Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Higher order approximate Markov chain filters

P.E. Kloeden[1] , E. Platen[2,3] , H. Schurz[3]

submitted: 20th October 1992

[1] Deakin University
Geelong
Victoria, 3217
Australia

[2] Australian National University
GPO Box 4
Canberra ACT, 2601
Australia

[3] Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany

Preprint No. 16
Berlin 1992

Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=dbp;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.dbp.de

Higher Order Approximate Markov Chain Filters

P.E. KLOEDEN, E. PLATEN and H. SCHURZ

Abstract. The aim of this paper is to construct higher order approximate discrete time filters for continuous time finite-state Markov chains with observations that are perturbed by the noise of a Wiener process.

1. Introduction.

The systematic construction and investigation of filters for Markov chains goes back to Wonham [11], Zakai [12] and Fujisaki, Kallianpur and Kunita [2]. Later the question of finding discrete time approximations for the optimal filter was considered by Clark and Cameron [1] and Newton [7], [8].

At first we introduce in the following filters for continuous time finite state Markov chains. Let $(\Omega,\ \mathcal{A},\ P)$ be the underlying probability space and suppose that the state process $\xi = \{\xi_t,\ t \in [0,T]\}$ is a continuous time homogeneous Markov chain on the finite state space $\mathcal{S} = \{a_1,\ a_2,\ \ldots,\ a_d\}$. Its d-dimensional probability vector $p(t)$, with components

$$p_i(t) = P\left(\xi_t = a_i\right) \tag{1}$$

for each $a_i \in \mathcal{S}$, then satisfies the vector ordinary differential equation

$$\frac{dp}{dt} = A\,p \tag{2}$$

where A is the intensity matrix. In addition, suppose that the m-dimensional observation process $W = \{W_t,\ t \in [0,T]\}$ is the solution of the stochastic equation

$$W_t = \int_0^t h\left(\xi_s\right)\, ds + W_t^*, \tag{3}$$

where $W^* = \{W_t^*,\ t \in [0,T]\}$ with $W_0^* = 0$ is an m-dimensional standard Wiener process with respect to the probability measure P, which is independent of the process ξ. Finally, let $\mathcal{Y}_t$ denote the σ-algebra generated by the observations W_s for $0 \le s \le t$. In what follows we shall use superscripts to label the components of vector-valued stochastic processes.

Our task is to filter as much information about the state process ξ as we can from the observation process W. With this aim we shall evaluate the conditional expectation

$$E\left(g\left(\xi_T\right) \mid \mathcal{Y}_T\right)$$

with respect to P for a given function $g : \mathcal{S} \rightarrow \Re$.

By application of the Girsanov transformation we obtain a probability measure $\dot{P}$ where

$$d\dot{P} = L_T^{-1}\, dP \tag{4}$$

with

$$L_T = \exp\left(-\frac{1}{2}\int_0^T \left|h\left(\xi_s\right)\right|^2 ds + \int_0^T h\left(\xi_s\right)^\top dW_s\right) \tag{5}$$

such that W is a Wiener process with respect to $\dot{P}$.

Let us introduce the un-normalized conditional probability X_t^i for the state $a_i \in \mathcal{S}$ at time t as the conditional expectation

$$X_t^i = \dot{E}\left(I_{\{a_i\}}\left(\xi_t\right) L_t \mid \mathcal{Y}_t\right), \tag{6}$$

$i \in \{1, \ldots, d\}$, $t \in [0, T]$, with respect to the probability measure $\dot{P}$, where $I_{\{a_i\}}(x)$ is the indicator function taking the value 1 when $x = a$ and the value 0 otherwise. It follows from a basic assertion in Fujisaki, Kallianpur and Kunita [2], also known as Kallianpur-Striebel formula, that the conditional probabilities of ξ_t given $\mathcal{Y}_t$ are

$$P\left(\xi_t = a_i \mid \mathcal{Y}_t\right) = E\left(I_{\{a_i\}}\left(\xi_t\right) \mid \mathcal{Y}_t\right) = X_t^i \Big/ \sum_{k=1}^d X_t^k \tag{7}$$

