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Sturm-Liouville boundary value problems with operator potentials and unitary equivalence

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Abstract

Consider the minimal Sturm-Liouville operator $A = A_{\min}$ generated by the differential expression $\mathcal{A} := -\frac{d^2}{dt^2} + T$ in the Hilbert space $L^2(\mathbb{R}_+, \mathcal{H})$ where $T = T^* \geq 0$ in $\mathcal{H}$. We investigate the absolutely continuous parts of different self-adjoint realizations of $\mathcal{A}$. In particular, we show that Dirichlet and Neumann realizations, A^D and A^N, are absolutely continuous and unitary equivalent to each other and to the absolutely continuous part of the Krein realization. Moreover, if $\inf \sigma_{\rm ess}(T) = \inf \sigma(T) \geq 0$, then the part $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma(A^D))$ of any self-adjoint realization $\widetilde{A}$ of $\mathcal{A}$ is unitarily equivalent to A^D. In addition, we prove that the absolutely continuous part $\widetilde{A}^{ac}$ of any realization $\widetilde{A}$ is unitarily equivalent to A^D provided that the resolvent difference $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1}$ is compact. The abstract results are applied to elliptic differential expression in the half-space.

Contents

1 Introduction

Let T be a non-negative self-adjoint operator in an infinite dimensional separable Hilbert space $\mathcal{H}$. We consider the minimal Sturm-Liouville operator A generated by the differential expression

$$\mathcal{A} := -\frac{d^2}{dt^2} + T \tag{1.1}$$

in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$ of $\mathcal{H}$-valued square summable vector-valued functions. Following [18, 19] the minimal operator $A := A_{\min}$ is defined as the closure of the operator A' defined by

$$A' := \mathcal{A} \upharpoonright \mathcal{D}_0, \quad \mathcal{D}_0 := \left\{ \sum_{1\le j\le k} \phi_j(t) h_j : \begin{array}{c} \phi_j \in W_0^{2,2}(\mathbb{R}_+) \\ h_j \in \mathrm{dom}\,(T), \quad k \in \mathbb{N} \end{array} \right\}, \tag{1.2}$$

where $W_0^{2,2}(\mathbb{R}_+) := \{\phi \in W^{2,2}(\mathbb{R}_+) : \phi(0) = \phi'(0) = 0\}$, that is, $A_{\min} := \overline{A'}$. It is easily seen that A is a closed non-negative symmetric operator in $\mathcal{H}$ with equal deficiency indices $n_\pm(A) = \dim(\mathcal{H})$. The adjoint operator A^* of $A = A_{\min}$ is the maximal operator denoted by $A_{\max}$ Extensions of A are usually called realizations of $\mathcal{A}$, self-adjoint extensions are called self-adjoint realizations. Self-adjoint realizations of $\mathcal{A}$ were firstly investigated by M. L. Gorbachuk [18] in the case of finite intervals I. Namely, he showed that the traces of vector-functions $f \in \mathrm{dom}\,(A_{\max})$ belong to the space $\mathcal{H}_{-1/4}(T)$, cf. (5.2). In particular, $\mathrm{dom}\,(A_{\max})$ is not contained in the Sobolev space $W^{2,2}(I, \mathcal{H})$. Based on this result he constructed a boundary triplet for the operator $A_{\max} = A^*_{\min} = A^*$ in the Hilbert space $L^2(I, \mathcal{H})$. These results are similar to those for elliptic operators in domains with smooth boundaries, cf. [3, 21, 29], and go back to classical papers of M.I. Višik [37] and G. Grubb [20].

After the pioneering work [18] the spectral theory of self-adjoint and dissipative realizations of $\mathcal{A}$ in $L^2(I, \mathcal{H})$ has intensively been investigated by several authors for bounded intervals. Their results have been summarized in the book of M.L. and V.I. Gorbachuk [19, Section 4] where one finds, in particular, discreteness criterion, asymptotic formulas for the eigenvalues, resolvent comparability results, etc. Some results from [19] including the construction of a boundary triplet were extended in [11, Section 9] to the case of the semi-axis.

However neither the absolutely continuous spectrum (in short ac-spectrum) nor the unitary equivalence of self-adjoint realizations of $\mathcal{A}$ have been investigated in previous papers. We show, cf. Lemma 5.1, that the domain $\mathrm{dom}\,(A)$ of the minimal operator A coincides algebraically and topologically with the Sobolev space $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : f(0) = f'(0) = 0\}$, where $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ consists of $\mathcal{H}$-valued functions $f(\cdot) \in W^{2,2}(\mathbb{R}_+, \mathcal{H})$ satisfying

$$\|f\|^2_{W^{2,2}_T} := \int_{\mathbb{R}_+} \big(\|f''(t)\|^2_{\mathcal{H}} + \|f(t)\|^2_{\mathcal{H}} + \|Tf(t)\|^2_{\mathcal{H}}\big) dt < \infty.$$

This statement is similar to the classical regularity result for minimal elliptic operators with smooth coefficients, see [3, 21, 29]. Besides we show that the Dirichlet and Neumann realizations defined by

$$\begin{aligned} \mathrm{dom}\,(A^D) &:= \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : \ f(0) = 0\}, \\ \mathrm{dom}\,(A^N) &:= \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : \ f'(0) = 0\} \end{aligned}$$

are self-adjoint, cf. Proposition 5.2. This statement is similar to that of the regularity of Dirichlet and Neumann realizations in elliptic theory (cf. [3, 21, 29]). It looks surprising, that these regularity statements were not obtained in previous papers even in the case of finite intervals.

Moreover, we show that the realizations A^D and A^N are absolutely continuous and unitarily equivalent for any T. We note that these results can easily be obtained using the tensor product structure of A^D and A^N, see Appendix A.2. However, the method fails if the special tensor product structure is missing. We investigate the spectral properties of arbitrary self-adjoint realizations of $\mathcal{A}$ by investigating the corresponding Weyl functions.

We point out that the results substantially differ from those for Dirichlet and Neumann extensions A_I^D and A_I^N of $\mathcal{A}$ on a finite interval I. In the later case *the spectral properties of A_I^D and A_I^N strongly correlate with those of T*, cf. Appendix A.1. In particular, we show that, in contrast to the case of a finite interval, for any $T = T^* \geq 0$ none of the realizations of $\mathcal{A}$ on the semi-axis is pure point, purely singular or discrete. Moreover, we show that for any $T \geq 0$ the Dirichlet and the Neumann realizations A^D and A^N are ac-minimal in the following sense.

Definition 1.1 ([33, Definition 3.5, Definition 5.1]) Let A be a closed symmetric operator and let A_0 be a self-adjoint extension of A.

(i) We say that A_0 is ac-minimal if for any self-adjoint extension $\widetilde{A}$ of A the absolutely continuous part A_0^{ac} is unitarily equivalent to a part of $\widetilde{A}$.

(ii) Let $\sigma_0 := \sigma_{ac}(A_0)$. We say that A_0 is strictly ac-minimal if for any self-adjoint extension $\widetilde{A}$ of A the part $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma_0)$ of $\widetilde{A}$ is unitarily equivalent to the absolutely continuous part A_0^{ac} of A_0.

One of our main results, which follows from Theorem 5.6, Theorem 5.7 and Corollary 5.8, can be summarized as follows:

Theorem 1.2 *Let T be a non-negative self-adjoint operator in the infinite dimensional Hilbert space $\mathcal{H}$ with $t_0 = \inf \sigma(T)$ and $t_1 = \inf \sigma_{\rm ess}(T)$. Further, let $\widetilde{A}$ be a self-adjoint realization of $\mathcal{A}$. Then the following holds:*

(i) *The Dirichlet and the Neumann realizations A^D and A^N of $\mathcal{A}$ are unitarily equivalent, absolutely continuous and $\sigma(A^D) = \sigma_{ac}(A^D) = \sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*

(ii) *The Dirichlet, Neumann and Krein realizations A^D, A^N and A^K of $\mathcal{A}$ are ac-minimal.*

(iii) *These realizations are strictly ac-minimal if and only if $t_0 = t_1$.*

(iv) *If one of the following conditions*

$$(\widetilde{A} - i)^{-1} - (A^D - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \textit{or} \quad (\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$$

is satisfied, then the absolutely continuous part $\widetilde{A}^{ac}$ of $\widetilde{A}$ is unitarily equivalent to the Dirichlet realization A^D.

(v) *If $t_0 = t_1$, then the absolutely continuous part $\widetilde{A}^{ac}$ of $\widetilde{A}$ is unitarily equivalent to the Dirichlet realization A^D provided that*

$$(\widetilde{A} - i)^{-1} - (A^N - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}).$$

At first glance it seems that the ac-minimality of A^D contradicts the classical Weyl-v. Neumann theorem, cf. [22, Theorem X.2.1], which guarantees the existence of a Hilbert-Schmidt perturbation $C = C^*$ such that the spectrum $\sigma(A^D + C)$ of the perturbed operator $A^D + C$ is pure point. But, in fact, Theorem 1.2 presents an explicit example showing that the analog of the Weyl-v.Neumann theorem does not hold for non-additive classes of perturbations. Indeed, Theorem 1.2 shows that for the class of self-adjoint extensions of A the absolutely continuous part can never be eliminated. Moreover, if $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1}$ is compact, then even unitary equivalence holds.

We apply Theorem 1.2 and other abstract results to Schrödinger operators

$$\mathcal{L} := -\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2} + q(x) = -\frac{\partial^2}{\partial t^2} - \Delta_x + q, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

considered in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$, $n \in \mathbb{N}$. Here q is a bounded non-negative potential, $q = \bar{q} \in L^\infty(\mathbb{R}^n),\ q \geq 0$. In this case the minimal elliptic operator $L := L_{\min}$ generated in $L^2(\mathbb{R}^{n+1}_+)$ by the differential expression $\mathcal{L}$ can be identified with the minimal operator $A = A_{\min}$ generated in $\mathfrak{H} = L^2(\mathbb{R}_+, \mathcal{H})$, $\mathcal{H} := L^2(\mathbb{R}^n)$, by the differential expression (1.1) with $T = -\Delta_x + q = T^*$. Therefore and due to the regularity theorem (see [21, 29]) the Dirichlet L^D and the Neumann L^N realizations of the elliptic expression $\mathcal{L}$ are identified, respectively, with the realizations A^D and A^N of the expression $\mathcal{A}$. Moreover, the Krein realization L^K of $\mathcal{L}$ is identical with A^K. This leads to statements on realizations of $\mathcal{L}$ which are similar to those of Theorem 1.2. In fact, one has only to replace A by L in Theorem 1.2. In addition, if the condition

$$\lim_{|x|\to\infty} \int_{|x-y|\leq 1} q(y)dy = 0 \tag{1.3}$$

is satisfied, then L^D and L^N are absolutely continuous and strictly ac-minimal. In particular, $\sigma(L^D) = \sigma_{ac}(L^D) = \sigma(L^N) = \sigma_{ac}(L^N) = [0, \infty)$.

To prove Theorem 1.2 we consider the minimal symmetric operator A associated with the differential expression $\mathcal{A}$ in the framework of extension theory, more precisely, in the framework of boundary triplets intensively developed during the last three decades, see for instance [11, 12, 19] or [9] and references therein. The key role in this theory plays the so-called abstract Weyl function introduced and investigated in [10, 11, 12]. Moreover, the proofs invoke techniques elaborated in [2, 8] and our recent publication [33].

Namely, the proofs of unitary equivalence are based on some statements from [33], which allow to compute the spectral multiplicity function $N_{\widetilde{A}^{ac}}(\cdot)$ of the ac-part $\widetilde{A}^{ac}$ of an extension $\widetilde{A} = \widetilde{A}^*$ in terms of boundary values of the Weyl functions at the real axis, cf. Proposition 2.6 and Corollary 2.7.

We construct a special boundary triplet for the operator A^* (in the case of unbounded $T = T^* \geq 0$) representing A as a direct sum of minimal Sturm-Liouville operators S_n with bounded operator potentials $T_n := TE_T([n-1,n))$, $n \in \mathbb{N}$, where $E_T(\cdot)$ is the spectral measure of T. The corresponding Weyl function $M(\cdot)$ has weak boundary values

$$M(\lambda) := M(\lambda + i0) = \operatorname{w-lim}_{y\downarrow 0} M(\lambda + iy) \quad \text{for a.e.} \quad \lambda \in \mathbb{R}. \tag{1.4}$$

This boundary triplet differs from that used in [11, Section 9]. It is more suitable for the investigation of the ac-spectrum of realizations of $\mathcal{A}$ than that one of [11, Section 9]. Due to the property (1.4) the statement (iv) of Theorem 1.2 follows immediately from our recent result [33, Theorem 1.1]). We note that this is more than one can expect when applying the classical Kato-Rosenblum theorem [22, 36]. Indeed, in accordance with its generalization by Kuroda [26, 27], Birman [4] and Birman and Krein [6] it is required that the resolvent differences in (iv) and (v) of Theorem 1.2 belong to the trace class ideal and not to the compact one as actually assumed. We note also that although the limit (1.4) does not exist for the Weyl function of the Neumann realization A^N the conclusion (iv) of Theorem 1.2 still remains valid, cf. Theorem 1.2(v).

The paper is organized as follows. In Section 2 we give a short introduction into the theory of boundary triplets and the corresponding Weyl functions. We recall here some statements on spectral multiplicity functions and the main theorem from [33] used in the following.

In Section 3 we obtain some new results on symmetric operators $S := \bigoplus_{n=1}^{\infty} S_n$ being an infinite direct sum of closed symmetric operators S_n with equal deficiency indices. First, let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. In general, the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is not a boundary triplet for $S^* = \bigoplus_{n=1}^{\infty} S_n^*$, cf. [23]. Nevertheless, we show, cf. Theorem 3.3, that each boundary triplet Π_n can slightly be modified such that the new sequence $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ of boundary triplets possess the following properties:

(i) the direct sum

$$\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}, \quad \mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad \widetilde{\Gamma}_j := \bigoplus_{n=1}^{\infty} \widetilde{\Gamma}_{jn}, \quad j \in \{0.1\},$$

is already a boundary triplet for S^*;

(ii) the extension $\widetilde{S}_0 := S^* \upharpoonright \ker \widetilde{\Gamma}_0$ satisfies $\widetilde{S}_0 = \bigoplus_{n=1}^{\infty} \widetilde{S}_{0n}$ where

$$\widetilde{S}_{0n} := S_n^* \upharpoonright \ker \widetilde{\Gamma}_{0n} = S_n^* \upharpoonright \ker \Gamma_{0n} =: S_{0n}, \quad n \in \mathbb{N}.$$

Moreover, the Weyl function $\widetilde{M}(\cdot)$ corresponding to the triplet $\widetilde{\Pi}$ is block-diagonal, that is, $\widetilde{M}(\cdot) = \bigoplus_{n=1}^{\infty} \widetilde{M}_n(\cdot)$ where $\widetilde{M}_n(\cdot)$ is the Weyl function corresponding to the triplet $\widetilde{\Pi}_n$, $n \in \mathbb{N}$. This result plays an important role in the sequel. In particular, we show that the self-adjoint extension $S_0 = \bigoplus_{n=1}^{\infty} S_{0n}$ is ac-minimal provided that the deficiency indices $n_{\pm}(S_n)$ are equal and finite. We also prove in this section that if $S_n \geq 0, n \in \mathbb{N}$, then the Friedrichs and Krein extensions S^F and S^K of $S := \bigoplus_{n=1}^{\infty} S_n$, respectively, are the direct sums of Friedrichs and Krein extensions of the summands S_n, i.e., $S^F := \bigoplus_{n=1}^{\infty} S_n^F$ and $S^K := \bigoplus_{n=1}^{\infty} S_n^K$, cf. Corollary 3.5. In a recent paper [24] Theorem 3.3 has been applied to Schrödinger operators with local point interactions.

In Section 4 we consider Sturm-Liouville operators with bounded operator potentials. In this case it is easy to construct a boundary triplet for A^*. We prove here Theorem 1.2 in the case $T \in [\mathcal{H}]$ and establish some additional properties of Krein's realization as well as other realizations.

In Section 5 we extend the results to the case of Sturm-Liouville operators with unbounded non-negative operator potentials. We construct here a boundary triplet for A^* using results of both Sections 3 and 4 and compute the (block-diagonal) Weyl function. Based on this construction we prove Theorem 1.2 for unbounded T and establish some additional properties of Dirichlet, Neumann and other realizations as well. In particular, we prove here the regularity results mentioned above. Finally, we apply the abstract results to the elliptic partial differential expression $\mathcal{L}$ in the half-space.

In the Appendix we present some results on realizations of $\mathcal{A}$ admitting separation of variables, i.e., having a certain tensor product structure.

The main results of the paper have been announced (without proofs) in [32], a preliminary version has been published as a preprint [31]. Since the results of the paper are obvious if $\dim(\mathcal{H}) < \infty$ we consider the case when $\dim(\mathcal{H}) = \infty$.

Notations In the following we consider only separable Hilbert spaces which are denoted by $\mathfrak{H}$, $\mathcal{H}$ etc. A closed linear relation in $\mathcal{H}$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. The set of all closed linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. A graph $\mathrm{gr}\,(B)$ of a closed linear operator B belongs to $\widetilde{\mathcal{C}}(\mathcal{H})$. The symbols $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $[\mathfrak{H}_1, \mathfrak{H}_2]$ stand for the sets of closed and bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, respectively. We set $\mathcal{C}(\mathcal{H}) := \mathcal{C}(\mathcal{H}, \mathcal{H})$ and $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. We regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$ identifying an operator B with its graph $\mathrm{gr}\,(B)$.

The Schatten-v. Neumann ideals of compact operators are denoted by $\mathfrak{S}_p(\mathfrak{H})$, $p \in [1, \infty]$, where $\mathfrak{S}_1(\mathfrak{H})$, $\mathfrak{S}_2(\mathfrak{H})$ and $\mathfrak{S}_\infty(\mathfrak{H})$ are the ideals of trace, Hilbert-Schmidt and compact operators, respectively.

The symbols $\mathrm{dom}\,(T)$, $\mathrm{ran}\,(T)$, $\varrho(T)$ and $\sigma(T)$ stand for the domain, the range, the resolvent set and the spectrum of an operator $T \in \mathcal{C}(\mathcal{H})$, respectively; T^{ac} and $\sigma_{ac}(T)$ stand for the absolutely continuous part and the absolutely continuous spectrum of a self-adjoint operator $T = T^*$.

2 Preliminaries

2.1 Boundary triplets and proper extensions

In this section we briefly recall basic facts on boundary triplets and their Weyl functions, cf. [10, 11, 12, 19].

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim(\ker\,(A^* \mp i)) \leq \infty$.

Definition 2.1 ([19]) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \ \mathrm{dom}\,(A^*) \to \mathcal{H}$ are linear mappings, is called an boundary triplet for A^* if the äbstract Green's identity"

$$(A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}},\qquad f,g\in\mathrm{dom}\,(A^*), \tag{2.1}$$

holds and the mapping $\Gamma := (\Gamma_0,\Gamma_1) : \mathrm{dom}\,(A^*) \to \mathcal{H}\oplus\mathcal{H}$ is surjective.

Definition 2.2 ([19]) A closed extension A' of A is called a proper extension, in short $A' \in \mathrm{Ext}\,_A$, if $A \subset A' \subset A^*$. Two proper extensions A', A'' are called disjoint if $\mathrm{dom}\,(A')\cap\mathrm{dom}\,(A'') = \mathrm{dom}\,(A)$ and transversal if in addition $\mathrm{dom}\,(A') + \mathrm{dom}\,(A'') = \mathrm{dom}\,(A^*)$.

Clearly, any self-adjoint extension $\widetilde{A} = \widetilde{A}^*$ is proper, $\widetilde{A} \in \mathrm{Ext}\,_A$. A boundary triplet $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* exists whenever $n_+(A) = n_-(A)$. Moreover, the relations $n_\pm(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0)\cap\ker(\Gamma_1) = \mathrm{dom}\,(A)$ are valid. In addition one has $\Gamma_0,\Gamma_1\in[\mathfrak{H}_+,\mathcal{H}]$ where $\mathfrak{H}_+$ denotes the Hilbert space obtained by equipping $\mathrm{dom}\,(A^*)$ with the graph norm of A^*.

Using the concept of boundary triplets one can parameterize all proper, in particular, self-adjoint extensions of A. For this purpose we denote by $\widetilde{\mathcal{C}}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$, that is, the set of all closed linear subspaces of $\mathcal{H}\oplus\mathcal{H}$. A linear relation Θ is called *symmetric* if $\Theta\subset\Theta^*$ and self-adjoint if $\Theta = \Theta^*$ where Θ^* is the adjoint relation.For the definition of the inverse and the resolvent set of a linear relation Θ we refer to [13].

