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NUMERICAL REGULARIZATION FOR SDEs: CONSTRUCTION OF NONNEGATIVE SOLUTIONS

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ABSTRACT: In the numerical solution of stochastic differential equations (SDEs) such appearances as sudden, large fluctuations (explosions), negative paths or unbounded solutions are sometimes observed in contrast to the qualitative behaviour of the exact solution. To overcome this dilemma we construct regular (bounded) numerical solutions through implicit techniques without discretizing the state space. For discussion and classification, the notation of life time of numerical solutions is introduced. Thereby the task consists in construction of numerical solutions with lengthened life time up to eternal one. During the exposition we outline the role of implicitness for this 'process of numerical regularization'. Boundedness (Nonnegativity) of some implicit numerical solutions can be proved at least for a class of linearly bounded models. Balanced implicit methods (BIMs) turn out to be very efficient for this purpose. Furthermore, the local property of conditional positivity of numerical solutions is shown constructively (by special BIMs). The suggested approach also gives some motivation to use BIMs for the construction of numerical solutions for SDEs on bounded manifolds with 'natural conditions' on their boundaries. Finally we suggest to apply these methods to population dynamics in Biology, innovation diffusion in Marketing and to mean reverting processes in Finance, such as stochastic interest rates.

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Key words and phrases. Stochastic differential equations, Numerical methods, Uniform boundedness, Explosions, Life time of numerical solutions, Algebraic constraints.

1. INTRODUCTION

Frequently one encounters with practical model equations which require nonnegativity of the solution components. Such stochastic models can be found in population dynamics, financial models, marketing structures, in quantum optics or in the modelling of water resources. There stochastic differential equations (SDEs) often describe pathwisely their dynamical behaviour. Mostly exact solutions of these 'very erratic' objects are not known. Thereby one has to solve numerically these equations, but then negative solutions can occur what surely does not make any practical sense. Moreover, sudden large fluctuations which we will also call explosions at finite time are observed in contrast to the behaviour of the exact solution. Just, if the numerical solution takes negative values, such explosions occur. In this paper we want to examine numerical solutions of SDEs under this situation and study conditions which ensure nonnegativity of the components of corresponding numerical solutions too, without discretizing the state space. For the sake of discussion and classification we introduce the notion of life time of numerical solutions. The main work is aiming at the construction of numerical solutions with eternal life time or at least with the largest–possible life time while keeping convergence towards the exact solution. In particular, this aim makes sense if one encounters with natural boundaries, as e.g. zero for continuous time interest rates in Finance or innovation diffusion in Marketing. In analogy to continuous time processes (e.g. see Khas'minskij (1980)), we also term this aim as **regularization of numerical solutions**. This regularization is possible by using implicit numerical techniques. Implicit methods are usually introduced to treat certain problems of numerical stability caused by stiff differential systems, i.e. such systems where one observes at least two components with 'low' and 'high velocity'. This paper is to show that they are also appropriate for the construction of nonnegative numerical solutions for a quite general class of both linear and nonlinear SDEs. Besides, to some extent the investigation shall be useful to supply appropriate numerical solutions in the more general situation of SDEs on bounded manifolds with 'natural boundary conditions', such as absorbing or nonattainable boundaries.
The paper is organized as follows. After brief description of the object, recalling some basic facts on SDEs and their numerical analysis and introducing the notion of 'numerical life time' in sections 2 and 3 we start with a series of instructive examples to outline aspects of numerical regularization. The relation between the incorporation of implicitness and extension of life time of numerical solutions will turn out to

be crucial in it. In section 5 the exposition continues with two generalizations and their proofs. After that some simulation studies follow for stochastic interest rates governed by an extension of the model of Cox, Ingersoll & Ross (1985). Eventually we give conclusions and remarks in section 7. The paper is closed by an appendix on some numerical analysis for a SDE (pinned Brownian motion) with two–sided deterministic boundary conditions in section 8.

2. SDEs AND THEIR NUMERICAL SOLUTION

Stochastic differential equations driven by a random force $\{W(t) : t \geq 0\}$ which is often interpreted componentwisely as m–dimensional standard Gaussian process W^j have the general form

$$dX(t) = a(X(t))dt + \sum_{j=1}^{m} b^j(X(t))dW^j(t) \tag{2.1}$$

where a and b are Lipschitz continuous functions on $\mathbb{R}^d$. In general, in contrast to deterministic analysis, the solution of these SDEs strongly depends on the choice of the integration calculus for the stochastic integrals occuring in (2.1). In this paper we will only take into consideration the well–known Itô interpretation for the corresponding stochastic integration. Note that the different stochastic integral interpretations can be transformed into each other in a natural way, cf. Gardiner (1984) and Wong & Zakai (1965). To obtain nonexploding stochastic solution processes $\{X(t) : 0 \leq t \leq T\} \in \mathbb{R}^d$ up to a final time T we should additionally require some polynomial boundedness of the drift $a(x)$ and diffusion functions $b^j(x)$, i.e.

$$\exists K_1 > 0 : \forall x \in \mathbb{R}^d : \; \|a(x)\|^2 + \sum_{j=1}^{m} \|b^j(x)\|^2 \leq K_1^2(1 + \|x\|^2)\,. \tag{2.2}$$

Without loss of generality, we suppose that $\|(.)\|$ denotes the Euclidean vector norm. If (2.2) does not hold then it can happen that the solution $\{X(t) : t \geq 0\}$ only exists uniquely up to a finite stopping time τ. The global requirements of Lipschitz continuity and polynomial boundedness are rather restrictive for systems modelling reality. In this case we point to literature and mention that they can be weakened via 'localization techniques' or construction of stationary measures, e.g. see Khas'minskij (1980) or Ikeda & Watanabe (1981). Anyway, throughout this paper we assume existence and uniqueness of solutions of SDEs at least on a given manifold.

Some of systems (2.1) are explicitely solvable, but, in general, only numerical techniques lead to their solutions. For a massive collection of them see Kloeden & Platen (1992). Further details can be found, e.g. in Mil'shtein (1988), Talay (1990), Newton (1991), Artemiev (1993) or Kloeden, Platen & Schurz (1994). Throughout this paper we concentrate us on 'lower order numerical methods' in order to construct sequences $(Y(t_n))_{n\in\mathbb{N}}$ approximating the solution process $\{X(t) : t \geq 0\}$ at finite time–points t_n from a given time–discretization of a fixed time–interval $[0, T]$. 'Lower order numerical methods' provide values $Y(t_n)$ which are defined by an iterative scheme along this time–discretization and strongly converge to the exact solution with order $\gamma = 0.5$ or $\gamma = 1$ as the maximum step size Δ tends to zero. For the sake of simplicity we only consider deterministic time–discretizations

$$\tau = \tau^{\Delta}([0,T]) = \{t_i : i = 0, 1, ..., n_T; 0 = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots < t_{n_T} = T\}$$

of the interval $[0, T]$ with time step sizes $\Delta_n = t_{n+1} - t_n$. Define $\Delta := \max \Delta_n$, the largest step size for the fixed time–discretization. Then the criterion of strong convergence of sequences $(Y(t_n))_{\Delta>0}$ towards the exact solution $X(t)$ requires that

$$\exists K_2(T) > 0 \, : \, \forall \tau = \tau^{\Delta}([0,T]) \, : \, \forall t_n \in \tau : \mathbb{E} \, ||X(t_n) - Y(t_n)|| \leq K_2(T) \cdot \Delta^{\gamma} \tag{2.3}$$

with a fixed $\gamma > 0$ called the order of convergence (or methods). This error criterion 'pathwisely compares' the exact and numerical solution at the discretization points $t_n (= n \cdot \Delta$ for equidistant discretization) on finite time intervals (i.e. $T < +\infty$), and also allows to conclude their convergence in probability. Simple examples of numerical methods are given by the family of implicit Euler schemes (see Kloeden & Platen (1992))

$$Y^E_{n+1} = Y^E_n + \left(\alpha a(Y^E_{n+1}) + (1-\alpha)a(Y^E_n)\right) \Delta_n + \sum_{j=1}^{m} b^j(Y^E_n)\Delta W^j_n \tag{2.4}$$