for $a_i \in \mathcal{S}$ and $t \in [0, T]$, where the d-dimensional process $X_t = \{X_t^1, \ldots, X_t^d\}$ of the un-normalized conditional probabilities satifies the Zakai equation,

$$X_t = p(0) + \int_0^t A\, X_s\, ds + \sum_{j=1}^m \int_0^t H_j\, X_s\, dW_s^j \tag{8}$$

for $t \in [0, T]$, which is a homogeneous linear Ito equation. H_j is the $d \times d$ diagonal matrix with iith component $h_j(a_i)$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m$.

The optimal least squares estimate for $g(\xi_t)$ with respect to the observations W_s for $0 \leq s \leq t$, that is with respect to the σ-algebra $\mathcal{Y}_t$, is given by the conditional expectation

$$\begin{aligned} \Pi_t(g) &= E\left(g\left(\xi_t\right) \mid \mathcal{Y}_t\right) \\ &= \sum_{k=1}^d g\left(a_k\right) X_t^k \Big/ \sum_{k=1}^d X_t^k, \end{aligned} \tag{9}$$

which we call the optimal filter or Markov chain filter.

2. Approximate Filters.

To compute the optimal filter (9) we have to solve the Ito equation (8). In practice, however, it is impossible to detect W completely on $[0,T]$. Electronic devices are often used to obtain increments of integral observations over small time intervals, which in the simplest case are the increments of W in integral form

$$\int_{t_0}^{\tau_1} dW_s^j, \ldots, \int_{t_n}^{\tau_{n+1}} dW_s^j, \ldots,$$

for each $j = 1, \ldots, m$, $\tau_n = n\delta$ for $n = 0, 1, 2, \ldots$. We shall see in the next section that with such integral observations it is possible to construct strong discrete time approximations Y^δ with time step δ of the solution X of the Zakai equation (8). Then for the given function g we can evaluate the expression

$$\Pi_t^\delta(g) = \sum_{k=1}^{d} g(a_k)\, Y_t^{\delta,k} \Big/ \sum_{k=1}^{d} Y_t^{\delta,k} \tag{10}$$

for $t \in [0,T]$, which we shall define to be the corresponding approximate Markov chain filter.

We shall say that a discrete time approximation Y^δ with step size δ converges on the time interval $[0,T]$ with order $\gamma > 0$ to the corresponding solution X of the stochastic differential equation if there exists a finite constant K, not depending on δ, and a $\delta_0 \in (0,1)$ such that

$$\dot{E}\left(\left|X_{\tau_n} - Y_{\tau_n}^\delta\right|\right) \leq K\,\delta^\gamma \tag{11}$$

for all $\delta \in (0,\delta_0)$ and $\tau_n \in [0,T]$. We note that the expectation in (11) is with respect to the probability measure $\dot{P}$ under which W is a Wiener process. Analogously we say that an approximate Markov chain filter $\Pi^\delta(g)$ with step size δ converges on the time interval $[0,T]$ with order $\gamma > 0$ to the optimal filter $\Pi(g)$ for a given function g if there exists a finite constant K, not depending on δ, and a $\delta_0 \in (0,1)$ such that

$$E\left(\left|\Pi_{\tau_n}(g) - \Pi_{\tau_n}^\delta(g)\right|\right) \leq K\,\delta^\gamma \tag{12}$$

for all $\delta \in (0,\delta_0)$ and $\tau_n \in [0,T]$. In contrast with (11) we take the expectation in (12) with respect to the original probability measure P.

PROPOSITION. *An approximate Markov chain filter* $\Pi^\delta(g)$ *with step size* δ *converges on the time interval* $[0,T]$ *with order* $\gamma > 0$ *to the optimal filter*

$\Pi(g)$ for a given bounded function g if the discrete time approximation Y^δ used in it converges on $[0,T]$ to the solution X of the Zakai equation (8) with the same order γ.

