Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

On the convergence rate of grad-div stabilized Taylor-Hood to Scott-Vogelius solutions for incompressible flow problems

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submitted: January 18, 2011

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No. 1589
Berlin 2011

2010 *Mathematics Subject Classification.* 65M60, 65N30, 76D05 .

Key words and phrases. Navier-Stokes equations, Scott-Vogelius, Taylor-Hood, strong mass conservation, MHD, Leray-alpha.

This work was partially supported by NSF grant DMS0914478.

Edited by
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Abstract

It was recently proven in [2] that, under mild restrictions, grad-div stabilized Taylor-Hood solutions of Navier-Stokes problems converge to the Scott-Vogelius solution of that same problem. However, even though the analytical rate was only shown to be $\gamma^{-\frac{1}{2}}$ (where γ is the stabilization parameter), the computational results suggest the rate may be improvable γ^{-1}. We prove herein the analytical rate is indeed γ^{-1}, and extend the result to other incompressible flow problems including Leray-α and MHD. Numerical results are given that verify the theory.

1 Introduction

We prove that under mild restrictions, solutions to incompressible flow problems found with grad-div stabilized $((P_k)^d, P_{k-1})$ Taylor-Hood (TH) elements (with parameter γ) converge to the solution of the $((P_k)^d, P_{k-1}^{disc})$ Scott-Vogelius pair, with rate γ^{-1} as $\gamma \to \infty$. Provided the SV pair is LBB stable, for example if

(A1) in 2d, $k \geq 4$ and the mesh has no singular vertices [19],

(A2) in 3d, $k \geq 6$ [24],

(A3) when $k \geq d$ and the mesh is a barycenter refinement of a regular mesh [23, 19], or

(A4) on Powell-Sabin meshes when $k = 1$ and $d = 2$ or when $k = 2$ and $d = 3$ [25],

this convergence is proven in [2] with rate $\gamma^{-\frac{1}{2}}$ for Navier-Stokes problems, but their numerical experiments indicate an improved rate of γ^{-1}. We verify herein, with careful analysis and no further assumptions, the analytical rate is improvable to γ^{-1}, thus agreeing with the computations in [2]. We also extend the results to related problems including Leray-α model and magnetohydrodynamics.

TH elements are a popular choice for simulating incompressible flows, and many commercial software packages have them implemented. However, despite their popularity, solutions obtained with TH elements often suffer from poor mass conservation [2, 13], creating solutions with little physical plausibility. However, it has been shown in [17, 18, 10] that using TH elements with grad-div stabilization can improve mass conservation in solutions, and sometimes even overall accuracy. Yet, in general, the improvement in physical fidelity is limited because grad-div stabilization with $\gamma > O(1)$ can overstabilize [16, 2]. The results of [2], which we improve herein, show that in settings where SV elements are LBB stable, TH elements can be used with a large stabilization parameter *without overstabilizing*, since as $\gamma \to \infty$ the limit solution is the optimally accurate SV solution. Thus with a mild mesh restriction, TH elements can be used to find accurate solutions that are also physically plausible due to this improved mass conservation.

This paper is arranged as follows. In Section 2 we give notation and preliminaries, and prove a lemma for norm equivalence, which is fundamental for the analysis throughout. Section 3 shows the improved convergence rate for the steady and time dependent Navier-Stokes equations (NSE). Section 4 extends the results of Section 3 to the Leray-α model, and gives a numerical example (flow over a step) verifying the theory. In Section 5 the results are extended to MHD. Finally, in Section 6, we consider SV solution approximations by extrapolating 'small γ' TH solutions.

2 Preliminaries

We will denote the $L^2(\Omega)$ norm and inner product by $\|\cdot\|$ and $(\cdot,\cdot)$. All other norms and inner products will be clearly labeled.

We consider a domain Ω to be a convex polygon in 2D or polyhedra in 3D, discretized by a regular triangularization or tetrahedralization.