or the family of balanced methods (BIMs, see Mil'shtein, Platen & Schurz (1992))

$$\begin{aligned} Y^B_{n+1} \;\; &= \;\; Y^B_n + a(Y^B_n)\Delta_n + \sum_{j=1}^{m} b^j(Y^B_n)\Delta W^j_n \\ & \quad + \Big(C^0\Delta_n + \sum_{j=1}^{m} C^j|\Delta W^j_n|\Big)(Y^B_n - Y^B_{n+1}) \end{aligned} \tag{2.5}$$

where $\alpha \in [0, 1]$ and C^0, C^1 are bounded matrices depending on Y^B_n such that $(I + C^0\Delta_n + \sum_{j=1}^{m} C^j|\Delta W^j_n|)^{-1}$ always exists and is uniformly bounded. I represents the $d \times d$ unit matrix with real entries throughout the paper. Both methods have strong convergence order $\gamma = \frac{1}{2}$. With Y_n we denote $Y(t_n)$ above, i.e. the value of the approximate solution Y using integration step sizes $\Delta_i > 0 (i = 0, 1, 2, ...)$ at time points t_n. $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$ is the abbreviation for the current increment of the corresponding Wiener process. For very specific systems as systems with additive noise, i.e. the diffusion functions $b^j(.)$ of (2.1) do not depend on the state variable x, one even obtains an improvement of the order of strong convergence up to $\gamma = 1.0$.

There are three major reasons why we are not willing to consider numerical methods of higher order here. 'Higher order methods' require too much smoothness of the drift $a(.)$ and diffusion functions b^j as well as more information about the σ-algebra generated by the underlying Wiener process. This can be illustrated by simple examples from finance or population dynamics, see also in section 4. In those cases one cannot prove the convergence rate as predicted for very smooth drift and diffusion functions with respect to fixed terminal times T. Moreover Clark & Cameron (1980) have proved that the order of strong convergence cannot exceed the value 1 provided that one only makes use of the local increments of the Wiener process. Thirdly, the stability behaviour of 'higher order methods' is not clarified so far, except for very simple equations, due to the lack of knowledge about appropriate test equations for stochastic stability of numerical methods.

3. LIFE TIME OF NUMERICAL SOLUTIONS

By numerical experiments confirmed, it seems to exist a relation between nonnegativity and explosions of numerical solutions (In some cases both features exclude each another!). Such explosions we consider as unnatural as long as it is not typical for the analytic solution. Thus we are motivated to introduce the following notion in order to classify the numerical solutions with respect to leaving of natural boundaries, as e.g. the bounded domain of definition of an analytic solution. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ denote the underlying probability space (sometimes called as stochastic basis).

Definition 3.1 *(Life time of numerical solutions)*
Assume that the process $\{X(t) : t \geq 0\} \in \mathbb{R}^d$ satisfying (2.1) has only nonnegative values a.s. provided that $X(0) \geq 0$, i.e. it holds

$$\mathbb{P}(\{\omega \in \Omega : X(t, \omega) < 0\}) = 0 \quad \forall t > 0 . \qquad (*)$$

Then a numerical solution $(Y_n)_{n\in\mathbb{N}}$ *has a* **finite life time** *if there is a finite stopping time* $\tau_n(w) \geq 0$ *such that*

$$Y_n = Y(\tau_n) < 0 \quad (a.s.). \qquad (**)$$

Otherwise we call it a numerical solution with **eternal life time.**

The nonnegativity (inequality sign in the definition) is understood in terms of the components of those vectors involved. Of course, with this definition we do not answer what we should do with the continuation of the numerical solution in the case of negative components. We are only interested to prevent this situation by numerical techniques or at least to reduce the frequency of such appearances. Thus we follow the rule beyond a finite life time it makes no sense to look further at the numerical solution, for the sake of practical requirements. It may be mentioned that conditions $(*)$ and $(**)$ are replaced by

$$\mathbb{P}(\{\omega \in \Omega : X(t,\omega) \notin \overline{\mathbb{D}}\}) = 0 \quad \forall t > 0\,. \qquad (*')$$

and

$$Y_n = Y(\tau_n) \notin \overline{\mathbb{D}} \quad \text{(a.s.)}, \qquad (**')$$

respectively, in case of more general consideration of SDEs on manifolds $\mathbb{D} \in \mathbb{R}^d$ (assume that $X(0) \in \overline{\mathbb{D}}$ and process $X(t)$ is well–defined on the domain $\overline{\mathbb{D}}$), e.g. $\mathbb{D} = (0,1)^d$. Thereby we also possess the possibility to characterize and investigate the more complex situation of bounded domains of definition. However, a detailed discussion on this topic we omit here. We will rather deal with domains of the form $\mathbb{D} = (0,+\infty)$ or their d–dimensional products.

4. CONSTRUCTION OF NONNEGATIVE SOLUTIONS IN $\mathbb{R}^1$

4.1 A deterministic one–dimensional model (Motivation)

In deterministic numerical analysis a very simple example is well–known. Consider the equation

$$\dot{x} = \lambda x \quad \text{with} \quad x(0) = x_0 \geq 0$$

and its nonnegative exact solution $x(t) = \exp(\lambda t) \cdot x_0$. Then the Euler scheme gives

$$y_{n+1} = y_n + \lambda y_n \Delta_n = (1 + \lambda\Delta_n) y_n = y_0 \prod_{i=0}^{n} (1 + \lambda\Delta_i)\,. \qquad (4.1)$$

Obviously, this solution is always positive if $y_0 > 0$ and $\lambda \geq 0$ or $|\lambda| < \frac{1}{\Delta_i}$ for all $i = 0, 1, ..., n$. Thus negative values may occur under the assumption $y_0 > 0, \lambda < 0$ and Δ_i large enough. In contrast to this scheme, in the case $y_0 > 0, \lambda < 0$, we can always prevent negative outcomes or even 'explosions' in numerical methods with arbitrary step sizes Δ_i for that linear differential equation. For this purpose we introduce implicit Euler schemes with

$$\begin{aligned} y_{n+1} &= y_n + (\alpha y_{n+1} + (1-\alpha) y_n) \lambda \Delta_n \\ &= \frac{1 + (1-\alpha)\lambda\Delta_n}{1 - \alpha\lambda\Delta_n} y_n = y_0 \prod_{i=0}^{n} \frac{1 + (1-\alpha)\lambda\Delta_i}{1 - \alpha\lambda\Delta_i}, \end{aligned} \tag{4.2}$$

hence it gives positive values if $1 + (1-\alpha)\lambda\Delta_i > 0$ for all $i \in \mathbb{N}$. A generalization of these schemes is presented by the deterministic balanced methods

$$\begin{aligned} y_{n+1} &= y_n + \lambda y_n \Delta_n + c\Delta_n (y_n - y_{n+1}) \\ &= \frac{1 + (\lambda + c)\Delta_n}{1 + c\Delta_n} y_n = y_0 \prod_{i=0}^{n} \frac{1 + (\lambda + c)\Delta_i}{1 + c\Delta_i} \end{aligned} \tag{4.3}$$

for an appropriate constant $c > 0$. Consequently, numerical solutions generated by (4.3) with $c \geq |\lambda|$ or by (4.2) with $\alpha = 1$ are positive and monotonically decreasing for all $y_0 > 0$, $\lambda < 0$ and arbitrary step sizes $\Delta_i \geq 0$. They do not have any explosions, and do not vanish for positive start values as well. Hence they possess eternal life time.

4.2 A stochastic bilinear one–dimensional model

In the following we consider the one–dimensional stochastic equation

$$dX_t = \lambda X_t dt + \gamma X_t dW_t \tag{4.4}$$

driven by the Wiener process W_t. The exact solution of (4.4) is known with

$$X(t) = \exp((\lambda - \gamma^2/2)t + \gamma W_t) \cdot X(0),$$

hence it is always nonnegative for nonnegative initial values $X(0) = x_0$ and does not change its sign. Without loss of generality we suppose $\gamma > 0$ for the further consideration.

