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# Well-posedness and long-time behavior for a nonstandard viscous Cahn-Hilliard system


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**Abstract**

We study a diffusion model of phase field type, consisting of a system of two partial differential equations encoding the balances of microforces and microenergy; the two unknowns are the order parameter and the chemical potential. By a careful development of uniform estimates and the deduction of certain useful boundedness properties, we prove existence and uniqueness of a global-in-time smooth solution to the associated initial/boundary-value problem; moreover, we give a description of the relative $\omega$-limit set.

# 1 Problem setting

The Cahn-Hilliard system:

$$\partial_t \rho - \kappa \Delta \mu = 0 \,, \qquad \mu = -\Delta \rho + f'(\rho), \tag{1.1}$$

describes diffusion-driven phase-segregation processes in a two-phase material body. Here $\rho$, with $\rho(x,t) \in [0,1]$, is an *order parameter* field interpreted as the scaled volumetric density of one of the two phases, $\kappa > 0$ is a *mobility* coefficient, and $\mu$ is the *chemical potential*; $f'$ stands for the derivative of a double-well potential $f$. Customarily, the two equations (1.1) are combined so as to obtain the *Cahn-Hilliard equation*:

$$\partial_t \rho = \kappa \Delta(-\Delta \rho + f'(\rho)), \tag{1.2}$$

a nonlinear high-order parabolic PDE for the order parameter that has been studied extensively. With this procedure – we note for later reference – the chemical potential is left in the background; in particular, there is no need to take an a priori decision about its sign.

To achieve their generalization of (1.2), Fried & Gurtin and Gurtin [8, 10] propose: (i) to regard the second of (1.1) as a *balance of microforces:*

$$\operatorname{div} \boldsymbol{\xi} + \pi + \gamma = 0, \tag{1.3}$$

where the distance microforce per unit volume is split into an internal part $\pi$ and an external part $\gamma$, and the contact microforce per unit area of a surface oriented by its normal $\boldsymbol{n}$ is measured by $\boldsymbol{\xi} \cdot \boldsymbol{n}$ in terms of the *microstress* vector $\boldsymbol{\xi}$;[1] (ii) to interpret the first equation as a *balance law for the order parameter*:

$$\partial_t \rho = -\operatorname{div} \boldsymbol{h} + \sigma, \tag{1.4}$$

where the pair $(\boldsymbol{h}, \sigma)$ is the *inflow* of $\rho$; (iii) to restrict the admissible constitutive choices for $\pi, \boldsymbol{\xi}, \boldsymbol{h}$, and the *free energy density* $\psi$, to those consistent in the sense of Coleman & Noll [4]

[1] In [7], the microforce balance is stated under the form of a principle of virtual powers for microscopic motions.

with an *ad hoc* version of the Second Law of continuum thermodynamics, namely a postulated "dissipation inequality that accomodates diffusion":

$$\partial_t\psi + (\pi - \mu)\partial_t\rho - \boldsymbol{\xi}\cdot\nabla(\partial_t\rho) + \boldsymbol{h}\cdot\nabla\mu \leq 0 \tag{1.5}$$

(cf. eq. (3.6) in [10]). Within this framework, the following set of constitutive prescriptions is shown acceptable:

$$\begin{aligned} \psi &= \widehat{\psi}(\rho, \nabla\rho), \\ \widehat{\pi}(\rho, \nabla\rho, \mu) &= \mu - \partial_\rho\widehat{\psi}(\rho, \nabla\rho), \\ \widehat{\boldsymbol{\xi}}(\rho, \nabla\rho) &= \partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho), \end{aligned} \tag{1.6}$$

together with

$$\boldsymbol{h} = -\boldsymbol{M}\nabla\mu, \quad \text{with } \boldsymbol{M} = \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu); \tag{1.7}$$

moreover, it is shown that the tensor-valued *mobility mapping* $\boldsymbol{M}$ must satisfy the inequality:

$$\nabla\mu\cdot\widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

It follows from (1.3), (1.4), (1.6), and $(1.7)_1$ that:

$$\partial_t\rho = \operatorname{div}\Big(\boldsymbol{M}\nabla\Big(\partial_\rho\widehat{\psi}(\rho, \nabla\rho) - \operatorname{div}\big(\partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho)\big) - \gamma\Big)\Big) + \sigma$$

(cf. eq. (3.17) in [10]); in particular, the Cahn-Hilliard equation (1.2) is arrived at by taking:

$$\widehat{\psi}(\rho, \nabla\rho) = f(\rho) + \frac{1}{2}|\nabla\rho|^2, \qquad \boldsymbol{M} = \kappa\boldsymbol{1}, \tag{1.8}$$

and both the external distance microforce $\gamma$ and the order-parameter source term $\sigma$ identically null.

One of us proposed in [15] a modified version of Fried & Gurtin's derivation, in which their step (i) is retained, but the order-parameter balance (1.4) and the dissipation inequality (1.5) are both dropped and replaced, respectively, by the *microenergy balance*

$$\partial_t\varepsilon = e + w, \quad e := -\operatorname{div}\overline{\boldsymbol{h}} + \overline{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho) \tag{1.9}$$

and the *microentropy imbalance*

$$\partial_t\eta \geq -\operatorname{div}\boldsymbol{h} + \sigma, \quad \boldsymbol{h} := \mu\overline{\boldsymbol{h}}, \quad \sigma := \mu\,\overline{\sigma}. \tag{1.10}$$

The salient new feature of this approach to phase-segregation modeling is that the *microentropy inflow* $(\boldsymbol{h}, \sigma)$ is deemed proportional to the *microenergy inflow* $(\overline{\boldsymbol{h}}, \overline{\sigma})$ through the *chemical potential* $\mu$, a positive field; consistently, the free energy is defined to be

$$\psi := \varepsilon - \mu^{-1}\eta, \tag{1.11}$$

with chemical potential playing the same role as *coldness* in the deduction of the heat equation.[2] Combination of (1.9)-(1.11) gives:

$$\partial_t\psi \leq -\eta\partial_t(\mu^{-1}) + \mu^{-1}\,\overline{\boldsymbol{h}}\cdot\nabla\mu - \pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{1.12}$$

[2] As much as absolute temperature is a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*; likewise, as argued in [15], chemical potential can be seen as a macroscopic measure of microscopic *organization*.

an inequality that replaces (1.5) in restricting *à la* Coleman & Noll the possible constitutive choices.

On taking all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\overline{\boldsymbol{h}}$, dependent in principle on $\rho, \nabla\rho, \mu, \nabla\mu$, and on choosing

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\,\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{1.13}$$

compatibility with (1.12) implies that we must have:

$$\begin{aligned} \widehat{\pi}(\rho, \nabla\rho, \mu) &= \partial_\rho\widehat{\psi}(\rho, \nabla\rho, \mu) = \mu - f'(\rho), \\ \widehat{\boldsymbol{\xi}}(\rho, \nabla\rho, \mu) &= \partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho, \mu) = \nabla\rho, \\ \widehat{\eta}(\rho, \nabla\rho, \mu) &= \mu^2\partial_\mu\widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu^2\rho, \end{aligned} \tag{1.14}$$

together with

$$\widehat{\overline{\boldsymbol{h}}}(\rho, \nabla\rho, \mu, \nabla\mu) = -\widehat{\boldsymbol{H}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu, \quad \nabla\mu \cdot \widehat{\boldsymbol{H}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

If we now choose for $\widehat{\boldsymbol{H}}$ the simplest expression $\boldsymbol{H} = \kappa\boldsymbol{1}$, implying a constant and isotropic mobility, and if we once again assume that the external distance microforce $\gamma$ and the source $\overline{\sigma}$ are null, then, with the use of (1.14) and (1.11), the microforce balance (1.3) and the energy balance (1.9) become, respectively,

$$\operatorname{div}(\nabla\rho) + \mu - f'(\rho) = 0 \tag{1.15}$$

and

$$2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \kappa\Delta\mu = 0, \tag{1.16}$$

a nonlinear system for the unknowns $\rho$ and $\mu$ that we supplement with homogeneous Neumann conditions at the body's boundary:

$$\partial_\nu\rho = \partial_\nu\mu = 0 \tag{1.17}$$

(here $\partial_\nu$ denotes the outward normal derivative), and with the initial conditions:

$$\rho|_{t=0} = \rho_0\,, \quad \mu|_{t=0} = \mu_0\,. \tag{1.18}$$

Needless to say, (1.15) is the same 'static' relation between $\mu$ and $\rho$ as $(1.1)_2$. Instead, (1.16) is rather different from $(1.1)_1$, for a number of reasons:

- (1.16) is nonlinear (whereas $\partial_t\rho - \kappa\Delta\mu = 0$ is a linear equation);
- the time derivatives of $\rho$ and $\mu$ are both present in (1.16);
- there are nonconstant factors in front of both $\partial_t\mu$ and $\partial_t\rho$.

Moreover, it should be possible to show that the initial/boundary-value problem (1.15)-(1.18) has solutions $\rho \in [0, 1]$ and $\mu > 0$.

We must confess that we boldly attacked this problem as is, prompted to optimism by the successful outcome of a previous joint research effort [5, 6], in which we tackled mathematically the system of Allen-Cahn type one arrives at via the approach in [15] for processes of phase segregation in the absence of diffusion. Unfortunately, system (1.15)–(1.18) turned out to be too difficult for us. Therefore, we decided to study a regularized version of it, obtained by introducing two extra terms, $\varepsilon\,\partial_t\mu$ in (1.16) and $\delta\,\partial_t\rho$ in the left-hand side of (1.15), for small positive coefficients $\varepsilon$ and $\delta$.

The introduction of the first term is motivated by the desire to have a strictly positive coefficient as a factor of $\partial_t\mu$ in (1.16), in order to guarantee the parabolic structure of this equation. As to the other term, on the one hand it gives (1.15) the form of an Allen-Cahn equation with right-hand side $\mu$; on the other hand, it assimilates our present model to the so-called *viscous Cahn-Hilliard equations* (see, e.g., [2, 14, 16] and references therein). With these measures, and taking $\kappa = 1$ for simplicity, we write the following modified version of problem (1.15)–(1.18), with inversion of the order of the differential equations:

$$(\varepsilon + 2\rho)\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0 \qquad \text{in } \Omega \times (0, +\infty), \tag{1.19}$$

$$\delta\partial_t\rho - \Delta\rho + f'(\rho) = \mu \qquad \text{in } \Omega \times (0, +\infty), \tag{1.20}$$

$$\partial_\nu\mu = \partial_\nu\rho = 0 \qquad \text{on } \Gamma \times (0, +\infty), \tag{1.21}$$

$$\mu(\,\cdot\,, 0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,, 0) = \rho_0 \qquad \text{in } \Omega, \tag{1.22}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary $\Gamma$. We remark that such a regularized system has the typical features of a phase field model, but with a nonstandard equation (1.19) for the chemical potential $\mu$, while quite often phase field systems feature temperature and order parameter as variables.

