(x)\dot{w}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_T} \widetilde{\kappa}_n^c \nabla\vartheta_n(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \vartheta^0(x)\xi(x)w(0)\,\mathrm{d}x, \end{aligned} \tag{6.1}$$

with $\widetilde{\kappa}_n^c \overset{\text{def}}{=} \kappa^c\big(\mathrm{e}(u_n), z_n, \theta_n\big)$ and (u_n, z_n) solutions of (P_{uz}) with $\theta_n \overset{\text{def}}{=} \zeta(\widetilde{\vartheta}_n)$. Since $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ converges to $\widetilde{\vartheta}_*$ in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we infer from Lemma 4.3 that $(u_n, z_n)_{n\in\mathbb{N}}$ converges to the solution (u_*, z_*) of (P_{uz}) with $\theta_* = \zeta(\widetilde{\vartheta}_*)$ in $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega){\times}\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega){\times}\mathrm{H}^1(\Omega))$. Let us recall that the mapping $\phi_1 : \widetilde{\vartheta} \mapsto \theta = \zeta(\widetilde{\vartheta})$ is Lipschitz continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, which implies, possibly after extracting another subsequence, that

$$\theta_n, u_n, z_n \to \theta_*, u_*, z_* \text{ a.e. } (x,t) \in \mathcal{Q}_T.$$

Note that the continuity of the mapping κ^c gives

$$\widetilde{\kappa}_n^c = \kappa^c(\mathrm{e}(u_n), z_n, \theta_n) \to \widetilde{\kappa}_*^c = \kappa^c(\mathrm{e}(u_*), z_*, \theta_*) \text{ a.e. } (x,t) \in \mathcal{Q}_T,$$

and due to the boundedness assumption on κ^c, we obtain with the Lebesgue's theorem that

$$\widetilde{\kappa}_n^c \nabla\xi \to \widetilde{\kappa}_*^c \nabla\xi \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)).$$

Therefore it is possible to pass to the limit in all the terms of the left hand side of (6.1) to get

$$\begin{aligned} \forall \xi \in \mathrm{H}^1(\Omega)\ \forall w \in \mathcal{V}_T :\ &-\int_{\mathcal{Q}_T} \vartheta(x,t)\xi(x)\dot{w}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_T} \widetilde{\kappa}_*^c \nabla\vartheta(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{n\to+\infty} \int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \vartheta^0(x)\xi(x)w(0)\,\mathrm{d}x. \end{aligned}$$

Recalling that $p \in [4,6]$ and $q > 8$, we infer that $\widetilde{\vartheta} \mapsto f^{\widetilde{\vartheta}}$ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{r_1}(0,T;\mathrm{L}^{r_2}(\Omega))$ with $\frac{1}{r_1} = \frac{3}{q} + \frac{1}{2}$ and $\frac{1}{r_2} = \frac{1}{p} + \frac{1}{2}$. Indeed, for any $\widetilde{\vartheta}_i$ in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, let (u_i, z_i) be the solution of (P_{uz}) with $\theta_i = \zeta(\widetilde{\vartheta}_i)$, $i = 1,2$, we find

$$\begin{aligned} f^{\widetilde{\vartheta}_1} - f^{\widetilde{\vartheta}_2} &= \mathbb{L}\mathrm{e}(\dot{u}_1{+}\dot{u}_2){:}\mathrm{e}(\dot{u}_1{-}\dot{u}_2) + (\theta_1{-}\theta_2)(\alpha\mathrm{tr}(\mathrm{e}(\dot{u}_1)){+}\mathrm{D}_z H_2(z_1){:}\dot{z}_1) \\ &+ \theta_2(\alpha\mathrm{tr}(\mathrm{e}(\dot{u}_1{-}\dot{u}_2)){+}\mathrm{D}_z H_2(z_1){:}\dot{z}_1{-}\mathrm{D}_z H_2(z_2){:}\dot{z}_2) + \Psi(\dot{z}_1) - \Psi(\dot{z}_2) + \mathbb{M}(\dot{z}_1{+}\dot{z}_2){:}(\dot{z}_1{-}\dot{z}_2). \end{aligned}$$

On the other hand, (3.10c) and (3.10b) imply

$$|\Psi(\dot z_1)-\Psi(\dot z_2)| \le C^{\Psi}|\dot z_1-\dot z_2|,$$

and (3.12) and (3.11b) give

$$\begin{aligned}|\mathrm{D}_z H_2(z_1):\dot z_1-\mathrm{D}_z H_2(z_2):\dot z_2| &\le |\mathrm{D}_z H_2(z_1)||\dot z_1-\dot z_2| + |\mathrm{D}_z H_2(z_1)-\mathrm{D}_z H_2(z_2)||\dot z_2|\\ &\le C_z^{H_2}(1+|z_1|)|\dot z_1-\dot z_2| + C_{zz}^{H_2}|z_1-z_2||\dot z_2|.\end{aligned}$$