Lemma 4.1 *Suppose $X(t)$ satisfies (4.4) with $X(0) > 0$, $\gamma > 0$ and $(1-\alpha\lambda\Delta) > 0$. Then the Euler approximation (2.4) started in $Y_0^E = X(0)$ has finite life time.*

Proof. For simplicity we only consider equidistant approximations. The scheme (2.4) takes for model (4.4) the form

$$\begin{aligned} Y_{n+1}^E &= Y_n^E + \alpha\lambda Y_{n+1}^E\Delta + (1-\alpha)\lambda Y_n^E\Delta + \gamma Y_n^E\Delta W_n \\ &= \frac{1+(1-\alpha)\lambda\Delta+\gamma\Delta W_n}{1-\alpha\lambda\Delta} Y_n^E = Y_0^E\prod_{i=0}^{n}\left(\frac{1+(1-\alpha)\lambda\Delta+\gamma\sqrt{\Delta}\xi_i}{1-\alpha\lambda\Delta}\right). \end{aligned}$$

Define the events $E_i \subseteq \Omega$ (whereas $(\Omega, \mathcal{F}, \mathbb{P})$ represents the Gaussian probability space attached to the random variable W_t, $i \in \mathbb{N}$) with

$$E_i := \{w \in \Omega : 1 + (1-\alpha)\lambda\Delta + \gamma\sqrt{\Delta}\xi_i(w) < 0\}$$

for i.i.d. $\xi_i(w) \in N(0,1)$ (standard Gaussian distributed). Then the event

$$E := \{w \in \Omega : \exists \tau(w) < +\infty, \tau(w) \in \mathbb{N} : Y_{\tau(w)}^E < 0\}$$

can be substituted by the events E_i, and with

$$\mathbb{P}(E_0) = \mathbb{P}\left(\{w \in \Omega : \xi_0(w) < -\frac{1+(1-\alpha)\lambda\Delta}{\gamma\sqrt{\Delta}}\}\right) =: p$$

one obtains

$$\begin{aligned} \mathbb{P}(E) &= \mathbb{P}(E \cap (E_0 \cup \overline{E_0})) \\ &= \mathbb{P}(E|E_0)\mathbb{P}(E_0) + \mathbb{P}(E|\overline{E}_0)\mathbb{P}(\overline{E}_0) \\ &= p + (1-p)\mathbb{P}(E|\overline{E}_0) \\ &= p + (1-p)\mathbb{P}(E \cap (E_1 \cup \overline{E}_1)|\overline{E}_0) \\ &= p + (1-p)\left(\mathbb{P}(E|\overline{E}_0, \overline{E}_1)\mathbb{P}(E_1) + \mathbb{P}(E|\overline{E}_0, \overline{E}_1)\mathbb{P}(\overline{E}_1)\right) \\ &= p + p(1-p) + (1-p)^2\mathbb{P}(E|\overline{E}_0, \overline{E}_1) \\ &= p + p(1-p) + p(1-p)^2 + (1-p)^3\mathbb{P}(E|\overline{E}_0, \overline{E}_1, \overline{E}_2) \\ &\dots \\ &= p\sum_{i=0}^{\infty}(1-p)^i = p\left(\frac{1}{1-(1-p)}\right) = \frac{p}{p} = 1. \end{aligned}$$

Note that it always holds $0 < p < 1$. Thus it must exist (a.s.) a finite stopping time $\tau_n = n(w)\Delta$ such that $Y^E(\tau_n) = Y_n^E < 0$. $\diamond$

Remark (Alternative proof). With the help of the well-known lemma of Borel–Cantelli (or Kolmogorov's $0-1$–law) one also finds a short proof of lemma 4.1. For this purpose we define

$$A_n = \{w \in \Omega : \exists i \leq n : Y_i^E < 0\}$$

for $n \in \mathbb{N}^+$. Then it follows

$$\overline{E} = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \overline{A}_k = \overline{\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k}\,.$$

Because of $\mathbb{P}(\overline{A}_n) = (1 - \mathbb{P}(E_0))^n \quad (n = 1, 2, ...)$ we obtain

$$\sum_{n=0}^{\infty} \mathbb{P}(\overline{A}_n) = \frac{1}{\mathbb{P}(E_0)} < +\infty$$

where

$$\begin{aligned} 1 > \mathbb{P}(E_0) &= \mathbb{P}(\{w \in \Omega : 1 + (1-\alpha)\lambda\Delta + \gamma\sqrt{\Delta}\xi_0(w) < 0\}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\left(-\frac{1+(1-\alpha)\lambda\Delta}{\gamma\sqrt{\Delta}}\right)} \exp(-x^2/2)dx \\ &= \Phi\left(-\frac{1+(1-\alpha)\lambda\Delta}{\gamma\sqrt{\Delta}}\right) > 0 \quad \forall \Delta \in (0, \infty) \end{aligned}$$

and Φ denotes the probability distribution function of the standard Gaussian distribution, hence the assertion $\mathbb{P}(\overline{E}) = 0$. Thus, for the implicit Euler schemes there is always a trajectory with negative outcome under the assumptions of the lemma 4.1. In contrast to that fact, we find numerical methods which only possess nonnegative values. A corresponding assertion is formulated in lemma 4.2.

Lemma 4.2 *Suppose $X(t)$ satisfies (4.4) with $\gamma > 0$ and $X(0) \geq 0$. Then the balanced methods with constants c^0 and $c^1 \geq 0$ have eternal life time provided that*

$$1 + (c^0 + \lambda)\Delta \geq 0 \quad \textit{and} \quad c^1 \geq \gamma\,. \tag{4.5}$$

Proof. This claim follows immediately from the structure of the methods (2.5) applied to the equation (4.4). One receives then

$$\begin{aligned} Y_{n+1}^B &= Y_n^B + \lambda Y_n^B \Delta + \gamma Y_n^B \Delta W_n + (c^0\Delta + c^1|\Delta W_n|)(Y_n^B - Y_{n+1}^B) \\ &= \frac{1 + (c^0 + \lambda)\Delta + \gamma\Delta W_n + c^1|\Delta W_n|}{1 + c^0\Delta + c^1|\Delta W_n|} Y_n^B\,. \end{aligned}$$

Thereby, $Y_{i+1}^B \geq 0$ iff

$$1 + (c^0 + \lambda)\Delta + \gamma\Delta W_i + c^1|\Delta W_i| \geq 0$$

for all $i \in \mathbb{N}$. Obviously, this is true under (4.5). $\diamond$

4.3 A nonlinear diffusion in population dynamics

Often in population dynamics one encounters with diffusion parts of the form

$$b(x) = \sigma\sqrt{x(1-x)}$$

where σ positive. For the sake of illustration we only consider the one–dimensional diffusion process governed by SDE

$$dX_t = \sigma\sqrt{X_t(1-X_t)}dW_t \tag{4.6}$$

where $\sigma > 0$ starting in $X_0 \geq 0$. Sometimes model (4.6) is also refered to Fisher–Wright diffusion. It is not hard to verify that for positive initial values X_0 the corresponding solution process $\{X(t) : t \geq 0\}$ remains still nonnegative with probability one. Due to problems with the positivity of the term structure under the square root of the diffusion part, the system can only live with reasonable interpretation on the interval $[0,1]$, without loosing the space of real numbers. Thereby it is natural to require $X_0 \in [0,1]$. Assume $X_0 \in (0,1)$. Then a corresponding BIM is given by the scheme

$$Y_{n+1} = Y_n + \sigma\sqrt{Y_n(1-Y_n)}\Delta W_n + \sigma c(Y_n)|\Delta W_n|(Y_n - Y_{n+1}) \tag{4.7}$$

with specifically choosen $c(\cdot)$. Natural convergence requirements lead to nonnegativity and some boundedness conditions for this parameter function $c(\cdot)$ to be specified. Suppose $c(.) \geq 0$ is bounded. Then the scheme can be rewritten to

$$Y_{n+1} = \frac{Y_n + \sigma\sqrt{Y_n(1-Y_n)}\Delta W_n + \sigma c(Y_n)Y_n|\Delta W_n|}{1+\sigma c(Y_n)|\Delta W_n|}\,. \tag{4.8}$$

Let us fix a parameter ε with $0 < \varepsilon < 0.5$. This parameter is to express the interest that one does not consider such approximate values which fall into a small neighbourhood of the boundaries. Thus a reasonable choice would be $\varepsilon << 0.5$. In cases of this neighbourhood around the boundaries the process is absorbed, hence it becomes stationary. This replicates the behaviour of the exact solution in the neighbourhood of the boundary points 0 and 1. Now we choose the parameter function

$$c_\varepsilon(y) = \begin{cases} \sqrt{\varepsilon^{-1}} & \text{if} \quad y < \varepsilon \\ \sqrt{\frac{1-y}{y}} & \text{if} \quad \varepsilon \leq y \leq 1-\varepsilon \\ \sqrt{\frac{1-\varepsilon}{\varepsilon}} & \text{if} \quad y > 1-\varepsilon \end{cases}\,. \tag{4.9}$$

In contrast to the previous linear examples here it seems to be very difficult to verify global life of approximations within reasonable boundaries. Thus we restrict the main interest to look for an optimizing of the one–step probabilities to live in given boundaries. It turns out that one can work out a so–called ε-technique.