Proposition 2.3 *Let* $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ *be a boundary triplet for* A^**. Then the mapping*

$$\mathrm{Ext}\,_A \ni \widetilde{A} \to \Gamma\mathrm{dom}\,(\widetilde{A}) = \{\{\Gamma_0 f,\Gamma_1 f\}:\ f\in\mathrm{dom}\,(\widetilde{A})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.2}$$

establishes a bijective correspondence between the sets $\mathrm{Ext}\,_A$ *and* $\widetilde{\mathcal{C}}(\mathcal{H})$*. We put* $A_\Theta := \widetilde{A}$ *where* Θ *is defined by* (2.2)*. Moreover, the following holds:*

(i) $A_\Theta = A_\Theta^*$ *if and only if* $\Theta = \Theta^*$*;*

(ii) *The extensions* A_Θ *and* A_0 *are disjoint if and only if there is an operator* $B\in\mathcal{C}(\mathcal{H})$ *such that* $\mathrm{gr}\,(B) = \Theta$*. In this case* (2.2) *takes the form*

$$A_\Theta = A^*\upharpoonright\ker(\Gamma_1 - B\Gamma_0);$$

(iii) *The extensions* A_Θ *and* A_0 *are transversal if and only if* A_Θ *and* A_0 *are disjoint and* $\Theta = \mathrm{gr}\,(B)$ *where* B *is bounded.*

With any boundary triplet Π one associates two special extensions $A_j := A^*\upharpoonright\ker(\Gamma_j),\ j\in\{0,1\}$, which are self-adjoint in view of Proposition 2.3. Indeed, we have $A_j := A^*\upharpoonright\ker(\Gamma_j) = A_{\Theta_j},\ j\in\{0,1\}$, where $\Theta_0 := \{0\}\times\mathcal{H}$ and $\Theta_1 := \mathcal{H}\times\{0\}$. Hence $A_j = A_j^*$ since $\Theta_j = \Theta_j^*$. In the sequel the extension A_0 is usually regarded as a reference self-adjoint extension.

Moreover, if Θ is the graph of a closed operator B, i.e. $\Theta = \mathrm{gr}\,(B)$, then the operator A_Θ is denoted by A_B.

Conversely, for any extension $A_0 = A_0^* \in \mathrm{Ext}\,_A$ there exists a boundary triplet $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* such that $A_0 := A^*\upharpoonright\ker(\Gamma_0)$.

2.2 Weyl functions and γ-fields

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [10, 11, 12] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A)$. Following [10, 11, 12] we recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.4 ([10, 11]) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The functions $\gamma(\cdot) : \varrho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \varrho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := \bigl(\Gamma_0 \upharpoonright \mathfrak{N}_z\bigr)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \qquad z \in \varrho(A_0), \tag{2.3}$$

are called the *γ-field* and the *Weyl function*, respectively, corresponding to Π.

It follows from the identity $\mathrm{dom}\,(A^*) = \ker\,(\Gamma_0) \dot{+} \mathfrak{N}_z$, $z \in \varrho(A_0)$, where $A_0 = A^* \upharpoonright \ker\,(\Gamma_0)$, and $\mathfrak{N}_z := \ker\,(A^* - z)$, that the γ-field $\gamma(\cdot)$ is well defined and takes values in $[\mathcal{H}, \mathfrak{H}]$. Since $\Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, it follows from (2.3) that $M(\cdot)$ is well defined too and takes values in $[\mathcal{H}]$. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\varrho(A_0)$. It turns out than the Weyl function $M(\cdot)$ is in fact a $R_{\mathcal{H}}$-function (Nevanlinna or Herglotz function), that is, $M(\cdot)$ is a $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ satisfying

$$M(z) = M(\overline{z})^* \qquad \text{and} \qquad \frac{\mathrm{Im}\,(M(z))}{\mathrm{Im}\,(z)} \geq 0, \qquad z \in \mathbb{C}\backslash\mathbb{R},$$

which in addition satisfies the condition $0 \in \varrho(\mathrm{Im}\,(M(z))),\ z \in \mathbb{C}\backslash\mathbb{R}$.

If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence (see [12, 25]). Therefore $M(\cdot)$ contains (implicitly) full information on spectral properties of A_0. We recall that a symmetric operator is said to be *simple* if there is no non-trivial subspace which reduces it to a self-adjoint operator.

For a fixed $A_0 = A_0^*$ extension of A the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ satisfying $\mathrm{dom}\,(A_0) = \ker\,(\Gamma_0)$ is not unique. If $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is another boundary triplet for A^* satisfying $\ker\,(\Gamma_0) = \ker\,(\widetilde{\Gamma}_0)$, then the corresponding Weyl functions $M(\cdot)$ and $\widetilde{M}(\cdot)$ are related by

$$\widetilde{M}(z) = R^* M(z) R + R_0, \tag{2.4}$$

where $R_0 = R_0^* \in [\widetilde{\mathcal{H}}]$ and $R \in [\widetilde{\mathcal{H}}, \mathcal{H}]$ is boundedly invertible.

2.3 Krein type formula for resolvents and resolvent comparability

With any boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and any proper (not necessarily self-adjoint) extension $A_\Theta \in \mathrm{Ext}\,_A$ it is naturally associated the following (unique) Krein type formula (cf. [10, 11, 12])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^*, \quad z \in \varrho(A_0) \cap \varrho(A_\Theta). \tag{2.5}$$

Formula (2.5) is a generalization of the known Krein formula for resolvents. We note also, that all objects in (2.5) are expressed in terms of the boundary triplet Π (cf. [10, 11, 12]). The following result is deduced from formula (2.5) (cf. [11, Theorem 2]).

Proposition 2.5 *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta_i = \Theta_i^* \in \widetilde{\mathcal{C}}(\mathcal{H}),\ i \in \{1,2\}$. Then for any Schatten-v. Neumann ideal $\mathfrak{S}_p$, $p \in (0,\infty]$, and any $z \in \mathbb{C} \setminus \mathbb{R}$ the following equivalence holds*

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \big(\Theta_1 - z\big)^{-1} - \big(\Theta_2 - z\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$$

In particular, $(A_{\Theta_1} - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \big(\Theta_1 - i\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$.

If in addition $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then for any $p \in (0,\infty]$ the equivalence holds

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longleftrightarrow \Theta_1 - \Theta_2 \in \mathfrak{S}_p(\mathcal{H}).$$

2.4 Spectral multiplicity function and unitary equivalence

Let as above A be a densely defined *simple* closed symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function $M(\cdot)$ and $A_0 = A^* \restriction \ker(\Gamma_0) = A_0^*$.

In our recent publication [33] using some results from [30] we expressed the spectral multiplicity function $N_{A_0^{ac}}(\cdot)$ of A_0^{ac} by means of the limit values of the Weyl function $M(\cdot)$. In general, the limit $M(t) := \operatorname{s-lim}_{y\downarrow 0} M(t+iy)$, $t \in \mathbb{R}$, does not exist. However, for any $D \in \mathfrak{S}_2(\mathcal{H})$ satisfying $\ker(D) = \ker(D^*) = \{0\}$ the "sandwiched" Weyl function,

$$M^D(z) := D^* M(z) D, \quad z \in \mathbb{C}_\pm,$$

admits limit values $M^D(t) := \operatorname{s-lim}_{y\downarrow 0} M^D(t+iy)$ for a.e. $t \in \mathbb{R}$, even in $\mathfrak{S}_2$-norm (cf. [5], [16]). We set

$$d_{M^D}(t) := \dim(\operatorname{ran}(\operatorname{Im}(M^D(t)))),$$

which is well-defined for a.e. $t \in \mathbb{R}$. The function $d_{M^D}(\cdot)$ is Lebesgue measurable and takes values in the set of extended natural numbers $\{0\} \cup \mathbb{N} \cup \{\infty\} = \{0,1,2,\ldots,\infty\}$. The set $\operatorname{supp}_{d_{M^D}} := \{t \in \mathbb{R} : d_{M^D}(t) > 0\}$ is called the support of $d_{M^D}(\cdot)$ and is, of course, a Lebesgue measurable set of $\mathbb{R}$. If the limit $M(t) := \operatorname{s-lim}_{y\downarrow 0} M(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then we set $d_M(t) := \dim(\operatorname{ran}(\operatorname{Im}(M(t))))$.

To state the next result we introduce the notion of the absolutely continuous closure $\operatorname{cl}_{ac}(\delta)$ of a Borel subset $\delta \subset \mathbb{R}$ (see for definition [33, Appendix] as well as [8, 14]). The use of this notion for the investigation of the ac-spectrum of Schrödinger operators etc. see the recent publication [15].

Proposition 2.6 ([33, Proposition 3.2]) *Let A be as above and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function. If D is a Hilbert-Schmidt operator such that $\ker(D) = \ker(D^*) = \{0\}$, then $N_{A_0^{ac}}(t) = d_{M^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_0) = \operatorname{cl}_{ac}(\operatorname{supp}(d_{M^D}))$.*

If, in addition, the limit $M(t) := \text{s-lim}_{y\downarrow 0} M(t+iy)$ *exists for a.e.* $t \in \mathbb{R}$*, then* $N_{A_0^{ac}}(t) = d_M(t)$ *for a.e.* $t \in \mathbb{R}$ *and* $\sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_M))$.

If $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}\,_A$ and is disjoint with A_0, then by Proposition 2.3(ii) there is a self-adjoint operator B acting in $\mathcal{H}$ such that $\widetilde{A} = A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. In this case the multiplicity function $N_{A_B^{ac}}(\cdot)$ is expressed by means of the generalized Weyl function $M_B(\cdot)$ of $\widetilde{A} = A_B$ defined by

$$M_B(z) := (B - M(z))^{-1}, \quad z \in \mathbb{C}_\pm, \tag{2.6}$$

Corollary 2.7 ([33, Corollary 3.3]) *Let* $A, \Pi, M(\cdot)$ *and* D *be as in Proposition 2.6 and let* $B = B^* \in \mathcal{C}(\mathcal{H})$*. Then* $N_{A_B^{ac}}(t) = d_{M_B^D}(t)$ *for a.e.* $t \in \mathbb{R}$ *and* $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_{M_B^D}))$.

If, in addition, the limit $M_B(t) := \text{s-lim}_{y\downarrow 0} M_B(t+iy)$ *exists for a.e.* $t \in \mathbb{R}$*, then* $N_{A_B^{ac}}(t) = d_{M_B}(t)$ *for a.e.* $t \in \mathbb{R}$ *and* $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}\,(d_{M_B}))$.

Finally, we can retranslate the unitary equivalence of ac-parts of two self-adjoint extensions in terms of the limit values of the Weyl functions.

Theorem 2.8 ([33, Theorem 3.4]) *Let* $A,\ \Pi,\ M(\cdot)$ *and* D *be as in Proposition 2.6 and* $B = B^* \in \mathcal{C}(\mathcal{H})$*. Let also* $E_{A_B}(\cdot)$ *and* $E_{A_0}(\cdot)$ *be the spectral measures of* $A_B = A_B^*$ *and* A_0*, respectively. If* δ *is a Borel subset of* $\mathbb{R}$*, then*

(i) $A_0 E_{A_0}^{ac}(\delta)$ *is unitarily equivalent to a part of* $A_B E_{A_B}^{ac}(\delta)$ *if and only if* $d_{M^D}(t) \le d_{M_B^D}(t)$ *for a.e.* $t \in \delta$*;*

(ii) $A_0 E_{A_0}^{ac}(\delta)$ *and* $A_B E_{A_B}^{ac}(\delta)$ *are unitarily equivalent if and only if* $d_{M^D}(t) = d_{M_B^D}(t)$ *for a.e.* $t \in \delta$.

Theorem 2.8 reduces the problem of unitary equivalence of ac-parts of certain self-adjoint extensions of A to the computation of the functions $d_{M^D}(\cdot)$ and $d_{M_B^D}(\cdot)$. If $\delta = \mathbb{R}$, then the absolutely continuous part A_0^{ac} is unitarily equivalent to $\widetilde{A}^{ac} = A_B^{ac}$ if and only if $d_{M^D}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$.

If $M(\cdot)$ is the Weyl function of a boundary triplet Π, then we introduce the maximal normal function

$$m^+(t) := \sup_{y \in (0,1]} \|M(t+iy)\|\,, \quad t \in \mathbb{R}.$$

Theorem 2.9 ([33, Theorem 4.3, Corollary 4.6]) *Let* $A,\ \Pi,\ M(\cdot)$ *and* D *be as in Proposition 2.6. Let* $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}\,_A$ *and* $A_0 := A^* \upharpoonright \ker(\Gamma_0)$*. Assume also that there is a Borel subset* δ *of* $\mathbb{R}$ *such that the maximal normal function* $m^+(t)$ *is finite for a.e.* $t \in \delta$ *and the condition*

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}), \tag{2.7}$$

is satisfied. Then the ac*-parts* $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta)$ *of* $\widetilde{A} E_{\widetilde{A}}(\delta)$ *and* $A_0 E_{A_0}(\delta)$*, respectively, are unitarily equivalent. In particular, if* $m^+(t)$ *is finite for a.e.* $t \in \mathbb{R}$*, then absolutely continuous parts* $\widetilde{A}^{ac}$ *and* A_0^{ac} *are unitarily equivalent.*

One easily verifies that $m^+(t) < \infty$ for a.e. $t \in \delta$ if and only if limit (1.4) exists for a.e. $t \in \delta$. Thus, condition $m^+(t) < \infty$ for a.e. $t \in \delta$ in Theorem 2.9 can be replaced by the assumption that the limit (1.4) exists for a.e. $t \in \delta$, cf. [33, Theorem 1.1].

However, the function $m^+(\cdot)$ depends on the chosen boundary triplet. In [31]-[33] we introduced the invariant maximal normal function $\mathfrak{m}^+(\cdot)$ defined by

$$\mathfrak{m}^+(t) := \sup_{y\in(0,1]} \left\| \frac{1}{\sqrt{\mathrm{Im}(M(i))}} \left(M(t+iy) - \mathrm{Re}(M(i))\right) \frac{1}{\sqrt{\mathrm{Im}(M(i))}} \right\|, \tag{2.8}$$

$t \in \mathbb{R}$. It follows from (2.4) that the invariant maximal normal functions for two boundary triplets $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\widetilde{\Pi} = \{\widetilde{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* coincide whenever $A^* \upharpoonright \ker(\Gamma_0) = A^* \upharpoonright \ker(\widetilde{\Gamma}_0)$. Clearly, $\mathfrak{m}^+(t) < \infty$ if and only if $m^+(t) < \infty$ for any $t \in \mathbb{R}$. However, the invariant maximal normal function is more convenient in applications. We demonstrate this fact in the next section applying this concept to infinite direct sums of symmetric operators.

3 Direct sums of symmetric operators

3.1 Boundary triplets for direct sums

Let S_n be a closed densely defined symmetric operators in $\mathfrak{H}_n,\ n_+(S_n) = n_-(S_n)$, and let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for $S_n^*,\ n \in \mathbb{N}$. Let

$$A := \bigoplus_{n=1}^{\infty} S_n, \quad \mathrm{dom}\,(A) := \bigoplus_{n=1}^{\infty} \mathrm{dom}\,(S_n). \tag{3.1}$$

Clearly, A is a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$ with $n_\pm(A) = \infty$. Obviously, we have

$$A^* = \bigoplus_{n=1}^{\infty} S_n^*, \quad \mathrm{dom}\,(A^*) = \bigoplus_{n=1}^{\infty} \mathrm{dom}\,(S_n^*). \tag{3.2}$$

Let us consider the direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n =: \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of boundary triplets defined by

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad \Gamma_0 := \bigoplus_{n=1}^{\infty} \Gamma_{0n} \quad \text{and} \quad \Gamma_1 := \bigoplus_{n=1}^{\infty} \Gamma_{1n}. \tag{3.3}$$

We note that the Green's identity

$$(S_n^* f_n, g_n) - (f_n, S_n^* g_n) = (\Gamma_{1n} f_n, \Gamma_{0n} g_n)_{\mathcal{H}_n} - (\Gamma_{0n} f_n, \Gamma_{1n} g_n)_{\mathcal{H}_n},$$

$f_n, g_n \in \mathrm{dom}\,(S_n^*)$, holds for every $S_n^*,\ n \in \mathbb{N}$. This yields that the Green's identity (2.1) holds for $A_* := A^* \upharpoonright \mathrm{dom}\,(\Gamma), \mathrm{dom}\,(\Gamma) := \mathrm{dom}\,(\Gamma_0) \cap \mathrm{dom}\,(\Gamma_1) \subseteq \mathrm{dom}\,(A^*)$, that is, for $f = \bigoplus_{n=1}^{\infty} f_n$, $g = \bigoplus_{n=1}^{\infty} g_n \in \mathrm{dom}\,(\Gamma)$ we have

$$(A_* f, g) - (f, A_* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}\,(\Gamma), \tag{3.4}$$

where A^* and Γ_j are defined by (3.2) and (3.3), respectively. However, the Green's identity (3.4) cannot extend to $\mathrm{dom}\,(A^*)$ in general, since $\mathrm{dom}\,(\Gamma)$ is smaller than $\mathrm{dom}\,(A^*)$ generically. It might even happen that Γ_j are not bounded as mappings from $\mathrm{dom}\,(A^*)$ equipped with the graph norm into $\mathcal{H}$. Counterexamples such that $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is not a boundary triplet firstly appeared in [23]).

In this section we show that it is always possible to modify the boundary triplets Π_n in such a way that the new sequence $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ of boundary triplets for S_n^* such that $\widetilde{\Pi} = \bigoplus_{n=1}^\infty \widetilde{\Pi}_n$ defines a boundary triplet for A^* and the relations

$$\widetilde{S}_{0n} := S_n^* \upharpoonright \ker(\widetilde{\Gamma}_{0n}) = S_n^* \upharpoonright \ker(\Gamma_{0n}) =: S_{0n}, \quad n \in \mathbb{N}, \tag{3.5}$$

are valid. Hence $\widetilde{A}_0 := \bigoplus_{n=1}^\infty \widetilde{S}_{0n} = \bigoplus_{n=1}^\infty S_{0n} =: A_0$. We note that the existence of a boundary triplet $\Pi' = \{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ for A^* satisfying $\ker(\Gamma_0') = \mathrm{dom}\,(A_0)$ is known (see [11, 19]). However, in applications we need a special boundary triplet for A^* which respects the direct sum structure and which leads therefore to a block-diagonal form of the corresponding Weyl function. We start with a simple technical lemma.