The boundedness and the continuity properties proved in Lemma 4.4 and Lemma 4.3, respectively, allow us to deduce the desired result. Therefore, we may infer that

$$\forall \xi\in \mathrm{H}^1(\Omega)\ \forall w\in\mathcal{V}_T:\ \lim_{n\to+\infty}\int_{\mathcal{Q}_T} f^{\widetilde\vartheta_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_T} f^{\widetilde\vartheta_*}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t.$$

We conclude that ϑ is solution of problem (P_ϑ) with the data $\widetilde\kappa_*^c$ and $f^{\widetilde\vartheta_*}$. Moreover by uniqueness of the solution, it follows that $\vartheta=\vartheta_*$ and the whole sequence $(\vartheta_n)_{n\in\mathbb{N}}$ converges to $\vartheta_*=\zeta(\widetilde\vartheta_*)$. $\square$

We establish now that the mapping ϕ fulfills the other assumptions of the Schauder's fixed point theorem. To this aim, we introduce some notations: let $R^0, R^\vartheta>0$ be any given positive real numbers such that $\max\big(\|u^0\|_{\mathrm{V}^p(\Omega)},\|z^0\|_{\mathrm{X}_{q,p}(\Omega)}\big)\le R^0$ and $\|\widetilde\vartheta\|_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))}\le R^\vartheta$. Clearly, we have

$$\begin{aligned}\|\zeta(\widetilde\vartheta)\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))} = \|\theta\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))} &\le \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta_1}}|\Omega|^{\frac{\beta_1\bar p-p}{\beta_1 p\bar p}}\|\widetilde\vartheta\|^{\frac{1}{\beta_1}}_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))}\\ &\le R^\theta \overset{\text{def}}{=} \big(\tfrac{\beta_1}{c^c}R^\vartheta\big)^{\frac{1}{\beta_1}}|\Omega|^{\frac{\beta_1\bar p-p}{\beta_1 p\bar p}}.\end{aligned}$$

Therefore once again the results of Section 4 are used, which imply that there exists a constant $R^f \overset{\text{def}}{=} R^f(R^0,R^\theta,\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))})>0$, depending only on R^0, R^θ and $\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}$, such that

$$\|f^{\widetilde\vartheta}\|_{\mathrm{L}^{q/4}(0,T;\mathrm{L}^{p/2}(\Omega))}\le R^f(R^0,R^\theta,\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}).$$

The results of Section 5 imply that there exists a generic constant $C>0$ such that

$$\begin{aligned}\|\vartheta\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} &\le C(\|f^{\widetilde\vartheta}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}+\|\vartheta^0\|_{\mathrm{L}^2(\Omega)})\\ &\le C\big(|\Omega|^{\frac{p-4}{2p}}T^{\frac{q-8}{2q}}\|f^{\widetilde\vartheta}\|_{\mathrm{L}^{q/4}(0,T;\mathrm{L}^{p/2}(\Omega))}+\|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big)\\ &\le C\big(|\Omega|^{\frac{p-4}{2p}}T^{\frac{q-8}{2q}}R^f(R^0,R^\theta,\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))})+\|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big).\end{aligned}$$

Now let $0<\tau\le T$ and let us introduce the following functional space

$$\mathcal{W}_\tau \overset{\text{def}}{=} \big\{\vartheta\in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega))\cap \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega)):\ \dot\vartheta\in \mathrm{L}^2(0,\tau;(\mathrm{H}^1(\Omega))')\big\}.$$

For any $\widetilde\vartheta\in \mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$, we define its extension $\widetilde\vartheta_{\mathsf{ext}}$ by $\widetilde\vartheta_{\mathsf{ext}}=\widetilde\vartheta$ on $[0,\tau]$ and $\widetilde\vartheta_{\mathsf{ext}}=0$ on $(\tau,T]$. It is clear that $\widetilde\vartheta_{\mathsf{ext}}\in \mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ and the mapping $\widetilde\vartheta\mapsto\widetilde\vartheta_{\mathsf{ext}}$ is a contraction from $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$ into $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$. For any $\widetilde\vartheta\in \mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$, we define $\phi_\tau(\widetilde\vartheta)$ as the restriction on $[0,\tau]$ of $\phi(\widetilde\vartheta_{\mathsf{ext}})$. We infer immediately from Proposition 6.1 that ϕ_τ is continuous from $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$. Furthermore, for any $\widetilde\vartheta\in \mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$, we have