Lemma 4.3 *For the BIM (4.8) with (4.9) it holds*

$$\mathbb{P}(1 \geq Y_{n+1} \geq 0 | 1 - \varepsilon \geq Y_n \geq \varepsilon) = 1$$

while the order $\gamma = 0.5$ *of strong convergence towards the exact solution of (4.6) is preserved for fixed* ε *with* $0 < \varepsilon < 0.5$.

Remark. Thereby one is able to construct one–step approximations ranging within reasonable boundaries, i.e. without leaving the domain of definition of the diffusion function. The proof of this lemma is obvious and can be omitted, because the parameter function c_ε is positive and bounded, and the convergence is justified by a general proof from Mil'shtein et al. (1992).

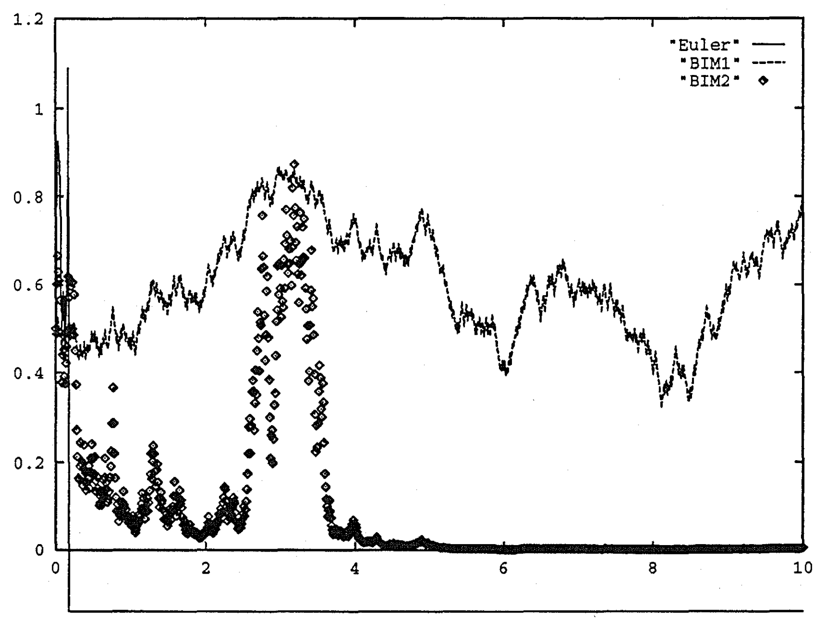

Figure 1. Paths of two BIMs and Euler method with $\sigma = 5$, step size $\Delta = 0.01$ started in $Y_0 = 0.5$.

In figure 1 the trajectories of two specifically chosen BIMs are compared with that of the corresponding Euler method. The BIMs could completely control their movement on the bounded domain of definition of the exact solution for a given path of the underlying Wiener process, whereas the corresponding 'Euler path' immediately explodes and leaves the interval $[0, 1]$.

4.4 A nonlinear diffusion process with linear drift

The following nonlinear diffusion process is given by the solution of the SDE

$$dX_t = [a - bX_t]dt + \sigma\sqrt{\varepsilon_0 + X_t^2}dW_t \tag{4.10}$$

where $a, b, \sigma \geq 0$ and $\varepsilon_0 > 0$. This process belongs to the class of more general mean–reverting processes and has positive solutions.

Once again we suggest a BIM with

$$\begin{aligned} Y_{n+1} = Y_n \quad &+ \quad [a - bY_n]\Delta_n + \sigma\sqrt{\varepsilon_0 + Y_n^2}\Delta W_n \\ &+ \quad \left(b\Delta_n + \sigma(1 + \frac{2\varepsilon_0}{\varepsilon + |Y_n|})|\Delta W_n|\right)(Y_n - Y_{n+1}) \end{aligned} \tag{4.11}$$

where $Y_0 = y_0 > 0$. $\varepsilon > 0$ is again a fixed, small parameter.

Consequently, we obtain the scheme

$$Y_{n+1} = \frac{Y_n + a\Delta_n + \sigma\sqrt{\varepsilon_0 + Y_n^2}\Delta W_n + \sigma(1 + \frac{2\varepsilon_0}{\varepsilon + |Y_n|})Y_n|\Delta W_n|}{1 + b\Delta_n + \sigma(1 + \frac{2\varepsilon_0}{\varepsilon + |Y_n|})|\Delta W_n|}. \tag{4.12}$$

Thus we can construct a numerical solution which has positive outcomes with high probability provided that $Y_0 \geq 0$. This assertion is supported by the following lemma (for a proof in a more general situation, see Theorem 5.2).

Lemma 4.4 *The BIM (4.11) maximizes the conditional one–step ε-probabilities of positivity, i.e. it holds*

$$I\!P(Y_{n+1} > 0 | Y_n \geq \varepsilon) = 1 .$$

4.5 A mean–reverting process with cubic diffusion

Mean–reverting processes in Finance satisfy SDEs of type

$$dX_t = [a - bX_t]dt + \psi(X_t)dW_t \tag{4.13}$$

where $\psi(x) \geq 0$ if $x \geq 0$. By choosing the special function

$$\psi(x) = \frac{\sigma x^3}{1 + x^2}$$

with some positive constant σ it leads to SDE

$$dX_t = [a - bX_t]dt + \frac{\sigma X_t^3}{1 + X_t^2}dW_t . \tag{4.14}$$

For this specific process we recommend to implement the BIM following

$$Y_{n+1} = Y_n + [a - bY_n]\Delta_n + \frac{\sigma Y_n^3}{1+Y_n^2}\Delta W_n + (b\Delta_n + \sigma|\Delta W_n|)(Y_n - Y_{n+1}). \quad (4.15)$$

Resolving this algebraic equation one obtains

$$Y_{n+1} = \frac{Y_n + a\Delta_n + \sigma Y_n|\Delta W_n| + \sigma \frac{Y_n^3}{1+Y_n^2}\Delta W_n}{1 + b\Delta_n + \sigma|\Delta W_n|}. \quad (4.16)$$

It is relatively easy to see that with probability one these numerical solutions possess only positive outcomes when Y_0 starts with a positive value. Take $a, \sigma, b \geq 0$. In contrast to the two previous examples this BIM even allows to construct numerical solutions with eternal life time. We note that a corresponding 'Euler solution' would always provide numerical solutions with finite life time.