Lemma 3.1 *Let S be a densely defined closed symmetric operator with equal deficiency indices, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, and let $M(\cdot)$ be the corresponding Weyl function. Then there exists a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for S^* such that $\ker(\widetilde{\Gamma}_0) = \ker(\Gamma_0)$ and the corresponding Weyl function $\widetilde{M}(\cdot)$ satisfies $\widetilde{M}(i) = i$.*

Proof. Let $M(i) = Q + iR^2$ where $Q := \mathrm{Re}(M(i)),\ R := \sqrt{\mathrm{Im}(M(i))}$. We set

$$\widetilde{\Gamma}_0 := R\Gamma_0 \quad \text{and} \quad \widetilde{\Gamma}_1 := R^{-1}(\Gamma_1 - Q\Gamma_0). \tag{3.6}$$

A straightforward computation shows that $\widetilde{\Pi} := \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for A^*. Clearly, $\ker(\widetilde{\Gamma}_0) = \ker(\Gamma_0)$. The Weyl function $\widetilde{M}(\cdot)$ of $\widetilde{\Pi}$ is given by $\widetilde{M}(\cdot) = R^{-1}(M(\cdot) - Q)R^{-1}$ which yields $\widetilde{M}(i) = i$. □

If S is a densely defined closed symmetric operator in $\mathfrak{H}$, then by the first v. Neumann formula the direct decomposition $\mathrm{dom}\,(S^*) = \mathrm{dom}\,(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i}$ holds, where $\mathfrak{N}_{\pm i} := \ker(S^* \mp i)$. Equipping $\mathrm{dom}\,(S^*)$ with the inner product

$$(f,g)_+ := (S^*f, S^*g) + (f,g), \quad f,g \in \mathrm{dom}\,(S^*), \tag{3.7}$$

one obtains a Hilbert space denoted by $\mathfrak{H}_+$. The first v. Neumann formula leads to the following orthogonal decomposition

$$\mathfrak{H}_+ = \mathrm{dom}\,(S) \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i}.$$

Lemma 3.2 *Let S, Π and $M(\cdot)$ be as in Lemma 3.1. If $M(i) = i$, then $\Gamma : \mathfrak{H}_+ \longrightarrow \mathcal{H} \oplus \mathcal{H}$, $\Gamma := (\Gamma_0, \Gamma_1)$ is a contraction. Moreover, Γ isometrically maps $\mathfrak{N} := \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$ onto $\mathcal{H}$.*

Proof. We show that

$$\|\Gamma(f + f_i + f_{-i})\|^2_{\mathcal{H}\oplus\mathcal{H}} = \|f_i + f_{-i}\|^2_+ \tag{3.8}$$

where $f \dotplus f_i \dotplus f_{-i} \in \mathrm{dom}\,(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i} = \mathrm{dom}\,(S^*)$. Since $\mathrm{dom}\,(S) = \ker\,(\Gamma_0) \cap \ker\,(\Gamma_1)$ we find

$$\|\Gamma(f + f_i + f_{-i})\|^2_{\mathcal{H}\oplus\mathcal{H}} = \|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}}.$$

Clearly,

$$\|\Gamma_j(f_i + f_{-i})\|^2_{\mathcal{H}} = \|\Gamma_j f_i\|^2 + 2\,\mathrm{Re}((\Gamma_j f_i, \Gamma_j f_{-i})) + \|\Gamma_j f_{-i}\|^2, \quad j \in \{0, 1\}. \tag{3.9}$$

Using $\Gamma_1 f_i = M(i)\Gamma_0 f_i = i\Gamma_0 f_i$ and $\Gamma_1 f_{-i} = M(-i)\Gamma_0 f_{-i} = -i\Gamma_0 f_{-i}$ we obtain

$$\|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = (\Gamma_0 f_i, \Gamma_0 f_i) - 2\,\mathrm{Re}((\Gamma_0 f_i, \Gamma_0 f_{-i})) + (\Gamma_0 f_{-i}, \Gamma_0 f_{-i}) \tag{3.10}$$

Taking a sum of (3.9) and (3.10) we get

$$\|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = 2\|\Gamma_0 f_i\|^2_{\mathcal{H}} + 2\|\Gamma_0 f_{-i}\|^2_{\mathcal{H}}. \tag{3.11}$$

Combining equalities $\Gamma_1 f_{\pm i} = \pm i\Gamma_0 f_{\pm i}$ with Green's identity (2.1) we obtain $\|\Gamma_0 f_i\|_{\mathcal{H}} = \|f_i\|$ and $\|\Gamma_0 f_{-i}\|_{\mathcal{H}} = \|f_{-i}\|$. Therefore (3.11) takes the form

$$\|\Gamma_0(f_i + f_{-i})\|^2_{\mathcal{H}} + \|\Gamma_1(f_i + f_{-i})\|^2_{\mathcal{H}} = 2\|f_i\|^2 + 2\|f_{-i}\|^2. \tag{3.12}$$

A straightforward computation shows $\|f_i + f_{-i}\|^2_+ = 2\|f_i\|^2 + 2\|f_{-i}\|^2$ which together with (3.12) proves (3.8). Since $\|f_i + f_{-i}\|^2_+ \le \|f\|^2_+ + \|f_i + f_{-i}\|^2_+ = \|f + f_i + f_{-i}\|^2_+$, we get from (3.8) that Γ is a contraction.

Obviously, Γ is an isometry from $\mathfrak{N}$ into $\mathcal{H}\oplus\mathcal{H}$. Since Π is a boundary triplet for S^*, $\mathrm{ran}\,(\Gamma) = \mathcal{H}\oplus\mathcal{H}$. Hence Γ is an isometry acting from $\mathfrak{N}$ onto $\mathcal{H}\oplus\mathcal{H}$. ☐

Passing to the direct sum (3.1), we equip $\mathrm{dom}\,(S_n^*)$ and $\mathrm{dom}\,(A^*)$ with their graph's norms and obtain the Hilbert spaces $\mathfrak{H}_{+n}$ and $\mathfrak{H}_+$, respectively. Clearly, the corresponding inner products $(f,g)_{+n}$ and $(f,g)_+$ are defined by (3.7) where S^* is replaced by S_n^* and A^*, respectively. Obviously, $\mathfrak{H}_+ = \bigoplus_{n=1}^\infty \mathfrak{H}_{+n}$.

Theorem 3.3 *Let $\{S_n\}_{n=1}^\infty$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and let $S_{0n} = S_{0n}^* \in \mathrm{Ext}_{S_n}$. Further, let A and A_0 be given by* (3.1) *and*

$$A_0 := \bigoplus_{n=1}^{\infty} S_{0n}, \tag{3.13}$$

respectively. Then there exist boundary triplets $\Pi_n := \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker\,(\Gamma_{0n})$, $n \in \mathbb{N}$, and the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ defined by (3.3) *forms a boundary triplet for A^* satisfying $A_0 = A^* \upharpoonright \ker\,(\Gamma_0)$. Moreover, the corresponding Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ are given by*

$$M(z) = \bigoplus_{n=1}^{\infty} M_n(z) \quad \textit{and} \quad \gamma(z) = \bigoplus_{n=1}^{\infty} \gamma_n(z) \tag{3.14}$$

where $M_n(\cdot)$ and $\gamma_n(\cdot)$ are the Weyl functions and the γ-field corresponding to Π_n, $n \in \mathbb{N}$. In addition, the condition $M(i) = iI$ holds.

Proof. For every $S_{0n} = S_{0n}^* \in \mathrm{Ext}_{S_n}$ there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_{0n} := S_n^* \upharpoonright \ker(\Gamma_{0n})$ (see [11]). By Lemma 3.1 we can assume without loss of generality that the corresponding Weyl function $M_n(\cdot)$ satisfies $M_n(i) = i$. By Lemma 3.2 the mapping $\Gamma^n := (\Gamma_{0n}, \Gamma_{1n}) : \mathfrak{H}_{+n} \longrightarrow \mathcal{H}_n \oplus \mathcal{H}_n$, is contractive for each $n \in \mathbb{N}$. Hence $\|\Gamma_j\| = \sup_n \|\Gamma_{jn}\| \le 1, j \in \{0, 1\}$, where Γ_0 and Γ_1 are defined by (3.3). It follows that the mappings Γ_0 and Γ_1 are well-defined on $\mathrm{dom}(\Gamma) = \mathrm{dom}(A^*) = \bigoplus_{n=1}^\infty \mathrm{dom}(S_n^*)$. Thus, the Green's identity (3.4) holds for all $f, g \in \mathrm{dom}(A^*)$.

Further, we set $\mathfrak{N}_{\pm in} := \ker(S_n^* \mp i)$, $\mathfrak{N}_n := \mathfrak{N}_{in} \dot{+} \mathfrak{N}_{-in}$, $\mathfrak{N}_{\pm i} := \ker(A^* \mp i)$ and $\mathfrak{N} := \mathfrak{N}_i \dot{+} \mathfrak{N}_{-i}$. By Lemma 3.2 the restriction $\Gamma^n \upharpoonright \mathfrak{N}_n$ is an isometry from $\mathfrak{N}_n$, regarded as a subspace of $\mathfrak{H}_{+n}$, onto $\mathcal{H}_n \oplus \mathcal{H}_n$. Since $\mathfrak{N}$ regarded as a subspace of $\mathfrak{H}_+$ admits the representation $\mathfrak{N} = \bigoplus_{n=1}^\infty \mathfrak{N}_n$, the restriction $\Gamma \upharpoonright \mathfrak{N}$, $\Gamma := \bigoplus_{n=1}^\infty \Gamma^n$, isometrically maps $\mathfrak{N}$ onto $\mathcal{H} \oplus \mathcal{H}$. Hence $\mathrm{ran}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. Equalities (3.14) are follow from Definition 2.4. $\square$

Remark 3.4 Theorem 3.3 generalizes a result of Kochubei [23, Theorem 3] which states that for any sequence of pairwise unitarily equivalent closed symmetric operators $\{S_n\}_{n\in\mathbb{N}}$ there are boundary triplets Π_n for S_n^*, $n \in \mathbb{N}$ such that $\Pi = \bigoplus_{n\in\mathbb{N}} \Pi_n$ defines a boundary triplet for $A^* = \bigoplus_{n\in\mathbb{N}} S_n^*$.

Recall, that for any non-negative symmetric operator A the set of its non-negative self-adjoint extensions $\mathrm{Ext}_A(0,\infty)$ is non-empty (see [1, 22]). The set $\mathrm{Ext}_A(0,\infty)$ contains the Friedrichs (the biggest) extension A^F and the Krein (the smallest) extension A^K. These extensions are uniquely determined by the following extremal property in the class $\mathrm{Ext}_A(0,\infty)$:

$$(A^F + x)^{-1} \le (\widetilde{A} + x)^{-1} \le (A^K + x)^{-1}, \quad x > 0, \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty).$$

Corollary 3.5 *Let the assumptions of Theorem 3.3 be satisfied. Further, let $S_n \ge 0$, $n \in \mathbb{N}$, and let S_n^F and S_n^K be the Friedrichs and Krein extensions of S_n, respectively. Then*

$$A^F = \bigoplus_{n=1}^\infty S_n^F \qquad \textit{and} \qquad A^K = \bigoplus_{n=1}^\infty S_n^K. \tag{3.15}$$

Proof. Let us prove the second relation. The first one is proved similarly. By Theorem 3.3 there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* such that $S_n^K = S_{0n}$ and $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is a boundary triplet for A^*.

Fix any $x_2 \in \mathbb{R}_+$ and put $C_2 := \|M(-x_2)\|$. Then any $h = \bigoplus_{n=1}^\infty h_n \in \mathcal{H}$ can be decomposed by $h = h^{(1)} \oplus h^{(2)}$ with $h^{(1)} \in \oplus_{n=1}^p \mathcal{H}_n$ and $h^{(2)} \in \oplus_{n=p+1}^\infty \mathcal{H}_n$ such that $\|h^{(2)}\| < C_2^{-1/2}$. Hence $|(M(-x_2)h^{(2)}, h^{(2)})| < 1$. Due to the monotonicity of $M(\cdot)$ we get

$$\Big(M(-x)h^{(2)}, h^{(2)}\Big) > \Big(M(-x_2)h^{(2)}, h^{(2)}\Big) > -1, \qquad x \in (0, x_2).$$

Since $S_{0n} = S_n^K$, the Weyl function $M_n(\cdot)$ satisfies

$$\lim_{x\downarrow 0} \Big(M_n(-x)g_n, g_n\Big) = +\infty, \qquad g_n \in \mathcal{H}_n \setminus \{0\}, \tag{3.16}$$

cf. [11, Proposition 4]. Because $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$ is block-diagonal, cf. (3.14), we get from (3.16) that for any $N > 0$ there exists $x_1 > 0$ such that

$$\Big(M(-x)h^{(1)}, h^{(1)}\Big) = \sum_{n=1}^{p} \Big(M_n(-x)h_n, h_n\Big) > N \quad \text{for} \quad x \in (0, x_1). \tag{3.17}$$

Combining (3.16) with (3.17) and using the diagonal form of $M(\cdot)$, we get

$$(M(-x)h, h) = (M(-x)h^{(1)}, h^{(1)}) + (M(-x)h^{(2)}, h^{(2)}) > N - 1$$

for $0 < x \le \min(x_1, x_2)$. Thus, $\lim_{x \downarrow 0}(M(-x)h, h) = +\infty$ for $h \in \mathcal{H} \setminus \{0\}$. Applying [11, Proposition 4] we prove the second relation of (3.15). □

Remark 3.6 Another proof can be obtained by using characterization of A^F and A^K by means of the respective quadratic forms.

3.2 Direct sums of symmetric operators with arbitrary deficiency indices

We start with some simple spectral observations for direct sums of symmetric operators where the symmetric operators may have arbitrary deficiency indices.

Proposition 3.7 *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and let $S_{0n} = S_{0n}^* \in \operatorname{Ext}_{S_n}$. Further, let A and A_0 be given by* (3.1) *and* (3.13)*, respectively. If $\widetilde{A}$ is a self-adjoint extension of A such that condition*

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \tag{3.18}$$

is satisfied, then

$$\sigma_{ac}(A_0) = \overline{\bigcup \sigma_{ac}(S_{0n})} \subseteq \sigma(\widetilde{A}) \quad \textit{and} \quad \sigma_{ac}(\widetilde{A}) \subseteq \overline{\bigcup \sigma(S_{0n})} = \sigma(A_0). \tag{3.19}$$

Proof. By the Weyl theorem, condition (3.18) yields $\sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A_0)$. Hence

$$\overline{\bigcup \sigma_{ac}(S_{0n})} = \sigma_{ac}(A_0) \subseteq \sigma_{\rm ess}(A_0) = \sigma_{\rm ess}(\widetilde{A}) \subseteq \sigma(\widetilde{A})$$

and

$$\sigma_{ac}(\widetilde{A}) \subseteq \sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A_0) \subseteq \sigma(A_0) = \overline{\bigcup \sigma(S_{0n})}$$

which completes the proof. □

Applying Theorem 2.9 the results of Proposition 3.7 can be improved as follows.

Theorem 3.8 *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and let $S_{0n} = S_{0n}^* \in \operatorname{Ext}_{S_n}$. Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and let $M_n(\cdot)$ be the corresponding Weyl function.*

Moreover, let $\mathfrak{m}_n^+(t)$, $n \in \mathbb{N}$, be the invariant maximal normal function for Π_n. Further, let A and A_0 be given by (3.1) *and* (3.13)*, respectively.*

If δ is a Lebesgue measurable subset of $\mathbb{R}$ such that $\sup_{n\in\mathbb{N}} \mathfrak{m}_n^+(t) < +\infty$ for a.e. $t \in \delta$, then for any self-adjoint extension $\widetilde{A}$ of A satisfying the condition (3.18)*, the absolutely continuous parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta)$ and $A_0^{ac}E_{A_0}(\delta)$ are unitarily equivalent. In particular, if $\delta = \mathbb{R}$, then the parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent and* (3.19) *is replaced by $\sigma_{ac}(A_0) = \sigma_{ac}(\widetilde{A})$.*

Proof. Let $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$, defined according to (3.6), that is $\widetilde{\Gamma}_{0n} := R_n\Gamma_{0n}$ and $\widetilde{\Gamma}_{1n} := R_n^{-1}\big(\Gamma_{1n} - \mathrm{Re}(M_n(i))\Gamma_{0n}\big)$, where $R_n := \sqrt{\mathrm{Im}\, M_n(i))}$. The corresponding Weyl function $\widetilde{M_n}(\cdot)$ is

$$\widetilde{M_n}(z) = R_n^{-1}\big(M_n(z) - \mathrm{Re}\, M_n(i)\big)R_n^{-1}, \quad n \in \mathbb{N}.$$

Since $\widetilde{M_n}(i) = i$, $n \in \mathbb{N}$, by Theorem 3.3, $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n =: \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $A^* \upharpoonright \ker \widetilde{\Gamma}_0 = A_0 := \bigoplus_{n=1}^{\infty} S_{0n}$. By the definition of $\mathfrak{m}_n^+(\cdot)$ one has $\mathfrak{m}_n^+(t) = \widetilde{m}_n^+(t) := \sup_{y\in(0,1]} \|\widetilde{M_n}(t + iy)\|$ for $t \in \mathbb{R}$, $n \in \mathbb{N}$. Since $A_0 = \bigoplus_{n=1}^{\infty} S_{0n}$ we get that $\widetilde{m}^+(t) = \sup_n m_n^+(t)$, where $\widetilde{m}^+(t) := \sup_{y\in(0,1]} \|\widetilde{M}(t+iy)\|$, $t \in \mathbb{R}$. By assumption, the maximal normal function $\widetilde{m}^+(t)$ is finite for a.e. $t \in \delta$. Hence we obtain from Theorem 2.9 that $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta)$ and $A_0^{ac}E_{A_0}(\delta)$ are unitarily equivalent. $\square$

Let T and T' be densely defined closed symmetric operators in $\mathfrak{H}$ and let T_0 and T_0' be self-adjoint extensions of T and T', respectively. The pairs $\{T, T_0\}$ and $\{T', T_0'\}$ are called unitarily equivalent if there exists a unitary operator U in $\mathfrak{H}$ such that $T' = UTU^{-1}$ and $T_0' = UT_0U^{-1}$.

Corollary 3.9 *Let the assumptions of Theorem 3.8 be satisfied. Moreover, let the pairs $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, be unitarily equivalent to the pair $\{S_1, S_{01}\}$. If the maximal normal function $m_1^+(t) := \sup_{0<y\le 1} \|M_1(t+iy)\|$ is finite for a.e. $t \in \delta$ and if the condition* (3.18) *is satisfied, then the absolutely continuous parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta)$ and $A_0^{ac}E_{A_0}(\delta)$ are unitarily equivalent.*

Proof. Since the symmetric operators S_n are unitarily equivalent, we assume without loss of generality that $\mathcal{H}_n = \mathcal{H}$ for each $n \in \mathbb{N}$. Let U_n be a unitary operator such that $A_1 = U_nS_nU_n^{-1}$ and $A_{01} = U_nS_{0n}U_n^{-1}$. A straightforward computation shows that $\Pi_n' := \{\mathcal{H}, \Gamma_{0n}', \Gamma_{1n}'\}$, $\Gamma_{0n}' := \Gamma_{01}U_n$ and $\Gamma_{1n}' := \Gamma_{1n}U_n$, defines a boundary triplet for S_n^*. The Weyl function $M_n'(\cdot)$ corresponding to Π_n' is $M_n'(z) = M_1(z)$. Hence $\mathfrak{m}_n^+(\cdot) = \mathfrak{m}_n'^+(\cdot)$ and $\mathfrak{m}_1^+(t) = \mathfrak{m}_n'^+(t)$ for $t \in \mathbb{R}$, where $\mathfrak{m}_n^+(t)$ and $\mathfrak{m}_n'^+(t)$ are the invariant maximal normal functions corresponding to the triplets Π_n and Π_n', respectively. Since $\mathfrak{m}_1^+(t) = \mathfrak{m}_n^+(t)$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ we complete the proof applying Theorem 3.8. $\square$

3.3 Direct sums of symmetric operators with finite deficiency

Here we improve the previous results assuming that $n_\pm(S_n) < \infty$. First, we show that extensions $A_0 = \bigoplus_{n=1}^{\infty} S_{0n} (\in \mathrm{Ext}\,_A)$ of the form (3.13) possess a certain spectral minimality property. To this end we start with the following lemma.

Lemma 3.10 *Let H be a bounded non-negative self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let L be a bounded operator in $\mathfrak{H}$. Then*

(i) $\dim(\overline{\mathrm{ran}\,(H)}) = \dim(\overline{\mathrm{ran}\,(\sqrt{H})})$;

(ii) *If $L^*L \le H$, then* $\dim(\overline{\mathrm{ran}\,(L)}) \le \dim(\overline{\mathrm{ran}\,(H)})$;

(iii) *If P is an orthogonal projection, then* $\dim(\overline{\mathrm{ran}\,(PHP)}) \le \dim(\overline{\mathrm{ran}\,(H)})$.

Proof. The assertion (i) is obvious.

(ii) If $L^*L \le H$, then there is a contraction C such that $L = C\sqrt{H}$. Hence $\dim(\overline{\mathrm{ran}\,(L)}) = \dim(\overline{\mathrm{ran}\,(C\sqrt{H})}) \le \dim(\overline{\mathrm{ran}\,(\sqrt{H})}) = \dim(\overline{\mathrm{ran}\,(H)})$.

(iii) Clearly, $\dim(\overline{\mathrm{ran}\,(PHP)}) \le \dim(\overline{\mathrm{ran}\,(\sqrt{H}P)}) \le \dim(\overline{\mathrm{ran}\,(\sqrt{H})})$. Applying (i) we complete the proof. $\square$

We are going to show that if the summands have only finite deficiency indices, then the absolutely spectrum of extensions of the direct sum can only increase comparing with the absolutely continuous spectrum of those extensions which are direct sums of extensions.

Theorem 3.11 *Let $\{S_n\}_{n=1}^\infty$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and let $S_{0n} = S_{0n}^* \in \mathrm{Ext}\,_{S_n}$. Further, let A and A_0 be given by* (3.1) *and* (3.13)*, respectively.*

If the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then A_0 is ac-minimal, in particular, $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ for any self-adjoint extension $\widetilde{A}$ of A.