PROOF. In view of (12) we need to estimate the error

$$F^\delta_{\tau_n}(g) = E\left(\left|\Pi_{\tau_n}(g) - \Pi^\delta_{\tau_n}(g)\right|\right) = \dot{E}\left(L_{\tau_n}\left|\Pi_{\tau_n}(g) - \Pi^\delta_{\tau_n}(g)\right|\right) \tag{13}$$

for all $\tau_n \in [0,T]$. We shall write

$$G_{\tau_n}(f) = \sum_{k=1}^{d} f(a_k)\, X^k_{\tau_n} \tag{14}$$

and

$$G^\delta_{\tau_n}(f) = \sum_{k=1}^{d} f(a_k)\, Y^{\delta,k}_{\tau_n} \tag{15}$$

for any bounded function $f : \mathcal{S} \to \Re$, $\delta \in (0,\delta_0)$ and $\tau_n \in [0,T]$. Then similarly to Picard [9] we can use (6), (9) and (10) to rewrite the error (13) in the form

$$\begin{aligned} F^\delta_{\tau_n}(g) &= \dot{E}\left(G_{\tau_n}(1)\left|\Pi_{\tau_n}(g) - \Pi^\delta_{\tau_n}(g)\right|\right) \\ &= \dot{E}\left(G_{\tau_n}(1)\left|\frac{1}{G_{\tau_n}(1)}\left(G_{\tau_n}(g) - G^\delta_{\tau_n}(g)\right.\right.\right. \\ &\qquad \left.\left.\left. + \Pi^\delta_{\tau_n}(g)\left(G^\delta_{\tau_n}(1) - G_{\tau_n}(1)\right)\right)\right|\right) \\ &\le \dot{E}\left(\left|G_{\tau_n}(g) - G^\delta_{\tau_n}(g)\right|\right) + \dot{E}\left(\left|\Pi^\delta_{\tau_n}(g)\right|\,\left|\,G^\delta_{\tau_n}(1) - G_{\tau_n}(1)\right|\right) \\ &\le K_1 \sum_{k=1}^{d} \dot{E}\left(\left|\,Y^{\delta,k}_{\tau_n} - X^k_{\tau_n}\right|\right). \end{aligned} \tag{16}$$

Finally, using (11) in (16) gives the estimate $F^\delta_{\tau_n}(g) \le K_2\,\delta^\gamma$ and hence the desired convergence rate. □

3. Explicit Filters.

It remains to describe discrete time approximations converging with a given order $\gamma > 0$ to the solution of the Zakai equation (8) which can be used in a corresponding approximate filter. A systematic presentation of such discrete time approximations can be found in Kloeden and Platen [3]. Given an equidistant time discretization of the interval $[0, T]$ with step size $\delta = \Delta = T/N$ for some $N = 1, 2, \ldots$, we define the partition σ-algebra $\mathcal{P}_N^1$ as the σ-algebra generated by the increments

$$\Delta W_0 = \int_0^{\Delta} dW_s^j, \quad \ldots, \quad \Delta W_{N-1} = \int_{(N-1)\Delta}^{N\Delta} dW_s^j \tag{17}$$

for all $j = 1, \ldots, m$. Thus $\mathcal{P}_N^1$ contains the information about the increments of W for this time discretization. The simplest discrete time approximation obtained from the Euler scheme (see Maruyama [5]) has for the Zakai equation (8) the form

$$Y_{\tau_{n+1}}^{\delta} = \left[I + A\,\Delta + G_n\right] Y_{\tau_n}^{\delta} \tag{18}$$

with

$$G_n = \sum_{j=1}^{m} H_j\,\Delta W_n^j \tag{19}$$

and initial value $Y_0 = X_0$, where I is the $d{\times}d$ unit matrix. The scheme (18) converges under the given assumptions with order $\gamma = 0.5$. For a general stochastic differential equation this is the maximum order of convergence that can be achieved under the partition σ-algebra $\mathcal{P}_N^1$, as was shown by Clark and Cameron [1]. However, the special multiplicative noise structure of the Zakai equation (8) allows the order $\gamma = 1.0$ to be attained with the information contained in $\mathcal{P}_N^1$. Milstein [6] proposed a scheme of order $\gamma = 1.0$, which for equation (8) has the form