By assuming, as we did in [5, 6], that $f'$ is the sum of a strictly increasing $C^1$ function $f_1'$ with domain $(0, 1)$ that is singular at the endpoints, and of a smooth bounded perturbation $f_2'$ (to allow for a double- or multi-well potential $f$), we prove the *existence of a strong solution* $(\mu, \rho)$ *to* (1.19)–(1.22) *satisfying* $\mu \geq 0$ *and* $0 < \rho < 1$ *almost everywhere in* $\Omega \times (0, +\infty)$ (of course, the initial data have to meet the same requirements in $\Omega$). Our existence proof is rather standard; it is based on an approximation $\to$ a priori estimates $\to$ passage-to-the-limit procedure. Under additional assumptions, by using certain delicate iterative estimates, we also show that the component $\mu$ *is bounded above*; this is probably the most difficult and technical part of the present paper. Boundedness of $\mu$ is expedient to deduce that $f'(\rho)$ is bounded as well; as a consequence, $\rho$ *stays away from the threshold values* $0$ *and* $1$. These boundedness properties are very useful in proving *uniqueness* of such solutions, since $f'(\rho)$ can be treated as a Lipschitz-continuous function of $\rho$.

As a final step, we deal with the long-time behavior of the system. We prove that *each element* $(\mu_\omega, \rho_\omega)$ *of the* $\omega$*-limit set is a steady state solution of* (1.19)–(1.22); therefore, in particular, $\mu_\omega$ *is a constant* (cf. (1.19) and (1.21)). This concludes our description of the contents of this paper. Needless to say, it would be interesting and challenging to study the singular limit of the solutions to (1.19)–(1.22) as $\varepsilon$ or $\delta$ tends to zero, or both parameters do. We plan to undertake such a study in the near future.

## 2 Main results

In this section, we describe the mathematical problem under investigation, make our assumptions precise, and state our results. First of all, we assume $\Omega$ to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary $\Gamma$ (to treat the lower-dimensional cases would only require minor changes). Moreover, for convenience we set:

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega) : \partial_\nu v = 0 \text{ on } \Gamma\}, \tag{2.1}$$

and we endow these spaces with their standard norms, for which we use the self-explanatory notation $\|\cdot\|_V$ (but $\|\cdot\|_H$ denotes the norm of any power of $H$). We remark that the embeddings $W \subset V \subset H$ are compact, because $\Omega$ is bounded and smooth. Since $V$ is dense in $H$, we can identify $H$ with a subspace of $V^*$ in the usual way (i.e., so as to have that ${}_{V^*}\langle u, v\rangle_V = (u, v)_H$ for every $u \in H$ and $v \in V$); the embedding $H \subset V^*$ is also compact. As to the potential $f$, we assume that

$$f = f_1 + f_2, \quad \text{where functions} \quad f_1, f_2 : (0,1) \to \mathbb{R} \text{ are such that} \tag{2.2}$$

$$f_1 \text{ is } C^1 \text{ and convex}, \quad f_2 \text{ is } C^2, \quad f_2'' \text{ is bounded}, \tag{2.3}$$

$$\lim_{r \searrow 0} f_1'(r) = -\infty\,, \quad \text{and} \quad \lim_{r \nearrow 1} f_1'(r) = +\infty. \tag{2.4}$$

For the initial data, we stipulate that

$$\mu_0 \in V \quad \text{and} \quad \mu_0 \geq 0 \quad \text{a.e. in } \Omega; \tag{2.5}$$

$$\rho_0 \in W, \quad 0 < \rho_0 < 1 \quad \text{in } \Omega; \quad \text{and} \quad f'(\rho_0) \in H. \tag{2.6}$$

We stress that the conditions in (2.6) imply that

$$\rho_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad f(\rho_0) \in H. \tag{2.7}$$

Indeed, $W \subset C^0(\overline{\Omega})$, assumptions (2.3) hold, and, by convexity, $-c \leq f_1(\rho_0) \leq f_1(1/2) + f_1'(\rho_0)(\rho_0 - 1/2)$ for some $c \in \mathbb{R}$.

Our aim is to solve problem (1.19)–(1.22) in a strong sense, i.e., we want to find a pair $(\mu, \rho)$ of such smooth functions satisfying suitable summability conditions and unilateral constraints that (1.19)–(1.22) are made fully meaningful. Precisely, we fix a final time $T > 0$, we set $Q := \Omega \times (0, T)$, and we require that:

$$\mu \in H^1(0,T;H) \cap L^2(0,T;W), \tag{2.8}$$

$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.9}$$

$$\mu \geq 0 \quad \text{a.e. in } Q, \tag{2.10}$$

$$0 < \rho < 1 \quad \text{a.e. in } Q \quad \text{and} \quad f'(\rho) \in L^\infty(0,T;H). \tag{2.11}$$

Note that the boundary conditions (1.21) follow from (2.8)–(2.9), due to the definition of $W$ in (2.1). In conclusion, we look for $(\mu, \rho)$ satisfying (2.8)–(2.11) and fulfilling the system

$$(\varepsilon + 2\rho)\partial_t \mu + \mu\, \partial_t \rho - \Delta \mu = 0 \qquad \text{a.e. in } Q, \tag{2.12}$$

$$\delta \partial_t \rho - \Delta \rho + f'(\rho) = \mu \qquad \text{a.e. in } Q, \tag{2.13}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{2.14}$$

Here is our main result.

**Theorem 2.1.** *Assume that* (2.2)–(2.4) *and* (2.5)–(2.6) *are satisfied. Then, there exists a pair* $(\mu, \rho)$ *satisfing* (2.8)–(2.11) *and solving problem* (2.12)–(2.14).

Once existence is secured, one wonders about uniqueness. We are able to prove it for solutions having the following additional properties:

$$\mu \in L^\infty(Q); \quad \inf \rho > 0 \quad \text{and} \quad \sup \rho < 1. \tag{2.15}$$

**Theorem 2.2.** *Assume that* (2.2)–(2.4) *and* (2.5)–(2.6) *are satisfied. Then, any two solutions to problem* (2.12)–(2.14) *satisfing* (2.8)–(2.11) *and* (2.15) *coincide.*

Interestingly, the additional boundedness conditions for $(\mu, \rho)$ postulated above are fulfilled whenever the data of the problem have similar boundedness properties, in addition to (2.5)–(2.6).

**Theorem 2.3.** *Assume that* (2.2)–(2.6) *and the following conditions are satisfied:*

$$\mu_0 \in L^\infty(\Omega); \quad \inf \rho_0 > 0 \quad \textit{and} \quad \sup \rho_0 < 1. \tag{2.16}$$

*Then, any pair* $(\mu, \rho)$ *satisfing* (2.8)–(2.11) *and solving problem* (2.12)–(2.14) *satisfies* (2.15) *as well.*

**Remark 2.4.** Even though the regularity of the solution given by (2.8)–(2.11) and (2.15) is completely satisfactory for our purposes, we observe that some further smoothness can be proved once the properties (2.15) are established. In that case, equation (2.13) can be read in the form $\partial_t \rho - \Delta \rho = g$ with $g \in L^\infty(Q)$, whence further regularity for $\rho$ can be derived, and a bootstrap procedure can start. Indeed, further regularity for $\rho$ implies that stronger properties for $\mu$ can be proved by (2.12). This improves the regularity of $g$ and leads to an increase of the regularity of $\rho$.

Once well-posedness on every finite time interval is ensured, one can study the long-time behavior of the solution. In particular, one can try to characterize the $\omega$-limit of any trajectory $(\mu, \rho)$ in some topology. We choose the weak topology of $H \times V$ and define such an $\omega$-limit as follows:

$$\begin{aligned} \omega(\mu, \rho) \quad &= \big\{(\mu_\omega, \rho_\omega) : \big(\mu(t_n), \rho(t_n)\big) \to (\mu_\omega, \rho_\omega) \\ &\qquad \text{weakly in } H \times V \text{ for some sequence } t_n \nearrow +\infty\big\}. \end{aligned} \tag{2.17}$$

Our last result gives a relationship between such an $\omega$-limit and the set of *steady states*, i.e., the set of the time-independent solutions $(\mu_s, \rho_s)$ to (2.12)–(2.13) with homogeneous Neumann boundary condition satisfying natural regularity properties. Note that in such a case $\mu_s$ must be harmonic, thus constant, since $\Omega$ is connected. Therefore, a steady state is a pair $(\mu_s, \rho_s)$ such that $\mu_s$ is a nonnegative constant and $\rho_s$ solves the following problem:

$$\rho_s \in W, \quad 0 < \rho_s < 1, \quad f'(\rho_s) \in H, \quad \text{and} \quad -\Delta \rho_s + f'(\rho_s) = \mu_s \quad \text{a.e. in } \Omega \tag{2.18}$$

(there is no reason for $\rho_s$ to be constant, since $f$ is not required to be convex).

**Theorem 2.5.** *Assume that conditions* (2.2)–(2.4), (2.5)–(2.6), *and* (2.16), *are satisfied. Let* $(\mu, \rho)$ *be the corresponding solution satisfing* (2.8)–(2.11) *and* (2.15). *Then, the* $\omega$*-limit* $\omega(\mu, \rho)$ *is nonempty, compact, and connected in the weak topology of* $H \times V$*; moreover, each of its elements coincides with a steady state* $(\mu_s, \rho_s)$ *(that is to say,* $\mu_s$ *is a nonnegative constant and* $\rho_s$ *solves* (2.18)).

Our paper is organized as follows. In the next section, we prove Theorem 2.1, while Theorems 2.2 and 2.3 are proved in Section 4. Our last section is devoted to the proof of Theorem 2.5.

Throughout the paper, we account for the well-known embedding $V \subset L^q(\Omega)$ for $1 \le q \le 6$ and the related Sobolev inequality:

$$\|v\|_{L^q(\Omega)} \le C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \le q \le 6, \tag{2.19}$$

where $C$ depends on $\Omega$ only, since sharpness is not needed (the embedding $V \subset L^q(\Omega)$ is compact if $q < 6$). Furthermore, we repeatedly make use of the well-known Hölder inequality, the interpolation inequality

$$\begin{aligned} &\|v\|_{L^r(\Omega)} \le \|v\|^{\vartheta}_{L^p(\Omega)} \|v\|^{1-\vartheta}_{L^q(\Omega)} \quad \text{for } v \in L^p(\Omega) \cap L^q(\Omega), \\ &\qquad \text{where} \quad p, q, r \in [1, +\infty], \quad \vartheta \in [0, 1], \quad \text{and} \quad \frac{1}{r} = \frac{\vartheta}{p} + \frac{1-\vartheta}{q}, \end{aligned} \tag{2.20}$$

and the elementary Young inequality

$$ab \le \sigma a^2 + \frac{1}{4\sigma} b^2 \quad \text{for every } a, b \ge 0 \text{ and } \sigma > 0. \tag{2.21}$$

Finally, throughout the paper we use a small-case italic $c$ for different constants, that may only depend on $\Omega$, the final time $T$, the shape of $f$, the properties of the data involved in the statements at hand, and the coefficients $\varepsilon$ and $\delta$; a notation like $c_\sigma$ signals a constant that depends also on the parameter $\sigma$. The reader should keep in mind that the meaning of $c$ and $c_\sigma$ might change from line to line and even in the same chain of inequalities, whereas those constants we need to refer to are always denoted by capital letters, just like $C$ in (2.19).