Proof. By Theorem 3.3 there is a sequence of boundary triplets $\Pi_n := \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$, $n \in \mathbb{N}$, for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and the direct sum $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} = \bigoplus_{n=1}^\infty \Pi_n$ of the form (3.1) is a boundary triplet for A^* satisfying $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. By Proposition 2.3, any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}\,_A$ admits a representation $\widetilde{A} = A_\Theta$ with $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$. By [33, Corollary 4.2(i)], we can assume that $\widetilde{A}$ and A_0, are disjoint, that is $\Theta = B = B^* \in \mathcal{C}(\mathcal{H})$. Consider the generalized Weyl function $M_B(\cdot) := (B - M(\cdot))^{-1}$, where $M(\cdot) = \bigoplus_{n=1}^\infty M_n(\cdot)$, cf. (3.14). Clearly,

$$\mathrm{Im}\,(M_B(z)) = M_B(z)^*\mathrm{Im}\,(M(z))M_B(z), \quad z \in \mathbb{C}_+.$$

Denote by P_N, $N \in \mathbb{N}$, the orthogonal projection from $\mathcal{H}$ onto the subspace $\mathcal{H}_N := \bigoplus_{n=1}^N \mathcal{H}_n$. Setting $M_B^{P_N}(z) := P_N M_B(z) \upharpoonright \mathcal{H}_N$, and taking into account the block-diagonal form of $M(\cdot)$ and the inequality $\mathrm{Im}\,(M(z)) > 0$ we obtain

$$\begin{aligned} \mathrm{Im}\,(M_B^{P_N}(z)) &= \mathrm{Im}\,(P_N M_B(z) P_N) \\ &= P_N M_B(z)^*\mathrm{Im}\,(M(z))M_B(z)P_N \ge M_B^{P_N}(z)^*\mathrm{Im}\,(M^{P_N}(z))M_B^{P_N}(z), \end{aligned} \tag{3.20}$$

where $M^{P_N}(z) := P_N M(z) \upharpoonright \mathcal{H}_N = \bigoplus_{n=1}^N M_n(z)$. Since P^N is a finite dimensional projection the limits $M_B^{P_N}(t) := \text{s-lim}_{y\downarrow 0} M_B^{P_N}(t+iy)$ and $M^{P_N}(t) := \text{s-lim}_{y\downarrow 0} M^{P_N}(t+iy)$ exists for a.e. $t \in \mathbb{R}$. From (3.20) we get

$$\mathrm{Im}\,(M_B^{P_N}(t)) \ge M_B^{P_N}(t)^*\mathrm{Im}\,(M^{P_N}(t))M_B^{P_N}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{3.21}$$

Since $M_B(\cdot)$ is a generalized Weyl function, it is a strict $R_{\mathcal{H}}$-function, that is, $\ker(\mathrm{Im}\,(M_B(z))) = \{0\},\ z \in \mathbb{C}_+$. Therefore, $M_B^{P_N}(\cdot)$ is also strict. Hence $0 \in \varrho(M_B^{P_N}(z))$, $z \in \mathbb{C}_+$, and $G_N(\cdot) := -(M_B^{P_N}(\cdot))^{-1}$ is strict. Since both $G_N(\cdot)$ and $M_B^{P_N}(\cdot)$ are matrix-valued R-functions, the limits $M_B^{P_N}(t+i0) := \lim_{y\downarrow 0} M_B^{P_N}(t+iy)$ and $G_N(t+i0) := \lim_{y\downarrow 0} G_N(t+iy)$ exist for a.e. $t \in \mathbb{R}$. Therefore, passing to the limit in the identity $M_B^{P_N}(t+iy)G_N(t+iy) = -I$ as $y \to 0$, we get $M_B^{P_N}(t+i0)G_N(t+i0) = -I$ for a.e. $t \in \mathbb{R}$. Hence $M_B^{P_N}(t) := M_B^{P_N}(t+i0)$ is invertible for a.e. $t \in \mathbb{R}$.

Further, combining (3.21) with Lemma 3.10(ii) we get

$$\dim\left(\overline{\mathrm{ran}\,\left(\sqrt{\mathrm{Im}\,M^{P_N}(t)}M_B^{P_N}(t)\right)}\right) \le d_{M_B^{P_N}}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Since $M_B^{P_N}(t)$ is invertible for a.e. $t \in \mathbb{R}$, we find

$$d_{M^{P_N}}(t) := \dim\left(\overline{\mathrm{ran}\,\left(\sqrt{\mathrm{Im}\,M^{P_N}(t)}\right)}\right) \le d_{M_B^{P_N}}(t) \quad \text{for a.e.} \quad t \in \mathbb{R}. \tag{3.22}$$

Let $D_N = P_N \oplus D_0$ where $D_0 \in \mathfrak{S}_2(\mathcal{H}_N^\perp)$ and satisfy $\ker(D_0) = \ker(D_0^*) = \{0\}$. Then $\ker(D_N) = \ker(D_N^*) = \{0\}$ and $P_N = P_N D_N = D_N P_N$. By Lemma 3.10(iii), $d_{M^{P_N}}(t) \le d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Further, for any $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfying $\ker(D) = \ker(D^*) = \{0\}$, $d_{M_B^D}(t) = d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Combining this equality with (3.22) we get $d_{M^{P_N}}(t) \le d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Since

$$d_{M^{P_N}}(t) = \sum_{n=1}^{N} d_{M_n(t)} \quad \text{and} \quad d_{M^D}(t) = \sum_{n=1}^{\infty} d_{M_n}(t) \tag{3.23}$$

for a.e. $t \in \mathbb{R}$, we finally prove that $d_{M^D}(t) \le d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$. One completes the proof by applying Theorem 2.8(i). □

Taking into account Proposition 2.6 and Corollary 2.7 the proof of Theorem 3.11 shows us that in fact the spectral multiplicity function $N_{\widetilde{A}^{ac}}(t)$ can only be increase with respect to $N_{A_0^{ac}}(t)$, that is, one always has $N_{\widetilde{A}^{ac}}(t) \ge N_{A_0^{ac}}(t)$ for a.e. $t \in \mathbb{R}$ and any self-adjoint extension $\widetilde{A}$ of A.

Corollary 3.12 *Let the assumptions of Theorem 3.11 be satisfied. If $S_n \ge 0$, $n \in \mathbb{N}$ and if the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then the Friedrichs and the Krein extensions A^F and A^K of A are ac-minimal. In particular, $(A^F)^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. Combining Theorem 3.11 and Corollary 3.5 one immediately proves the assertions. □

Corollary 3.13 *Let the assumptions of Theorem 3.8 be satisfied. Further, let the deficiency indices of S_n be finite for each $n \in \mathbb{N}$.*

(i) *If*

$$\delta_\infty := \{t \in \mathbb{R} : \sum_{n\in\mathbb{N}} d_{M_n}(t) = \infty\}, \tag{3.24}$$

then for any self-adjoint extension $\widetilde{A}$ of A the parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta_\infty)$ and $A_0^{ac}E_{A_0}(\delta_\infty)$ are unitarily equivalent.

(ii) *If δ is a Lebesgue measurable subset of $\mathbb{R}$ such that $\sup_n \mathfrak{m}_n^+(t) < \infty$ for a.e. $t \in \delta$, then for any self-adjoint extension $\widetilde{A}$ of A the parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta_\infty \cup \delta)$ and $A_0^{ac}E_{A_0}(\delta_\infty \cup \delta)$ are unitarily equivalent.*

Proof. (i) By (3.23) and (3.24) we find $d_{M^D}(t) = +\infty$ for a.e. $t \in \delta_\infty$. Since by Theorem 3.11 the spectral multiplicity function can only be increase for self-adjoint extensions $\widetilde{A}$ one gets that $N_{\widetilde{A}^{ac}}(t) = N_{A_0^{ac}}(t)$ for a.e. $t \in \delta$ which immediately yields the unitary equivalence of the parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta_\infty)$ and $A_0^{ac}E_{A_0}(\delta_\infty)$.

(ii) By Theorem 3.8 the parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta)$ and $A_0^{ac}E_{A_0}(\delta)$ are unitarily equivalent. Using (i) we immediately obtain the unitary equivalence of the parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta_\infty \cup \delta)$ and $A_0^{ac}E_{A_0}(\delta_\infty \cup \delta)$. $\square$

Corollary 3.14 *Let the assumptions of Theorem 3.11 be satisfied. If the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then $\overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})} \subseteq \sigma_{ac}(\widetilde{A})$ for any self-adjoint extension $\widetilde{A}$ of A. If in addition condition* (3.18) *is valid and the extensions S_{0n} are purely absolutely continuous for each $n \in \mathbb{N}$, then*

$$\sigma_{ac}(\widetilde{A}) = \overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})}. \tag{3.25}$$

Proof. The first statement immediately follows from Theorem 3.11. Relation (3.25) is implied by Proposition 3.7. $\square$

Corollary 3.15 *Let the assumptions of Theorem 3.11 be satisfied. Further, let the pairs $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, be unitarily equivalent to $\{S_1, S_{01}\}$. If the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, holds, then for any self-adjoint extension $\widetilde{A}$ of A satisfying condition* (3.18) *the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent.*

Proof. The proof follows immediately from Corollary 3.9. $\square$

Remark 3.16 (i) For the special case $n_\pm(S_n) = 1, n \in \mathbb{N}$, Theorem 3.11 complements [2, Corollary 5.4] where the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ was proved. Moreover, Corollary 3.15 might be regarded as a substantial generalization of [2, Theorem 5.6(i)] to the case $n_\pm(S_n) > 1$. However, in the case $n_\pm(S_n) = 1$, Corollary 3.15 is implied by [2, Theorem 5.6(i)] where the unitary equivalence of $\widetilde{A}^{ac} = \widetilde{A}_B^{ac}$ and A_0^{ac} was proved under the weaker assumption that B is purely singular. Indeed, by Proposition 2.5 condition (3.18) with $\widetilde{A} = A_B$ is equivalent to the discreteness of B.

(ii) The inequality $N_{A_0^{ac}}(t) \le N_{\widetilde{A}^{ac}}(t)$ in Theorem 3.11 might be strict even for $t \in \sigma_{ac}(A_0)$. Indeed, assume that (α, β) is a gap for all except for the operators $S_1, \dots, S_N$. Set $S_1 := \bigoplus_{n=1}^N S_n$ and $S_2 := \bigoplus_{n=N+1}^\infty S_n$. Then $n_\pm(S_2) = \infty$ and (α, β) is a gap for S_2.

By [7] there exists $\widetilde{S}_2 = \widetilde{S}_2^* \in \operatorname{Ext}_{S_2}$ having ac-spectrum within (α, β) of arbitrary multiplicity. Moreover, even for operators $A = \bigoplus_{n=1}^{\infty} S_n$ satisfying assumptions of Corollary 3.15 with $n_{\pm}(S_n) = 1$ the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ might be strict whenever condition (3.18) is violated, cf. [7] or [2, Theorem 4.4] which guarantees the appearance of prescribed spectrum either within one gap or within several gaps of A_0.

4 Sturm-Liouville operators with bounded operator potentials

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. As usual, $L^2(\mathbb{R}_+, \mathcal{H})$ stands for the Hilbert space of (weakly) measurable vector-functions $f(\cdot) : \mathbb{R}_+ \to \mathcal{H}$ satisfying $\int_{\mathbb{R}_+} \|f(t)\|_{\mathcal{H}}^2 dt < \infty$. Denote also by $W^{2,2}(\mathbb{R}_+, \mathcal{H})$ the Sobolev space of vector-functions taking values in $\mathcal{H}$.

Let $T = T^* \geq 0$ be a bounded operator in $\mathcal{H}$. Denote by $A := A_{\min}$ the minimal operator generated by $\mathcal{A}$, cf. (1.1), in $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$. It is known (see [19, 35]) that the minimal operator A is given by

$$(Af)(x) = -\frac{d^2}{dx^2} f(x) + T f(x), \quad f \in \operatorname{dom}(A) = W_0^{2,2}(\mathbb{R}_+, \mathcal{H}), \tag{4.1}$$

where $W_0^{2,2}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : \ f(0) = f'(0) = 0\}$.

The operator A is closed, symmetric and non-negative. It can be proved similarly to [8, Example 5.3] that A is simple. The adjoint operator A^* is given by [19, Theorem 3.4.1]

$$(A^* f)(x) = -\frac{d^2}{dx^2} f(x) + T f(x), \quad f \in \operatorname{dom}(A^*) = W^{2,2}(\mathbb{R}_+, \mathcal{H}). \tag{4.2}$$

The Dirichlet realization A^D is defined by $A^D f := \mathcal{A} f$, $f \in \operatorname{dom}(A^D) := \{g \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : g(0) = 0\}$. Similarly, the Neumann realization A^N is defined by $A^N f := \mathcal{A} f$, $f \in \operatorname{dom}(A^N) := \{g \in W^{2,2}(\mathbb{R}_+, \mathcal{H} : g'(0) = 0\}$. Since $\operatorname{dom}(A) \subseteq \operatorname{dom}(A^D), \operatorname{dom}(A^N) \subseteq \operatorname{dom}(A^*)$ one gets that A^D and A^N are proper extensions of A. One easily verifies that A^D and A^N are symmetric extensions.

By [29, Theorem 1.3.1] the trace operators $\Gamma_0, \ \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$,

$$\Gamma_0 f = f(0) \quad \text{and} \quad \Gamma_1 f = f'(0), \quad f \in \operatorname{dom}(A^*), \tag{4.3}$$

are well defined. Moreover, the deficiency subspace $\mathfrak{N}_z(A)$ is

$$\mathfrak{N}_z(A) = \{e^{ix\sqrt{z-T}} h : \ h \in \mathcal{H}\}, \qquad z \in \mathbb{C}_{\pm}, \tag{4.4}$$

with the cut along $\mathbb{R}_+$.

Lemma 4.1 *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 are defined by* (4.3)*, forms a boundary triplet for A^*. The corresponding Weyl function $M(\cdot)$ is*

$$M(z) = i\sqrt{z-T} = i \int \sqrt{z-\lambda}\, dE_T(\lambda), \quad z \in \mathbb{C}_+. \tag{4.5}$$

Proof. One obtains the Green formula integrating by parts. The surjectivity of the mapping $\Gamma := (\Gamma_0, \Gamma_1) : \mathrm{dom}\,(A^*) \to \mathcal{H} \oplus \mathcal{H}$ follows from (4.3) and [29, Theorem 1.3.2]. Formula (4.5) is implied by (4.4). $\square$

Lemma 4.2 *Let T be a bounded non-negative self-adjoint operator in $\mathcal{H}$ and let A and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be defined by* (4.1) *and* (4.3)*, respectively. Then*

(i) *the invariant maximal normal function $\mathfrak{m}^+(t)$ of the Weyl function $M(\cdot)$ is finite for all $t \in \mathbb{R}$ and satisfies*

$$\mathfrak{m}^+(t) \le (1 + \sqrt{2})(1 + t^2)^{1/4}, \quad t \in \mathbb{R}. \tag{4.6}$$

(ii) *The limit $M(t + i0) := \text{s-lim}_{y \downarrow 0} M(t + iy)$ exists, is bounded and equals*

$$M(t + i0) = i \int_{\mathbb{R}} \sqrt{t - \lambda}\, dE_T(\lambda) \quad \textit{for any } t \in \mathbb{R}. \tag{4.7}$$

(iii) $d_M(t) = \dim(\mathrm{ran}\,(E_T([0, t))))$ *for any $t \in \mathbb{R}$.*

Proof. (i) It follows from (4.5) and definition (2.8) that

$$\mathfrak{m}^+(t) \le \sup_{y \in (0,1]} \sup_{\lambda \ge 0} \left| \frac{\sqrt{t + iy - \lambda} - \mathrm{Re}\,(\sqrt{i - \lambda})}{\mathrm{Im}\,(\sqrt{i - \lambda})} \right|.$$

Clearly, $\sqrt{i - \lambda} = (1 + \lambda^2)^{1/4} e^{i(\pi - \varphi)/2}$ where $\varphi := \arccos\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$. Hence

$$\left| \frac{\mathrm{Re}(\sqrt{i - \lambda})}{\mathrm{Im}(\sqrt{i - \lambda})} \right| = \tan\left(\frac{\varphi}{2}\right) = \frac{1}{\lambda + \sqrt{1 + \lambda^2}} \le 1, \quad \lambda \ge 0.$$

Furthermore, we have

$$\left| \frac{\sqrt{t + iy - \lambda}}{\mathrm{Im}(\sqrt{i - \lambda})} \right| \le \sqrt{2} \sqrt{\frac{\sqrt{(\lambda - t)^2 + y^2}}{\lambda + \sqrt{1 + \lambda^2}}} \le \sqrt{2}(1 + t^2)^{1/4}$$

for $\lambda \ge 0, t \in \mathbb{R}$ and $y \in (0, 1]$ which yields (4.6).

(ii) From (4.5) we find $M(t) := M(t + i0) := \text{s-lim}_{y \downarrow 0}\, i\sqrt{t + iy - T} = i\sqrt{t - T}$, for any $t \in \mathbb{R}$, which proves (4.7). Clearly, $M(t) \in [\mathcal{H}]$ since $T \in [\mathcal{H}]$.

(iii) It follows that $\mathrm{Im}(M(t)) = \sqrt{t - T} E_T([0, t))$, which yields $d_M(t) = \dim(\mathrm{ran}\,(\mathrm{Im}(M(t)))) = \dim(\mathrm{ran}\,(E_T([0, t))))$. $\square$

With $A = A_{\min}$ one associates a closable quadratic form $\mathfrak{t}'_F[f] := (Af, f)$, $\mathrm{dom}\,(\mathfrak{t}') = \mathrm{dom}\,(A)$. Its closure $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_F[f] := \int_{\mathbb{R}_+} \left\{ \|f'(x)\|^2_{\mathcal{H}} + \|\sqrt{T} f(x)\|^2_{\mathcal{H}} \right\} dx, \tag{4.8}$$

$f \in \mathrm{dom}\,(\mathfrak{t}_F) = W^{1,2}_0(\mathbb{R}_+, \mathcal{H})$, where $W^{1,2}_0(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{1,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$. By definition, the Friedrichs extension A^F of A is a self-adjoint operator associated with $\mathfrak{t}_F$. Clearly, $A^F = A^* \restriction (\mathrm{dom}\,(A^*) \cap \mathrm{dom}\,(\mathfrak{t}_F))$.

Theorem 4.3 *Let $T \geq 0, T = T^* \in [\mathcal{H}]$, and $t_0 := \inf \sigma(T)$. Let A be defined by* (4.1) *and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ the boundary triplet for A^* defined by* (4.3)*. Then the following holds:*

(i) *The Dirichlet realization A^D coincides with $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ which is identical with the Friedrichs extension A^F. Moreover, A^D is absolutely continuous and its spectrum is given by $\sigma(A^D) = \sigma_{ac}(A^D) = [t_0, \infty)$.*

(ii) *The Neumann realization A^N coincides with $A_1 := A^* \upharpoonright \ker(\Gamma_1)$. A^N is absolutely continuous $(A^N)^{ac} = A^N$ and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*

(iii) *The Krein realization (or extension) A^K is given by*

$$\operatorname{dom}(A^K) = \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : \ f'(0) + \sqrt{T} f(0) = 0\}. \tag{4.9}$$

Moreover, $\ker(A^K) = \mathfrak{H}_0 := \overline{\mathfrak{H}_0'}$, $\mathfrak{H}_0' := \{e^{-x\sqrt{T}}h : \ h \in \operatorname{ran}(T^{1/4})\}$ and the restriction $A^K \upharpoonright \operatorname{dom}(A^K) \cap \mathfrak{H}_0^\perp$ is absolutely continuous, that is, $\mathfrak{H}_0^\perp = \mathfrak{H}^{ac}(A^K)$ and $A^K = 0_{\mathfrak{H}_0} \oplus (A^K)^{ac}$. In particular, $\sigma(A^K) = \{0\} \cup \sigma_{ac}(A^K)$ and $\sigma_{ac}(A^K) = [t_0, \infty)$.