Two element pairs are studied herein, Taylor-Hood $(X_h, Q_h) := ((P_k)^d, P_{k-1})$, and Scott-Vogelius $(X_h, \widetilde{Q_h}) := ((P_k)^d, P_{k-1}^{disc})$ [21, 22]. We will always consider the elements with the same polynomial approximating degree k and on the same mesh, and thus the only difference between discretizations with the different elements is the pressure space for Scott-Vogelius is discontinuous.

Throughout the report, the constant C will be used to denote a data-dependent constant, whose value can change at each occurrence. However, C will always be independent of the grad-div stabilization parameter γ.

We assume conditions on the mesh and polynomial degree so that the SV element is LBB stable (e.g. any of A1-A4), and thus admits optimal convergence properties.

Denote the discretely divergence-free spaces for TH and SV elements, respectively, by

$$\begin{aligned} V_h &:= \{v_h \in X_h \,:\, (\nabla \cdot v_h, q_h) = 0 \;\forall q_h \in Q_h\}, \\ V_h^0 &:= \{v_h \in X_h \,:\, (\nabla \cdot v_h, q_h) = 0 \;\forall q_h \in \widetilde{Q_h}\}. \end{aligned}$$

Note that V_h^0 is also the divergence free subspace of V_h,

$$V_h^0 = \{v_h \in V_h \,:\, \nabla \cdot v_h = 0\}.$$

Define R_h to be the orthogonal complement of V_h^0 in V_h,

$$V_h =: V_h^0 \oplus R_h,$$

with respect to the X_h inner product which is defined to be $(\cdot,\cdot)_{X_h} := (\nabla \cdot, \nabla \cdot)$, due to the Poincare inequality.

The skew-symmetric operator $b^* : X_h \times X_h \times X_h \to \mathbb{R}$ is defined by

$$b^*(u,v,w) := \frac{1}{2}(u\cdot\nabla v, w) - \frac{1}{2}(u\cdot\nabla w, v).$$

The following bounds on b^* will be used.

Lemma 1. *There exists a constant C_s dependent only on the size of Ω satisfying $\forall u, v, w \in X_h$,*

$$\begin{aligned} b^*(u,v,w) &\leq C_s\|\nabla u\|\|\nabla v\|\|\nabla w\| \\ b^*(u,v,w) &\leq C_s\|\nabla u\|\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2} \end{aligned}$$

Proof. This well known lemma is proven, e.g., in [9]. □

The following lemma shows an equivalence of norms on R_h which is used throughout this article.

Lemma 2. *There exists a constant $M < \infty$ satisfying $\forall r_h \in R_h$,*

$$\|\nabla r_h\| \leq M\|\nabla\cdot r_h\|.$$

Proof. Define

$$M := \max_{v_h\in R_h, \|\nabla v_h\|=1} \frac{1}{\|\nabla\cdot v_h\|}$$

Observe $M < \infty$ since $v_h \in R_h$, $\|\nabla\cdot v_h\| > 0$, and the max is taken over a compact set of $\mathbf{R^n}$. For any $r_h \in R_h$, there is an $e_h \in R_h$ satisfying $\|\nabla e_h\| = 1$ and

$$r_h = \|\nabla r_h\| e_h.$$

Taking divergence of both sides, then L^2 norms gives

$$\|\nabla\cdot r_h\| = \|\nabla r_h\|\|\nabla\cdot e_h\|,$$

which implies

$$\|\nabla r_h\| = \frac{\|\nabla\cdot r_h\|}{\|\nabla\cdot e_h\|} \leq M\|\nabla\cdot r_h\|.$$

□

The discrete Gronwall Lemma is used in our analysis, when analyzing semi-discrete formulations.