$$Y_{\tau_{n+1}}^{\delta} = \left[I + \underline{A}\,\Delta + G_n\left(I + \frac{1}{2}\,G_n\right)\right] Y_{\tau_n}^{\delta} \tag{20}$$

where

$$\underline{A} = A - \frac{1}{2}\sum_{j=1}^{m} H_j^2. \tag{21}$$

Newton [7] searched for a scheme which is asymptotically the "best" in the class of order 1.0 schemes in the sense that it has the smallest leading error coefficient in an error estimate similar to (11). He obtained the scheme

$$Y_{\tau_{n+1}}^{\delta} = \left[I + \underline{A}\,\Delta + G_n + \frac{\Delta^2}{2}A^2 + \frac{\Delta}{2}\,A\,G_n - \frac{\Delta}{2}\,G_n\,A + G_n\underline{A}\,\Delta + \frac{1}{2}\,G_n^2 + \frac{1}{6}G_n^3\right] Y_{\tau_n}^{\delta} \tag{22}$$

which is called asymptotically efficient under $\mathcal{P}_N^1$.

We can obtain higher order convergence by exploiting additional information about the observation process such as contained in the integral observations

$$\text{(23)}\quad \Delta Z_0^j = \int_0^\Delta \int_0^s dW_r^j\, ds, \quad \ldots, \quad \Delta Z_{N-1}^j = \int_{(N-1)\Delta}^{N\Delta} \int_{(N-1)\Delta}^s dW_r^j\, ds$$

for all $j = 1, \ldots, m$, easily measured in practice by digital devices. We shall define as the partition σ-algebra $\mathcal{P}_N^{1.5}$ the σ-algebra generated by $\mathcal{P}_N^1$ together with the multiple integrals $\Delta Z_0^j, \ldots, \Delta Z_{N-1}^j$ for all $j = 1, \ldots, m$. The order 1.5 strong Taylor scheme described in Platen [10] and Kloeden and Platen [3] uses for the Zakai equation (8) only the information contained in $\mathcal{P}_N^{1.5}$. It takes the form

$$Y_{\tau_{n+1}}^\delta = \left[I + \underline{A}\,\Delta + G_n + \frac{\Delta^2}{2} A^2 + A\, M_n - M_n\, A + G_n\, \underline{A}\,\Delta + \frac{1}{2}\, G_n^2 + \frac{1}{6} G_n^3\right] Y_{\tau_n}^\delta$$

(24)
where

$$\text{(25)}\qquad M_n = \sum_{j=1}^m H_j\, \Delta Z_n^j.$$

We note that we obtain the order 1.0 scheme (22) from (24) if we replace the ΔZ_n^j by their conditional expectations under $\mathcal{P}_N^1$ with respect to the probability measure $\dot{P}$, that is we substitute $\frac{1}{2} G_n\, \Delta$ for M_n in (24).

In order to form a scheme of order $\gamma = 2.0$ we need the information from the observation process expressed in the partition σ-algebra $\mathcal{P}_N^2$ which is generated by $\mathcal{P}_N^{1.5}$ together with the multiple Stratonovich integrals

$$\begin{aligned} J_{(j_1,j_2,0),n} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ dW_{s_1}^{j_1} \circ dW_{s_2}^{j_2}\, ds_3, \\ J_{(j_1,0,j_2),n} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ dW_{s_1}^{j_1}\, ds_2 \circ dW_{s_3}^{j_2}, \end{aligned} \qquad (26)$$

for all $n = 0, 1, \ldots, N-1$ and $j_1, j_2 = 1, \ldots, m$. Here the symbol "$\circ$" denotes the Stratonovich integration. Electronic devices can extract these Stratonovich integrals from the observation measurements in practical filtering situations. Using this information we can apply the order 2.0 strong Taylor scheme in Kloeden and Platen [3] to the Zakai equation (8) to obtain the approximation