(iv) *The realizations A^D, A^N and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. (i) It follows from (4.2) and (4.3) that $\operatorname{dom}(A^D) = \operatorname{dom}(A_0)$ which yields $A^D = A_0$. Since $\operatorname{dom}(A_0) \subseteq W_0^{1,2}(\mathbb{R}_+, \mathcal{H}) = \operatorname{dom}(\mathfrak{t}_F)$ we have $A^F = A_0$ (see [1, Section 8] and [22, Theorem 6.2.11]). It follows from (4.7) and [8, Theorem 4.3] that $\sigma_p(A_0) = \sigma_{sc}(A_0) = \emptyset$. Hence A_0 is absolutely continuous. Taking into account Lemma 4.2(iii) and Proposition 2.6 we get $\sigma(A_0) = \sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\operatorname{supp}(d_M)) = [t_0, \infty)$ which proves (i).

(ii) Obviously we have $\operatorname{dom}(A^N) = \operatorname{dom}(A_1) := \ker(\Gamma_1)$ which proves $A^N = A_1$. It follows from Lemma 4.1 and (2.6) that the Weyl function corresponding to A_1 is given by

$$M_0(z) := (0 - M(z))^{-1} = i(z - T)^{-1/2} = i \int \frac{1}{\sqrt{z - \lambda}} dE_T(\lambda), \quad z \in \mathbb{C}_+. \tag{4.10}$$

Since $M_0(\cdot)$ is regular within $(-\infty, t_0)$, we have $(-\infty, t_0) \subset \varrho(A_1)$. Further, let $\tau > t_0$. We set $\mathcal{H}_\tau := E_T([t_0, \tau))\mathcal{H}$ and note that for any $h \in \mathcal{H}_\tau$ and $t > \tau$

$$\big(M_0(t + i0)h, h\big) = i\big((t - T)^{-1/2}h, h\big) = i \int_{t_0}^{\tau} \frac{1}{\sqrt{t - \lambda}} d(E_T(\lambda)h, h). \tag{4.11}$$

Hence for any $h \in \mathcal{H}_\tau \setminus \{0\}$ and $t > \tau$

$$0 < (t - t_0)^{-1/2} \|h\|^2 \leq \operatorname{Im}(M_0(t + i0)h, h) = \int_{t_0}^{\tau} (t - \lambda)^{-1/2} d(E_T(\lambda)h, h) < \infty.$$

By [8, Proposition 4.2], $\sigma_{ac}(A_1) \supseteq [\tau, \infty)$ for any $\tau > t_0$, which yields $\sigma_{ac}(A_1) = [t_0, \infty)$. It remains to show that A_1 is purely absolutely continuous. Since $M_0(t + i0) \notin [\mathcal{H}]$ we cannot apply [8, Theorem 4.3]. Fortunately, to we can use [8, Corollary 4.7]. For any $t \in \mathbb{R},\ y > 0$, and $h \in \mathcal{H}$ we set

$$V_h(t + iy) := \operatorname{Im}(M_0(t + iy)h, h) = \int \operatorname{Im}\left(\frac{1}{\sqrt{\lambda - t - iy}}\right) d(E_T(\lambda)h, h).$$

Obviously, one has

$$V_h(t+iy) \le \int \frac{1}{((\lambda-t)^2+y^2)^{1/4}} d(E_T(\lambda)h,h), \quad t\in\mathbb{R}, \quad y>0, \quad h\in\mathcal{H}.$$

Hence

$$V_h(t+iy)^p \le \|h\|^{2(p-1)} \int \frac{1}{((\lambda-t)^2+y^2)^{p/4}} d(E_T(\lambda)h,h), \quad p\in(1,\infty).$$

We show that for $p\in(1,2)$ and $-\infty<a<b<\infty$

$$C_p(h;a,b) := \sup_{y\in(0,1]} \int_a^b V_h(t+iy)^p\, dt < \infty.$$

Clearly,

$$\begin{aligned}\int_a^b V_h(t+iy)^p dt &\le \|h\|^{2(p-1)} \int_0^{\|T\|} d(E(\lambda)h,h) \int_a^b \frac{1}{((\lambda-t)^2+y^2)^{p/4}} dt \\ &= \|h\|^{2(p-1)} \int_0^{\|T\|} d(E(\lambda)h,h) \int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2+y^2)^{p/4}} dt.\end{aligned}$$

Note, that for $p\in(1,2)$ and $-\infty<a<b<\infty$

$$\int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2+y^2)^{p/4}} dt \le \int_{a-\|T\|}^{b} \frac{1}{t^{p/2}} dt =: \varkappa_p(b, a-\|T\|) < \infty,$$

Hence $C_p(h;a,b) \le \varkappa_p(b,a-\|T\|)\|h\|^{2p} < \infty$ for $p\in(1,2)$, $-\infty<a<b<\infty$ and $h\in\mathcal{H}$. By [8, Corollary 4.7], A_1 is purely absolutely continuous on any bounded interval (a,b). Hence A_1 is purely absolutely continuous.

(iii) By [11, Proposition 5] A^K is defined by $A^K = A^* \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$. It follows from (4.5) that $M(0) = -\sqrt{T}$. Therefore, A^K is defined by (4.9).

It follows from the extremal property of the Krein extension that $\ker(A^K) = \ker(A^*)$. Clearly, $f_h(x) := \exp(-x\sqrt{T})h \in L^2(\mathbb{R}_+,\mathcal{H})$, $h\in\operatorname{ran}(T^{1/4})$, since

$$\begin{aligned}&\int_0^\infty \|\exp(-x\sqrt{T})h\|^2_{\mathcal{H}} dx \\ &\qquad = \int_0^{\|T\|} d\rho_h(t) \int_0^\infty e^{-2x\sqrt{t}} dx = \int_0^{\|T\|} \frac{1}{2\sqrt{t}} d\rho_h(t) < \infty,\end{aligned}$$

where $\rho_h(t) := \big(E_T(t)h,h\big)$. Thus, $\mathfrak{H}_0' \subset \ker(A^*)$. It is easily seen that $\mathfrak{H}_0'$ is dense in $\mathfrak{H}_0$. To investigate the rest of the spectrum of A^K consider the Weyl function $M_K(\cdot)$ corresponding to A^K. It follows from (4.5) and (2.6) that

$$\begin{aligned}M_K(z) &= M_{-\sqrt{T}}(z) = -\big(\sqrt{T}+M(z)\big)^{-1} \\ &= -(\sqrt{T}+i\sqrt{z-T})^{-1} = \frac{1}{z}(i\sqrt{z-T}-\sqrt{T}) = -\frac{2\sqrt{T}}{z} + \Phi(z).\end{aligned}$$

where $\Phi(z) := \frac{1}{z}[i\sqrt{z-T}+\sqrt{T}]$. For $t>0$ we get

$$\operatorname{Im} M_K(t+i0) = \operatorname{Im}\Phi(t+i0) = t^{-1}\sqrt{t-T}E_T([0,t)). \tag{4.12}$$

Hence, by [8, Theorem4.3], $\sigma_p(A^K)\cap(0,\infty) = \sigma_{sc}(A^K)\cap(0,\infty) = \emptyset$. It follows from (4.12) that $\operatorname{Im}(M_K(t+i0)) > 0$ for $t > t_0$. By Corollary 2.7 we find $\sigma_{ac}(A^K) = [t_0,\infty)$.

(iv) It follows from (4.7) and (4.12) that $d_M(t) = d_{M_K}(t) = \dim(\operatorname{ran}(E_T([0,t))))$ for $t > t_0$. Combining this equality with $\sigma_{ac}(A^K) = \sigma_{ac}(A^F) = [t_0,\infty)$, we conclude from Theorem 2.8(ii) that A^F and $(A^K)^{ac}$ are unitarily equivalent.

Passing to A_1, we assume that $1 \le \dim(\operatorname{ran}(E_T([0,s)))) = p_1 < \infty$ for some $s > 0$. Let $\lambda_k,\ k\in\{1,\dots,p\},\ p\le p_1$, be the set of distinct eigenvalues within $[0,s)$. Since $M_0(t+iy)E_T([0,t))$ is the $p\times p$ matrix-function, the limit $M_0(t+i0)E_T([0,t))$ exists for $t\in[0,s)\setminus\bigcup_{k=1}^p\{\lambda_k\}$. It follows from (4.11) that

$$\operatorname{Im}(M_0(t)) = |T-t|^{-1/2}E_T([0,t)), \qquad t\in[0,s)\setminus\bigcup_{k=1}^p\{\lambda_k\}.$$

This yields

$$d_{M_0(t))} := \dim(\operatorname{ran}(\operatorname{Im}(M_0(t)))) = \dim(\operatorname{ran}(E_T([0,t)))) = d_M(t)$$

for a.e $t\in[0,s)\setminus\bigcup_{k=1}^p\{\lambda_k\}$, that is, for a.e. $t\in[0,s)$.

If $\dim(E_T([t_0,s))) = \infty$, then there exists a point $s_0\in(0,s)$, such that $\dim(E_T([0,s_0])) = \infty$ and $\dim(E_T([0,s))) < \infty$ for $s\in[0,s_0)$. For any $t\in(s_0,s)$ choose $\tau\in(s_0,t)$ and note that $\dim(\operatorname{ran}(E_T([0,\tau)))) = \infty$. We set $\mathcal{H}_\tau := E_T([0,\tau))\mathcal{H}$ and $\mathcal{H}_\infty := E_T([\tau,\infty))\mathcal{H}$ as well as $T_\tau := TE_T([0,\tau))$ and $T_\infty := TE_T([\tau,\infty))$. Further, we choose Hilbert-Schmidt operators D_τ and D_∞ in $\mathcal{H}_\tau$ and $\mathcal{H}_\infty$, respectively, such that $\ker(D_\tau) = \ker(D_\tau^*) = \ker(D_\infty) = \ker(D_\infty^*) = \{0\}$. According to the decomposition $\mathcal{H} = \mathcal{H}_\tau\oplus\mathcal{H}_\infty$ we have $M_0 = M_\tau\oplus M_\infty$, $D = D_\tau\oplus D_\infty$ and $d_{M_0^D}(t) = d_{M_\tau^{D_\tau}}(t) + d_{M_\infty^{D_\infty}}(t)$ for a.e. $t\in[0,\infty)$. Hence $d_{M_0^D}(t)\ge d_{M_\tau^{D_\tau}}(t)$ for a.e. $t\in[0,\infty)$. Clearly, $M_\tau(t+iy) = i(t+iy-T_\tau)^{-1/2}$. If $t>\tau$, then $t\in\varrho(T_\tau)$ and $M(t) := \operatorname{s-lim}_{y\downarrow 0} M(t+i0)$ exists and

$$M_\tau(t) := \operatorname*{s-lim}_{y\to 0} M_\tau(t+iy) = i(t-T_\tau)^{-1/2}E_T([0,\tau)).$$

Hence $d_{M_\tau^{D_\tau}}(t) = \dim(\operatorname{ran}(E_T([0,\tau)))) = \infty$ for $t > s_0$. Hence $d_{M_0^D}(t) = d_M(t) = \infty$ for a.e. $t > s_0$ which yields $d_{M_0^D}(t) = d_M(t)$ for a.e. $t\in[0,\infty)$. Using Theorem 2.8(ii) we obtain that A_0^{ac} and A_1^{ac} are unitarily equivalent which shows A_0 and A_1 are unitarily equivalent. □

Remark 4.4 The statements on A^D, A^N and A^K are proved self-consistently in the framework of boundary triplets. However, the unitary equivalence of A^D and A^N can be proved much simpler. In fact, the Dirichlet and Neumann realizations l_D and l_N of the differential expression $l := -\frac{d^2}{dt^2}$ in $L^2(\mathbb{R}_+)$ are unitary equivalent. If $U:\ L^2(\mathbb{R}_+)\longrightarrow L^2(\mathbb{R}_+)$ is such a unitary operator, i.e. $Ul_D = l_NU$, then we have

$$\begin{aligned}A^N &= l_N\otimes I_{\mathcal{H}} + I_{\mathfrak{H}}\otimes T = \\ &(U\otimes I_{\mathcal{H}})[l_D\otimes I_{\mathcal{H}} + I_{\mathfrak{H}}\otimes T](U^*\otimes I_{\mathcal{H}}) = (U\otimes I_{\mathcal{H}})A^D(U^*\otimes I_{\mathcal{H}}).\end{aligned}$$

The proof can be extended to any non-negative realization l_h of l fixed by the domain $\mathrm{dom}\,(l_h) = \{f \in W^{1,2}(\mathbb{R}_+) : f'(0) = hf(0), \quad h \geq 0\}$. Moreover, a proof of the absolutely continuity of A^D and A^N, which does not used boundary triplets, can be found in Appendix A.2. For the Krein realization A^K we do not know such proofs.

Next we describe the spectral properties of any self-adjoint extension of A. In particular, we show that the Friedrichs extension A^F of A is ac-minimal, though A does not satisfy conditions of Theorem 3.11.

Theorem 4.5 *Let $T \geq 0, T = T^* \in [\mathcal{H}]$, and $t_1 := \inf \sigma_{\mathrm{ess}}(T)$. Let also A be the symmetric operator defined by* (4.1) *and $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}\,_A$. Then*

(i) *the absolutely continuous part $\widetilde{A}^{ac} E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to the part $A^D E_{A^D}([t_1,\infty))$;*

(ii) *the Dirichlet, Neumann and Krein realizations are ac-minimal and $\sigma(A^D) = \sigma(A^N) = \sigma_{ac}(A^K) \subseteq \sigma_{ac}(\widetilde{A})$;*

(iii) *the absolutely continuous part $\widetilde{A}^{ac}$ is unitarily equivalent to A^D whenever either $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ or $(\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$.*

Proof. By [33, Corollary 4.2] it suffices to assume that the extension $\widetilde{A} = \widetilde{A}^*$ is disjoint with A_0, that is, by Proposition 2.3(ii) it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$.

(i) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* defined by (4.3). In accordance with Theorem 2.8 we calculate $d_{M_B^K}(t)$ where $M_B(\cdot) := (B - M(\cdot))^{-1}$ is the generalized Weyl function of the extension A_B, cf. (2.6). Clearly,

$$\mathrm{Im}(M_B(z)) = M_B(z)^* \,\mathrm{Im}(M(z)) M_B(z), \quad z \in \mathbb{C}_+. \tag{4.13}$$

Since $\mathrm{Re}(\sqrt{z-\lambda}) > 0$ for $z = t + iy,\ y > 0$, it follows from (4.5) that

$$\mathrm{Im}(M(z)) = \int_{[0,\infty)} \mathrm{Re}(\sqrt{z-\lambda})\, dE_T(\lambda) \geq \int_{[0,\tau)} \mathrm{Re}(\sqrt{z-\lambda})\, dE_T(\lambda), \tag{4.14}$$

where $z = t + iy$. It is easily seen that

$$\mathrm{Re}(\sqrt{z-\lambda}) \geq \sqrt{t-\lambda} \geq \sqrt{t-\tau}, \quad \lambda \in [0,\tau), \quad t > \tau. \tag{4.15}$$

Combining (4.13) with (4.14) and (4.15) we get

$$\mathrm{Im}(M_B(t+iy)) \geq \sqrt{t-\tau} M_B(t+iy)^* E_T([0,\tau)) M_B(t+iy), \quad t > \tau > 0.$$

Let Q be a finite-dimensional orthogonal projection, $Q \leq E_T([0,\tau))$. Hence

$$\mathrm{Im}(M_B(t+iy)) \geq \sqrt{t-\tau} M_B(t+iy)^* Q M_B(t+iy), \quad t > \tau > 0, \quad y > 0.$$

Setting $\mathcal{H}_1 = \operatorname{ran}(Q)$, $\mathcal{H}_2 := \operatorname{ran}(Q^\perp)$, and choosing $K_2 \in \mathfrak{S}_2(\mathcal{H}_2)$ and satisfying $\ker(K_2) = \ker(K_2^*) = \{0\}$, we define a Hilbert-Schmidt operator $K := Q \oplus K_2 \in \mathfrak{S}_2(\mathcal{H})$. Clearly, $\ker(K) = \ker(K^*) = \{0\}$ and,

$$\begin{aligned}&\operatorname{Im}(K^* M_B(t+iy)K) \geq \\ &\qquad \sqrt{t-\tau} K^* M_B(t+iy)^* Q M_B(t+iy)K, \quad t > \tau > 0.\end{aligned} \tag{4.16}$$

Since $M_B(\cdot) \in (R_{\mathcal{H}})$ and $Q,\ K \in \mathfrak{S}_2(\mathcal{H})$, the limits

$$\begin{aligned}K^* M_B(t)^* Q &:= \operatorname*{s-lim}_{y\downarrow 0} K^* M_B(t+iy)^* Q \quad \text{and}\\ (QM_BK)(t) &:= \operatorname*{s-lim}_{y\downarrow 0} QM_B(t+iy)K\end{aligned}$$

exist for a.e. $t \in \mathbb{R}$ (see [5]). Therefore passing to the limit as $y \to 0$ in (4.16), we arrive at the inequality

$$\operatorname{Im}(M_B^K(t)) \geq \sqrt{t-\tau}(K^* M_B(t)^* Q)(QM_BK(t)), \quad t > \tau > 0, \quad y > 0.$$

It follows that

$$\dim(\operatorname{ran}\,((QM_BK)(t))) \leq \dim(\operatorname{ran}\,\left(\operatorname{Im} M_B^K(t)\right)) = d_{M_B^K}(t), \quad t > \tau. \tag{4.17}$$

We set $\widetilde{M}_B^Q(z) := QM_B(z)Q \restriction \mathcal{H}_1$. Since $\dim(\mathcal{H}_1) < \infty$ the limit $\widetilde{M}_B^Q(t) := \operatorname{s-lim}_{y\downarrow 0} \widetilde{M}_B^Q(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Since $(QM_BK)(t) \restriction \mathcal{H}_1 = \operatorname{ran}\left((\widetilde{M}_B^Q)(t)\right)$, (4.17) yields the inequality

$$\dim(\operatorname{ran}\,\left(\widetilde{M}_B^Q(t)\right)) \leq \dim(\operatorname{ran}\,((QM_BK)(t))) \leq d_{M_B^K}(t) \tag{4.18}$$

for a.e. $t \in [\tau, \infty)$.

Since $\dim(\mathcal{H}_1) < \infty$ and $\ker(\widetilde{M}_B^Q(z)) = \{0\}, z \in \mathbb{C}$, we easily get by repeating the corresponding reasonings of the proof of Theorem 3.11 that $\operatorname{ran}\left(\widetilde{M}_B^Q(t)\right) = \mathcal{H}_1$ for a.e. $t \in \mathbb{R}$. Therefore (4.18) yields $\dim(\mathcal{H}_1) \leq d_{M_B^K}(t)$ for a.e. $t \in [\tau, \infty)$.

If $\tau > t_1$, then $\dim(E_T([0,\tau))\mathcal{H}) = \infty$ and the dimension of a projection $Q \leq E_T([0,\tau))$ can be arbitrary. Thus, $d_{M_B^K}(t) = \infty$ for a.e. $t > \tau$. Since $\tau > t_1$ is arbitrary we get $d_{M_B^K}(t) = \infty$ for a.e. $t > t_1$. By Theorem 2.8(ii) the operator $\widetilde{A}^{ac} E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to $A_0 E_{A_0}([t_1,\infty))$.

(ii) If $\tau \in (t_0, t_1)$, then $\dim(E_T([0,\tau))\mathcal{H}) =: p(\tau) < \infty$. Hence, $\dim(Q\mathcal{H}) \leq p(\tau)$ which shows that $d_{M_B^K}(t) \geq p(\tau)$ for a.e. $t \in (\tau, t_1)$. Since τ is arbitrary, we obtain $d_{M_B^K}(t) \geq p(\tau)$ for a.e. $t \in [0, t_1)$. Using Theorem 2.8(i) we prove A^D is ac-minimal. Using Theorem 4.3(iv) we complete the proof of (ii).

(iii) By Lemma 4.2 the invariant maximal normal function $\mathfrak{m}^+(t)$ is finite for $t \in \mathbb{R}$. By Theorem 2.9 $\widetilde{A}^{ac}$ and $(A^F)^{ac}$ are unitarily equivalent. Similarly we prove that $\widetilde{A}^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent. To complete the proof it remains to apply Theorem 4.3(i). ☐

Using Definition 1.1 one gets the following corollary.

Corollary 4.6 *Let the assumptions of Theorem 4.5 be satisfied. If* $\dim(\mathcal{H}) = \infty$ *and* $t_0 := \inf\sigma(T) = \inf\sigma_{\rm ess}(T) =: t_1$, *then*

(i) *the Dirichlet, Neumann and Krein realizations are strictly* ac*-minimal;*

(ii) *the absolutely continuous part* $\widetilde{A}^{ac}$ *of* $\widetilde{A}$ *is unitarily equivalent to* A^D, *whenever*

$$(\widetilde{A} - i)^{-1} - (A^N - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{4.19}$$

Proof. (i) This statement follows from Theorem 4.5(i) and Theorem 4.3.