$$\begin{aligned}\|\phi_\tau(\widetilde\vartheta)\|_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))} = \|\phi_\tau(\widetilde\vartheta)\|_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^2(\Omega))} &= \Big(\int_0^\tau \|\phi(\widetilde\vartheta_{\mathsf{ext}}(\cdot,t))\|^{\bar q}_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac{1}{\bar q}}\\ &\le \tau^{\frac{1}{\bar q}}\|\phi(\widetilde\vartheta_{\mathsf{ext}})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))},\end{aligned}$$

and the previous estimates allow us to show that, for any $R^\vartheta > 0$, there exists $\tau \in (0,T]$ such that ϕ_τ maps the closed ball $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ into itself. Note that the image of $\bar{B}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ by ϕ is a bounded subset of $\mathcal{W}$ and thus it is relatively compact in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$. It follows that the image of $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ by ϕ_τ is also relatively compact in $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$. Consequently, we may conclude that the problem (3.6)–(3.8) possesses a local solution (u,z,ϑ) defined on $[0,\tau]$ such that $u \in \mathrm{W}^{1,q}(0,\tau;\mathrm{V}_0^p(\Omega))$, $z \in \mathrm{L}^\infty(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{H}^1(0,\tau;\mathrm{L}^2(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{H}^2(\Omega))$, $\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{L}^{p/2}(\Omega))$ and $\vartheta \in \mathcal{W}_\tau$.

We have to go back to the problem (3.1)–(3.3). First we observe that g and ζ define a C^1-diffeomorphism from $(0,\infty)$ to $(0,\infty)$ and any solution of (3.6)–(3.8) provides a solution of (3.1)–(3.3) as soon as the enthalpy ϑ remains strictly positive. So we assume now that the initial enthalpy is strictly positive almost everywhere on Ω, i.e., there exists $\bar{\vartheta} > 0$ such that

$$g(\theta^0(x)) = \vartheta^0(x) \geq \bar{\vartheta} > 0 \;\text{ a.e. }\; x \in \Omega. \tag{6.2}$$

Therefore it is possible to use the Stampacchia's truncation method and to prove a local existence result for the problem (3.1)–(3.3).

Theorem 6.2 (Local existence result) *Assume that* (3.10), (3.11), (3.13), (3.14), (3.16) *and* (3.17) *hold. Then, for any initial data $u^0 \in \mathrm{V}_0^p(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ and $\vartheta^0 \in \mathrm{L}^2(\Omega)$ satisfying* (6.2)*, there exists $\tau \in (0,T]$ such that the problem* (3.1)–(3.3) *admits a solution on $[0,\tau]$.*

Proof. Let (u,z,ϑ) be a solution of (3.6)–(3.8) on $\big[0,\tau\big]$. We prove now that

$$\vartheta(x,t) > 0 \;\text{ a.e. }\; (x,t) \in \mathcal{Q}_\tau.$$

To do so, we introduce some notations: let $C \overset{\text{def}}{=} \frac{(3\alpha)^2}{2c^{\mathbb{L}}} + \frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}$ and let $\varphi : [0,\tau] \to \mathbb{R}$ such that

$$\forall t \in [0,\tau] : \; \varphi(t) \overset{\text{def}}{=} \bar{\vartheta} \mathrm{e}^{-\frac{\beta_1}{c^c}\int_0^t (C + \frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}\|z(\cdot,s)\|^2_{\mathrm{L}^\infty(\Omega)})\,\mathrm{d}s}. \tag{6.3}$$

We use the classical Stampacchia's truncation method. Let us define $G \in \mathrm{C}^1(\mathbb{R})$ satisfying

(i) $\forall \sigma \in \mathbb{R}\; \exists C^G > 0 : \; |G'(\sigma)| \leq C^G$,

(ii) G is strictly increasing on $(0,\infty)$,

(iii) $\forall \sigma \leq 0 : \; G(\sigma) = 0$.

Let us define also $H(\sigma) \overset{\text{def}}{=} \int_0^\sigma G(s)\,\mathrm{d}s$ for all $\sigma \in \mathbb{R}$, $\vartheta_1 \overset{\text{def}}{=} -\vartheta + \varphi$ and $h(t) \overset{\text{def}}{=} \int_\Omega H(\vartheta_1)\,\mathrm{d}x$. Clearly, we have $H \in \mathrm{C}^2(\mathbb{R};\mathbb{R})$ and $H(\sigma) > 0$ for all $\sigma > 0$. Furthermore, $\vartheta_1(0) = -\vartheta^0 + \bar{\vartheta} \leq 0$ almost everywhere on Ω implies that $h(0) = 0$. Since $\vartheta \in \mathcal{W}_\tau$ and $\varphi \in \mathrm{H}^1(0,\tau;\mathbb{R})$, we infer that h is absolutely continuous and