$$Y_{\tau_{n+1}}^\delta = \left[I + \underline{A}\,\Delta \left(I + \frac{1}{2} \underline{A}\,\Delta\right) - M_n \underline{A} + \underline{A} M_n\right.$$

$$+G_n\left(I+\underline{A}\,\Delta+\frac{1}{2}\,G_n\left(I+\frac{1}{3}G_n\left(I+\frac{1}{4}G_n\right)\right)\right) \tag{27}$$

$$+\sum_{j_1,j_2=1}^{m}\Big(\underline{A}H_{j_2}H_{j_1}\,J_{(j_1,j_2,0),n}+H_{j_2}\underline{A}H_{j_1}\,J_{(j_1,0,j_2),n}$$

$$+H_{j_2}H_{j_1}\underline{A}\,(\Delta\,J_{(j_1,j_2)}-J_{(j_1,j_2,0)}-J_{(j_1,0,j_2)})\Big)\Big]\,Y^{\delta}_{\tau_n}.$$

We remark that the corresponding orders of strong convergence of the schemes described above follow from a convergence theorems in Platen [10] or Kloeden, Platen [3].

4. Implicit Filters.

Explicit discrete time approximations can sometimes behave numerically unstable. In such a situation control is lost over the propagation of errors and the approximation is rendered useless. We can then use an implicit discrete time scheme to obtain a numerically stable approximation. Here we state some of the implicit discrete time schemes from Kloeden and Platen [3], [4] applied to the Zakai equation (8). These express an iterate in terms of itself and its predecessor, but since the Zakai equation is linear they can all be rearranged algebraically to express the next iterate just in terms of its predecessor. After rearranging we have from the family of implicit Euler schemes

$$Y^{\delta}_{\tau_{n+1}}=(I-\alpha A\,\Delta)^{-1}\left[I+(1-\alpha)\,A\,\Delta+G_n\right]Y^{\delta}_{\tau_n} \tag{28}$$

where $\alpha\in[0,1]$ denotes the degree of implicitness. The scheme (28) converges with order $\gamma=0.5$. The family of implicit Milstein schemes, all of which converge with order $\gamma=1.0$, gives us

$$Y^{\delta}_{\tau_{n+1}}=(I-\alpha\underline{A}\,\Delta)^{-1}\left[I+(1-\alpha)\underline{A}\,\Delta+G_n\left(I+\frac{1}{2}\,G_n\right)\right]Y^{\delta}_{\tau_n}. \tag{29}$$

In principle to each explicit scheme there corresponds a family of implicit schemes by making implicit the terms involving the nonrandom multiple stochastic integrals such as Δ or $\frac{1}{2}\Delta^2$. As a final example we mention the order 1.5 implicit Taylor scheme yielding

$$Y^{\delta}_{\tau_{n+1}} = \left(I-\frac{1}{2}\underline{A}\,\Delta\right)^{-1}\Big[I+\frac{1}{2}\underline{A}\,\Delta+G_n\,\underline{A}\,\Delta-M_n\underline{A}+\underline{A}M_n$$

$$+G_n\left(I+\frac{1}{2}\,G_n\left(I+\frac{1}{3}G_n\right)\right)\Big]\,Y^{\delta}_{\tau_n}. \tag{30}$$

5. A Numerical Example.

We consider the random telegraphic noise process, that is the two state continuous time Markov chain ξ on the state space $\mathcal{S} = \{-1, +1\}$ with intensity matrix

$$A = \begin{bmatrix} -50.0 & 50.0 \\ 50.0 & -50.0 \end{bmatrix}$$

and initial probability vector $p(0) = (0.9, 0.1)$. Further, we suppose that the observation process W satisfies the stochastic equation (3) with $h(1) = 5$ and $h(-1) = 0$.