# 3 Existence

In this section, we prove Theorem 2.1. Our method uses an approximation scheme based on a time delay in the right-hand side of (2.13). Namely, we define the translation operator $\mathcal{T}_\tau : L^1(0, T; H) \to L^1(0, T; H)$ depending on a time step $\tau > 0$ by setting, for $v \in L^1(0, T; H)$ and for a.a. $t \in (0, T)$,

$$(\mathcal{T}_\tau v)(t) := v(t - \tau) \quad \text{if } t > \tau \quad \text{and} \quad (\mathcal{T}_\tau v)(t) := \mu_0 \quad \text{if } t < \tau, \tag{3.1}$$

and consider the problem obtained by replacing the right-hand side of (2.13) by $\mathcal{T}_\tau\mu$, i.e., we look for a pair $(\mu_\tau, \rho_\tau)$ such that

$$(\mu_\tau, \rho_\tau) \text{ satisfies (2.8)–(2.11)} \tag{3.2}$$

$$(\varepsilon + 2\rho_\tau)\partial_t\mu_\tau - \Delta\mu_\tau + \mu_\tau\,\partial_t\rho_\tau = 0 \qquad \text{a.e. in } Q \tag{3.3}$$

$$\delta\partial_t\rho_\tau - \Delta\rho_\tau + f'(\rho_\tau) = \mathcal{T}_\tau\mu_\tau \qquad \text{a.e. in } Q \tag{3.4}$$

$$\mu_\tau(0) = \mu_0 \quad \text{and} \quad \rho_\tau(0) = \rho_0 \qquad \text{a.e. in } \Omega. \tag{3.5}$$

For convenience, we allow $\tau$ to take just discrete values, namely, $\tau = T/N$, where $N$ is any positive integer. Our existence proof consists in two parts. Firstly, we check that problem (3.2)–(3.5) is well-posed (see the next lemma). Secondly, we let $\tau$ tend to $0$. This is done by proving a number of a priori estimates and using compactness and monotonicity arguments.

**Lemma 3.1.** *There exists a unique pair* $(\mu_\tau, \rho_\tau)$ *solving problem* (3.2)–(3.5).

**Proof.** Recall that $\tau = T/N$. Hence, if we set $t_n := n\tau$ for $n = 0, \dots, N$, we see that problem (3.2)–(3.5) becomes equivalent to a finite sequence of $N$ problems that can be solved step by step. However, instead of considering the natural time intervals $[t_{n-1}, t_n]$, $n = 1, \dots, N$, and glueing the solutions together, we solve $N$ problems on the time intervals $I_n = [0, t_n]$, $n = 1, \dots, N$, by constructing the solution directly on the whole of $I_n$ at each step. These problems are the following:

$$(\varepsilon + 2\rho_n)\partial_t\mu_n - \Delta\mu_n + (\partial_t\rho_n)\mu_n = 0 \quad \text{and} \quad \mu_n \geq 0 \quad \text{a.e. in } \Omega \times I_n \tag{3.6}$$

$$\partial_\nu\mu_n(t)|_\Gamma = 0 \quad \text{for a.a. } t \in I_n \quad \text{and} \quad \mu_n(0) = \mu_0 \tag{3.7}$$

$$0 < \rho_n < 1 \quad \text{and} \quad \delta\partial_t\rho_n - \Delta\rho_n + f'(\rho_n) = \mathcal{T}_\tau\mu_{n-1} \quad \text{a.e. in } \Omega \times I_n \tag{3.8}$$

$$\partial_\nu\rho_n(t)|_\Gamma = 0 \quad \text{for a.a. } t \in I_n \quad \text{and} \quad \rho_n(0) = \rho_0. \tag{3.9}$$

Their solutions are required to satisfy the regularity properties induction obtained by taking $t_n$ in place of $T$ in (2.8)–(2.11). The operator $\mathcal{T}_\tau$ that appears on the right-hand side of (3.8) acts on functions that are not defined in the whole of $(0, T)$. However, its meaning is still given by (3.1) if $n > 1$, while we simply set $\mathcal{T}_\tau\mu_{n-1} = \mu_0$ if $n = 1$.

Clearly, the solution $(\mu_\tau, \rho_\tau)$ we are looking for is simply given by $(\mu_N, \rho_N)$. The above problems can be solved inductively, because the right-hand side of (3.8) is known at each step in the next lemma: one first solves problem (3.8)–(3.9) for $\rho_n$, and then problem (3.6)–(3.7) for $\mu_n$. We note that the former problem is quite standard; the latter is a regular linear parabolic problem (the coefficient of $\partial_t\mu_n$ is $\geq \varepsilon$) provided that $\partial_t\rho_n$ is sufficiently smooth. That the inequality $\mu_n \geq 0$ holds is not obvious. The uniqueness of a solution $(\mu_n, \rho_n)$ satisfying smoothness properties analogous to (2.8)–(2.11) is clear, and the existence of a variational solution is expected. However, not even the desired regularity is obviously guaranteed. Therefore, we provide a few arguments in this direction.

Proceeding at an as-low-as-possible level of formality, we introduce a problem depending on a positive parameter $\lambda$ and approximating problem (3.8)–(3.9). To begin with, we regularize $f_1$ and $f_2$ by constructing certain suitable $C^2$ approximations $f_{1,\lambda}, f_{2,\lambda}$ having bounded first and second derivatives. Precisely, we assume that $f''_{2,\lambda}$ is bounded uniformly with respect to $\lambda$;

moreover, on thinking of $f_1'$ as a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, we assume that $f_{1,\lambda}$ is convex and that $f_{1,\lambda}'$ is similar to the Yosida regularization of $f_1'$ (see, e.g., [3, p. 28]), in order to preserve the main properties of the latter (such a regularization is detailed, e.g., in [9, Section 3]). Finally, we set $f_\lambda = f_{1,\lambda} + f_{2,\lambda}$. The approximating problem is:

$$\delta \partial_t \rho_n^\lambda - \Delta \rho_n^\lambda + f_\lambda'(\rho_n^\lambda) = \mathcal{T}_\tau \mu_{n-1} \quad \text{a.e. in } \Omega \times I_n, \tag{3.10}$$

$$\partial_\nu \rho_n^\lambda(t)|_\Gamma = 0 \quad \text{for a.a. } t \in I_n, \quad \text{and} \quad \rho_n^\lambda(0) = \rho_0; \tag{3.11}$$

it has a unique smooth solution, which satisfies sufficiently strong a priori estimates to allow letting $\lambda$ tend to zero in (3.10)–(3.11). This leads to a solution $\rho_n$ to problem (3.8)–(3.9), which can be used to solve problem (3.6)–(3.7). Needless to say, the desired regularity for $\rho_n$ will follow once we prove suitable estimates uniformly with respect to $\lambda$. We confine ourselves to derive the highest-order estimate, the others being quite standard.

With a view toward assembling a proof by induction, we assume that

$$\mu_{n-1} \in H^1(I_{n-1}; H) \cap L^\infty(I_{n-1}; V) \quad \text{and} \quad \mu_{n-1} \geq 0 \quad \text{for} \quad n > 1 \tag{3.12}$$

and we prove that

$$\|\rho_n^\lambda\|_{W^{1,\infty}(I_n;H)\cap H^1(I_n;V)\cap L^\infty(I_n;W)} + \|f_{1,\lambda}'(\rho_n^\lambda)\|_{L^\infty(I_n;H)} \leq c_\tau, \tag{3.13}$$

$$\mu_n \in H^1(I_n; H) \cap L^\infty(I_n; V) \cap L^2(I_n; W) \quad \text{and} \quad \mu_n \geq 0 \tag{3.14}$$

(as anticipated in closing Section 2, in (3.13) as well as in the following the symbol $c_\tau$ stands for one or another of a list of different constants that do not depend on $\lambda$, but are allowed to depend on $\tau$). We remark that the induction procedure can actually start, because $\mathcal{T}_\tau \mu_{n-1} = \mu_0$ if $n = 1$ and, moreover, properties (2.5) and (2.6) for $\mu_0$ and $\rho_0$ are fulfilled; these properties are also used at each step.

We omit stressing the dependences on $n$ and $\lambda$, and write simply $u$ and $u_0$ for, respectively, $\partial_t \rho_n^\lambda$ and $\partial_t \rho_n^\lambda(0)$. By differentiating (3.10) with respect to time, we see that $u$ solves the equation:

$$\delta \partial_t u - \Delta u + f_{1,\lambda}''(\rho_n^\lambda)\, u = \partial_t(\mathcal{T}_\tau \mu_{n-1}) - f_{2,\lambda}''(\rho_n^\lambda)\, u \quad \text{a.e. in } \Omega \times I_n, \tag{3.15}$$

and satisfies both the Cauchy condition $u(0) = u_0$ and homogeneous Neumann boundary condition. Hence, by testing (3.15) by $u$ and using the convexity of $f_{1,\lambda}$, we immediately obtain for $t \in I_n$ that

$$\frac{\delta}{2}\|u(t)\|_H^2 + \int_0^t \int_\Omega |\nabla u|^2 \leq \frac{\delta}{2}\|u_0\|_H^2 + \big(1 + \sup|f_{2,\lambda}''|\big) \int_0^t \int_\Omega u^2 + \|\partial_t(\mathcal{T}_\tau \mu_{n-1})\|_{L^2(I_n;H)}^2. \tag{3.16}$$

Now, we observe that the last norm is finite, in view of our assumption (3.12), and that $|f_{2,\lambda}''| \leq c$. Moreover, due to (3.10), we have that $\delta u_0 = \mu_0 + \Delta \rho_0 - f_\lambda'(\rho_0)$. Hence, $u_0$ is bounded in $H$, by (2.5)–(2.6) and our choice of the approximation $f_\lambda$ of $f$. Therefore, thanks to the Gronwall lemma, we obtain:

$$\|u\|_{L^\infty(I_n;H)\cap L^2(I_n;V)} \leq c_\tau\,, \quad \text{whence} \quad \|\rho_n^\lambda\|_{W^{1,\infty}(I_n;H)\cap H^1(I_n;V)} \leq c_\tau\,.$$

Next, coming back to (3.10), we deduce that $-\Delta \rho_n^\lambda + f_{1,\lambda}'(\rho_n^\lambda)$ is bounded in $L^\infty(I_n; H)$ and hence, by a standard argument (for instance, by testing (3.10) by $f_{1,\lambda}'(\rho_n^\lambda)$), that each of