Lemma 4.5 *The numerical solution $(Y_n)_{n\in I\!N}$ generated by (4.15) has eternal life time if $Y_0 = y_0 \geq 0$.*

4.6 Diffusion of innovation in Marketing Sciences

Bass (1969) suggested to model how a product, technology, news, ideas, etc. diffuse in a given deterministic media. This model admits to describe the number of adoptions X_t in terms of nonlinear differential equations. Stochastic generalizations of that model have recently led to the class

$$dX_t = [(p + \frac{q}{M}X_t)(M - X_t)]\,dt + \sigma X_t^{\alpha}(M - X_t)^{\beta} dW_t \quad (4.17)$$

started in $X_0 \in [0, M]$, where p, q, M, σ are positive parameters, e.g. see Karmeshu, Lal & Schurz (1995). p can be understood as coefficient of innovation, q as coefficient of immitation, M as total adoption size. Under the presumption $\alpha \geq 0, \beta \geq 1$ one is able to prove

$$\forall t \geq 0 \quad I\!P(X_t \in [0, M]) = 1.$$

The following BIM solves the problem of numerical regularization on the bounded domain $[0, M]$, provided that $\alpha \geq 1, \beta \geq 1$. Take

$$\begin{aligned} Y_{n+1} &= Y_n + (p + \frac{q}{M}Y_n)(M - Y_n)\Delta_n + \sigma Y_n^{\alpha}(M - Y_n)^{\beta}\Delta W_n \\ &\quad + \sigma K(M) Y_n^{\alpha-1}(M - Y_n)^{\beta-1}|\Delta W_n|(Y_n - Y_{n+1}), \end{aligned} \quad (4.18)$$

where $K = K(M)$ is an appropriate positive constant. Then it holds

Lemma 4.6 *The numerical solution $(Y_n)_{n\in I\!N}$ governed by (4.18) has eternal life time if $Y_0 \in [0, M]$ and*

$$K(M) \geq M > 0, \alpha \geq 1, \beta \geq 1, 0 < \Delta_n \leq \frac{1}{p+q} \quad (\forall n \in I\!N).$$

Proof. Use induction on $n \in \mathbb{N}$. Then, after explicit rewriting of (4.18), one finds the following estimation of an upper bound

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{(p + \frac{q}{M}Y_n)\Delta_n + \sigma Y_n^\alpha (M - Y_n)^{\beta-1}\Delta W_n}{1 + \sigma K(M) Y_n^{\alpha-1}(M - Y_n)^{\beta-1}|\Delta W_n|}(M - Y_n) \\ &= Y_n + \rho \cdot (M - Y_n) \;\leq\; Y_n + M - Y_n \;=\; M\,, \end{aligned}$$

since $\rho \leq 1$ if $K(M) \geq M$. Otherwise, nonnegativity of Y_{n+1} follows from $Y_{n+1} =$

$$\frac{Y_n + (p + \frac{q}{M}Y_n)(M - Y_n)\Delta_n + \sigma Y_n^\alpha (M - Y_n)^{\beta-1}((M - Y_n)\Delta W_n + K(M)|\Delta W_n|)}{1 + \sigma K(M) Y_n^{\alpha-1}(M - Y_n)^{\beta-1}|\Delta W_n|}$$

if $K = K(M) \geq M$. Consequently, we have

$$\forall n \in \mathbb{N} \quad \mathbb{P}\,(\,0 \leq Y_n \leq M\,) \;=\; 1\,.\diamond$$

Remark. The boundedness of this sequence of numerical values turns out to be essential for both the interpretability within in the framework of Marketing issues and the proof of rates of convergence.

5. TWO THEOREMS FOR LINEARLY BOUNDED DIFFUSIONS

The numerous examples of the previous section have already indicated that for a quite general class of SDEs one is capable to construct nonnegative numerical solutions. The following result generalizes all the ideas presented before in this respect. In stating the result below we componentwisely understand nonnegativity of vectors, hence the occuring inequality signs between vectors. Define $b^0(x) \equiv a(x)$.

Theorem 5.1 *Assume that there are bounded, real–valued $d \times d$ matrices $C^0, ..., C^m$ with nonnegative entries and positive constants K_3 and K_4 such that*
for all real–valued vectors x with nonnegative components
(1). $[a(x) + C^0(x)x]_i \geq 0$ for all $i = 1, 2, ..., d$,
(2). $[C^j(x)x]_i \geq |[b^j(x)]_i|$ for all $i = 1, 2, ..., d; j = 1, 2, .., m$,
for all real–valued vectors $x \in I\!R^d$

(3). $\sum_{j=0}^{m} \|C^j(x)b^j(x)\|^2 \leq K_3^2(1+\|x\|^2)$

(4). $\forall(\alpha_j \geq 0)_{j=0,1,\dots,m}, \alpha_0 \leq \hat{\alpha}$

$\exists M^{-1}$ *with* $M(x) = I + \sum_{j=0}^{m} \alpha_j C^j(x)$ *and* $\|M^{-1}(x)\| \leq K_4$ *and*

(5). M^{-1} *has only nonnegative entries for nonnegative vectors* x.

Then, for any choice of step sizes $(\Delta_n > 0)_{n\in\mathbb{N}}$, *a numerical method exists which only gives nonnegative approximate values provided that it starts with nonnegative initial vectors* Y_0. *Furthermore these numerical solutions strongly converge towards the exact solution of system (2.1) with order* $\gamma = 0.5$.

Proof. (Constructive)

Consider the balanced implicit method (BIM) generated by the scheme (2.5) with matrices $C^j(.)$. Suppose these matrices satisfy the conditions $(1)-(5)$. Then these methods provide numerical solutions converging strongly towards the exact solution with order $\gamma = 0.5$ for general SDEs (2.1). This can be immediately concluded from the exposition Mil'shtein et al. (1992). Under the condition (4) of Theorem 5.1 the scheme of BIM (2.5) is rewritten to

$$Y_{n+1} = M_n^{-1}(Y_n)\left(Y_n + \sum_{j=0}^{m}\left(b^j(Y_n)\Delta W_n^j + C^j(Y_n)Y_n|\Delta W_n^j|\right)\right) \tag{5.1}$$

where $\Delta W_n^0 = \Delta_n$ and $M_n(x) = I + \sum_{j=0}^{m} C^j(x)|\Delta W_n^j|$. Suppose that $[Y_n]_i \geq 0$. Matrix M_n^{-1} preserves the nonnegativity because of requirement (5). Thereby we only have to check whether the random vector–valued function $\phi(x)$ with

$$\phi(x) = x + \sum_{j=0}^{m}\left(b^j(x)\Delta W_n^j + C^j(x)x|\Delta W_n^j|\right) \tag{5.2}$$

takes nonnegative values for nonnegative vectors $x \in \mathbb{R}^d$. Now we obtain the componentwise estimate

$$\begin{aligned}\phi_i(x) &\geq \sum_{j=0}^{m}\left([b^j(x)]_i\Delta W_n^j + [C^j(x)x]_i|\Delta W_n^j|\right)\\ &\geq \left([a(x)+C^0(x)x]_i\right)\Delta_n + \sum_{j=1}^{m}\left([C^j(x)x]_i - |[b^j(x)]_i|\right)|\Delta W_n^j| \;\geq\; 0\,.\end{aligned}$$

Each component of this random sum is nonnegative under assumptions $(1)-(2)$, hence function ϕ takes nonnegative values for any random input ΔW_n^j. Therefore the new vector Y_{n+1} only possesses nonnegative components. Consequently, the proof of Theorem 5.1 can be completed by induction and we have found a numerical solution with eternal life time. $\diamond$

Remark. At the first glance, it seems that the problem of an appropriate choice of weight matrices is very complicated while requiring condition (4) of Theorem 5.1. However, positive semi–definite matrices C^j or even simpler nonnegative diagonal matrices C^j trivially fulfil requirement (4). Condition (5) seems to be more restrictive, but it is satisfied in case of nonnegative diagonal matrices.

A further result is deduced for one–step approximations of multi–dimensional SDEs with specific drift and diffusion functions. But, in stating this assertion below, we note that the condition of componentwise, linear boundedness of these functions by their components themself turns out to be rather restrictive for multi–dimensional SDEs.

Theorem 5.2 *Assume that the drift and diffusion functions of SDE (2.1) are linearly bounded with nonnegative constants K_j^0 and K_j^1 $(j = 0, 1, ..., m)$ such that*

$$|[b^j(x)]_i| \leq (K_j^0 + K_j^1 |x_i|) \qquad \forall i = 1, 2, ..., d\,.$$

Then there are numerical solutions $(Y_n)_{n \in I\!N}$ which strongly converge with order $\gamma = 0.5$ and maximize the one–step ε–probabilities of positivity, i.e.

$$I\!P(Y_{n+1} > 0 | [Y_n]_i \geq \varepsilon, i = 1, 2, ..., d) = 1$$

for fixed, small values $\varepsilon > 0$.