Our task is to determine the actual state of the chain on the basis of these observations. We could say that ξ_t has most likely the value $+1$ if $P\left(\xi_t = +1 \,|\mathcal{Y}_t\right) \geq 0.5$. We evaluate the conditional probability

$$p_1(t) = P\left(\xi_t = +1 \,|\mathcal{Y}_t\right) = E\left(I_{\{+1\}}\left(\xi_t\right) |\mathcal{Y}_t\right) = \Pi_t\left(I_{\{+1\}}\right),$$

which is the optimal filter here. To obtain an approximation of $\Pi_t\left(I_{\{+1\}}\right)$ we can use a filter $\Pi_t^\delta\left(I_{\{+1\}}\right)$ based on a discrete time approximation. For a comparison of approximate filters we shall suppose that we have here a scenario of a realization of the Markov chain on the interval $[0,4]$ with $\xi_t = 1$ for $0 \leq t < 0.5$ and $\xi_t = -1$ for $0.5 \leq t \leq 4.0$. Using this realization of the Markov chain we computed the approximate filters $\Pi_t^\delta\left(I_{\{+1\}}\right)$ for the same realization of the Wiener process W^* using the above mentioned schemes with equidistant step size $\delta = \Delta = 2^{-7}$.

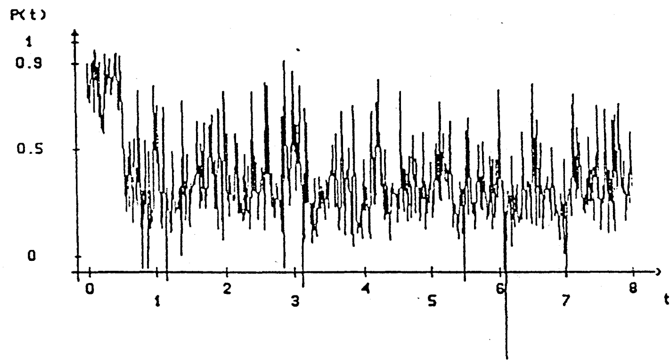

Figure 1. $p_1(t)$ for the explicit order 1.5 strong Taylor filter.

Two calculated $p_1(t)$ paths are plotted in Figures 1 and 2 respectively. The result for the order 1.5 Taylor filter which is an explicit one is plotted in Figure 1. Considerably more sensitive detections of the jump of the Markov chain from state 1 to state -1 at $t = 0.5$ were obtained by implicit schemes.

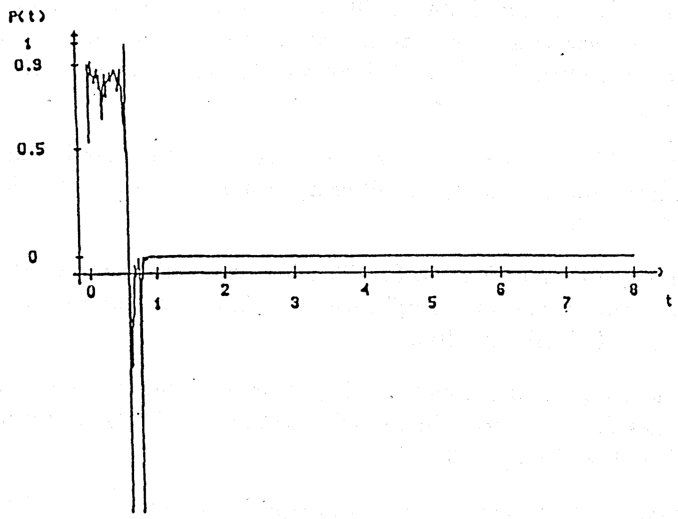

Figure 2. $p_1(t)$ for the implicit order 1.5 Taylor filter.

Figure 2 shows the result with the implicit order 1.5 Taylor scheme. The above numerical example underlines the importance of implicit stochastic numerical schemes. Also the additional information given by multiple stochastic integrals turns out to be substantial for a sensitive detection of a signal.