$-\Delta\rho_n^\lambda$ and $f'_{1,\lambda}(\rho_n^\lambda)$ is bounded. With this, given that the $W$-estimate follows from elliptic theory, (3.13) is established, and we can let $\lambda$ tend to zero. We obtain:

$$\rho_n \in W^{1,\infty}(I_n;H)\cap H^1(I_n;V)\cap L^\infty(I_n;W), \quad 0<\rho_n<1, \quad \text{and} \quad f_1'(\rho_n)\in L^\infty(I_n;H).$$

At this point, we should prove (3.14). However, we confine ourselves to derive a formal estimate that clearly shows that the desired regularity for $\mu_n$ can be deduced by regularizing the linear problem (3.6)–(3.7) (if the coefficient $\partial_t\rho_n$ is replaced by a smooth function and the initial datum is regularized, by the same token $\partial_t\mu_n$ is an admissible test function). For convenience, we write (3.6) in the form:

$$(\varepsilon+2\rho_n)\partial_t\mu_n+\mu_n-\Delta\mu_n=(1-\partial_t\rho_n)\mu_n;$$

next, we multiply this relation by $\partial_t\mu_n$ and use the result in the calculation given below. Since $\rho_n\geq 0$, we find, for $t\in I_n$,

$$\begin{aligned}
&\varepsilon\int_0^t\!\!\int_\Omega|\partial_t\mu_n|^2+\frac{1}{2}\,\|\mu_n(t)\|_V^2\leq\int_0^t\!\!\int_\Omega(\varepsilon+2\rho_n)|\partial_t\mu_n|^2+\frac{1}{2}\,\|\mu_n(t)\|_V^2\\
&=\int_0^t\!\!\int_\Omega(\varepsilon+2\rho_n)|\partial_t\mu_n|^2+\frac{1}{2}\,\|\mu_0\|_V^2+\frac{1}{2}\int_0^t\!\!\int_\Omega\partial_t\big(|\mu_n|^2+|\nabla\mu_n|^2\big)\\
&=\frac{1}{2}\,\|\mu_0\|_V^2+\int_0^t\!\!\int_\Omega\big((\varepsilon+2\rho_n)|\partial_t\mu_n|^2+\mu_n\,\partial_t\mu_n+\nabla\mu_n\cdot\nabla\partial_t\mu_n\big)\\
&=\frac{1}{2}\,\|\mu_0\|_V^2+\int_0^t\!\!\int_\Omega(1-\partial_t\rho_n)\,\mu_n\,\partial_t\mu_n\\
&\leq\frac{1}{2}\,\|\mu_0\|_V^2+\int_0^t\|1+|\partial_t\rho_n(s)|\|_{L^4(\Omega)}\,\|\mu_n(s)\|_{L^4(\Omega)}\,\|\partial_t\mu_n(s)\|_{L^2(\Omega)}\,ds\\
&\leq\frac{1}{2}\,\|\mu_0\|_V^2+\frac{\varepsilon}{2}\int_0^t\!\!\int_\Omega|\partial_t\mu_n|^2+\frac{C^2}{2\varepsilon}\int_0^t\|1+|\partial_t\rho_n(s)|\|_V^2\,\|\mu_n(s)\|_V^2\,ds,
\end{aligned}$$

by the Sobolev and Young inequalities (2.19) and (2.21). Then, the Gronwall lemma yields that

$$\|\partial_t\mu_n\|_{L^2(I_n;H)}+\|\mu_n\|_{L^\infty(I_n;V)}\leq c_M, \tag{3.17}$$

where $M$ is a constant satisfying $M\geq\|u_0\|_V+\|\partial_t\rho_n\|_{L^2(I_n;V)}$. By comparison in (3.8), even $\Delta\mu_n$ is estimated in $L^2(I_n;H)$, since a bound for $\mu_n\partial_t\rho_n$ in the same space follows from (3.17). By elliptic regularity, we derive the desired estimate for $\mu_n$ in $L^2(I_n;W)$. So, the first assertion in (3.14) is established; it remains for us to show that $\mu_n\geq 0$. This is done by testing (3.6) by $-\mu_n^-$. We obtain, for $t\in I_n$, that

$$\begin{aligned}
&\frac{1}{2}\int_0^t\!\!\int_\Omega\partial_t\big((\varepsilon+2\rho_n)|\mu_n^-|^2\big)+\int_0^t\!\!\int_\Omega|\nabla\mu_n^-|^2\\
&=\int_0^t\!\!\int_\Omega\big((\varepsilon+2\rho_n)\partial_t\mu_n(-\mu_n^-)+(\partial_t\rho_n)\mu_n(-\mu_n^-)+\nabla\mu_n\cdot\nabla(-\mu_n)^-\big)=0.
\end{aligned}$$

As $\rho_n\geq 0$ and $\mu_0\geq 0$, we deduce that

$$\varepsilon\int_\Omega|\mu_n^-(t)|^2\leq\int_\Omega(\varepsilon+2\rho_n(t))|\mu_n^-(t)|^2\leq\int_\Omega(\varepsilon+2\rho_0)|\mu_0^-|^2=0,$$

whence it immediately follows that $\mu_n^- = 0$, i.e., that $\mu_n \geq 0$. Thus, the lemma is proved. $\square$

Now that the well-posedness of problem (3.2)–(3.5) is established, we perform a number of a priori estimates of its solution. These estimates allow us to let $\tau$ tend to zero, so as to prove our existence result for problem (2.12)–(2.14). In order to make the formulas to come more readable, we shall omit the index $\tau$ in the calculations, waiting for writing $(\mu_\tau, \rho_\tau)$ only when each estimate is established.

**First a priori estimate.** We observe that $\partial_t\big((\varepsilon/2)\mu^2 + \rho\mu^2\big) = \big((\varepsilon + 2\rho)\partial_t\mu + \mu\,\partial_t\rho\big)\mu$. Thus, testing (3.3) by $\mu$ and integrating, we obtain, for $t \in (0, T)$, that

$$\int_\Omega \Big(\frac{\varepsilon}{2}\,\mu^2 + \rho\mu^2\Big)(t) + \int_0^t\!\!\int_\Omega |\nabla\mu|^2 = \int_\Omega \Big(\frac{\varepsilon}{2}\,\mu_0^2 + \rho_0\mu_0^2\Big) = c. \tag{3.18}$$

This implies that

$$\|\mu_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c. \tag{3.19}$$

**Second a priori estimate.** This standard estimate for phase field equations can be derived by testing (3.4) by $\partial_t\rho$. We get:

$$\|\rho_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|f(\rho_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \tag{3.20}$$

**Third a priori estimate.** We rewrite (3.4) as

$$-\Delta\rho + f_1'(\rho) = -\delta\partial_t\rho - f_2'(\rho) + \mathcal{T}_\tau\mu, \tag{3.21}$$

and notice that the right-hand side is bounded in $L^2(0, T; H)$. Then, by applying a standard procedure (for instance, testing by $f_1'(\rho)$), and counting on elliptic regularity, we deduce that

$$\|\rho_\tau\|_{L^2(0,T;W)} + \|f_1'(\rho_\tau)\|_{L^2(0,T;H)} \leq c. \tag{3.22}$$

**Fourth a priori estimate.** To derive the next inequality, we prefer to proceed formally, avoiding the $\lambda$-regularization we used in the proof of Lemma 3.1. As for (3.16), we obtain the following estimate:

$$\begin{aligned}
&\frac{\delta}{2}\,\|\partial_t\rho(t)\|_H^2 + \int_0^t\!\!\int_\Omega |\nabla\partial_t\rho|^2 \\
&\leq \frac{\delta}{2}\,\|\Delta\rho_0 - f_1'(\rho_0) + \mu_0\|_H^2 + \sup|f_{2,\lambda}''| \int_0^t\!\!\int_\Omega |\partial_t\rho|^2 + \int_0^t\!\!\int_\Omega (\partial_t\mathcal{T}_\tau\mu)\,\partial\rho.
\end{aligned} \tag{3.23}$$

Once this inequality is established, our procedure is rigorous. The estimate of the last term requires now more care than before, because we aim to obtain bounds that are uniform with respect to $\tau$. We have:

$$\int_0^t\!\!\int_\Omega (\partial_t\mathcal{T}_\tau\mu)\,\partial_t\rho = \int_\tau^t \int_\Omega \partial_t\mu(s-\tau)\,\partial_t\rho(s)\,ds = \int_0^{t-\tau} \int_\Omega \partial_t\mu(s)\,\partial_t\rho(s+\tau)\,ds,$$

and we compute $\partial_t\mu$ from (3.6). On recalling that $\rho \geq 0$, we can continue as follows:

$$
\begin{aligned}
&\int_0^t\!\!\int_\Omega (\partial_t \mathcal{T}_\tau \mu)\, \partial_t\rho = \int_0^{t-\tau}\!\!\int_\Omega \Big(\frac{1}{\varepsilon+2\rho}\,\big(\Delta\mu - \mu\,\partial_t\rho\big)\Big)(s)\,\partial_t\rho(s+\tau)\,ds \\
&= \int_0^{t-\tau}\!\!\int_\Omega \Big\{-\Big(\frac{\nabla\mu}{\varepsilon+2\rho}\Big)(s)\cdot\nabla\partial_t\rho(s+\tau) + 2\,\frac{\partial_t\rho(s+\tau)}{(\varepsilon+2\rho(s))^2}\,\nabla\mu(s)\cdot\nabla\rho(s) \\
&\qquad\qquad - \partial_t\rho(s)\,\mu(s)\,\partial_t\rho(s+\tau)\,\frac{1}{\varepsilon+2\rho(s)}\Big\}\,ds \\
&\leq \frac{1}{4}\int_0^t\!\!\int_\Omega |\nabla\partial_t\rho|^2 + c\|\mu\|^2_{L^2(0,T;V)} \\
&\quad + c\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_{L^4(\Omega)}\,\|\nabla\mu(s)\|_H\,\|\nabla\rho(s)\|_{L^4(\Omega)}\,ds \\
&\quad + c\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_{L^4(\Omega)}\,\|\mu(s)\|_{L^4(\Omega)}\,\|\partial_t\rho(s)\|_H\,ds.
\end{aligned}
\tag{3.24}
$$