Proof. (Constructive)

Take the BIM with diagonal matrices $C^j(x) = (c_{i,i}^j)$ and elements

$$c_{i,i}^j(x) = \frac{2K_j^0}{\varepsilon + |x_i|} + K_j^1\,, \quad x = (x_1, ..., x_d)^T; j = 0, 1, ..., m; i = 1, 2, ..., d\,.$$

Thus these functions are bounded and satisfy the conditions for the strong convergence stated in Mil'shtein et al. (1992) with the order 0.5. The nonnegativity of the one–step approximation a.s. follows by means of the same analysis as in the proof above. $\diamond$

6. AN EXTENDED MODEL OF COX–INGERSOLL–ROSS

For modelling of the behaviour of interest rates one often makes use of special mean–reverting processes. Such processes have been considered by many authors, e.g. Vasiček (1977), Courtadon (1982) or Cox, Ingersoll & Ross (1985). Somehow it seems to be natural that one exclusively models with nonnegative interest rates $r(t)$. Nonnegativity of them ensures monotonicity of the pricing functional of an asset, option, etc., e.g. when

$$\text{Price}(t) \; = \; \exp\left(-\int_t^T r(s)\,ds\right) \; , \; 0 \leq t \leq T$$

as price of zero coupon bond normalized to one at maturity–time T. In one factor models one encounters with SDEs for interest rates governed by the general class

$$dr(t) = [a_0 - a_1 r(t)]dt + \sigma(r(t))^p dW(t) \tag{6.1}$$

starting in $r(0) = r_0 \geq 0$ where a_0, a_1, σ and $p \geq 0$. Vasiček (1977) started with analytic examinations of effects on corresponding term structures under the presumption $p = 0$. The drawback of his model lies in the presence of negative interest rates. After Courtadon (1982) with $p = 1.0$, Cox, Ingersoll & Ross (1985) have investigated term structures using interest rates (6.1) with $p = 0.5$. The latter two models only possess nonnegative outcomes for the interest rate. Meanwhile, some real data analysis has shown that exponents close to $p = 1.5$ are somehow more realistic within the framework of model (6.1), cf. remarks in Chan et al. (1992) and Jaschke (1994). Anyway, for the sake of illustration, our numerical studies shall mainly concern with the situation $p = 0.5$.

An analytic solution of (6.1) uniquely exists for almost all parameter constellations, but it is very complicated. For $p = 0.5$ the solution process $\{r(t) : t \geq 0\}$ belongs to the class of Bessel–type diffusions. This can be motivated by computation of stationary solutions and their probability density. In general (i.e. except for the special cases $p = 0$, $p = 1$ or $a_0 = \frac{1}{4}\sigma^2$ and $p = 0.5$), an explicit expression for the pathwise solution of this SDE is not known up to now. Note that for $a_0 = \frac{1}{4}\sigma^2$ and $p = 0.5$, via Itô formula, the solution is found to be a squared Ornstein–Uhlenbeck process, namely

$$r(t) = \exp(-a_1 t)\left(\sqrt{r(0)} + \frac{\sigma}{2}\int_0^t \exp(\frac{a_1}{2}s)\,dW(s)\right)^2 \; .$$

Even in this special case, although the probability distribution is completely known, one has to approximate pathwisely a stochastic integral for given increments of the underlying Wiener process.

In Ikeda & Watanabe (1981) one finds a corresponding justification (argumentation) that this model only lives on the nonnegative real axis. Thus the equation is well–defined. Consequently, the requirement of nonnegativity on numerical solutions makes sense and is an urgent task for construction procedures. Following the ideas of the previous sections one tries to find an appropriate BIM for the construction of nonnegative numerical solutions. Naturally, one should make use of the very specific structure. After analyzing the scheme

$$r_{n+1} = \frac{r_n + a_0\Delta_n + \sigma(c_\varepsilon(r_n)r_n|\Delta W_n| + (r_n)^p\Delta W_n)}{1 + a_1\Delta_n + \sigma c_\varepsilon(r_n)|\Delta W_n|} \tag{6.2}$$

we find several recommendations, among them

$$c_\varepsilon^1(y) = \begin{cases} y^{p-1} & \text{if} \quad y \geq \varepsilon \\ \varepsilon^{p-1} & \text{if} \quad y < \varepsilon \end{cases} \tag{6.3}$$

$$c_\varepsilon^2(y) = 1 + \frac{2}{\varepsilon + |y|}. \tag{6.4}$$

The first recommendation exploits the specific structure of the given SDE, whereas the second corresponds to the general one arising from Theorem 5.2. This example shows how difficult it is to control the 'erratic' behaviour of nonlinear diffusion terms, although they are linearly bounded. Thus we cannot generally expect to provide nonnegative numerical solutions with probability one through Balanced implicit methods with constant integration step size. However, we succeed in maximizing the one–step ε–probabilities, as predicted by Theorem 5.2. Moreover, one gains numerical solutions with nonnegative outcomes with larger probability than that of the usual Euler method. This suspicion can be supported by some numerical experiments where one estimates the corresponding probabilities at discrete times t_n. We receive the inequality

$$\mathbb{P}(Y_n^E < 0) \geq \mathbb{P}(Y_n^{B_1} < 0) \geq \mathbb{P}(Y_n^{B_2} < 0)$$

for all time points t_n provided that $Y_0^E = Y_0^{B_1} = Y_0^{B_2} \geq 0$.

Numerical Experiments for an Explicitly Solvable SDE.

A very simple example with known pathwise solution is given by

$$dr(t) = dt + 2\sqrt{r(t)}\, dW(t) \tag{6.5}$$

within the class of processes mentioned above. After using the Itô formula one encounters with its solution expression

$$r(t) = \left(\sqrt{r(0)} + W(t)\right)^2, \quad r(0) = r_0 \geq 0\,. \tag{6.6}$$

In passing, we remark that this SDE seems to be the only SDE of type (6.1) where its solution expression does not have to be approximated for its pathwise description in any form, while the underlying Wiener path is given at discrete time points. Only for the sake of some illustration we carry out numerical experiments for this example. In the following two figures we compare the numerical behaviour of the explicit Euler method with that of BIMs B_1 and B_2 with recommendations c_ε^1 and c_ε^2, respectively.

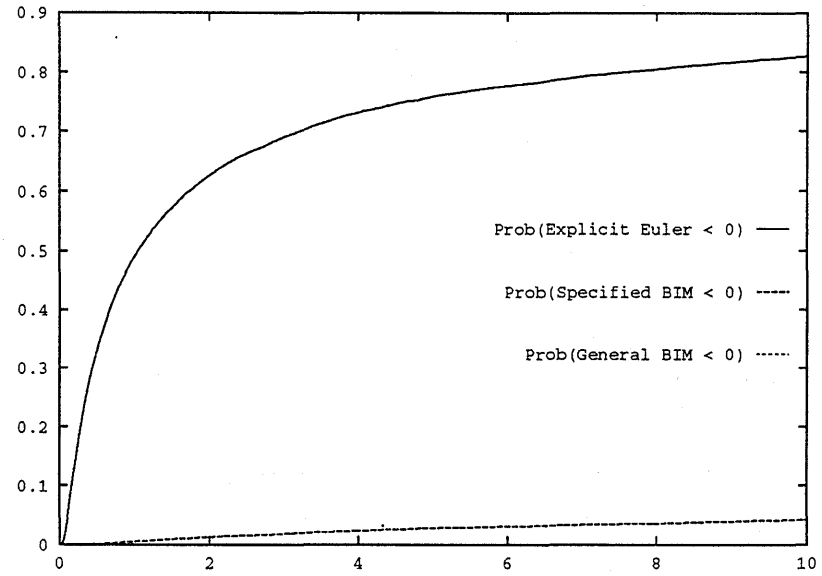

Figure 2. Evolution of estimates for the probability falling into negative approximation values started in $r_0 = 0.5$ and using step size $\Delta = 0.01$, $\varepsilon = 0.03$ and sample size $N = 10000$.