Lemma 3. *(Gronwall's inequality) Let $f(x)$ and $B(x)$ be functions which are piecewise continuous on the interval $[a,b]$ and let K be a nonnegative scalar. Further, assume that $f(x)$ and $B(x)$ satisfy $\forall t \in [a,b]$*

$$\int_a^t g(s)ds + f(t) \leq K + \int_a^t B(s)f(s)ds. \tag{1}$$

Then, $\forall t \in [a,b]$ we have the following upper bound

$$\int_a^s g(s)ds + f(t) \leq Ke^{\int_a^t B(s)ds}. \tag{2}$$

3 Order of convergence for NSE approximations

In this section we consider the rate of convergence of finite element approximations of the NSE using grad-div stabilized Taylor-Hood formulations to the solution of Scott-Vogelius elements, as the grad-div stabilization parameter γ tends to zero. We show first for the steady case, then for the time-dependent case, that the rate is $O(\gamma^{-1})$.

3.1 The steady NSE case

Consider the discrete steady convective NSE formulation: Find $(u_h, p_h) \in (X_h, P_h)$ such that $\forall (v_h, q_h) \in (X_h, P_h)$, where $P_h = Q_h$ (Taylor-Hood) or $\widetilde{Q_h}$ (Scott-Vogelius),

$$\begin{aligned} b^*(u_h, u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) + \gamma(\nabla \cdot u_h, \nabla \cdot v_h) &= (f, v_h) && (3) \\ (\nabla \cdot u_h, q_h) &= 0. && (4) \end{aligned}$$

We note that for the case of Scott-Vogelius elements, the grad-div term trivially vanishes.

Define $\alpha := 1 - C_s \nu^{-2} \|f\|_{-1}$. The formulation (3)-(4) is known to be well-posed under the small data condition $\alpha > 0$ [9], for either element choice, due to assumptions on the mesh and polynomial degree.

Lemma 4. *Solutions to* (3)-(4) *exist and satisfy*

$$\begin{aligned} \nu \|\nabla u_h\|^2 + 2\gamma \|\nabla \cdot u_h\|^2 &\leq \nu^{-1} \|f\|_{-1}^2 && (5) \\ \text{If } P_h = Q_h\text{: } \|p_h - \gamma(\nabla \cdot u_h)\| &\leq \|f\|_{-1} \left(1 + C_s \nu^{-2} \|f\| + \nu^{-1}\right) && (6) \\ \text{If } P_h = \widetilde{Q_h}\text{: } \|p_h\| &\leq \|f\|_{-1} \left(1 + C_s \nu^{-2} \|f\| + \nu^{-1}\right) && (7) \end{aligned}$$

If $\alpha > 0$, then solutions are unique.

Proof. Taking $v_h = u_h$ in (3) and using Cauchy-Schwarz and Young's inequalities gives (5). The pressure bounds follow directly from the discrete LBB condition and the bound (5). The Scott-Vogelius pressure bound does not include the term with γ since the grad-div term is trivially zero in this case. □

Remark 5. *We consider limiting behavior as $\gamma \to \infty$, and thus the bound* (6) *seems insufficient to guarantee stability of the pressure in the limit. However, the following theorem implies that $\|\nabla \cdot u_h\| \leq \frac{C}{\gamma}$, and the Taylor-Hood pressure solution is indeed bounded by a data-dependent constant, independent of γ.*

Theorem 6. *On a fixed mesh and with data satisfying $\alpha > 0$, the Taylor-Hood velocity solutions to* (3)-(4) *converge to the Scott-Vogelius velocity solution with convergence order γ^{-1} in the energy norm, as $\gamma \to \infty$: if u_h is the Taylor-Hood solution and u_h^0 is the Scott-Vogelius solution, then*

$$\|\nabla(u_h - u_h^0)\| \leq \frac{C}{\gamma}.$$

Remark 7. *From the a priori bound* (5)*, one might suspect the convergence is only of the order* $\gamma^{-1/2}$*.*