We need to estimate the last two integrals. As to the first, we begin by using the Sobolev inequality (2.19) and the elementary Young inequality (2.21). We find that

$$
\begin{aligned}
&\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_{L^4(\Omega)}\,\|\nabla\mu(s)\|_H\,\|\nabla\rho(s)\|_{L^4(\Omega)}\,ds \\
&\leq \frac{1}{8}\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_V^2\,ds + c\int_0^t \|\nabla\mu(s)\|_H^2\,\|\nabla\rho(s)\|_V^2\,ds \\
&\leq \frac{1}{8}\int_0^t\!\!\int_\Omega |\nabla\partial_t\rho|^2 + \frac{1}{8}\,\|\partial_t\rho\|^2_{L^2(0,T;H)} \\
&\quad + c\int_0^t \|\mu(s)\|_V^2\,\big(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\big)\,ds,
\end{aligned}
$$

the last inequality holding because, thanks to elliptic regularity, $\|v\|_W \leq c(\|v\|_V + \|\Delta v\|_H)$ for any $v \in V$ such that $\Delta v \in H$ and $\partial_\nu v|_\Gamma = 0$. At this point, we recall that $\rho$ is bounded in $H^1(0,T;H)\cap L^\infty(0,T;V)$, and $\mu$ in $L^2(0,T;V)$, by (3.20) and (3.19). Moreover, we notice that (3.21) entails (formally, by testing it by $-\Delta\rho(s)$):

$$
\begin{aligned}
\|\Delta\rho(s)\|_H^2 &\leq \delta^2\|\partial_t\rho(s)\|_H^2 + c\big(1 + \|\mathcal{T}_\tau\mu(s)\|_H^2\big) \\
&\leq \delta^2\|\partial_t\rho(s)\|_H^2 + c \quad \text{for a.a. } s \in (0,T),
\end{aligned}
$$

the last inequality being a consequence of (3.19). Therefore, we can infer that

$$
\begin{aligned}
&\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_{L^4(\Omega)}\,\|\nabla\mu(s)\|_H\,\|\nabla\rho(s)\|_{L^4(\Omega)}\,ds \\
&\leq \frac{1}{8}\int_0^t\!\!\int_\Omega |\nabla\partial_t\rho|^2 + c + c\int_0^t \|\mu(s)\|_V^2\,\|\partial_t\rho(s)\|_H^2\,ds.
\end{aligned}
\tag{3.25}
$$

Passing now to estimate the last integral in (3.24), we have that

$$
\begin{aligned}
&\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_{L^4(\Omega)} \|\mu(s)\|_{L^4(\Omega)} \|\partial_t\rho(s)\|_H \, ds \\
&\le \frac{1}{8}\int_0^{t-\tau} \|\partial_t\rho(s+\tau)\|_V^2 \, ds + c\int_0^t \|\mu(s)\|_V^2 \|\partial_t\rho(s)\|_H^2 \, ds \\
&\le \frac{1}{8}\int_0^t\int_\Omega |\nabla\partial_t\rho|^2 + \frac{1}{8}\|\partial_t\rho\|_{L^2(0,T;H)}^2 + c\int_0^t \|\mu(s)\|_V^2 \|\partial_t\rho(s)\|_H^2 \, ds \\
&\le \frac{1}{8}\int_0^t\int_\Omega |\nabla\partial_t\rho|^2 + c + c\int_0^t \|\mu(s)\|_V^2 \|\partial_t\rho(s)\|_H^2 \, ds.
\end{aligned}
$$

With this and (3.25), we see that (3.24) yields:

$$
\int_0^t\int_\Omega (\partial_t\mathcal{T}_\tau\mu)\,\partial_t\rho \le c + \frac{1}{2}\int_0^t\int_\Omega |\nabla\partial_t\rho|^2 + c\int_0^t \|\mu(s)\|_V^2 \|\partial_t\rho(s)\|_H^2 \, ds,
$$

so that (3.23) takes the form:

$$
\frac{\delta}{2}\|\partial_t\rho(t)\|_H^2 + \frac{1}{2}\int_0^t\int_\Omega |\nabla\partial_t\rho|^2 \le c + c\int_0^t \|\mu(s)\|_V^2 \|\partial_t\rho(s)\|_H^2 \, ds.
$$

Since $\mu$ has been estimated in $L^2(0,T;V)$, the Gronwall lemma can be applied, so as to have that

$$
\|\partial_t\rho_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{3.26}
$$

Finally, the same argument as in the derivation of (3.22) yields that

$$
\|\rho_\tau\|_{L^\infty(0,T;W)} + \|f_1'(\rho_\tau)\|_{L^\infty(0,T;H)} \le c. \tag{3.27}
$$

**Fifth a priori estimate.** We formally test (3.3) by $\partial_t\mu$, and obtain:

$$
\begin{aligned}
&\varepsilon\int_0^t\int_\Omega |\partial_t\mu|^2 + \frac{1}{2}\int_\Omega |\nabla\mu(t)|^2 \\
&\le \frac{1}{2}\|\nabla\mu_0\|_H^2 - \int_0^t\int_\Omega \partial_t\rho\,\mu\,\partial_t\mu \\
&\le c + \frac{1}{2\varepsilon}\int_0^t \|\partial_t\rho(s)\|_{L^4(\Omega)}^2 \|\mu(s)\|_{L^4(\Omega)}^2 \, ds + \frac{\varepsilon}{2}\int_0^t \|\partial_t\mu(s)\|_H^2 \, ds \\
&\le c + c\int_0^t \|\partial_t\rho_s(s)\|_V^2 \left(\|\mu(s)\|_H^2 + \|\nabla\mu(s)\|_H^2\right) ds + \frac{\varepsilon}{2}\int_0^t \|\partial_t\mu(s)\|_H^2 \, ds \\
&\le c + c\int_0^t \|\partial_t\rho_s(s)\|_V^2 \|\nabla\mu(s)\|_H^2 \, ds + \frac{\varepsilon}{2}\int_0^t \|\partial_t\mu(s)\|_H^2 \, ds,
\end{aligned}
$$

where the last inequality follows from (3.26) and (3.19). Using (3.26) once more, we can apply the Gronwall lemma and conclude that

$$
\|\mu_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c. \tag{3.28}
$$

**Sixth a priori estimate.** Recalling that $0 < \rho < 1$, and using the Sobolev inequality (2.19), we get:

$$\begin{aligned}
&\|(\varepsilon + 2\rho)\partial_t \mu + \mu\, \partial_t \rho\|_{L^2(0,T;H)} \\
&\leq (\varepsilon + 2)\|\partial_t \mu\|_{L^2(0,T;H)} + \|\mu\|_{L^\infty(0,T;L^4(\Omega))}\, \|\partial_t \rho\|_{L^2(0,T;L^4(\Omega))} \\
&\leq c\big(\|\partial_t \mu\|_{L^2(0,T;H)} + \|\mu\|_{L^\infty(0,T;V)}\, \|\partial_t \rho\|_{L^2(0,T;V)}\big).
\end{aligned}$$

Since the right-hand side is bounded by (3.28) and (3.26), a comparison in (3.3) shows that $\Delta\mu$ is bounded in $L^2(0,T;H)$. Consequently, by elliptic regularity we deduce that

$$\|\mu_\tau\|_{L^2(0,T;W)} \leq c. \tag{3.29}$$

**Conclusion.** Collecting all the estimates we have proved, we see that

$$\begin{aligned}
&\mu_\tau \to \mu \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \\
&\rho_\tau \to \rho \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \\
&f_1'(\rho_\tau) \to \xi \quad \text{weakly star in } L^\infty(0,T;H),
\end{aligned}$$

at least for some subsequence $\tau_k \searrow 0$. Thanks to the Aubin-Lions lemma (cf. [13, Thm. 5.1, p. 58]) and to similar results to be found in [17, Sect. 8, Cor. 4], we also deduce the following strong convergences:

$$\begin{aligned}
&\mu_\tau \to \mu \quad \text{strongly in } C^0([0,T];H) \cap L^2(0,T;V) \\
&\rho_\tau \to \rho \quad \text{strongly in } C^0([0,T];V).
\end{aligned}$$

In particular, having recourse to a well-known monotonicity technique (see, e.g., [1, Lemma 1.3, p. 42]), we conclude that $0 < \rho < 1$ and $\xi = f_1'(\rho)$ a.e. in $Q$. The strong convergence shown above also entails that $f_2'(\rho_\tau)$ converges to $f_2'(\rho)$, e.g., strongly in $C^0([0,T];H)$ (because $f_2'$ is Lipschitz continuous), and that $\mathcal{T}_\tau \mu_\tau$ converges to $\mu$, e.g., strongly in $L^2(0,T;H)$. Finally, a combination of the above weak and strong convergence results with the Hölder and Sobolev inequalities yields that

$$\begin{aligned}
&\mu_\tau\, \partial_t \rho_\tau \to \mu\, \partial_t \rho \quad \text{weakly in } L^1(0,T;H), \\
&\rho_\tau\, \partial_t \mu_\tau \to \rho\, \partial_t \mu \quad \text{weakly in } L^2(0,T;L^{3/2}(\Omega)).
\end{aligned}$$

Indeed, $\mu_\tau \to \mu$ strongly in $L^2(0,T;L^4(\Omega))$, $\partial_t \rho_\tau \to \partial_t \rho$ weakly in $L^2(0,T;L^4(\Omega))$, $\rho_\tau \to \rho$ strongly in $C^0([0,T];L^6(\Omega))$, and $\partial_t \mu_\tau \to \partial_t \mu$ weakly in $L^2(0,T;L^2(\Omega))$. Therefore, it is straightforward to conclude that the pair $(\mu, \rho)$ is a solution to problem (2.12)–(2.14) having the desired regularity (2.8)–(2.11), that is to say, Theorem 2.1 is proved.

# 4 Uniqueness and boundedness

In this section, we prove Theorem 2.2 and Theorem 2.3. We first show our uniqueness result.

**Proof of Theorem 2.2.** We take two solutions to problem (2.12)–(2.14) satisfying (2.15) in addition to (2.8)–(2.11) and label their components with the subscripts $1$ and $2$; in the following, the values of constants $c$ may depend on these solutions. Moreover, we choose constants such $M \geq 0$ and $r_*, r^* \in (0,1)$ that $\mu \leq M$ a.e. in $Q$ and $r_* \leq \rho_i \leq r^*$ a.e. in $Q$, for $i = 1, 2$. Finally, we denote by $L$ the Lipschitz constant of the function $r \mapsto r - f'(r)$, $r \in [r_*, r^*]$. Having done this, we write (2.12) for both solutions and take the difference. Then, we set $\mu := \mu_1 - \mu_2$ and $\rho := \rho_1 - \rho_2$, test the resulting equality by $\mu$, and integrate, using the boundary condition. Due to the identity:

$$\left\{(\partial_t\rho_1)\mu_1 + 2\rho_1\partial_t\mu_1 - (\partial_t\rho_2)\mu_2 - 2\rho_2\partial_t\mu_2\right\}\mu = \partial_t(\rho_1\mu^2) + 2(\partial_t\mu_2)\,\rho\,\mu + \mu_2\,(\partial_t\rho)\,\mu, \quad (4.1)$$

we obtain:

$$\int_\Omega \Big(\frac{\varepsilon}{2} + \rho_1(t)\Big)\,|\mu(t)|^2 + \int_0^t\!\!\int_\Omega |\nabla\mu|^2 = -2\int_0^t\!\!\int_\Omega (\partial_t\mu_2)\,\rho\,\mu - \int_0^t\!\!\int_\Omega \mu_2\,(\partial_t\rho)\,\mu. \quad (4.2)$$