Figure 2 shows the estimates for the corresponding probabilities falling into negative real axis. There a significant difference between those methods is visible. The method B_2 seems to have no paths with negative outputs, whereas B_1 possesses negative outcomes with positive probability. However, the explicit Euler method has the highest probability of negativity. This obviously contradicts to the behaviour of the exact solution. The basic assertion of figure 2 can be confirmed by any other choice of step sizes, but in detail we observe a strong dependence of results on the amount of initial value r_0. For very small step sizes one receives the predicted convergence, and the three methods above approximately merge into the same scheme evaluation. Of course, because the class of BIMs tries to correct the explicit Euler method with terms of 'neighboured order', we do not expect a decisive improvement of the strong error $e(t_n) = \mathbb{E}\ |r(t_n) - Y_n|$ itself. In these experiments we obtain a slight worsening of the estimates of this error behaviour at an initial period of time, whereas the long term error even becomes smaller than that of Euler method. This is visualized in figure 3.

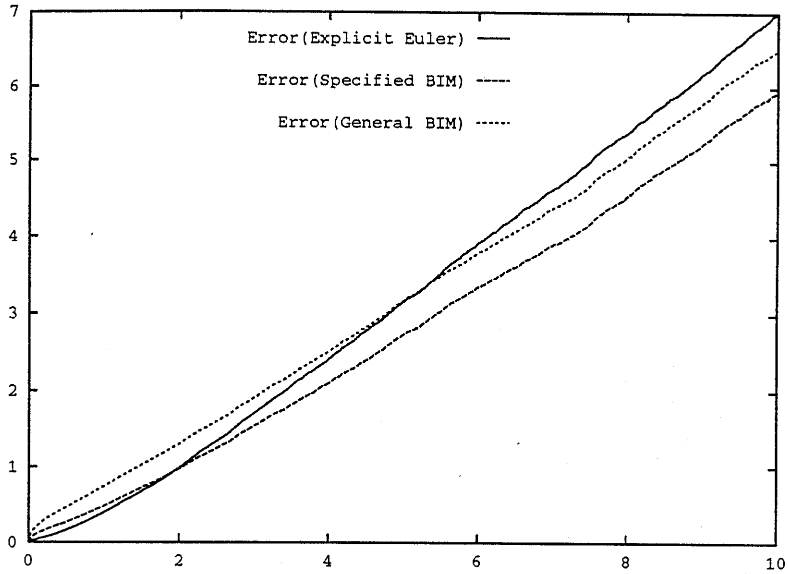

Figure 3. Evolution of strong error estimates for the Euler method and BIMs started in $r_0 = 0.5$ and using step size $\Delta = 0.01$, $\varepsilon = 0.03$ and sample size $N \doteq 10000$.

Adequate incorporation of implicitness plays an essential role for the process of numerical regularization. It is reasonable to take into account the requirements of controlling both error and nonnegativity of numerical solutions. This can be also made visible as follows. We consider once again BIMs (6.2) with weights

$$c_\rho(y) = \rho, \rho \geq 0$$

and estimate probabilities of the corresponding numerical solutions leaving the natural boundary zero. As in figure 4 we observe that increasing implicitness, i.e. increasing parameter $\rho \geq 0$, reduces the probabilities of 'numerical negativity' with fixed step size. Thereby effects of incorporated implicitness on the process of numerical regularization have been confirmed, as already underlined in previous sections. Now it is worth noting that such increasing parameter ρ of implicitness (at least from a certain critical value on) can also effect large strong errors up to their divergence. For confirmation of this fact one could repeat numerical experiments in analogy to figure 3. Thus the 'art of numerical regularization' consists of finding appropriate parameters or weights to control both error and nonnegativity!

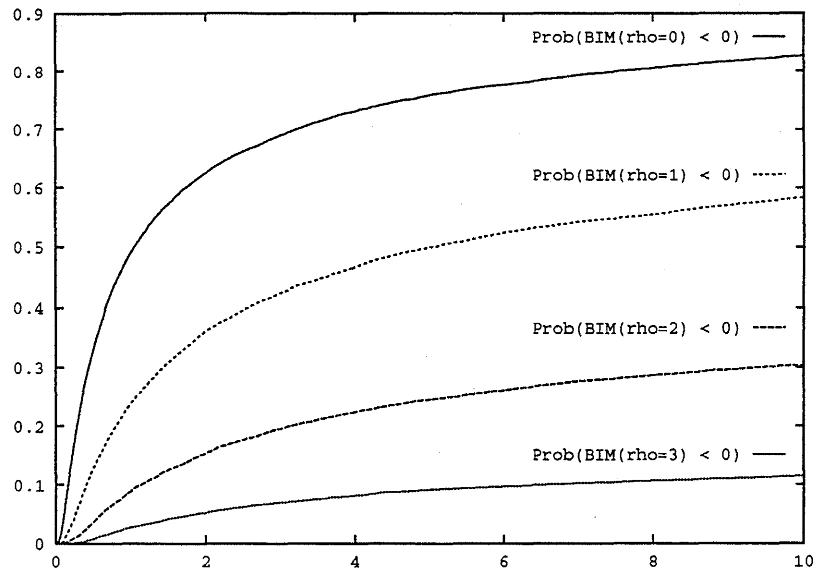

Figure 4. Evolution of estimates for the probability of BIMs falling into negative approximation values started in $r_0 = 0.5$ and using step size $\Delta = 0.01$, parameters $\rho = 0, 1, 2, 3$ and sample size $N = 10000$.

All in all we succeeded in improving the qualitative behaviour of the paths of numerical solutions. Which method has to be finally prefered depends on the model issue and task to be solved. Consequently, this decision is left to the user himself. For the simulation of the class of special mean–reverting processes discussed in this section we suggest to take the BIMs using correction weights c_ε^1 or c_ε^2 with a small ε. Once again we stress that $\varepsilon > 0$ is to be fixed, due to convergence requirements. In passing we note, for more systematic investigation of numerical methods with lower order of convergence applied to model (6.1), we point to a forthcoming paper of Küchler & Schurz (1994). There aspects of stochastic numerical analysis of term structures of interest rates following equation (6.1) with general nonnegative exponent p will be collected, as well as some simulation studies for this specific model in view of its application in Mathematical Finance. For an analytic investigation using stochastic Lyapunov–type methods we refer to a forthcoming manuscript of the author. There pathwise existence, uniqueness, regularity, stationarity and nonnegativity will be enlightened within a general framework. Besides, another approach to discrete time analysis of interest rates is presented in Pfann, Schotman & Tschernig (1994). They use nonlinear, autoregressive models to explore effects of nonlinearity on discrete time term structures.

7. CONCLUSIONS AND REMARKS

For very specific SDEs one is able to construct positive numerical solutions (a.s.) without loss of the order of strong convergence and without discretizing the domain of definition of exact solutions. The class of **B**alanced **I**mplicit **M**ethods (**BIMs**) turns out to be already efficient to solve this task. The role of implicitness for the construction of adequate numerical solutions on the positive real half space has been outlined in this exposition (cf. figure 4). As a by-product, we have found the hand rule 'Increasing implicitness in the numerical solution implies increasing time staying on the positive axis' for a certain class of SDEs.

For qualitative judgement about numerical solutions we introduced the notion of life time. The well–known **Euler method fails** to provide approximations with eternal life time, i.e. there are simple examples where this method leads to values outside the domain of definition of the exact solution at a finite stopping time with probability one. This message is important in so far that the Euler method represents the most–known, simplest and frequently used, stochastic numerical method.

Usual discrete time methods (as strong Taylor or explicit methods, even methods with higher order of convergence) would mostly lead to the problem of having approximate values outside the domain of definition of the exact solution. This is particularly precarious for the construction of numerical solutions for SDEs on bounded manifolds or with algebraic constraints. To prevent this appearance or reduce its frequency we recommend to work out corresponding implicit techniques, above all appropriate BIMs. In this respect this paper exhibits a first trial to handle with numerical analysis for **S**tochastic **D**ifferential **A**lgebraic **E**quations (**SDAEs**) with nonanticipating algebraic constraints (see also in appendix).