(ii) To prove this statement we note that by the Weyl theorem the inclusion (4.19) yields $\sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A^N)$. Since $\sigma_{\rm ess}(A^N) = \sigma_{ac}(A^N) = [t_0,\infty)$ we have $\sigma_{\rm ess}(\widetilde{A}) = [t_0,\infty)$. On the other hand, by Theorem 4.5(i) we get $[t_0,\infty) = \sigma_{\rm ess}(\widetilde{A}) \subseteq \sigma_{ac}(\widetilde{A})$. Thus, $\sigma_{ac}(\widetilde{A}) = [t_0,\infty)$ and $\widetilde{A}^{ac} = \widetilde{A}^{ac}E_{\widetilde{A}}\big([t_0,\infty)\big)$. Using Theorem 4.3(i) and again Theorem 4.5(i) we find that $\widetilde{A}^{ac}$ is unitarily equivalent to A^D. $\square$

Remark 4.7 According to (4.10) the condition $\mathfrak{m}^+(t) < \infty$, $t \in \mathbb{R}$ (cf. (2.8)) is not satisfied for the Weyl function $M_0(\cdot)$ of the Neumann extension A^N. Thus, the statement (ii) of Corollary 4.6 shows that the assumption $\mathfrak{m}^+(t) < \infty$ of Theorem 2.9, which is a generalization of the classical Kato-Rosenblum theorem, is sufficient but not necessary for validity of the conclusions.

Corollary 4.8 *Let the assumptions of Theorem 4.5 be satisfied and let* $\dim(\mathcal{H}) = \infty$. *Then* A^D *is strictly* ac*-minimal if and only if* $t_0 = t_1$.

Proof. Let $t_0 < t_1$. Then there is a decomposition $T = T_{\rm fin} \oplus T_\infty$ such that $T_{\rm fin}$ acts in a finite dimensional Hilbert space $\mathcal{H}_{\rm fin}$ and $t_0 = \inf\sigma(T_{\rm fin})$ and $T_\infty = T_\infty^* \in \mathcal{C}(\mathcal{H}_\infty)$ and $t_0 < t_\infty := \inf\sigma(T_\infty) \le t_1$. This leads to the decomposition $A = A_{\rm fin} \oplus A_\infty$ where $A_{\rm fin}$ and A_∞ are defined analogously to (4.1). Clearly $A^D = A^D_{\rm fin} \oplus A^D_\infty$. By Theorem 4.3 both extensions $A^D_{\rm fin}$ and A^D_∞ are absolutely continuous and their spectra are given by $\sigma(A^D_{\rm fin}) = [t_0,\infty)$ and $\sigma(A^D_\infty) = [t_\infty,\infty)$. Since $\dim(\mathcal{H}_\infty) = \infty$ the deficiency indices of A_∞ are infinite. We note that $(-\infty, t_\infty)$ is a spectral gap for A_∞. Using a result of Brasche [7] there exists an extension $\widetilde{A}_\infty = \widetilde{A}_\infty^* \in \operatorname{Ext} A_\infty$ such that $\sigma(\widetilde{A}_\infty) \subseteq [t_0,\infty)$, the part $\widetilde{A}_\infty E_{\widetilde{A}_\infty}([t_0,t_\infty))$ is absolutely continuous and $N_{\widetilde{A}^{ac}_\infty}(t) = \infty$ for $t \in [t_0,t_1)$.

Let $\widetilde{A} := A^D_{\rm fin} \oplus \widetilde{A}_\infty$. The operator $\widetilde{A}$ is a self-adjoint extension of A such that $\sigma(\widetilde{A}) = \sigma(A^D) = [t_0,\infty)$. The parts $A^D E_{A^D}([t_0,t_\infty))$ and $\widetilde{A}E_{\widetilde{A}}([t_0,t_\infty))$ are absolutely continuous. However, the absolutely continuous parts of both extensions are not unitarily equivalent. Indeed, for a.e. $t \in [t_0,t_\infty)$ one has $N_{A^D}(t) < \infty$ but $N_{\widetilde{A}^{ac}}(t) = \infty$, by construction. Hence A^D is not strictly ac-minimal which yields $t_0 = t_1$. The converse follows from Corollary 4.6(i). $\square$

5 Sturm-Liouville operators with unbounded operator potentials

5.1 Regularity properties

In this subsection we consider the differential expression (4.1) with unbounded non-negative $T = T^*(\in \mathcal{C}(\mathcal{H}))$ in $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$. The minimal operator $A := A_{\min} := \overline{\mathcal{A}}$, cf.(1.1) and (1.2), is densely defined and non-negative. If T is bounded, then A coincides with (4.1).

Let $\mathcal{H}_1(T)$ be the Hilbert space which is obtained equipping the set $\mathrm{dom}\,(T)$ with the graph norm of T. Moreover, for any $s \geq 0$ we equip $\mathrm{dom}\,(T^s)$ with the graph norm

$$\|u\|_s = (\|u\|^2_{\mathcal{H}} + \|T^s u\|^2_{\mathcal{H}})^{1/2}, \qquad s \geq 0, \quad u \in \mathcal{H}, \tag{5.1}$$

and denote by $\mathcal{H}_s(T)$ the corresponding the Hilbert space. Following [29, Definition I.2.1] the intermediate spaces $[X, Y]_\theta$, $\theta \in [0,1]$, of $X = \mathcal{H}_1(T)$ and $Y = \mathcal{H}_0(T) := \mathcal{H}$ are defined by $[X, Y]_\theta = \mathcal{H}_{1-\theta}(T)$, $\theta \in [0,1]$.

Furthermore, by $\mathcal{H}_s(T)$, $s < 0$, we denote the completion of $\mathcal{H}$ with respect to the "negative"norm

$$\|u\|s = \|(I + T^{-2s})^{-1/2} u\|_{\mathcal{H}}, \qquad s < 0, \quad u \in \mathcal{H}. \tag{5.2}$$

At first, we describe the domain $\mathrm{dom}\,(A)$ of the minimal operator A. For this purpose, following [29] we introduce the Hilbert spaces $W^{k,2}_T(\mathbb{R}_+, \mathcal{H}) := W^{k,2}(\mathbb{R}_+, \mathcal{H}) \cap L^2(\mathbb{R}_+, \mathcal{H}_1(T))$, $k \in \mathbb{N}$, equipped with the Hilbert norms

$$\|f\|^2_{W^{k,2}_T} = \int_{\mathbb{R}_+} \big(\|f^{(k)}(t)\|^2_{\mathcal{H}} + \|f(t)\|^2_{\mathcal{H}} + \|Tf(t)\|^2_{\mathcal{H}}\big) dt.$$

Obviously we have $\mathcal{D}_0 \subseteq W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ where is given by (1.2). The closure of $\mathcal{D}_0$ in $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ coincides with $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : \; f(0) = f'(0) = 0\}$ which yields $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) \subseteq \mathrm{dom}\,(A)$.

Lemma 5.1 *Let $T = T^*$ be a non-negative operator in $\mathcal{H}$. Then the domain $\mathrm{dom}\,(A)$ equipped with the graph norm coincides with the Hilbert space $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H})$ algebraically and topologically.*

Proof. Obviously, for any $f \in \mathcal{D}_0$ we have

$$\begin{aligned}\|\mathcal{A}f\|^2_{\mathfrak{H}} &= \int_{\mathbb{R}_+} \|f''(x)\|^2_{\mathcal{H}}\, dx \\ &+ \int_{\mathbb{R}_+} \|Tf(x)\|^2_{\mathcal{H}} dx - 2\mathrm{Re}\left\{\int_{\mathbb{R}_+} (f''(x), Tf(x))_{\mathcal{H}}\, dx\right\}.\end{aligned}$$

Integrating by parts we find

$$\int_{\mathbb{R}_+} (f''(x), Tf(x))\, dx = -\int_{\mathbb{R}_+} \left\|\sqrt{T} f'(x)\right\|^2_{\mathcal{H}} dx.$$

Hence

$$\|\mathcal{A}f\|_{\mathfrak{H}}^2 = \int_{\mathbb{R}_+} \|f''(x)\|^2\,dx + \int_{\mathbb{R}_+} \|Tf(x)\|^2 dx + 2\int_{\mathbb{R}_+} \left\|\sqrt{T}f'(x)\right\|_{\mathcal{H}}^2 dx$$

for any $f \in \mathcal{D}_0$ which yields

$$\|f\|_{W_T^{2,2}}^2 \le \|\mathcal{A}f\|_{\mathfrak{H}}^2 + \|f\|^2, \quad f \in \mathcal{D}_0.$$

Furthermore, by the Schwartz inequality,

$$2\left|\mathrm{Re}\left\{\int_{\mathbb{R}_+} (f'(x), Tf(x))_{\mathcal{H}}\,dx\right\}\right| \le \|f\|_{W_T^{2,2}}^2, \quad f \in \mathcal{D}_0.$$

which gives

$$\|\mathcal{A}f\|_{\mathfrak{H}}^2 + \|f\|^2 \le 2\|f\|_{W_T^{2,2}}^2, \quad f \in \mathcal{D}_0.$$

Thus, we arrive at the two-sided estimate

$$\|f\|_{W_T^{2,2}}^2 \le \|\mathcal{A}f\|_{\mathfrak{H}}^2 + \|f\|_{\mathfrak{H}}^2 \le 2\|f\|_{W_T^{2,2}}^2, \qquad f \in \mathcal{D}_0.$$

Since $\mathcal{D}_0$ is dense in $W_{0,T}^{2,2}(\mathbb{R}_+, \mathcal{H})$ we obtain that $\mathrm{dom}\,(A)$ coincides with $W_{0,T}^{2,2}(\mathbb{R}_+, \mathcal{H})$ algebraically and topologically. $\square$

In opposite to the case of the minimal operator $A = A_{\min}$ the maximal operator $A_{\max} = A_{\min}^*$ obviously satisfies $W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) \subset \mathrm{dom}\,(A_{\max})$, though $\mathrm{dom}\,(A_{\max}) \ne W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$ if T is not bounded. Moreover, it was firstly shown in [18] (see also [19, Section 4.1]) that the trace mapping

$$\{\gamma_0, \gamma_1\}:\ W_T^{2,2}(I, \mathcal{H}) \longrightarrow \mathcal{H}_{3/4}(T) \oplus \mathcal{H}_{1/4}(T), \quad \{\gamma_0, \gamma_1\}f = \{f(a), f'(a)\},$$

can be extended to a continuous (non-surjective) mapping

$$\{\gamma_0, \gamma_1\}:\ \mathrm{dom}\,(A_{\max}) \to \mathcal{H}_{-1/4}(T) \oplus \mathcal{H}_{-3/4}(T).$$

It is also shown in [19, Theorem 4.1.1] that $y(\cdot) \in \mathrm{dom}\,(A_{\max})$ if and only if the following conditions are satisfied:

(i) $y'(\cdot)$ exists and is an absolutely continuous function on I into $\mathcal{H}_{-1}(T)$;

(ii) $\mathcal{A}y \in L^2(I, \mathcal{H})$.

This result is similar to that for elliptic operators with smooth coefficients in domains with smooth boundary, cf. [21, 28]. A similar statement holds also for the operator $A_{\max} = A_{\min}^*$ considered in $L^2(\mathbb{R}_+, \mathcal{H})$, cf. [11, Section 9].

Next, we investigate the Friedrichs extension A^F and the Krein extension A^K of the operator $A \ge 0$. We define also the Neumann realization A^N as the self-adjoint operator associated with the closed quadratic form $\mathfrak{t}_N$,

$$\mathfrak{t}_N[f] := \int_0^\infty \left\{\|f'(x)\|_{\mathcal{H}}^2 + \|\sqrt{T}f(x)\|_{\mathcal{H}}^2\right\} dx = \|f\|_{W_{\sqrt{T}}^{1,2}}^2 - \|f\|_{L^2(\mathbb{R}_+,\mathcal{H})}^2, \tag{5.3}$$

$f \in \mathrm{dom}\,(\mathfrak{t}_N) := W^{1,2}_{\sqrt{T}}(\mathbb{R}_+, \mathcal{H})$. Clearly, $A^N \in \mathrm{Ext}\,_A$. In the case of bounded T one has $A^N = A_1$ where A_1 is defined in Theorem 4.3(ii).

We note that the closed quadratic $\mathfrak{t}_F$ associated with Friedrich extensions A^F is given by $\mathfrak{t}_F := \mathfrak{t}_N \upharpoonright \mathrm{dom}\,(\mathfrak{t}_F), \mathrm{dom}\,(\mathfrak{t}_F) := \{f \in W^{1,2}_{\sqrt{T}}(\mathbb{R}_+, \mathcal{H}) :\ f(0) = 0\}$.

Proposition 5.2 *Let $T = T^* \in \mathcal{C}(\mathcal{H}), T \geq 0$, and let $A := \overline{\mathcal{A}}$ Let also $\mathcal{H}_n := \mathrm{ran}\,\big(E_T([n-1, n))\big)$, $T_n := TE_T([n-1, n))$, $n \in \mathbb{N}$, and let S_n be the closed minimal symmetric operator defined by* (4.1) *in $\mathfrak{H}_n := L^2(\mathbb{R}_+, \mathcal{H}_n)$ with T replaced by T_n. Then*

(i) *the following decompositions hold*

$$A = \bigoplus_{n=1}^{\infty} S_n, \quad A^F = \bigoplus_{n=1}^{\infty} S_n^F, \quad A^K = \bigoplus_{n=1}^{\infty} S_n^K, \quad A^N = \bigoplus_{n=1}^{\infty} S_n^N; \tag{5.4}$$

(ii) *the domain $\mathrm{dom}\,(A^F)$ equipped with the graph norm is a closed subspace of $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ is given by $\mathrm{dom}\,(A^F) = \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$;*

(iii) *the domain $\mathrm{dom}\,(A^N)$ equipped with the graph norm is a closed subspace of $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$, is give by $\mathrm{dom}\,(A^N) = \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : f'(0) = 0\}$.*

Proof. (i) Since Lemma 5.1 is valid for bounded T we find that the graph $\mathrm{gr}\,(S_n)$ of S_n equipped with usual graph norm is algebraically and topologically equivalent to $W^{2,2}_{T_n}(\mathbb{R}_+, \mathcal{H}_n)$, $n \in \mathbb{N}$. Obviously, we have

$$W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) = \bigoplus_{n \in \mathbb{N}} W^{2,2}_{T_n}(\mathbb{R}_+, \mathcal{H}_n)$$

which yields

$$\mathrm{gr}\,(A) = \bigoplus_{n \in \mathbb{N}} \mathrm{gr}\,(S_n).$$

However, the last relation proves the first relation of (5.4).

The second and the third relations are implied by Corollary 3.5. To prove the last relation of (5.4) we set $S^N := \bigoplus_{n=1}^{\infty} S_n^N$. Since $S_n^N = (S_n^N)^* \in \mathrm{Ext}\,_{S_n}$ and $A = \bigoplus_{n=1}^{\infty} S_n$, S^N is a self-adjoint extension of A, $S^N \in \mathrm{Ext}\,_A$. Let $f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H}$ where $\mathfrak{H} = \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$. Denoting by $\widetilde{\mathfrak{t}}_N$ the quadratic form associated with S^N we find $f = \bigoplus_{n=1}^{\infty} f_n \in \mathrm{dom}\,(\widetilde{\mathfrak{t}}_N)$ if and only if $f_n \in \mathrm{dom}\,(\mathfrak{t}_n)$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] < \infty$ where $\mathfrak{t}_n$ is the quadratic form associated with S_n^N, $n \in \mathbb{N}$. If $f \in \mathrm{dom}\,(\widetilde{\mathfrak{t}}_N)$, then

$$\begin{aligned}
\widetilde{\mathfrak{t}}_N[f] = \sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] &= \sum_{n=1}^{\infty} \int_0^{\infty} \Big\{\|f_n'(x)\|^2_{\mathcal{H}_n} + \|\sqrt{T}_n f_n(x)\|^2_{\mathcal{H}_n}\Big\}\, dx \\
&= \int_0^{\infty} \Big\{\|f'(x)\|^2_{\mathcal{H}} + \|\sqrt{T} f(x)\|^2_{\mathcal{H}}\Big\}\, dx = \mathfrak{t}_N[f]
\end{aligned}$$

which yields $f \in \mathrm{dom}\,(\mathfrak{t}_N)$. Conversely, if $f \in \mathrm{dom}\,(\mathfrak{t}_N)$ and $f = \bigoplus_{n=1}^{\infty} f_n$, then $f_n \in \mathrm{dom}\,(\mathfrak{t}_n)$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] < \infty$ which proves $f \in \mathrm{dom}\,(\widetilde{\mathfrak{t}}_N)$. Hence $S^N = A^N$.

(ii) Following the reasoning of Lemma 5.1 we find

$$\|f_n\|^2_{W^{2,2}_{T_n}} \leq \|S^F_n f_n\|^2_{\mathfrak{H}_n} + \|f_n\|^2_{\mathfrak{H}_n} \leq 2\|f_n\|^2_{W^{2,2}_{T_n}}, \qquad n \in \mathbb{N}, \tag{5.5}$$

where $f_n \in \mathrm{dom}\,(S^F_n) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n) : g_n(0) = 0\}$. Using representation (5.4) for A^F and setting $f^m := \bigoplus_{n=1}^m f_n$, $f_n \in \mathrm{dom}\,(F_n)$, we obtain from (5.5)

$$\|f^m\|^2_{W^{2,2}_T} \leq \|A^F f^m\|^2_{\mathfrak{H}} + \|f^m\|^2_{\mathfrak{H}} \leq 2\|f^m\|^2_{W^{2,2}_T}, \quad m \in \mathbb{N}. \tag{5.6}$$

Since the set $\{f^m = \bigoplus_{n=1}^m f_n : \ f_n \in \mathrm{dom}\,(S^F_n), \ m \in \mathbb{N}\}$, is a core for A^F, inequality (5.6) remains valid for $f \in \mathrm{dom}\,(A^F)$. This shows that $\mathrm{dom}\,(A^F) = \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}$. Moreover, due to (5.6) the graph norm of A^F and the norm $\|\cdot\|_{W^{2,2}_T}$ restricted to $\mathrm{dom}\,(A^F)$ are equivalent.

(iii) Similarly to (5.5) one gets

$$\|f_n\|^2_{W^{2,2}_{T_n}} \leq \|S^N_n f_n\|^2_{\mathfrak{H}_n} + \|f_n\|^2 \leq 2\|f_n\|^2_{W^{2,2}_{T_n}}$$

for $f_n \in \mathrm{dom}\,(S^N_n) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n) : g_n'(0) = 0\}$, $n \in \mathbb{N}$. It remains to repeat the reasonings of (ii). $\square$

In the following we denote by $C_b(\mathbb{R}_+, \mathcal{H}_s)$, $s \in [0,1]$, the space of bounded continuous functions $f : \mathbb{R}_+ \longrightarrow \mathcal{H}_s$.

Corollary 5.3 *Let the assumptions of Proposition 5.2 be satisfied. Further, let $\partial f := f'$ be the derivative of $f \in W^{2,2}(\mathbb{R}_+, \mathcal{H})$ in the distribution sense. If $f \in \mathrm{dom}\,(A^D) \cup \mathrm{dom}\,(A^N)$, then*

(i) $\partial f := f' \in L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T))$ *and the maps*

$$\begin{aligned} \partial : &\quad \mathrm{dom}\,(A^D) \ni f \longrightarrow f' \in L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T)), \\ \partial : &\quad \mathrm{dom}\,(A^N) \ni f \longrightarrow f' \in L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T)) \end{aligned}$$

are continuous;

(ii) $f(\cdot) \in C_b(\mathbb{R}_+, \mathcal{H}_{3/4}(T))$, $f'(\cdot) \in C_b(R_+, \mathcal{H}_{1/2}(T))$ *and the maps*

$$\begin{aligned} \partial^j : &\quad \mathrm{dom}\,(A^D) \ni f \longrightarrow f^{(j)} \in C_b(\mathbb{R}_+, \mathcal{H}_{3/4-j/2}(T)), \\ \partial^j : &\quad \mathrm{dom}\,(A^N) \ni f \longrightarrow f^{(j)} \in C_b(\mathbb{R}_+, \mathcal{H}_{3/4-j/2}(T)), \end{aligned}$$

$j = 0, 1$, *are continuous. In particular, one has $f(0) \in \mathcal{H}_{3/4}(T)$ and $f'(0) \in \mathcal{H}_{1/4}(T)$.*

Proof. (i) From Proposition 5.2(ii) and (iii) we get that $u \in L^2(\mathbb{R}_+, X)$, $X = \mathcal{H}_1(T)$. Applying the intermediate Theorem I.2.3 of [29] to $X \subseteq Y = \mathcal{H}_0 := \mathcal{H}$ we immediately obtain $f' \in L^2(\mathbb{R}_+, [X, Y]_{1/2})$ which yields $f' \in L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T))$. Moreover, it follows that the map ∂ is continuous.