Furthermore, we write (2.13) for both solutions, take the difference, test the resulting equality by $\partial_t\rho$, and add $\rho\,\partial_t\rho$ to both sides, for convenience. Then, we integrate, using the boundary condition, and easily obtain that

$$\begin{aligned}
&\delta\int_0^t\!\!\int_\Omega |\partial_t\rho|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 = \int_0^t\!\!\int_\Omega \big((\rho_1 - f'(\rho_1)) - (\rho_2 - f'(\rho_2)) + \mu\big)\,\partial_t\rho \\
&\leq L\int_0^t\!\!\int_\Omega |\rho|\,|\partial_t\rho| + \int_0^t\!\!\int_\Omega |\mu|\,|\partial_t\rho|. \qquad (4.3)
\end{aligned}$$

Now, adding (4.2) and (4.3) and taking into account that $\rho_1$ is nonnegative, we get

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\mu(t)|^2 + \int_0^t\!\!\int_\Omega |\nabla\mu|^2 + \delta\int_0^t\!\!\int_\Omega |\partial_t\rho|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 \\
&\leq 2\int_0^t \|\partial_t\mu_2(s)\|_H\,\|\rho(s)\|_{L^4(\Omega)}\,\|\mu(s)\|_{L^4(\Omega)}\,ds \\
&\quad + \int_0^t \|\mu_2(s)\|_{L^\infty(\Omega)}\,\|\partial_t\rho(s)\|_H\,\|\mu(s)\|_H\,ds \\
&\quad + L\int_0^t \|\rho(s)\|_H\,\|\partial_t\rho(s)\|_H\,ds + \int_0^t \|\mu(s)\|_H\,\|\partial_t\rho(s)\|_H\,ds. \qquad (4.4)
\end{aligned}$$

To estimate the first integral on the right-hand side, we use the Sobolev inequality (2.19) with $q = 4$ and $C$ the Sobolev constant, and we invoke the elementary Young inequality (2.21) to obtain that

$$\begin{aligned}
&\int_0^t \|\partial_t\mu_2(s)\|_H\,\|\rho(s)\|_{L^4(\Omega)}\,\|\mu(s)\|_{L^4(\Omega)}\,ds \leq C^2\int_0^t \|\partial_t\mu_2(s)\|_H\,\|\rho(s)\|_V\,\|\mu(s)\|_V\,ds \\
&\leq \frac{1}{2}\int_0^t \|\mu(s)\|_V^2\,ds + \frac{C^4}{2}\int_0^t \|\partial_t\mu_2(s)\|_H^2\,\|\rho(s)\|_V^2\,ds \\
&= \frac{1}{2}\int_0^t\!\!\int_\Omega |\nabla\mu|^2 + c\int_0^t\!\!\int_\Omega |\mu|^2 + c\int_0^t \|\partial_t\mu_2(s)\|_H^2\,\|\rho(s)\|_V^2\,ds.
\end{aligned}$$

The remainder of the right-hand side of (4.4) is estimated as follows:

$$\begin{aligned}
&\int_0^t \|\mu_2(s)\|_{L^\infty(\Omega)} \, \|\partial_t \rho(s)\|_H \, \|\mu(s)\|_H \, ds \\
&\quad + L \int_0^t \|\rho(s)\|_H \, \|\partial_t \rho(s)\|_H \, ds + \int_0^t \|\mu(s)\|_H \, \|\partial_t \rho(s)\|_H \, ds \\
&\le \frac{\delta}{2} \int_0^t \int_\Omega |\partial_t \rho|^2 + c \int_0^t \big( \|\mu(s)\|_H^2 + \|\rho(s)\|_V^2 \big) \, ds.
\end{aligned}$$

Combining these estimates with (4.4), we immediately get

$$\begin{aligned}
&\frac{\varepsilon}{2} \int_\Omega |\mu(t)|^2 + \frac{1}{2} \int_0^t \int_\Omega |\nabla \mu|^2 + \frac{\delta}{2} \int_0^t \int_\Omega |\partial_t \rho|^2 + \frac{1}{2} \|\rho(t)\|_V^2 \\
&\le c \int_0^t \big( 1 + \|\partial_t \mu_2(s)\|_H^2 \big) \, \|\rho(s)\|_V^2 \, ds + c \int_0^t \|\mu(s)\|_H^2 \, ds.
\end{aligned}$$

Since the function $s \mapsto \|\partial_t \mu_2(s)\|_H^2$ belongs to $L^1(0,T)$, we can apply the Gronwall lemma and deduce that both $\mu$ and $\rho$ vanish. Hence, the two solutions coincide. □

We now turn to proving our boundedness result.

**Proof of Theorem 2.3.** Let $(\mu, \rho)$ be any solution to problem (2.8)–(2.11) and (2.12)–(2.14) whose initial data have, in addition to (2.5)–(2.6) , the further properties (2.16). We show that the boundedness claims in (2.15) actually hold true.

With a view to proving that $\mu$ satisfies the specified lower bound, we set:

$$\mu_0^* := \|\mu_0\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \operatorname{ess} \mu_0(x) \, ; \tag{4.5}$$

we take any real constant $k$ such that $k \ge \mu_0^*$; and we introduce the auxiliary function $\chi_k \in L^\infty(Q)$ defined for a.a. $(x,t) \in Q$ by the formula:

$$\chi_k(x,t) = 1 \quad \text{if } \mu(x,t) > k, \quad \text{and} \quad \chi_k(x,t) = 0 \quad \text{otherwise.}$$

Then, we test (2.12) by $(\mu - k)^+$ and integrate over $\Omega \times (0,t)$ for any $t \in (0,T)$. The result is:

$$\begin{aligned}
&\int_\Omega \Big( \frac{\varepsilon}{2} + \rho(t) \Big) \, \big| \big( \mu(t) - k \big)^+ \big|^2 + \int_0^t \int_\Omega |\nabla (\mu - k)^+|^2 \\
&= \int_0^t \int_\Omega \partial_t \rho \, |(\mu - k)^+|^2 - \int_0^t \int_\Omega \partial_t \rho \, \mu \, (\mu - k)^+ = - \int_0^t \int_\Omega k \, \partial_t \rho \, (\mu - k)^+.
\end{aligned}$$

Given that $\rho$ is nonnegative, this equality and the Hölder inequality with ad hoc exponents lead to:

$$\begin{aligned}
&\frac{\varepsilon}{2} \, \|(\mu(t) - k)^+\|_H^2 + \int_0^t \int_\Omega |\nabla (\mu - k)^+|^2 \\
&\le k \int_0^t \|\chi_k(s)\|_{L^{7/2}(\Omega)} \, \|\partial_t \rho(s)\|_{L^{14/3}(\Omega)} \, \|(\mu - k)^+(s)\|_{L^2(\Omega)} \, ds.
\end{aligned}$$

Now, we use the Gronwall-Bellman lemma as in [3, Lemma A.4, p. 156], and find that

$$\begin{aligned}
&\Big\{\varepsilon\|(\mu-k)^+\|^2_{C^0([0,T];H)}+\int_0^T\int_\Omega|\nabla(\mu-k)^+|^2\Big\}^{1/2}\\
&\le\frac{k}{\sqrt{\varepsilon}}\int_0^T\|\chi_k(t)\|_{L^{7/2}(\Omega)}\,\|\partial_t\rho(t)\|_{L^{14/3}(\Omega)}\,dt\\
&\le\frac{k}{\sqrt{\varepsilon}}\,\|\partial_t\rho\|_{L^{7/3}(0,T;L^{14/3}(\Omega))}\,\|\chi_k\|_{L^{7/4}(0,T;L^{7/2}(\Omega))}\,.
\end{aligned}$$

Next, we observe that the interpolation inequality (2.20) (with $p=2$, $q=6$, $r=14/3$, and $\vartheta=1/7$), together with the Sobolev inequality (2.19), gives that

$$\begin{aligned}
&\|v\|_{L^{7/3}(0,T;L^{14/3}(\Omega))}\le\Big(\int_0^T\|v(t)\|^{1/3}_{L^2(\Omega)}\,\|v(t)\|^2_{L^6(\Omega)}\,dt\Big)^{3/7}\\
&=\|v\|^{1/7}_{L^\infty(0,T;H)}\Big(\int_0^T\|v(t)\|^2_{L^6(\Omega)}\Big)^{3/7}=\|v\|^{1/7}_{L^\infty(0,T;H)}\,\|v\|^{6/7}_{L^2(0,T;L^6(\Omega))}\\
&\le c\|v\|^{1/7}_{L^\infty(0,T;H)}\,\|v\|^{6/7}_{L^2(0,T;V)},
\end{aligned}$$

and we denote by $D_0$ the rightmost side of this inequality chain, when evaluated for $v=\partial_t\rho$. We also remark that

$$\begin{aligned}
&\|\chi_k\|_{L^{7/4}(0,T;L^{7/2}(\Omega))}=\Big\{\int_0^T\Big(\int_\Omega|\chi_k(x,t)|^{7/2}\,dx\Big)^{1/2}dt\Big\}^{4/7}\\
&=\Big\{\int_0^T\Big(\int_\Omega|\chi_k(x,t)|^4\,dx\Big)^{1/2}dt\Big\}^{\frac12\cdot\frac87}=\|\chi_k\|^{8/7}_{L^2(0,T;L^4(\Omega))}.
\end{aligned}$$

Hence, our estimate for $(\mu-k)^+$ yields the following basic inequality:

$$|\!|\!|(\mu-k)^+|\!|\!|\le kD_1\|\chi_k\|^{8/7}_{L^2(0,T;L^4(\Omega))}\quad\text{for every } k\ge\mu_0^*, \tag{4.6}$$

where $D_1=D_0/\min\{\varepsilon,1\}$, and where the norm $|\!|\!|\cdot|\!|\!|$ is defined by

$$|\!|\!|v|\!|\!|^2:=\sup_{t\in[0,T]}\|v(t)\|^2_H+\int_Q|\nabla v|^2\quad\text{for } v\in C^0([0,T];H)\cap L^2(0,T;V).$$

We notice that the Sobolev inequality (2.19) implies that

$$\|v\|_{L^2(0,T;L^4(\Omega))}\le D_2|\!|\!|v|\!|\!|\quad\text{for every } v\in C^0([0,T];H)\cap L^2(0,T;V), \tag{4.7}$$

where $D_2$ depends on $\Omega$ and $T$, only. At this point, we select a sequence $\{k_j\}$ depending on a real parameter $m>1$ as follows:

$$k_j:=M\big(2-2^{-j}\big)\quad\text{for } j=0,1,\dots,\quad\text{with}\quad M:=m\mu_0^*; \tag{4.8}$$

note that $k_0=M>\mu_0^*$. Then, owing to (4.6) and (4.7), it is not difficult to check that

$$\begin{aligned}
&\big(k_{j+1}-k_j\big)\,\|\chi_{k_{j+1}}\|_{L^2(0,T;L^4(\Omega))}\le\|(\mu-k_j)^+\|_{L^2(0,T;L^4(\Omega))}\le D_2|\!|\!|(\mu-k_j)^+|\!|\!|\\
&\le k_j\,D_1D_2\|\chi_{k_j}\|^{8/7}_{L^2(0,T;L^4(\Omega))}.
\end{aligned}\tag{4.9}$$