Balanced implicit methods (BIMs) are appropriate to treat at least the linear influence caused by drift and diffusion functions of SDEs in numerical methods. Besides, their justified application requires relatively low smoothness conditions on the coefficients of considered SDE (i.e. drift and diffusion) and the smallest information on the σ–algebra generated by the underlying Wiener process. Therefore their implementation costs lesser computational effort than that of corresponding methods of higher order of strong convergence. The gap how the weight matrices $C^j(x)$ have to be chosen in BIMs can be filled by using 'local linearization techniques', e.g. by the use of current Jacobi matrices of drift and diffusion functions or their positive semi–definite regularizations – a kind of stochastic Rosenbrock methods (sometimes called linear–implicit methods, cf. Artemiev (1993)). Thus we may encourage the reader to **apply** the techniques described herein to models for dynamics **in Biology**, in **Hydrology**, in **Marketing Sciences** or in **Finance**. For example, by them one is able to construct discrete time approximations which only possess nonnegative values (almost surely) for interest rates following the continuous time models of Dothan (1978) or Courtadon (1982), see Hull (1989) or Duffie (1992) for their definition and role in Finance.

In linearly bounded models, i.e. where drift and diffusion functions of the considered SDE are bounded by polynomials of degree one, one can carry out a so-called local ε–technique for probabilities of numerical positivity. However, in some models the suggested numerical methods lead to a slight worsening of the global errors at an initial period of time, whereas they reduce the 'long term error'. These facts can be easily seen within the class of stochastic processes based on an extended model of Cox, Ingersoll & Ross (1985) in Finance (cf. figure 3). Nevertheless, the rate of strong convergence could be preserved. Consequently, there is the challenge of getting a balance between the requirements of strong convergence and positivity of numerical solutions.

At the end we point out that all the examinations and results of this paper only make sense if the exact solutions have positive outcomes with high probability, or more general, if they move on bounded manifolds and do not leave them. We are also conscious that the present knowledge when one should discretize the domain of definition and when not is very little for such stochastic differential systems. Another alternative is performed by **stochastic adaptation** of step size selection depending on the current distance to the boundaries. However, this procedure without truncation rules assumes the possibility of 'almost infinite refinement' of discretization. Hence an arbitrary access to the source of random noise is required then. This can represent an impractical requirement (except for short term problems, pure simulation purposes or stochastic adaptive techniques with appropriate truncation rules). Thus the question 'Which method has to be prefered when?' has to be left to the experience of the reader.

8. APPENDIX: DISCUSSION ON BROWNIAN BRIDGES

In this appendix we briefly report about an observation during discussion on SDEs and their numerical analysis with two–sided boundary conditions, i.e. with deterministic initial and deterministic terminal conditions. Such solutions can be generated by discontinuities in the drift part of SDEs. As a simple example, we analyze approximations of **Brownian Bridges**, cf. Karatzas & Shreve (1991). This stochastic process – sometimes also called pinned Brownian motion – can be described by the one–dimensional equation

$$dX_t \;=\; \frac{b-X_t}{T-t}dt + dW_t\,, \tag{8.1}$$

started in $X_0 = a$, pinned to $X_T = b$ and defined on $t \in [0,T]$, where a and b are some fixed real numbers. According to Corollary 6.10 of Karatzas & Shreve (1991), the process

$$X_t \;=\; \begin{cases} a(1-\frac{t}{T}) + b\frac{t}{T} + (T-t)\int_0^t \frac{dW_s}{T-s} & \text{if} \quad 0 \le t < T \\ b & \text{if} \quad t = T \end{cases} \tag{8.2}$$

is the pathwise unique solution of (8.1) which is Gaussian distributed with continuous paths (a.s.) and expectation function

$$m(t) \;=\; \mathbb{E}\, X_t \;=\; a(1-\frac{t}{T}) + b\frac{t}{T} \tag{8.3}$$

on $[0,T]$.

Surely, up to any terminal time $T^* < T$, we can provide numerical solutions which are pathwisely converging towards the exact solution (8.2). But what happens with these numerical solutions when one takes the time–limit towards the terminal time T? Can we achieve a preservation of boundary condition $X_T = b$ in the numerical solution Y under nonboundedness of drift part of the underlying SDE?

For a first numerical approach, one may discuss the behaviour of numerical solutions given by the **family of implicit Euler methods** (2.4). This class of numerical methods applied to equation (8.1) lies in the class of Balanced implicit methods and is governed by the scheme

$$Y_{n+1} \;=\; Y_n + \left[\alpha\frac{b-Y_{n+1}}{T-t_{n+1}} + (1-\alpha)\frac{b-Y_n}{T-t_n}\right]\Delta_n + \Delta W_n \tag{8.4}$$

where $\alpha \in \mathbb{R}^+ = [0,+\infty), Y_0 = a$ and $n = 0,1,...,n_T - 1$. Obviously, in the case $\alpha = 0$, it holds

$$Y^0(T) \;:=\; Y_{n_T} \;=\; \lim_{n\to n_T} Y_n \;=\; b + \Delta W_{n_T-1}\,. \tag{8.5}$$

Thus, **explicit Euler method ends in random terminal values**, which is a contradiction to the behaviour of exact solution (8.2)! Otherwise, in the case $\alpha > 0$, one may explicitly rewrite (8.4) to $Y_{n+1} \;=$

$$\frac{T-t_{n+1}}{T-t_{n+1}+\alpha\Delta_n}Y_n - \frac{(1-\alpha)(T-t_{n+1})\Delta_n}{(T-t_n)(T-t_{n+1}+\alpha\Delta_n)}Y_n + \frac{T-t_{n+1}}{T-t_{n+1}+\alpha\Delta_n}\Delta W_n$$

$$+\frac{(1-\alpha)(T-t_{n+1})\Delta_n}{(T-t_n)(T-t_{n+1}+\alpha\Delta_n)}b+\frac{\alpha\Delta_n}{T-t_{n+1}+\alpha\Delta_n}b. \tag{8.6}$$

A simple analysis of (8.6) shows that

$$Y^\alpha(T) := Y_{n_T} = \lim_{n\to n_T} Y_n = b. \tag{8.7}$$

Thus, **implicit Euler methods can preserve (a.s.) the right terminal conditions!** Once again we have demonstrated that implicit techniques are favourable in order to guarantee algebraic constraints (Of course, this fact has been known in deterministic analysis since a long time!). Summarizing the displayed results, one arrives at the following assertion.

Theorem 8.1 *It holds*

$$\begin{array}{lrcl}
1. & \mathbb{E}\, Y_{n_T} & = & b & \textit{if} & \alpha \geq 0 \\
2. & \mathbb{E}\,(Y_{n_T}-b)^2 & = & \Delta_{n_T-1} & \textit{if} & \alpha = 0 \\
3. & \mathbb{P}(Y_{n_T}=b) & = & 0 & \textit{if} & \alpha = 0 \\
4. & \mathbb{P}(Y_{n_T}=b) & = & 1 & \textit{if} & \alpha > 0
\end{array}$$

for any choice of step sizes $\Delta_n > 0, n = 0, 1, ..., n_T - 1$, *where sequence* $(Y_n)_{n=0,1,...,n_T}$ *satisfies (8.4).*

Remark. Item 2 of Theorem 8.1 represents a remarkable fact. Discontinuities in the drift part of SDEs can significantly reduce the order γ of mean square convergence of the Euler method which is identical with the Mil'shtein method under additive noise (i.e. state–independent diffusion part) to $\gamma = 0.5$! Note, under classical existence and uniqueness conditions on coeffcients of SDEs with additive noise one proves order $\gamma = 1.0$ of both mean square and pathwise convergence. In contrast to that, implicit Euler methods can obviously preserve this order (even exact replication of boundary conditions, cf. item 4 of Theorem 8.1).

Same results hold for Brownian Bridges when one supposes random boundary values a and b which are independent of σ–algebras

$$\mathcal{F}_t = \sigma(\{W_s : 0 \leq s \leq t\}), \qquad t \in [0,T].$$

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