$$\begin{aligned}
\dot{h}(t) &= \int_\Omega G(\vartheta_1)\dot{\vartheta}_1\,\mathrm{d}x \\
&= -\int_\Omega G(\vartheta_1)\big(\mathrm{div}(\kappa^c\nabla\vartheta) + \mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u}) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot{u})) + \mathrm{D}_z H_2(z):\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}:\dot{z} - \dot{\varphi}\big)\,\mathrm{d}x \\
&= -\int_\Omega G'(\vartheta_1)\kappa^c\nabla\vartheta_1:\nabla\vartheta_1\,\mathrm{d}x \\
&\quad -\int_\Omega G(\vartheta_1)\big(\mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u}) + \theta(\alpha\mathrm{tr}(\mathrm{e}(\dot{u})) + \mathrm{D}_z H_2(z):\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}:\dot{z} - \dot{\varphi}\big)\,\mathrm{d}x,
\end{aligned}$$

for almost every $t \in [0,\tau]$. It follows from (3.12), (3.14) and Cauchy-Schwarz's inequality that

$$\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})) \geq c^{\mathbb{L}}|\mathrm{e}(\dot{u})|^2 - 3\alpha|\theta||\mathrm{e}(\dot{u})| \geq \tfrac{c^{\mathbb{L}}}{2}|\mathrm{e}(\dot{u})|^2 - \tfrac{(3\alpha)^2|\theta|^2}{2c^{\mathbb{L}}},$$

and

$$\begin{aligned}\mathbb{M}\dot{z}{:}\dot{z} + \theta\mathrm{D}_z H_2(z){:}\dot{z} &\geq c^{\mathbb{M}}|\dot{z}|^2 - |\theta||\mathrm{D}_z H_2(z){:}\dot{z}| \\ &\geq c^{\mathbb{M}}|\dot{z}|^2 - C_z^{H_2}|\theta|(1{+}|z|)|\dot{z}| \geq \tfrac{c^{\mathbb{M}}}{2}|\dot{z}|^2 - \tfrac{(C_z^{H_2})^2|\theta|^2}{c^{\mathbb{M}}}(1{+}|z|^2).\end{aligned}$$

Since $G'(\vartheta_1) \geq 0$ and $G(\vartheta_1) \geq 0$ almost everywhere and (3.10b) and (3.17d) hold, we get

$$\dot{h}(t) \leq \int_\Omega G(\vartheta_1)\big(|\theta|^2\big(C{+}\tfrac{(C_z^{H_2})^2}{c^{\mathbb{M}}}|z|^2\big){+}\dot{\varphi}\big)\,\mathrm{d}x \ \text{ a.e. } \ t \in [0,\tau].$$

But $\theta = \zeta(\vartheta)$, and reminding that $\beta_1 \geq 2$, we have with (3.18) that

$$|\theta| = |\zeta(\vartheta)| \leq \sqrt{\tfrac{\beta_1}{c^c}\vartheta^+{+}1} - 1 \leq \sqrt{\tfrac{\beta_1}{c^c}\vartheta^+} \ \text{ a.e. } \ (x,t) \in \mathcal{Q}_\tau.$$

On the other hand, $G(\vartheta_1)$ vanishes whenever $\vartheta \geq \varphi$, it follows that

$$\dot{h}(t) \leq \int_\Omega G(\vartheta_1)\big(\tfrac{\beta_1}{c^c}\varphi\big(C{+}\tfrac{(C_z^{H_2})^2}{c^{\mathbb{M}}}|z|^2\big){+}\dot{\varphi}\big)\,\mathrm{d}x \leq 0 \ \text{ a.e. } \ t \in [0,\tau].$$

We may deduce that $h(t) \leq h(0) = 0$ for all $t \in [0,\tau]$. Then we infer that

$$H(\vartheta_1) = 0 \ \text{ a.e. } \ (x,t) \in \Omega \times (0,\tau),$$

which implies that

$$\vartheta_1 = -\vartheta + \varphi \leq 0 \ \text{ a.e. } \ (x,t) \in \Omega \times (0,\tau).$$

This concludes the proof. □

7 Global existence result

We begin this section with some a priori estimates for the solutions of the problem (3.6)–(3.8). As usual, the result relies on an energy balance combined with Grönwall's lemma. Then the global existence result is proved by using a contradiction argument together with the results obtained in the previous sections.