Therefore, if we set

$$S_j := \|\chi_{k_j}\|_{L^2(0,T;L^4(\Omega))} \quad \text{for } j = 0, 1, \ldots,$$

then the following inequality holds:

$$S_{j+1} \leq \frac{k_j}{k_{j+1} - k_j} D_1 D_2 S_j^{8/7} \leq 4 D_1 D_2 \, 2^j S_j^{8/7} \quad \text{for } j = 0, 1, \ldots.$$

Using [12, Lemma 5.6, p. 95], we conclude that $S_j \to 0$ as $j \to \infty$, provided that

$$S_0 = \|\chi_{k_0}\|_{L^2(0,T;L^4(\Omega))} \leq (4 D_1 D_2)^{-7} 2^{-49}. \tag{4.10}$$

On the other hand, we notice that $\chi_{k_0} = \chi_M$, and we recall that $M > \mu_0^*$ and $m = M/\mu_0^*$, by (4.8). Moreover, we observe that $\chi_M = 1 < (\mu - \mu_0^*)/(M - \mu_0^*)$ when $\mu > M$, and that $\chi_M = 0$ otherwise. Therefore, using (4.7) and (4.6) with $k = k_0 = M$, we have:

$$\begin{aligned} S_0 &\leq \frac{1}{M - \mu_0^*} \|(\mu - \mu_0^*)^+\|_{L^2(0,T;L^4(\Omega))} \leq \frac{D_2}{M - \mu_0^*} \|\!|(\mu - \mu_0^*)^+|\!\| \\ \frac{D_1 D_2}{m - 1} &\|\chi_{\mu_0^*}\|^{8/7}_{L^2(0,T;L^4(\Omega))} \leq \frac{D_1 D_2}{m - 1} |\Omega|^{\frac{1}{4}\cdot\frac{8}{7}} T^{\frac{1}{2}\cdot\frac{8}{7}}. \end{aligned}$$

We are now in a position to choose $m := 1 + D_1 D_2 |\Omega|^{2/7} T^{4/7} (4 D_1 D_2)^7 2^{49}$. Then, $m > 1$ and (4.10) is satisfied. Consequently,

$$\|\chi_{2M}\|_{L^2(0,T;L^4(\Omega))} = \lim_{j \to \infty} S_j = 0,$$

due to Beppo Levi's Monotone Convergence Theorem. This implies that $\mu \leq 2M$ a.e. in $Q$, and the boundedness of $\mu$ claimed in (2.15) is established.

We are left with the task of proving that the limitations for $\rho$ in (2.15) do hold. We find it convenient to set: $\rho_* := \inf_\Omega \rho_0$ (recall that we assumed $\rho_*$ to be strictly positive). Moreover, we rewrite (2.13) in the form:

$$\delta \partial_t \rho - \Delta \rho + f_1'(\rho) = g, \quad \text{where} \quad g := \mu - f_2'(\rho), \tag{4.11}$$

and we notice that $g \in L^\infty(Q)$, in view of the above proof and (2.3). Consequently, in view also of (2.4), we can choose $r_* \in (0, \rho_*)$ such that $f_1'(r_*) \leq g$ a.e. in $Q$. Then, we test (4.11) by $-(\rho - r_*)^-$ and deduce that

$$\begin{aligned} &\frac{\delta}{2} \int_\Omega |(\rho - r_*)^-(t)|^2 + \int_0^t \int_\Omega |\nabla(\rho - r_*)^-|^2 - \int_0^t \int_\Omega \big(f_1'(\rho) - f_1'(r_*)\big)(\rho - r_*)^- \\ &= \int_0^t \int_\Omega \big(f_1'(r_*) - g\big)(\rho - r_*)^- \leq 0. \end{aligned}$$

We conclude that $(\rho - r_*)^- = 0$ and $\rho \geq r_*$ a.e. in $Q$. In a similar way, for a suitable $r^* < 1$, we show that $\rho \leq r^*$ a.e. in $Q$ by testing (4.11) by $(\rho - 1 + r^*)^+$. We conclude that solutions satisfy all of the requirements stated in (2.15). □

# 5 Long-time behavior

In this section, we prove Theorem 2.5. To this end, we fix any solution $(\mu, \rho)$ to problem (2.12)–(2.14). Our proof of the properties of the $\omega$-limit $\omega(\mu, \rho)$ relies on a number of a priori estimates for $(\mu, \rho)$, and on a well-known tool. For $(\mu_\omega, \rho_\omega)$ any element of $\omega(\mu, \rho)$, and $\{t_n\}$ a corresponding time sequence of type (2.17), we set

$$\mu_n(t) := \mu(t_n + t), \quad \rho_n(t) := \rho(t_n + t) \quad \text{for } t \geq 0, \tag{5.1}$$

and we study the sequence $\{(\mu_n, \rho_n)\}$ on a fixed finite time interval $[0, T]$. Clearly, the pair $(\mu_n, \rho_n)$ enjoys the same regularity as $(\mu, \rho)$, and solves the equations

$$(\varepsilon + 2\rho_n)\partial_t \mu_n + \mu_n \, \partial_t \rho_n - \Delta \mu_n = 0 \qquad \text{a.e. in } Q \tag{5.2}$$

$$\delta \partial_t \rho_n - \Delta \rho_n + f'(\rho_n) = \mu_n \qquad \text{a.e. in } Q; \tag{5.3}$$

moreover, it satisfies the homogeneous Neumann boundary conditions and the Cauchy conditions:

$$\mu_n(0) = \mu(t_n) \quad \text{and} \quad \rho_n(0) = \rho(t_n) \quad \text{a.e. in } \Omega. \tag{5.4}$$

Our argument also relies on two basic identities, to be proved in the next lemma.

**Lemma 5.1.** *The following identities hold:*

$$\delta(\partial_t \rho)^2 - \partial_t \rho \, \Delta \rho + f'(\rho) \, \partial_t \rho = \varepsilon \partial_t \mu + 2\partial_t(\rho\mu) - \Delta \mu \quad \textit{a.e. in } Q; \tag{5.5}$$

$$\begin{aligned} &\delta \int_0^t \|\partial_t \rho(s)\|_H^2 \, ds + \frac{1}{2} \|\nabla \rho(t)\|_H^2 + \int_\Omega f(\rho(t)) \\ &= \frac{1}{2} \|\nabla \rho_0\|_H^2 + \int_\Omega f(\rho_0) + \varepsilon \int_\Omega \mu(t) - \varepsilon \int_\Omega \mu_0 + 2 \int_\Omega (\rho\mu)(t) - 2 \int_\Omega \rho_0 \mu_0, \end{aligned} \tag{5.6}$$

*for every* $t \in [0, T]$.

**Proof.** We have from (2.12) that

$$\mu \, \partial_t \rho = \Delta \mu - \varepsilon \partial_t \mu - 2\rho \, \partial_t \mu = \Delta \mu - \varepsilon \partial_t \mu - 2\partial_t(\rho\mu) + 2\partial_t \rho \, \mu \, .$$

By a simple rearrangement, we deduce that

$$\mu \, \partial_t \rho = \varepsilon \partial_t \mu + 2\partial_t(\rho\mu) - \Delta \mu.$$

On the other hand, multiplication of (2.13) by $\partial_t \rho$ yields:

$$\delta(\partial_t \rho)^2 - \partial_t \rho \, \Delta \rho + f'(\rho) \, \partial_t \rho = \mu \, \partial_t \rho,$$

so that (5.5) immediately follows by comparison. Next, identity (5.6) is arrived at by integrating (5.5) over $\Omega \times (0, t)$ and by noting that, in view of the homogeneous Neumann boundary condition, $\Delta \mu$ does not contribute to the integral. $\square$

We proceed with proving some a priori estimates. In so doing, we depart from our general rule, and write $c$ for constants that do not depend on the final time $T$, although they are allowed to depend on the element $(\mu_\omega, \rho_\omega)$ of the $\omega$-limit under consideration. Whenever a dependence of $c$ on the parameter $T$ cannot be excluded, we stress this possibility by writing $c_T$. Morever, without any loss of generality, we assume that $\varepsilon \leq 1$.

**First a priori estimate.** Just as we did for (3.18), we immediately deduce that

$$\int_0^t \|\nabla\mu(s)\|_H^2\,ds + \frac{\varepsilon}{2}\|\mu(t)\|_H^2 + \int_\Omega (\rho\mu^2)(t) \leq c \quad \text{for every } t > 0. \tag{5.7}$$

This implies, in particular, that

$$\int_0^{+\infty} \|\nabla\mu(t)\|_H^2\,dt < +\infty \quad \text{and} \quad \|\mu_n\|_{L^\infty(0,T;H)} \leq c. \tag{5.8}$$

**Second a priori estimate.** We recall (5.6) and estimate some terms of its right-hand side. We have:

$$\varepsilon\int_\Omega \mu(t) + 2\int_\Omega (\rho\mu)(t) \leq \varepsilon^{1/2}\int_\Omega \mu(t) + 2\int_\Omega (\rho^{1/2}\mu)(t) \leq 2|\Omega| + \varepsilon\|\mu(t)\|_H^2 + \int_\Omega (\rho\mu^2)(t).$$

On the other hand, (5.7) holds and $f$ is bounded from below. Hence, (5.6) yields:

$$\delta\int_0^t \|\partial_t\rho(s)\|_H^2\,ds + \frac{1}{2}\|\nabla\rho(t)\|_H^2 \leq c \quad \text{for every } t > 0, \tag{5.9}$$

whence we have that

$$\int_0^{+\infty} \|\partial_t\rho(t)\|_H^2\,dt < +\infty \quad \text{and} \quad \|\rho_n\|_{L^\infty(0,T;V)} \leq c. \tag{5.10}$$

**Third a priori estimate.** We formally test (2.13) by $-\Delta\rho$ and integrate over $\Omega \times (t_n, t_n + t)$. Due to the convexity of $f_1$ and the boundedness of $f_2''$ (compare with the derivation of (3.16)), we get

$$\begin{aligned}
&\frac{\delta}{2}\|\nabla\rho(t_n+t)\|_H^2 + \int_{t_n}^{t_n+t}\int_\Omega |\Delta\rho|^2 \\
&\leq \frac{\delta}{2}\|\nabla\rho(t_n)\|_H^2 + c\int_{t_n}^{t_n+t}\int_\Omega |\nabla\rho|^2 + \frac{1}{2}\int_{t_n}^{t_n+t}\int_\Omega |\Delta\rho|^2 + \frac{1}{2}\int_{t_n}^{t_n+t}\int_\Omega |\mu|^2.
\end{aligned}$$