Authors' addresses

P.E. Kloeden, Deakin University, Geelong, Victoria, 3217, Australia

E. Platen, Australian National University, Canberra, ACT, 2601, Australia and Institute Appl. Anal. Stochastics, Mohrenstr. 39, Berlin, 1086

H. Schurz, Institute Appl. Anal. Stochastics, Mohrenstr. 39, Berlin, 1086

References.

[1] Clark, J.M.C., and Cameron, R.J. The maximum rate of convergence of discrete approximations for stochastic differential equations. Springer Lecture Notes in Control and Inform. Sc. Vol. 25, (1980), pp. 162-171.

[2] Fujisaki, M., Kallianpur, G., and Kunita, H. Stochastic differential equations for the nonlinear filtering problem. Osaka J. Math. **9**, (1972), 19-40.

[3] Kloeden, P.E., and Platen, E. The Numerical Solution of Stochastic Differential Equations. Applications of Mathematics Series, Nr. 23. Springer, Heidelberg, (1992).

[4] Kloeden, P.E., and Platen, E. Higher-order implicit strong numerical schemes for stochastic differential equations. J. Statist. Physics, Vol. 66, No. 1/2 (1992), 283-314.

[5] Maruyama, G. Continuous Markov processes and stochastic equations. Rend. Circolo Math. Palermo **4**, (1955), 48-90.

[6] Milstein, G.N. Approximate integration of stochastic differential equations. Theor. Prob. Appl. **19**, (1974), 557-562.

[7] Newton, N.J. An asymptotically efficient difference formula for solving stochastic differential equations. Stochastics **19**, (1986), 175-206.

[8] Newton, N.J. Asymptotically efficient Runge-Kutta methods for a class of Ito and Stratonovich equations. SIAM J. Appl. Math. 51, (1991), 542-567.

[9] Picard, J. Approximation of nonlinear filtering problems and order of convergence. Springer Lecture Notes in Control and Inform. Sc. Vol. 61, (1984), pp. 219-236.

[10] Platen, E. An approximation method for a class of Ito processes. Lietuvos Matem. Rink. **21**, (1981), 121-133.

[11] Wonham, W.M. Some applications of stochastic differential equations to optimal nonlinear filtering. SIAM J. Control **2**, (1965), 347-369.

[12] Zakai, M. On the optimal filtering of diffusion processes. Z. Wahrsch. verw. Gebiete **11**, (1969), 230-343.

Veröffentlichungen des Instituts für Angewandte Analysis und Stochastik

Preprints 1992

1. D.A. Dawson and J. Gärtner: Multilevel large deviations.
2. H. Gajewski: On uniqueness of solutions to the drift–diffusion–model of semiconductor devices.
3. J. Fuhrmann: On the convergence of algebraically defined multigrid methods.
4. A. Bovier and J.-M. Ghez: Spectral properties of one–dimensional Schrödinger operators with potentials generated by substitutions.
5. D.A. Dawson and K. Fleischmann: A super–Brownian motion with a single point catalyst.
6. A. Bovier, V. Gayrard: The thermodynamics of the Curie–Weiss model with random couplings.
7. W. Dahmen, S. Prößdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations I: stability and convergence.
8. A. Rathsfeld: Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries. Part I: The wedge, Part II: The cube.
9. G. Schmidt: Boundary element discretization of Poincaré–Steklov operators.
10. K. Fleischmann, F. I. Kaj: Large deviation probability for some rescaled superprocesses.
11. P. Mathé: Random approximation of finite sums.
12. C.J. van Duijn, P. Knabner: Flow and reactive transport in porous media induced by well injection: similarity solution.
13. G.B. Di Masi, E. Platen, W.J. Runggaldier: Hedging of options under discrete observation on assets with stochastic volatility.
14. J. Schmeling, R. Siegmund–Schultze: The singularity spectrum of self-affine fractals with a Bernoulli measure.
15. A. Koshelev: About some coercive inequalities for elementary elliptic and parabolic operators.