(ii) Combining Proposition 5.2(ii) and (iii) with the trace theorem [29, Theorem 1.3.1] one proves (ii). $\square$

Remark 5.4 Lemma 5.1, Proposition 5.2 and Corollary 5.3 also hold for realizations of the differential expression $\mathcal{A}$ considered on a finite interval I, i,e, in the space $L^2(I, \mathcal{H})$. For this case Corollary 5.3 has firstly been proved by M.L. Gorbachuk [18] (see also [19, Corollary 4.1.5], [19, Theorem 4.2.4]) by applying another method. Realizations $\widetilde{A} \in \mathrm{Ext}_A$ satisfying the condition $\mathrm{dom}\,(\widetilde{A}) \subset C(I, \mathcal{H}_{3/4}(T))$ are called maximally smooth (see [19, Section 4.2]).

We emphasize however, that Lemma 5.1 and Proposition 5.2 are new for the case of finite interval realizations too.

5.2 Operators on semi-axis: Spectral properties.

To extend Theorem 4.3 to the case of unbounded operators $T = T^* \geq 0$ we firstly construct a boundary triplet for A^*, using Theorem 3.3 and representation (5.4) for A.

Lemma 5.5 *Let the assumptions of Proposition 5.2 be satisfied. Then there is a sequence of boundary triplets $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S_n^* such that $\Pi := \bigoplus_{n=1}^{\infty} \widehat{\Pi}_n =: \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triplet for A^*. Moreover, $A^F = A^* \upharpoonright \ker(\widehat{\Gamma}_0)$ and the corresponding Weyl function is given by*

$$\widehat{M}(z) = \frac{i\sqrt{z-T} + \mathrm{Im}(\sqrt{i-T})}{\mathrm{Re}(\sqrt{i-T})}. \qquad z \in \mathbb{C}_+, \tag{5.7}$$

Proof. For any $n \in \mathbb{N}$ we choose a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S_n^* with Γ_{0n}, Γ_{1n} defined by (4.3). By Theorem 4.3(i) $S_n^F = S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$ and by Lemma 4.1 the corresponding Weyl function is $M_n(z) = i\sqrt{z - T_n}$.

Following Lemma 3.1, cf. (3.6), we define a sequence of regularized boundary triplets $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S_n^* by setting $R_n := (\mathrm{Re}(\sqrt{i-T_n}))^{1/2}$, $Q_n := -\mathrm{Im}(\sqrt{i-T_n})$ and

$$\widehat{\Gamma}_{0n} := R_n \Gamma_{0n}, \quad \widehat{\Gamma}_{1n} := R_n^{-1}(\Gamma_{1n} - Q_n \Gamma_{0n}), \qquad n \in \mathbb{N}. \tag{5.8}$$

Hence $S_n^F = S_{0n}$ and the corresponding Weyl function $\widehat{M}_n(\cdot)$ is given by

$$\widehat{M}_n(z) = \frac{i\sqrt{z-T_n} + \mathrm{Im}(\sqrt{i-T_n})}{\mathrm{Re}(\sqrt{i-T_n})}, \qquad z \in \mathbb{C}_+, \qquad n \in \mathbb{N}. \tag{5.9}$$

By Theorem 3.3 the direct sum $\widehat{\Pi} := \bigoplus_{n=1}^{\infty} \widehat{\Pi}_n = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triplet for A^* and the corresponding Weyl function is

$$\widehat{M}(z) = \bigoplus_{n\in\mathbb{N}} \widehat{M}_n(z), \qquad z \in \mathbb{C}_+. \tag{5.10}$$

Combining (5.10) with (5.9) we arrive at (5.7). From Theorem 3.3 (cf. (3.13)) and Corollary 3.5 we get

$$A_0 = A^* \upharpoonright \ker(\widehat{\Gamma}_0) = \bigoplus_{n=1}^{\infty} S_n^* \upharpoonright \ker(\widehat{\Gamma}_{0n}) = \bigoplus_{n=1}^{\infty} S_{0n} = \bigoplus_{n=1}^{\infty} S_n^F = A^F \tag{5.11}$$

which proves the second assertion. $\square$

Next we generalize Theorem 4.3 to the case of unbounded operator potentials.

Theorem 5.6 *Let $T = T^* \geq 0$, $t_0 := \inf \sigma(T)$. Let $A := A_{\min}$ be the minimal operator associated with $\mathcal{A}$, cf.* (1.1) *and let $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ be the boundary triplet for A^* defined by Lemma 5.5. Then the following holds:*

(i) *The Dirichlet realization $A^D f := \mathcal{A}f$, $f \in \mathrm{dom}\,(A^D) := \{g \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) : g(0) = 0\}$ coincides with $A_0 := A^* \upharpoonright \ker(\widehat{\Gamma}_0)$ which is identical with the Friedrichs extension A^F. Moreover, A^D is absolutely continuous and $\sigma(A^D) = \sigma_{ac}(A^D) = [t_0, \infty)$.*

(ii) *The Neumann realization $A^N := \mathcal{A}f$, $f \in \mathrm{dom}\,(A^N) := \{g \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H} : g'(0) = 0\}$ coincides with $A_{B^N} := A^* \upharpoonright \mathrm{dom}\,(A_{B^N})$ where $\mathrm{dom}\,(A_{B^N}) = \mathrm{dom}\,(\ker(\widehat{\Gamma}_1 - B^N \widehat{\Gamma}_0))$ and $B^N := \sqrt{T + \sqrt{I + T^2}}$. Moreover, A^N is absolutely continuous $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*

(iii) *The Krein realization (or extension) A^K is given by $A_{B^K} := A^* \upharpoonright \ker(\Gamma_1 - B^K\Gamma_0)$, where*

$$B^K = \frac{1}{\sqrt{2}\sqrt{T} + \sqrt{T + \sqrt{1+T^2}}} \frac{1}{\sqrt{T + \sqrt{1+T^2}}}. \tag{5.12}$$

Moreover, $\ker(A^K) = \mathfrak{H}_0 := \overline{\mathfrak{H}_0'}$, $\mathfrak{H}_0' := \{e^{-x\sqrt{T}}h : h \in \mathrm{ran}\,(T^{1/4})\}$, the restriction $A^K \upharpoonright \mathrm{dom}\,(A^K) \cap \mathfrak{H}_0^\perp$ is absolutely continuous, and $A^K = 0_{\mathfrak{H}_0} \bigoplus (A^K)^{ac}$. In particular, $\sigma(A^K) = \{0\} \cup \sigma_{ac}(A^K)$ and $\sigma_{ac}(A^K) = [t_0, \infty)$.

(iv) *The realizations A^D, A^N and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. (i) From Proposition 5.2(ii) we get $A^D = A^F$. Applying Lemma 5.5 we get $A^F = A_0$. Finally, using Proposition 5.2(i) and Theorem 4.3(i) we verify the remaining part.

(ii) It is easily seen that with respect to the boundary triplet $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ defined by (5.8) the extension A^N_n admits a representation $A^N_n = A_{B_n}$ where $B_n := \sqrt{T_n + \sqrt{1 + T_n^2}}$, $n \in \mathbb{N}$. By Proposition 5.2(i), $A^N = \bigoplus_{n=1}^\infty A^N_n = A_{B^N}$ where $B^N = \bigoplus_{n=1}^\infty B_n$. The remaining part of (ii) follows from the representation $A^N = \bigoplus_{n=1}^\infty A^N_n$ and Theorem 4.3(ii).

(iii) Using the polar decomposition $i - \lambda = \sqrt{1+\lambda^2}e^{i\theta(\lambda)}$ with $\theta(\lambda) = \pi - \arctan(1/\lambda)$, $\lambda \geq 0$ we get

$$\mathrm{Re}(\sqrt{i - T}) = \int_0^\infty \sqrt[4]{1+\lambda^2}\cos(\theta(\lambda)/2)dE_T(\lambda). \tag{5.13}$$

Setting $\varphi(\lambda) = \arctan(1/\lambda)$, $\lambda \geq 0$ and noting that $\cos(\varphi(\lambda)) = \lambda(1+\lambda^2)^{-1/2}$, we find $\cos(\theta(\lambda)/2) = 2^{-1/2}(1+\lambda^2)^{-1/4}(\lambda + \sqrt{1+\lambda^2})^{-1/2}$. Substituting this expression in (5.13) yields

$$\mathrm{Re}(\sqrt{i - T}) = 2^{-1/2}(T + \sqrt{1+T^2}\,)^{-1/2}. \tag{5.14}$$

Similarly, taking into account $\sin(\theta(\lambda)/2) = \cos(\varphi(\lambda)/2)$ and $\cos(\varphi(\lambda)/2) = 2^{-1/2}(1+\lambda^2)^{-1/4}(\lambda + \sqrt{1+\lambda^2})^{1/2}$, we get

$$\mathrm{Im}(\sqrt{i - T}) = \int_0^\infty \sqrt[4]{1+\lambda^2}\cos(\varphi(\lambda)/2)dE_T(\lambda) = \frac{1}{\sqrt{2}}\sqrt{T + \sqrt{1+T^2}}. \tag{5.15}$$

It follows from (5.7) with account of (5.14) and (5.15) that $M(0) := \text{s-lim}_{x\downarrow 0} M(-x) =: B^K$ where B^K is defined by (5.12). Therefore, by [11, Proposition 5(iv)] the Krein extension A^K is given by $A_{B^K} := A^* \upharpoonright \ker(\Gamma_1 - B^K\Gamma_0)$. The remaining statement follows from Proposition 5.2(i) and Theorem 4.3(iii).

(iv) The assertion follows from Theorem 4.3(iv) and (5.4). $\square$

Next we generalize Theorem 4.5 to the case of unbounded $T \geq 0$.

Theorem 5.7 *Let $T = T^* \geq 0$ and $t_1 := \inf \sigma_{\rm ess}(T)$. Further, let A be the minimal operator of $\mathcal{A}$, cf.* (1.1)-(1.2)*, and $\widetilde{A} = \widetilde{A}^* \in Ext_A$. Then*

(i) *the absolutely continuous part $\widetilde{A}^{ac}E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to the part $A^D E_{A^D}([t_1,\infty))$;*

(ii) *the Dirichlet, Neumann and Krein realizations are ac-minimal and $\sigma(A^D) = \sigma(A^N) = \sigma_{ac}(A^K) \subseteq \sigma_{ac}(\widetilde{A})$;*

(iii) *the ac-part $\widetilde{A}^{ac}$ is unitarily equivalent to A^D if either $(\widetilde{A}-i)^{-1} - (A^F - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$ or $(\widetilde{A}-i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$.*

Proof. By [33, Corollary 4.2] it suffices to assume that the extension $\widetilde{A} = \widetilde{A}^*$ is disjoint with A_0, that is, it admits a representation $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$.

(i) We consider the boundary triplet $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ defined in Lemma 5.5. In accordance with (2.6) the Weyl function corresponding to A_B is given by $\widehat{M}_B(z) = (B - \widehat{M}(z))^{-1}$, $z \in \mathbb{C}_+$, where $\widehat{M}(z)$ is given by (5.7). Clearly,

$$\mathrm{Im}(\widehat{M}_B(z)) = \widehat{M}_B(z)^* \,\mathrm{Im}(\widehat{M}(z))\, \widehat{M}_B(z), \qquad z \in \mathbb{C}_+. \tag{5.16}$$

It follows from (5.7) that $\left(\mathrm{Re}(\sqrt{i-T})\right)^{-1} \geq \sqrt{2}$. Therefore (5.14) yields

$$\mathrm{Im}(\widehat{M}(z)) \geq \sqrt{2}\,\mathrm{Im}(M(z)), \quad z \in \mathbb{C}_+, \quad \text{where} \quad M(z) = i\sqrt{z-T}, \tag{5.17}$$

cf. (4.5). Following the line of reasoning of the proof of Theorem 4.5(i) we obtain from (5.17) that $d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1,\infty)$, where $D = D^* \in \mathfrak{S}_2(\mathcal{H})$ and $\ker D = \{0\}$. Moreover, it follows from (5.16) that $d_{\widehat{M}_B^D}(t) = d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1,\infty)$. One completes the proof by applying Theorem 2.8.

(ii) To prove (ii) for A^D we use again estimates (5.17) and follow the proof of Theorem 4.5(ii). We complete the proof for A^D by applying Theorem 2.8. Taking into account Theorem 5.6(iv) we complete the proof of (ii).

(iii) The Weyl function $\widehat{M}(\cdot)$ is given by (5.7). Taking into account (5.10) one obtains $\sup_{n\in\mathbb{N}} \mathfrak{m}_n^+ < \infty$, where $\mathfrak{m}_n^+$ is the invariant maximal normal function defined by (2.8). Indeed, this follows from (4.6) because this estimate shows that $\mathfrak{m}_n^+$ does not depend on $n \in \mathbb{N}$. Applying Theorem 2.9 we complete the proof.

To prove the second statement we note that the operator B^K defined by (5.12) is bounded. Therefore, by (2.6) to A_{B^K} the Weyl function

$$\widehat{M}_{B^K}(z) = (B^K - \widehat{M}(z))^{-1}, \quad z \in \mathbb{C}_+.$$

corresponds. Inserting expression (5.12) into this formula we get

$$\widehat{M}_{B^K}(z) = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{T}+i\sqrt{z-T}}\frac{1}{\sqrt{T+\sqrt{1+T^2}}} = \frac{1}{z\sqrt{2}}\frac{\sqrt{T}-i\sqrt{z-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

It follows that the limit $\widehat{M}_{B^K}(t+i0)$ exists for any $t \in \mathbb{R} \setminus \{0\}$ and

$$\widehat{M}_{B^K}(t) := \operatorname{s-}\lim_{y\to+0} M_{B^K}(t+iy) = -\frac{1}{t\sqrt{2}}\frac{\sqrt{T}-i\sqrt{t-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

Clearly, $\widehat{M}_{B^K}(t) \in [\mathcal{H}]$ for any $t \in \mathbb{R} \setminus \{0\}$. By Theorem 2.9 the ac-parts of $\widetilde{A}$ and A^K are unitarily equivalent whenever $(\widetilde{A}-i)^{-1} - (A^K-i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$. This completes the proof. $\square$

Finally, we generalize Corollary 4.6 to unbounded operator potentials.

Corollary 5.8 *Let the assumptions of Theorem 5.7 be satisfied. If the conditions* $\dim(\mathcal{H}) = \infty$ *and* $t_0 := \inf\sigma(T) = \inf\sigma_{\rm ess}(T) =: t_1$ *are valid, then*

(i) *the Dirichlet, Neumann and Krein realizations are strictly* ac*-minimal;*

(ii) *the* ac*-part* $\widetilde{A}^{ac}$ *of* $\widetilde{A}$ *is unitarily equivalent to* A^D *whenever* (4.19) *is satisfied.*

Proof. Corollary 5.8 follows immediately from Theorem 5.7(i) and Theorem 5.6(iv). $\square$

5.3 Application

In this subsection we apply previous results to Schrödinger operators in the half-space. To this end we denote by $L = L_{\min}$ the minimal elliptic operator associated with the differential expression

$$\mathcal{L} := -\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n}\frac{\partial^2}{\partial x^2} + q(x), \quad q(x) = \overline{q(x)} \in L^\infty(\mathbb{R}^n),$$

in $L^2(\mathbb{R}^{n+1}_+)$, $\mathbb{R}^{n+1}_+ := \mathbb{R}_+ \times \mathbb{R}^n$. Recall that $L_{\min}$ is the closure of $\mathcal{L}$ defined on $C_0^\infty(\mathbb{R}^{n+1}_+)$. It holds $\operatorname{dom}(L_{\min}) = H_0^2(\mathbb{R}^{n+1}_+) := \{f \in H^2(\mathbb{R}^{n+1}_+) : f \restriction \partial\mathbb{R}^{n+1}_+ = 0, \ \frac{\partial f}{\partial \mathfrak{n}} \restriction \partial\mathbb{R}^{n+1}_+ = 0\}$ where $\mathfrak{n}$ stands for the interior normal to $\partial\mathbb{R}^{n+1}_+$. Clearly, L is symmetric. The maximal operator $L_{\max}$ is defined by $L_{\max} = (L_{\min})^*$. We emphasize that $H^2(\mathbb{R}^{n+1}_+) \subset \operatorname{dom}(L_{\max}) \subset H^2_{loc}(\mathbb{R}^{n+1}_+)$ but $\operatorname{dom}(L_{\max}) \neq H^2(\mathbb{R}^{n+1}_+)$. The trace mappings $\gamma_j : C^\infty(\overline{\mathbb{R}^{n+1}_+}) \longrightarrow C^\infty(\partial\mathbb{R}^{n+1}_+)$, $j \in \{0,1\}$ are defined by $\gamma_0 f := f \restriction \partial\mathbb{R}^{n+1}_+$ and $\gamma_1 f := \frac{\partial f}{\partial \mathfrak{n}} \restriction \partial\mathbb{R}^{n+1}_+$. Let $\mathfrak{L}_+$ be the domain $\operatorname{dom}(L_{\max})$ equipped with the graph norm. It is known (see [21, 29]) that γ_j can be extended by continuity to the operators mapping $\mathfrak{L}_+$ continuously onto $H^{-j-1/2}(\partial\mathbb{R}^{n+1}_+)$, $j \in \{0,1\}$.

Let us define the following realizations of $\mathcal{L}$:

(i) $L^D f := \mathcal{L}f$, $f \in \mathrm{dom}\,(L^D) := \{\varphi \in H^2(\mathbb{R}^{n+1}_+) :\ \gamma_0\varphi = 0\}$;

(ii) $L^N f := \mathcal{L}f$, $f \in \mathrm{dom}\,(L^N) := \{\varphi \in H^2(\mathbb{R}^{n+1}_+) : \gamma_1\varphi = 0\}$;

(iii) $L^K f := \mathcal{L}f$, $f \in \mathrm{dom}\,(L^K) := \{\varphi \in \mathrm{dom}\,(L_{\max}) : \gamma_1\varphi + \Lambda\gamma_0\varphi = 0\}$ where $\Lambda := \sqrt{-\Delta_x + q(\cdot)} :\ H^{-1/2}(\partial\mathbb{R}^{n+1}_+) \to H^{-3/2}(\partial\mathbb{R}^{n+1}_+)$.

To treat the operator $L_{\min}$ as the Sturm-Liouville operator with (unbounded) operator potential we denote by T the minimal operator associated with the Schrödinger expression

$$\mathcal{T} := -\Delta_x + q(x) := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + q(x), \quad \overline{q(x)} = q(x), \tag{5.18}$$

in $\mathcal{H} := L^2(\mathbb{R}^n)$. It turns out that T is Moreover, If $q(x) \geq 0$, then $T \geq 0$. Let $A := A_{\min}$ be the minimal operator associated with (1.1) where $T = T_{\min}$.