We note that the first term on the right-hand side is bounded, since $\rho(t_n)$ is weakly convergent to $\rho_\omega$ in $V$. Hence, owing to (5.9) and (5.7), we can conclude that

$$\|\nabla\rho(t_n+t)\|_H^2 + \int_{t_n}^{t_n+t}\int_\Omega |\Delta\rho|^2 \leq c_T \quad \text{for every } t \in [0,T] \text{ and every } n.$$

By comparison with (2.13), and by exploiting elliptic regularity, we deduce that

$$\int_{t_n}^{t_n+t} \|f_1'(\rho)\|_H^2 \, dt + \int_{t_n}^{t_n+t} \|\rho(t)\|_W^2 \, dt \le c_T \, .$$

In terms of $\rho_n$, all this reads:

$$\|\rho_n\|_{L^2(0,T;W)} \le c_T \quad \text{and} \quad \|f_1'(\rho_n)\|_{L^2(0,T;H)} \le c_T \, . \tag{5.11}$$

**Fourth a priori estimate.** By rewriting (5.2) in the form

$$\partial_t \mu_n = -\frac{\mu_n \, \partial_t \rho_n}{\varepsilon + 2\rho_n} + \frac{\Delta \mu_n}{\varepsilon + 2\rho_n}, \tag{5.12}$$

and owing to the homogeneous Neumann boundary condition, we are entitled to write the following equation in $V^*$ in the framework of the Hilbert triplet $(V, H, V^*)$:

$$\int_\Omega \partial_t \mu(t) \, v = -\int_\Omega \frac{\mu_n(t) \, \partial_t \rho_n(t) \, v}{\varepsilon + 2\rho_n(t)} - \int_\Omega \nabla \mu_n(t) \cdot \nabla \frac{v}{\varepsilon + 2\rho_n(t)} \, , \tag{5.13}$$

for a.a. $t \in (0,T)$ and for every $v \in V$. Starting from this equation, we can prove a bound for $\partial_t \mu_n$ in $L^p(0,T;V^*)$ for some $p > 1$. For a while, we argue for a.a. $t \in (0,T)$ and estimate each term on the right-hand side separately. Owing to the Sobolev inequality (2.19), we get

$$\begin{aligned} &\Big| \int_\Omega \frac{\mu_n(t) \, \partial_t \rho_n(t) \, v}{\varepsilon + 2\rho_n(t)} \Big| \le \varepsilon^{-1} \|\mu_n(t)\|_{L^4(\Omega)} \, \|\partial_t \rho(t)\|_{L^2(\Omega)} \, \|v\|_{L^4(\Omega)} \\ &\le c \|\mu_n(t)\|_{L^4(\Omega)} \, \|\partial_t \rho(t)\|_{L^2(\Omega)} \, \|v\|_V . \end{aligned} \tag{5.14}$$

On the other hand, using the interpolation inequality (2.20) and the Sobolev inequality once more, we obtain that

$$\|\mu_n(t)\|_{L^4(\Omega)} \le \|\mu_n(t)\|_{L^6(\Omega)}^{3/4} \, \|\mu_n(t)\|_{L^2(\Omega)}^{1/4} \le c \|\mu_n(t)\|_V^{3/4} \, \|\mu_n(t)\|_H^{1/4} .$$

Consequently, by accounting for the $L^\infty$ bound of (5.8), we derive from (5.14) that

$$\Big| \int_\Omega \frac{\mu_n(t) \, \partial_t \rho_n(t) \, v}{\varepsilon + 2\rho_n(t)} \Big| \le c \|\mu_n(t)\|_V^{3/4} \, \|\partial_t \rho(t)\|_H \, \|v\|_V . \tag{5.15}$$

This takes care of the first addendum in on the right-hand side of (5.13). As to the second, we have:

$$\Big| \int_\Omega \nabla \mu_n(t) \cdot \nabla \frac{v}{\varepsilon + 2\rho_n(t)} \Big| \le \|\mu_n(t)\|_V \, \|\nabla \big( (\varepsilon + 2\rho_n(t))^{-1} \, v \big) \|_H , \tag{5.16}$$

where it remains for us to estimate the last norm. By applying the Leibniz rule and making use of the Hölder interpolation and the Sobolev inequalities, we get:

$$\begin{aligned} &\|\nabla ((\varepsilon + 2\rho_n(t))^{-1} \, v)\|_H \le 2\varepsilon^{-2} \|v \nabla \rho_n(t)\|_H + \varepsilon^{-1} \|\nabla v\|_H \\ &\le c \big( \|v\|_{L^4(\Omega)} \, \|\nabla \rho_n(t)\|_{L^4(\Omega)} + \|\nabla v\|_H \big) \le c \big( \|\nabla \rho_n(t)\|_{L^4(\Omega)} + 1 \big) \|v\|_V \\ &\le c \big( \|\nabla \rho_n(t)\|_{L^6(\Omega)}^{3/4} \, \|\nabla \rho_n(t)\|_{L^2(\Omega)}^{1/4} + 1 \big) \|v\|_V \le c \big( \|\rho_n(t)\|_W^{3/4} \, \|\rho_n(t)\|_V^{1/4} + 1 \big) \|v\|_V \, . \end{aligned}$$

Hence, on accounting for the $L^\infty$ bound of (5.10), we see that (5.16) becomes:

$$\Big|\int_\Omega \nabla\mu_n(t)\cdot\nabla\frac{v}{\varepsilon+2\rho_n(t)}\Big| \le c\|\mu_n(t)\|_V\big(\|\rho_n(t)\|_W^{3/4}+1\big)\,\|v\|_V\,. \tag{5.17}$$

Since $v\in V$ is arbitrary, by combining (5.13), (5.15), and (5.17), we arrive at:

$$\|\partial_t\mu_n(t)\|_{V^*} \le c\big(\|\mu_n(t)\|_V^{3/4}\,\|\partial_t\rho(t)\|_H + \|\mu_n(t)\|_V\,\|\rho_n(t)\|_W^{3/4} + \|\mu_n(t)\|_V\big), \tag{5.18}$$

for a.a. $t\in(0,T)$ and for every $n$; by estimating each term on the right-hand side, we are going to find some $p>1$ such that

$$\|\partial_t\mu_n\|_{L^p(0,T;V^*)} \le c_T. \tag{5.19}$$

Now, due to (5.8), the last term is bounded in $L^2(0,T)$. As to the first, we observe that the functions

$$t\mapsto\|\mu_n(t)\|_V^{3/4} \quad\text{and}\quad t\mapsto\|\partial_t\rho(t)\|_H$$

are bounded, respectively, in $L^{8/3}(0,T)$ by (5.8), and in $L^2(0,T)$ by (5.10); hence, their product is bounded in $L^{8/7}(0,T)$, by the Hölder inequality. Finally, the middle term of (5.18) can be treated in a similar way, with the use of (5.8) and the first inequality in (5.11). Therefore, (5.19) does hold, with $p=8/7$.

**Conclusion.** Our estimates (5.8), (5.10), and (5.19) ensure that $(\mu,\rho)$ is a bounded and weakly continuous $(H,V)$-valued function. Hence, the first part of the statement in Theorem 2.5 follows from the general theory (see, e.g., [11, p. 12]). We pass to the study of the $\omega$-limit.

Recalling (5.11) and using standard weak and weak star compactness results, we see that there is a triplet $(\mu_\infty,\rho_\infty,\phi_\infty)$ such that

$$\mu_n\to\mu_\infty \qquad \text{weakly star in } L^\infty(0,T;H)\cap L^2(0,T;V), \tag{5.20}$$

$$\rho_n\to\rho_\infty \qquad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W), \tag{5.21}$$

$$f_1'(\rho_n)\to\phi_\infty \qquad \text{weakly in } L^2(0,T;H), \tag{5.22}$$

at least for some subsequence. Our first aim is to prove that $\mu_\infty$ is a nonnegative constant, i.e., that $\mu_\infty(x,t)=\mu_s$ for a.a. $(x,t)\in Q$ for some $\mu_s\in[0,+\infty)$; and, to prove that $\rho_\infty$ is time independent, i.e., that $\rho_\infty(t)=\rho_s$ for a.a. $t\in(0,T)$ for some $\rho_s\in W$. Secondly, we want to prove that the pair $(\mu_s,\rho_s)$ found in such a way is indeed a steady state and coincides with the given pair $(\mu_\omega,\rho_\omega)$.

From the first bounds of (5.8) and (5.10), we immediately deduce that

$$|\nabla\mu_n|\to 0 \quad\text{and}\quad \partial_t\rho_n\to 0 \quad\text{strongly in } L^2(0,T;H).$$

This implies that $\mu_\infty$ is space independent and $\rho_\infty$ is time independent. Thus, we can write $\rho_\infty(t)=\rho_s$ for a.a. $t\in(0,T)$, for some $\rho_s\in W$. Moreover, (5.21) implies strong convergence:

$$\rho_n\to\rho_\infty \quad\text{strongly in } C^0([0,T];H)\cap L^2(0,T;V) \tag{5.23}$$

(see, e.g., [17, Sect. 8, Cor. 4]). Therefore, $f_2'(\rho_n)$ converges to $f_2'(\rho_\infty)$, e.g., strongly in $L^2(0,T;H)$, and it is clear that

$$-\Delta\rho_\infty+\phi_\infty=\mu_\infty-f_2'(\rho_\infty) \quad\text{a.e. in } Q. \tag{5.24}$$

Collecting (5.23) and (5.22), and recalling Lemma 1.3, p. 42, in [1], we conclude that $0 < \rho_\infty < 1$ and $\phi_\infty = f_1'(\rho_\infty)$ a.e. in $Q$. Therefore, (5.24) becomes:

$$0 < \rho_s < 1 \quad \text{and} \quad -\Delta\rho_s + f'(\rho_s) = \mu_\infty \quad \text{a.e. in } Q,$$

and we deduce that $\mu_\infty$ is time independent as well. Thus, $\mu_\infty(x,t) = \mu_s$ for a.a. $(x,t) \in Q$ for some constant $\mu_s$. Furthermore, $\mu_s$ is nonnegative, since $\mu_n \geq 0$ for every $n$. This concludes the proof that $(\mu_s, \rho_s)$ is a steady state.

It remains for us to show that $(\mu_s, \rho_s)$ coincides with $(\mu_\omega, \rho_\omega)$. From (5.23) we see that $\rho_n(0)$ converges strongly in $H$ to $\rho_\infty(0) = \rho_s$; on the other hand, $\rho_n(0) = \rho(t_n)$ converges weakly in $V$ to $\rho_\omega$, by assumption; hence, $\rho_s = \rho_\omega$. A similar argument holds for $\mu_s$ and $\mu_\omega$, because $\mu_n$ converges strongly in $C^0([0,T];V^*)$. Indeed, $\mu_n$ is bounded in $L^\infty(0,T;H)$, by (5.8); on the other hand, (5.19) holds with $p = 8/7 > 1$, and the embedding $H \subset V^*$ is compact; hence, the desired convergence follows from [17, Sect. 8, Cor. 4]. This completes the proof of Theorem 2.5.

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