Proposition 7.1 (Global energy estimate) *Assume that* (3.10), (3.11), (3.13), (3.14), (3.16) *and* (3.17) *hold. Assume moreover that* $u^0 \in \mathrm{V}_0^p(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$, $\vartheta^0 \in \mathrm{L}^2(\Omega)$ *such that* (6.2) *holds. Then, there exists a constant* $\widetilde{C} > 0$, *depending only on* $\|u^0\|_{\mathrm{H}^1(\Omega)}$, $\|z^0\|_{\mathrm{H}^1(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}$ *and the data such that for any solution* (u,z,ϑ) *of problem* (3.6)–(3.8) *defined on* $[0,\tau]$, $\tau \in (0,T]$, *we have*

$$\forall \widetilde{\tau} \in [0,\tau]: \ \|u(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|\vartheta(\cdot,\widetilde{\tau})\|_{\mathrm{L}^1(\Omega)} \leq \widetilde{C}.$$

Proof. On the one hand, we multiply (3.6a) by $\dot{u}$ and we integrate this expression over $\mathcal{Q}_{\widetilde{\tau}}$, with $\widetilde{\tau} \in [0,\tau]$, to get

$$\int_{\mathcal{Q}_{\widetilde{\tau}}} (\mathbb{E}(\mathrm{e}(u){-}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u})){:}\mathrm{e}(\dot{u})\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_{\widetilde{\tau}}} \ell{\cdot}\dot{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.1}$$

On the other hand, by using the definition of the subdifferential $\partial\Psi(\dot z)$, we deduce from (3.6b) that

$$\int_{\mathcal{Q}_{\widetilde\tau}}(\mathbb{M}\dot z-\mathbb{E}(\mathrm{e}(u)-z)+\mathrm{D}_zH_1(z)+\theta\mathrm{D}_zH_2(z)-\nu\Delta z):\dot z\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}_{\widetilde\tau}}\Psi(\dot z)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{7.2}$$

Adding (7.1) and (7.2), we obtain

$$\begin{aligned}&\tfrac12\int_\Omega\mathbb{E}(\mathrm{e}(u(\cdot,\widetilde\tau))-z(\cdot,\widetilde\tau)):(\mathrm{e}(u(\cdot,\widetilde\tau))-z(\cdot,\widetilde\tau))\,\mathrm{d}x+\tfrac{\nu}{2}\|\nabla z(\cdot,\widetilde\tau)\|^2_{\mathrm{L}^2(\Omega)}\\&+\int_{\mathcal{Q}_{\widetilde\tau}}\mathbb{M}\dot z:\dot z\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}_{\widetilde\tau}}\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega H_1(z(\cdot,\widetilde\tau))\,\mathrm{d}x\\&+\int_{\mathcal{Q}_{\widetilde\tau}}\theta(\alpha\mathrm{tr}(\mathrm{e}(\dot u))+\mathrm{D}_zH_2(z):\dot z)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}_{\widetilde\tau}}\Psi(\dot z)\,\mathrm{d}x\,\mathrm{d}t=C_0^{u,z}+\int_{\mathcal{Q}_{\widetilde\tau}}\ell\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.3}$$

where $C_0^{u,z}\overset{\text{def}}{=}\frac12\int_\Omega\mathbb{E}(\mathrm{e}(u^0)-z^0):(\mathrm{e}(u^0)-z^0)\,\mathrm{d}x+\frac{\nu}{2}\|\nabla z^0\|^2_{\mathrm{L}^2(\Omega)}+\int_\Omega H_1(z^0)\,\mathrm{d}x$. We integrate (3.6c) over $\mathcal{Q}_{\widetilde\tau}$, by taking into account the boundary conditions (3.7), we find

$$\begin{aligned}\int_\Omega\vartheta(\cdot,\widetilde\tau)\,\mathrm{d}x&=\int_\Omega\vartheta^0\,\mathrm{d}x+\int_{\mathcal{Q}_{\widetilde\tau}}\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}_{\widetilde\tau}}\mathbb{M}\dot z:\dot z\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\mathcal{Q}_{\widetilde\tau}}\theta(\alpha\mathrm{tr}(\mathrm{e}(\dot u))+\mathrm{D}_zH_2(z):\dot z)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{Q}_{\widetilde\tau}}\Psi(\dot z)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We add this last equality to (7.3), and thanks to (3.11a), we obtain

$$\begin{aligned}&\tfrac12\int_\Omega\mathbb{E}(\mathrm{e}(u(\cdot,\widetilde\tau))-z(\cdot,\widetilde\tau)):(\mathrm{e}(u(\cdot,\widetilde\tau))-z(\cdot,\widetilde\tau))\,\mathrm{d}x+\tfrac{\nu}{2}\|\nabla z(\cdot,\widetilde\tau)\|^2_{\mathrm{L}^2(\Omega)}+c^{H_1}\|z(\cdot,\widetilde\tau)\|^2_{\mathrm{L}^2(\Omega)}\\&+\int_\Omega\vartheta(\cdot,\widetilde\tau)\,\mathrm{d}x\le C_0^{u,z}+\int_\Omega\vartheta^0\,\mathrm{d}x+\widetilde c^{H_1}|\Omega|+\int_{\mathcal{Q}_{\widetilde\tau}}\ell\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Clearly there exists $C_1>0$, depending only on $c^{\mathbb{E}}$, ν and c^{H_1} such that

$$\begin{aligned}&C_1\|u(\cdot,\widetilde\tau)\|^2_{\mathrm{H}^1(\Omega)}+C_1\|z(\cdot,\widetilde\tau)\|^2_{\mathrm{H}^1(\Omega)}+\int_\Omega\vartheta(\cdot,\widetilde\tau)\,\mathrm{d}x\\&\le C_0^{u,z}+\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}+\widetilde c^{H_1}|\Omega|+\int_{\mathcal{Q}_{\widetilde\tau}}\ell\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.4}$$