Proposition 5.9 *Let $q(\cdot) \in L^\infty(\mathbb{R})$, $q(\cdot) \geq 0$, and let T be the minimal (self-adjoint) operator associated with $\mathcal{T}$ in $L^2(\mathbb{R})$. Let also $t_0 := \inf \sigma(T)$ and $t_1 := \inf \sigma_{\mathrm{ess}}(T)$. Then:*

(i) *the minimal operator A coincides with the minimal operator L and $\mathrm{dom}\,(A) = H^2_0(\mathbb{R}^{n+1}_+)$;*

(ii) *the Dirichlet realization A^D coincides with L^D, hence, L^D is absolutely continuous and $\sigma(L^D) = \sigma_{ac}(L^D) = [t_0, \infty)$;*

(iii) *the Neumann realization A^N coincides with L^N, in particular, L^N is absolutely continuous and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$;*

(iv) *the Krein realization A^K coincides with L^K, in particular, L^K admits the decomposition $L^K = 0_{\mathcal{H}_0} \bigoplus (L^K)^{ac}$, $\mathcal{H}_0 := \ker\,(L^K)$, and $\sigma_{ac}(L^K) = [t_0, \infty)$;*

(v) *the self-adjoint realizations L^D, L^N, and L^K are ac-minimal, in particular, L^D, L^N, and $(L^K)^{ac}$ are unitarily equivalent to each other. If $t_0 = t_1$, then the operators L^D, L^N and L^K are strictly ac-minimal;*

(vi) *if $\widetilde{L}$ is a self-adjoint realization of $\mathcal{L}$ such that either $(\widetilde{L} - i)^{-1} - (L^D - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^{n+1}_+))$ or $(\widetilde{L} - i)^{-1} - (L^K - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^{n+1}_+))$ is satisfied, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent;*

(vii) *If $t_0 = t_1$ and if $\widetilde{L}$ is a self-adjoint realization of $\mathcal{L}$ such that $(\widetilde{L} - i)^{-1} - (L^N - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^{n+1}_+))$ is satisfied, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent.*

Proof. (i) We introduce the set

$$\mathcal{D}_\infty := \left\{ \sum_{1\leq j\leq k} \phi_j(x)h_j(\xi) :\ \phi_j \in C_0^\infty(\mathbb{R}_+),\ h_j \in C_0^\infty(\mathbb{R}^n),\ k \in \mathbb{N} \right\}$$

We note that $\mathcal{D}_\infty \subseteq \mathcal{D}_0$, which is given by (1.2), and $\mathcal{D}_\infty \subseteq C_0^\infty(\mathbb{R}^{n+1}_+)$. Moreover, $A \restriction \mathcal{D}_\infty = L \restriction \mathcal{D}_\infty$. Since $\mathcal{D}_\infty$ is a core for both minimal operators A and L we have $A = L$ which yields $\mathrm{dom}\,(A) = H_0^{2,2}(\mathbb{R}^{n+1}_+)$.

(ii) Since $A = L$ we have $A^F = L^F$. Using $L^F = L^D$ the proof of (ii) follows immediately from Theorem 5.6(i).

(iii) One verifies that $W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) = H^2(\mathbb{R}_+^{n+1})$, i.e, both spaces are isomorphic. A straightforward computation shows that

$$\mathfrak{t}^{\mathcal{L}}[f] := (\mathcal{L}f, f)_{\mathfrak{H}} = (\mathcal{A}f, f)_{L^2(\mathbb{R}_+^{n+1})} =: \mathfrak{t}^{\mathcal{A}}[f], \quad f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) = H^2(\mathbb{R}_+^{n+1}).$$

Since $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$ is dense in $W_{\sqrt{T}}^{1,2}(\mathbb{R}_+, \mathcal{H})$ the completion of $\mathfrak{t}^{\mathcal{A}}$ gives $\mathfrak{t}_N$ defined by (5.3) which is the closed quadratic form associated with A^N. Moreover, using that $H^{2,2}(\mathbb{R}_+^{n+1})$ is dense in $H^{1,2}(\mathbb{R}_+^{n+1})$ the completion of $\mathfrak{t}^{\mathcal{L}}$ gives the closed quadratic form associated with L^N. Since both completion coincide we get that $A^N = L^N$. The remaining part follows from Theorem 5.6(ii).

(iv) Since $A = L$ we have that A^K is identical with the Krein realization of $\mathcal{L}$. However, it was proved in [11, Section 9.7] that even L^K is the Krein extension of $\mathcal{L}$ The rest of the statements is implied by Theorem 5.6(iii).

(v) By Theorem 5.7(ii) the extension A^D, A^N and A^K are ac-minimal. Taking into account (i) - (iv) we find that L^D, L^N and L^K are ac-minimal. The second statement of (v) follows from Corollary 5.8(i).

(vi) This statement follows immediately from Theorem 5.7(iii) and (ii).

(vii) It follows from Corollary 5.8(ii). $\square$

Remark 5.10 Let T be the (closed) minimal non-negative operator associated in $\mathcal{H} := L^2(\mathbb{R}^n)$ with general uniformly elliptic operator

$$\widetilde{\mathcal{T}} := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_j} + q(x), \quad a_{jk} \in C^1(\overline{\mathbb{R}}_+^{n+1}), \quad q \in C(\overline{\mathbb{R}}_+^{n+1}) \cap L^\infty(\mathbb{R}_+^{n+1}),$$

where the coefficients $a_{jk}(\cdot)$ are bounded with their C^1-derivatives, $q \geq 0$. If the coefficients have some additional "good"properties, then $\mathrm{dom}\,(T) = H^2(\mathbb{R}^n)$ algebraically and topologically. By Lemma 5.1, $\mathrm{dom}\,(A_{\min}) = W_{0,T}^{2,2}(\mathbb{R}_+, \mathcal{H}) = H_0^{2,2}(\mathbb{R}_+^{n+1})$ and Proposition 5.9 remains valid with T in place of the Schrödinger operator (5.18).

Note also that the Dirichlet and the Neumann realizations L^D and L^N are always self-adjoint ((cf. [29, Theorem 2.8.1], [21])).

Corollary 5.11 *Let the assumptions of Proposition 5.9 be satisfied. If*

$$\lim_{|x|\to\infty} \int_{|x-y|\leq 1} q(y)dy = 0, \tag{5.19}$$

then the realizations L^D, L^N and L^K are strictly ac-minimal and

$$\sigma(L^D) = \sigma_{ac}(L^K) = \sigma(L^N) = \sigma_{ac}(L^N) = [0, \infty).$$

Proof. By [17, Section 60] condition (5.19) yields the equality $\sigma_c(T) = \mathbb{R}_+$, in particular $0 \in \sigma_c(T)$ and $t_1 = 0$. Since $q \geq 0$, we have $0 \leq t_0 \leq t_1 = 0$, that is $t_0 = t_1 = 0$. It remains to apply Proposition 5.9(i)-(iv). $\square$

Remark 5.12 Condition (5.19) is satisfied whenever $\lim_{|x|\to\infty} q(x) = 0$. Thus, in this case the conclusions of Corollary 5.11 are valid. However, it might happen that $\sigma(L^D) = \sigma(L^N) = \sigma_{ac}(L^K) = [t_0, \infty)$, $t_0 > 0$, though $\inf q(x) = 0$.

A Appendix: Operators admitting separation of variables

A.1 Finite interval

Here we consider the differential expression $\mathcal{A}$ with unbounded $T = T^* \geq 0$ (cf. (1.1)) on a finite interval $I = [0, \pi]$ and denote it by $\mathcal{A}_I$. The minimal operator $A := A_{I,\min} := \overline{A'}$ generated by $\mathcal{A}$ in the Hilbert space $\mathfrak{H}_I := L^2(I, \mathcal{H})$ is defined similarly to that of $A = A_{\min}$ in $L^2(\mathbb{R}_+, \mathcal{H})$. Obviously, $A_{I,\min}$ is densely defined and non-negative.

We briefly discuss the spectral properties of realizations of $\mathcal{A}_I$ which admit separating of variables. We set

$$
\begin{aligned}
A_I^D f &:= \mathcal{A}_I f, \quad f \in \mathrm{dom}\,(A_I^D) := \{f \in W_T^{2,2}(I, \mathcal{H}) : f(0) = f(\pi) = 0\} \\
A_I^N f &:= \mathcal{A}_I f, \quad f \in \mathrm{dom}\,(A_I^D) := \{f \in W_T^{2,2}(I, \mathcal{H}) : f'(0) = f'(\pi) = 0\}
\end{aligned}
$$

where $W_T^{2,2}(I, \mathcal{H}) = W^{2,2}(I, \mathcal{H}) \cap L^2(I, \mathcal{H}_1(T))$ with $\mathcal{H}_1(T)$ defined by (5.1).

To state the main result denote by l_D and l_N the Dirichlet and Neumann realization of the differential expression $l := -d^2/dx^2$ in the Hilbert space $L^2(I)$, i.e.

$$
\begin{aligned}
l_D &:= -\tfrac{d^2}{dx^2} \upharpoonright \mathrm{dom}\,(l_D),\ \mathrm{dom}\,(l_D) = \{f \in W^{2,2}[0, \pi] : f(0) = f(\pi) = 0\}, \\
l_N &:= -\tfrac{d^2}{dx^2} \upharpoonright \mathrm{dom}\,(l_N),\ \mathrm{dom}\,(l_N) = \{f \in W^{2,2}[0, \pi] : f'(0) = f'(\pi) = 0\}.
\end{aligned}
$$

Obviously, both spectra are discrete and given by $\sigma(l_D) = \{1, 4, \ldots, k^2, \ldots\}$, $k \in \mathbb{N}$ and $\sigma(l_N) = \{0, 1, 4, \ldots, k^2, \ldots\}$, $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Proposition A.1 *Let A_I^D and A_I^N be the Dirichlet and the Neumann realizations of $\mathcal{A}_I$ in $L^2(I, \mathcal{H})$ and let $T_k := T + k^2 I_{\mathcal{H}} \big(\in \mathcal{C}(\mathcal{H})\big)$. Then*

(i) *A_I^D is unitarily equivalent to the operator $\oplus_{k=1}^{\infty} T_k$;*

(ii) *A_I^N is unitarily equivalent to the operator $\oplus_{k=0}^{\infty} T_k$;*

(iii) *The spectrum of the operators A_I^D and A_I^N is discrete, pure point, purely singular and absolutely continuous if and only if the spectrum of T is so.*

(iv) *The spectral multiplicity functions $N_{A_I^D}(\cdot)$ and $N_{A_N^D}(\cdot)$ of the realizations A_I^D and A_I^N, respectively, are finite for each $\lambda \in \mathbb{R}$ whenever the multiplicity function $N_T(\cdot)$ is finite. Moreover, if $\sigma_{ac}(T) = [t_0, \infty)$, then $\sigma_{ac}(A_I^D) = [t_0 + 1, \infty)$ and*

$$N_{(A_I^D)^{ac}}(t) = pN_{T^{ac}}(t) \quad \textit{for a.e.} \quad t \in [t_0 + k^2, t_0 + (k+1)^2), \quad k \in \mathbb{N},$$

as well as $\sigma_{ac}(A_I^D) = [t_0, \infty)$ and

$$N_{(A_I^N)^{ac}}(t) = (p+1)N_{T^{ac}}(t) \quad \textit{for a.e.} \quad t \in [t_0 + k^2, t_0 + (k+1)^2),$$

$k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

(v) *The operators $(A_I^D)^{ac}$ and $(A_I^N)^{ac}$ are not unitarily equivalent.*

Proof. (i) By the spectral theorem, the operator $l_D = l_D^*$ is unitarily equivalent to the diagonal operator $\Lambda_D = \operatorname{diag}(1^2, 2^2, \ldots, k^2, \ldots)$ acting in $\mathfrak{H}_D = l^2(\mathbb{N})$. Namely, $U_D l_D = \Lambda_D U_D$ where U_D is the unitary map from $L^2[0, \pi]$ onto $l^2(\mathbb{N})$,

$$U_D : \ f = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} a_k \sin kx \to \{a_k\}_1^\infty \in l^2(\mathbb{N})$$

and $a_k = (f, \sqrt{2/\pi} \sin kx)$. Hence

$$(U_D \otimes I_{\mathcal{H}})A^D(U_D^* \otimes I_{\mathcal{H}}) = (U_D \otimes I_{\mathcal{H}})(l_D \otimes I_{\mathcal{H}} + I_{\mathfrak{H}_1} \otimes T)(U_D^* \otimes I_{\mathcal{H}}) =$$
$$\Lambda_D \otimes I_{\mathcal{H}} + I_{\mathfrak{H}_2} \otimes T = \bigoplus_{k=1}^{\infty}(k^2 I_{\mathcal{H}} + T) = \bigoplus_{k=1}^{\infty} T_k.$$

(ii) In this case, by the spectral theorem, the operator A^N is unitarily equivalent to the diagonal operator $\Lambda_N = \operatorname{diag}(0, 1^2, 2^2, \ldots, k^2, \ldots)$ in $\mathfrak{H}_N = l^2(\mathbb{N}_0)$, $U_N l_N = \Lambda_N U_N$ where

$$U_N : f = \frac{1}{\sqrt{\pi}} b_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} b_k \cos kx \to \{b_k\}_0^\infty \in l^2(\mathbb{N}_0)$$

and $b_k = (f, \sqrt{2/\pi} \cos kx)$. Repeating the previous reasonings we arrive at the required relation

$$(U_N \otimes I_{\mathcal{H}})A^N(U_N^* \otimes I_{\mathcal{H}}) = \oplus_{k=0}^{\infty} T_k.$$

(iii) This statement follows immediately from (i) and (ii) in view of the obvious relations $\sigma\big(\bigoplus_{k=1}^\infty T_k\big) = \bigcup_{k=1}^\infty \sigma(T_k)$ and $\sigma_\tau\big(\bigoplus_{k=1}^\infty T_k\big) = \bigcup_{k=1}^\infty \sigma_\tau(T_k)$, $\tau = pp, s, sc, ac$.

(iv) From (i) and (ii) and the obvious relations $\sigma_\tau(T_k) = k^2 + \sigma_\tau(T_k)$, $\tau = d, pp, s, sc, ac$, $k \in \mathbb{N}$ we verify (iv).

(v) From (i) and (ii) it follows that $\sigma_{ac}(A_I^N) = \bigcup_{k=0}^\infty \sigma_{ac}(T_k)$ and $\sigma_{ac}(A_I^D) = \bigcup_{k=1}^\infty \sigma_{ac}(T_k)$ which yields $\sigma_{ac}(A_I^N) \neq \sigma_{ac}(A_I^D)$ which proves (v). □

A.2 Semi-axis

Our next purpose is to show that the spectral properties of realizations of $\mathcal{A}$ admitting separation of variables can be investigated directly by applying elementary methods. In particular, we present a simple proof of Theorem 5.6(ii). let us at first prove a general statement.

Lemma A.2 *Let K and T be self-adjoint operators in the separable Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, respectively, and let $L_K := K \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ which is self-adjoint in $\mathcal{K} \otimes \mathcal{H}$.*

(i) *If the self-adjoint operators K_1 and K_2 are unitarily equivalent, then L_{K_1} and L_{K_2} are unitarily equivalent*

(ii) *If K is absolutely continuous, then L_K is absolutely continuous.*

Proof. (i) Let V be a unitary operator such that $K_2 = V^* K_1 V$. Then $U := V \otimes I_{\mathcal{H}}$ is unitary and

$$U^* L_{K_1} U = V^* \otimes I_{\mathcal{H}}(K_1 \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T)V \otimes I_{\mathcal{H}} = K_2 \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T = L_{K_2}.$$

(ii) Let $\mathfrak{h}$ be an auxiliary infinite dimensional separable Hilbert space. In $L^2(\mathbb{R}, \mathfrak{h})$ we consider the multiplication operator Q defined by

$$(Qf)(t) = tf(t), \quad t \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, \mathfrak{h}). \tag{A.1}$$

If K is absolutely continuous, then there is an isometry $\Phi_0 : \mathcal{K} \longrightarrow L^2(\mathbb{R}, \mathfrak{h})$ such that $Q\Phi_0 = \Phi_0 K$, $\Phi_0^*\Phi_0 = I_{\mathcal{K}}$. Hence the isometry $\Phi := \Phi_0 \otimes I_{\mathcal{H}} : \mathcal{K} \otimes \mathcal{H} \longrightarrow L^2(\mathbb{R}, \mathfrak{h}) \otimes \mathcal{H}$ intertwines L_K and $\widehat{L} := Q \otimes I_{\mathcal{H}} + I_{L^2(\mathbb{R},\mathfrak{h})} \otimes T$, i.e.

$$\widehat{L}\,\Phi = \Phi L_K.$$

Notice that $L^2(\mathbb{R}, \mathfrak{h}) \otimes \mathcal{H} = L^2(\mathbb{R}, \mathfrak{h} \otimes \mathcal{H})$. The operator $\widehat{L}$ has in $L^2(\mathbb{R}, \mathfrak{h}')$, $\mathfrak{h}' := \mathfrak{h} \otimes \mathcal{H}$, the representation $\widehat{L} := \widehat{Q} + \widehat{T}$ where $\widehat{Q}$ is a multiplication operator which is defined similarly as Q, cf. (A.1), and $\widehat{T}$ is given by

$$(\widehat{T}\,f)(t) := T'f(t), \quad f \in \mathrm{dom}\,(\widehat{T}\,) := \{f \in L^2(\mathbb{R}, \mathfrak{h}') : T'f(t) \in L^2(\mathbb{R}, \mathfrak{h}')\}$$

where $T' := I_{\mathfrak{h}} \otimes T$. Using the Fourier transform $\mathcal{F}$ one easily verifies that $\widehat{Q}$ is unitarily equivalent to the momentum operator $-i\frac{d}{dt}$ in $L^2(\mathbb{R}, \mathfrak{h}')$, i.e $\mathcal{F}^{-1}\,\widehat{Q}\,\mathcal{F} = -i\frac{d}{dt}$. This yields that

$$\mathcal{F}\,\widehat{L}\,\mathcal{F}^{-1} = -i\frac{d}{dt} + \widehat{H}\,.$$

Finally, using the gauge transform $(\mathcal{G}f)(t) = e^{-it\widehat{H}}f(t)$, $f \in L^2(\mathbb{R}, \mathfrak{h}')$, we find $\mathcal{G}\mathcal{F}\,\widehat{L}\,\mathcal{F}^{-1}\mathcal{G}^{-1} = -i\frac{d}{dt}$. Hence

$$-i\frac{d}{dt}\,\mathcal{G}\mathcal{F}\Phi = \mathcal{G}\mathcal{F}\Phi L_K \tag{A.2}$$

Since the momentum operator $-i\frac{d}{dt}$ is absolutely continuous the relation (A.2) immediately implies that L_K is absolutely continuous. $\square$

We consider the self-adjoint operator

$$l_\tau := -\frac{d^2}{dt^2} \upharpoonright \mathrm{dom}\,(l_\tau), \qquad \mathrm{dom}\,(l_\tau) = \{f \in W^{2,2}(\mathbb{R}_+) : \ f'(0) = \tau f(0)\},$$

in $\mathcal{K} := L^2(\mathbb{R}_+)$ where $\tau \in \mathbb{R}_+ \cup \{0\} \cup \{\infty\}$. The extensions $\tau = 0$ and $\tau = \infty$ are identified with the Neumann and the Dirichlet realizations of $-\frac{d^2}{dt^2}{}^2$, respectively. Further, let $T = T^* \geq 0, T \in \mathcal{C}(\mathcal{H})$. Consider the family of self-adjoint operators

$$A_\tau := l_\tau \otimes I_\mathcal{H} + I_\mathcal{K} \otimes T, \qquad \tau \in \mathbb{R}_+ \cup \{0\} \cup \{\infty\}, \tag{A.3}$$

in the Hilbert space $\mathcal{K} \otimes \mathcal{H} = L^2(\mathbb{R}_+, \mathcal{H})$. Note for each $\tau \in \mathbb{R}_+ \cup \{0\} \cup \{\infty\}$ the operator A_τ can be regarded as a self-adjoint extension of the minimal operator A defined by (1.1) and (1.2). In particular, we have $A_0 = A^N$ and $A_\infty = A^D$.

Corollary A.3 *Let $T = T^* \geq 0$.*

(i) *If $\tau_1 \geq 0$ and $\tau_2 \geq 0$, then A_{τ_1} and A_{τ_2} are unitarily equivalent. In particular, the extensions A^D and A^N are unitarily equivalent.*

(ii) *If $\tau \geq 0$, then A_τ is absolutely continuous. In particular, A^D and A^N are absolutely continuous.*

Proof. (i) From [34, Section 21.5] we get that the operators l_τ are unitarily equivalent to each other if $\tau \geq 0$. Applying Lemma A.2(i) we prove (i).

(ii) Using the Fourier transformation one easily proves that the operator l_0 is absolutely continuous. Taking into account Lemma A.2(ii) we verify (ii). $\square$

Remark A.4

(i) We note that the above reasonings cannot be applied to realizations of $\mathcal{A}$ which do not admit the tensor product structure (A.3).

(ii) Comparing Corollary A.3 with Proposition A.1 we obtain that there are substantial differences between spectral properties of realizations on the semi-axis $\mathbb{R}_+$ and on a finite interval I. Indeed, for self-adjoint realizations of $\mathcal{A}$ on $\mathbb{R}_+$ the ac-part can never be eliminated for any $T = T^* \geq 0$, cf. Theorem 5.7(ii). In contrast to that the spectral properties of self-adjoint realizations of $\mathcal{A}_I$ strongly depend on T.

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