Since $\ell\in\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$, we may integrate by parts the last term of (7.4), we get

$$\begin{aligned}&\tfrac{C_1}{2}\|u(\cdot,\widetilde\tau)\|^2_{\mathrm{H}^1(\Omega)}+C_1\|z(\cdot,\widetilde\tau)\|^2_{\mathrm{H}^1(\Omega)}+\int_\Omega\vartheta(\cdot,\widetilde\tau)\,\mathrm{d}x\le C_0^{u,z}+\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}+c^{H_1}|\Omega|\\&+\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}\|u^0\|_{\mathrm{L}^2(\Omega)}+\tfrac{1}{2C_1}\|\ell\|^2_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}+\tfrac12\|\dot\ell\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))}+\tfrac12\int_0^{\widetilde\tau}\|u\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t,\end{aligned}$$

which allows us to conclude with Grönwall's lemma since $\vartheta(x,t)\ge0$ almost everywhere on $\mathcal{Q}_\tau$. $\square$

Note that (3.18) enable us also to obtain a global estimate for the temperature. More precisely, under the assumptions of Proposition 7.1, we have

$$\|\theta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^{\beta_1}(\Omega))}\le\big(\tfrac{\beta_1}{c^c}\widetilde C\big)^{\frac{1}{\beta_1}}, \tag{7.5}$$

for any solution (u,z,θ) of problem (3.1)–(3.3) defined on $[0,\tau]$ with $\tau\in(0,T]$.

We assume that $\beta_1 \geq 4$. We define

$$\bar{R}^\theta \stackrel{\text{def}}{=} T^{\frac{1}{q}} |\Omega|^{\frac{\beta_1-4}{4\beta_1}} \big(\tfrac{\beta_1}{c^c} \widetilde{C}\big)^{\frac{1}{\beta_1}},$$

and using the notations of Section 6, we define

$$\bar{R}^f \stackrel{\text{def}}{=} R^f(R^0, \bar{R}^\theta, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}), \quad \bar{R}^\vartheta_\infty \stackrel{\text{def}}{=} C\big(T^{\frac{q-8}{2q}} \bar{R}^f + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big), \quad \bar{R}^\vartheta \stackrel{\text{def}}{=} T^{\frac{1}{\bar{q}}} \bar{R}^\vartheta_\infty + 1.$$

Then, the results of Section 6 allow us to infer that there exists $\tau \in (0, T]$ such that ϕ_τ admits a fixed point in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)$. Let us define

$$\bar{\tau} \stackrel{\text{def}}{=} \sup\big\{\tau \in (0,T] :\ \phi_\tau \text{ admits a fixed point in } \bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)\big\} \in (0, T].$$

It is clear that problem (3.6)–(3.8) admits a global solution if and only if $\bar{\tau} = T$. This identity is established below by a contradiction argument. To do so, we assume that $\bar{\tau} \in (0,T)$ and we choose $\epsilon > 0$ such that $\bar{\tau} - \epsilon \in (0, \bar{\tau})$. By definition of $\bar{\tau}$, there exists $\tau \in (\bar{\tau}-\epsilon, \bar{\tau}]$ such that ϕ_τ admits a fixed point $\vartheta = \phi_\tau(\vartheta)$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)$, i.e., the problem (3.6)–(3.8) admits a solution (u, z, ϑ) defined on $[0, \tau]$. We infer from the results of Section 7 that $\|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^1(\Omega))} \leq \widetilde{C}$ and $\|\theta = \zeta(\vartheta)\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \leq \bar{R}^\theta$. Then, the definition of $\bar{R}^\vartheta_\infty$ and the results of Section 5 imply that $\vartheta = \phi_\tau(\vartheta) \in \mathrm{L}^\infty(0, \tau; \mathrm{L}^2(\Omega))$ with $\|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \leq \bar{R}^\vartheta_\infty$.

Now let $\widetilde{\tau} \in (0, T-\bar{\tau}]$ and $\widetilde{R}^\vartheta \stackrel{\text{def}}{=} ((\bar{R}^\vartheta)^{\bar{q}} - \bar{\tau}(\bar{R}^\vartheta_\infty)^{\bar{q}})^{\frac{1}{\bar{q}}} > 0$. For any $\widetilde{\vartheta} \in \bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$, we define $\widetilde{\vartheta}_{\text{ext}}$ as follows $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} \vartheta$ on $[0, \tau]$, $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} \widetilde{\vartheta}$ on $(\tau, \tau+\widetilde{\tau}]$ and $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} 0$ on $(\tau+\widetilde{\tau}, T]$. Clearly, we have

$$\|\widetilde{\vartheta}_{\text{ext}}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^2(\Omega))} = \|\vartheta\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))} + \|\widetilde{\vartheta}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))} \leq \tau(\bar{R}^\vartheta_\infty)^{\bar{q}} + (\widetilde{R}^\vartheta)^{\bar{q}} \leq (\bar{R}^\vartheta)^{\bar{q}},$$

and the mapping $\widetilde{\vartheta} \mapsto \widetilde{\vartheta}_{\text{ext}}$ is a contraction on $\mathrm{L}^{\bar{q}}(\tau, \tau+\widetilde{\tau}; \mathrm{L}^2(\Omega))$. Let $\widetilde{\theta} = \zeta(\widetilde{\vartheta}_{\text{ext}})$. By definition of ζ, we have $\widetilde{\theta} = \zeta(\vartheta) = \theta$ on $[0, \tau]$, $\widetilde{\theta} = \zeta(\widetilde{\vartheta})$ on $(\tau, \tau + \widetilde{\tau}]$ and $\widetilde{\theta} = \zeta(0) = 0$ on $(\tau + \widetilde{\tau}, T]$. Hence $\widetilde{\theta} \in \mathrm{L}^q(0, T; \mathrm{L}^4(\Omega))$ and

$$\begin{aligned}
\|\widetilde{\theta}\|^q_{\mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))} &\leq (\bar{R}^\theta)^q + \int_\tau^{\tau+\widetilde{\tau}} \|\zeta(\widetilde{\vartheta})\|^q_{\mathrm{L}^4(\Omega)}\, \mathrm{d}t \\
&\leq (\bar{R}^\theta)^q + \big(\tfrac{\beta_1}{c^c}\big)^{\frac{q}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1} q} \|\widetilde{\vartheta}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))} \leq (\widetilde{R}^\theta)^q,
\end{aligned}$$

with $\widetilde{R}^\theta \stackrel{\text{def}}{=} \big((\bar{R}^\theta)^q + \big(\frac{\beta_1}{c^c}\big)^{\frac{q}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1} q} (\widetilde{R}^\vartheta)^{\bar{q}}\big)^{\frac{1}{q}}$. By definition of ϕ, we get immediately that the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ on $[0, \tau]$ coincide with $\phi_\tau(\vartheta) = \vartheta$ and we define $\widetilde{\phi}_{\widetilde{\tau}}(\widetilde{\vartheta})$ as the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ to $[\tau, \tau+\widetilde{\tau}]$. Furthermore, with the estimates of Section 6, we have $\phi(\widetilde{\vartheta}_{\text{ext}}) \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega))$ and

$$\|\phi(\widetilde{\vartheta}_{\text{ext}})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \leq C\big(T^{\frac{q-8}{2q}} R^f(R^0, \widetilde{R}^\theta, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}) + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big).$$

It follows that there exists $\widetilde{\tau} \in (0, T-\bar{\tau}]$, independent of τ, such that $\widetilde{\phi}_{\widetilde{\tau}}$ admits a fixed point $\widetilde{\vartheta}$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$. By construction of $\widetilde{\phi}_{\widetilde{\tau}}$, the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ to $[0, \tau+\widetilde{\tau}]$ is also a fixed point of $\phi_{\tau+\widetilde{\tau}}$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)$. But we may choose $\epsilon \in (0, \bar{\tau})$ such that $\tau + \widetilde{\tau} > \bar{\tau} - \epsilon + \widetilde{\tau} > \bar{\tau}$, which gives a contradiction with the definition of $\bar{\tau}$.

Hence we can conclude that $\bar{\tau} = T$. Consequently, we deduce the following theorem:

Theorem 7.2 (Global existence result) *Assume that* (3.10), (3.11), (3.13), (3.14), (3.16) *and* (3.17) *hold. Assume moreover that* $\beta_1 \geq 4$, $u^0 \in \mathrm{V}_0^p(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$, $\vartheta^0 \in \mathrm{L}^2(\Omega)$ *such that* (6.2) *holds. Then the problem* (3.6)–(3.8) *admits a global solution* (u, z, ϑ) *such that* $u \in \mathrm{W}^{1,q}(0,T; \mathrm{V}_0^p(\Omega))$, $z \in \mathrm{L}^\infty(0,T; \mathrm{H}^1(\Omega) \cap \mathrm{X}_{q,p}(\Omega)) \cap \mathrm{H}^1(0,T; \mathrm{L}^2(\Omega))$, $\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0,T; \mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T; \mathrm{L}^{p/2}(\Omega))$ *and* $\vartheta \in \mathcal{W}$. *Moreover* ϑ *remains strictly positive and* $(u, z, \theta = \zeta(\vartheta))$ *is a solution of problem* (3.1)–(3.3) *on* $[0,T]$.

Remark 7.3 *Let us assume furthermore that there exists* $\widetilde{C}_z^{H_2} > 0$ *and* $p_1 \in (0,1)$ *such that*

$$\forall z \in \mathbb{R}^{3\times 3}_{\mathsf{dev}} : \ |\mathrm{D}_z H_2(z)| \leq \widetilde{C}_z^{H_2}(1+|z|^{p_1}).$$

Then we may obtain a global existence result for any $\beta_1 > \max\big(3, \frac{6}{5(1-p_1)}\big)$. *Indeed, we can establish a global a priori estimate for the enthalpy by using the technique proposed by Boccardo and Gallouët ([BoG89]), i.e., by choosing the test-function* $\chi = 1 - \frac{1}{(1+\vartheta)^\xi}$ *for some* $\xi > 0$. *By reproducing the same computations as in [Rou10, Prop. 4.2], we may obtain an estimate of* $\nabla\vartheta$ *in* $\mathrm{L}^r(0,\tau; \mathrm{L}^r(\Omega))$, *independent of* τ, *for any* $r \in \big[1, \frac{d+2}{d+1}\big)$ *provided that* $\beta_1 > \frac{2d}{(d+2)(1-p_1)} = \frac{6}{5(1-p_1)}$ *since* $d = 3$. *Then we consider* $\alpha > 1$ *such that* $2\mu\alpha \leq r$ *and* $\frac{1}{\alpha} \geq \mu\big(\frac{1}{r} - \frac{1}{d}\big) + 1 - \mu$ *for some real number* $\mu \in (0,1)$. *Using Gagliardo-Nirenberg's inequality, we may deduce that there exists* $C_{\mathsf{GN}} > 0$ *such that*

$$\|\vartheta\|^{2\alpha}_{\mathrm{L}^{2\alpha}(0,\tau;\mathrm{L}^\alpha(\Omega))} \leq C_{\mathsf{GN}} \|\vartheta\|^{(1-\mu)2\alpha}_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^1(\Omega))} \int_0^\tau \big(\|\vartheta(t,\cdot)\|_{\mathrm{L}^1(\Omega)} + \|\nabla\vartheta(t,\cdot)\|_{\mathrm{L}^r(\Omega)}\big)^{2\mu\alpha} \,\mathrm{d}t,$$

and reminding the estimate of ϑ *in* $\mathrm{L}^\infty(0,\tau; \mathrm{L}^1(\Omega))$ *obtained at Proposition 7.1, we infer that there exists a constant* $C > 0$, *depending only on the data, such that*

$$\|\vartheta\|_{\mathrm{L}^{2\alpha}(0,\tau;\mathrm{L}^\alpha(\Omega))} \leq C\big(1 + \|\nabla\vartheta\|_{\mathrm{L}^r(0,\tau;\mathrm{L}^r(\Omega))}\big).$$

The three conditions

$$1 \leq r < \tfrac{d+2}{d+1}, \quad 2\mu\alpha \leq r, \quad \tfrac{1}{\alpha} \geq \mu\big(\tfrac{1}{r} - \tfrac{1}{d}\big) + 1 - \mu \text{ with } 0 < \mu < 1,$$

allow us to choose $\mu = \frac{rd}{d + r(d+1)} \in (0,1)$ *and thus* $\alpha = \frac{r}{2\mu} = \frac{1}{2} + \frac{r(d+1)}{2d} \in \big(\frac{7}{6}, \frac{4}{3}\big)$ *(here* $d = 3$*). It follows that, for any* $\beta_1 > \max\big(3, \frac{6}{5(1-p_1)}\big)$, *we will obtain a global estimate of* ϑ *in* $\mathrm{L}^{2\alpha}(0,\tau; \mathrm{L}^\alpha(\Omega))$ *for any* $\alpha \in \big(\frac{7}{6}, \frac{4}{3}\big)$ *and of* $\theta = \zeta(\vartheta)$ *in* $\mathrm{L}^{2\beta_1\alpha}(0,\tau; \mathrm{L}^{\beta_1\alpha}(\Omega))$. *Thus, with* α *such that* $\beta_1\alpha > 4$, *we may obtain a global existence result by the same contradiction argument as in Section 7.*

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