Proof. Let $(u_h^0, p_h^0) \in (V_h^0, \widetilde{Q_h})$ denote the solution of (3)-(4) using Scott-Vogelius elements, $(u_h, p_h) \in (V_h, Q_h)$ for the Taylor-Hood solution, and the difference between them to be $r_h \in V_h$, so that

$$u_h = u_h^0 + r_h.$$

For the Taylor-Hood solution u_h, setting $v_h = w_h^0 \in V_h^0$ and $s_h \in R_h$, respectively, in (3) gives the equations

$$\begin{aligned} b^*(u_h, u_h, w_h^0) + \nu(\nabla u_h, \nabla w_h^0) &= (f, w_h^0), && (8)\\ b^*(u_h, u_h, s_h) + \nu(\nabla u_h, \nabla s_h) + \gamma(\nabla\cdot u_h, \nabla\cdot s_h) &= (f, s_h). && (9)\end{aligned}$$

Similarly, the Scott-Vogelius solution $u_h^0 \in V_h^0$ satisfies

$$\begin{aligned} b^*(u_h^0, u_h^0, w_h^0) + \nu(\nabla u_h^0, \nabla w_h^0) &= (f, w_h^0), && (10)\\ b^*(u_h^0, u_h^0, s_h) - (p_h^0, \nabla\cdot s_h) &= (f, s_h). && (11)\end{aligned}$$

From (9) and (11), we have

$$b^*(u_h, u_h, s_h) + \nu(\nabla u_h, \nabla s_h) + \gamma(\nabla\cdot u_h, \nabla\cdot s_h) = b^*(u_h^0, u_h^0, s_h) - (p_h^0, \nabla\cdot s_h), \quad (12)$$

and since $(\nabla u_h, \nabla s_h) = (\nabla r_h, \nabla s_h)$ and $\nabla\cdot u_h = \nabla\cdot r_h$,

$$\begin{aligned}\nu(\nabla r_h, \nabla s_h) + \gamma(\nabla\cdot r_h, \nabla\cdot s_h) &= b^*(u_h^0, u_h^0, s_h) - b^*(u_h, u_h, s_h) - (p_h^0, \nabla\cdot s_h)\\ &= -b^*(r_h, u_h^0, s_h) - b^*(u_h, r_h, s_h) - (p_h^0, \nabla\cdot s_h). \quad (13)\end{aligned}$$

Orthogonally decompose $r_h =: r_h^0 + r_h'$, where $r_h^0 \in V_h^0$ and $r_h' \in R_h$. Now setting $s_h = r_h'$ in (13) gives, after reducing with orthogonality properties and using Lemma 2,

$$\begin{aligned}\nu\|\nabla r_h'\|^2 + \gamma\|\nabla\cdot r_h'\|^2 &= -b^*(r_h, u_h, r_h') - b^*(u_h^0, r_h, r_h') - (p_h^0, \nabla\cdot r_h')\\ &\le C\left(M\|\nabla r_h\|\|\nabla u_h^0\| + M\|\nabla r_h\|\|\nabla u_h\| + \|p_h^0\|\right)\|\nabla\cdot r_h'\| \quad (14)\end{aligned}$$

Since u_h, u_h^0 are uniformly bounded by the data by (5), independent of γ, r_h is also. Using this and (7) provides

$$\nu\|\nabla r_h'\|^2 + \gamma\|\nabla\cdot r_h'\|^2 \le C\|\nabla\cdot r_h'\|. \quad (15)$$

Dropping the first term on the left and dividing by $\|\nabla\cdot r_h'\|$ gives

$$\|\nabla\cdot r_h'\| \le \frac{C}{\gamma}, \quad (16)$$

which implies from Lemma 2 that

$$\|\nabla r_h'\| \le \frac{C}{\gamma}. \quad (17)$$

It remains to bound $\|\nabla r_h^0\|$. From (8), (10), and taking $w_h^0 = r_h^0$, we get

$$b^*(u_h, u_h, r_h^0) + \nu(\nabla u_h, \nabla r_h^0) = b^*(u_h^0, u_h^0, r_h^0) + \nu(\nabla u_h^0, \nabla r_h^0), \tag{18}$$

which reduces to

$$\begin{aligned} \nu(\nabla r_h, \nabla r_h^0) &= b^*(u_h^0, u_h^0, r_h^0) - b^*(u_h, u_h, r_h^0), \\ &= -b^*(u_h, r_h, r_h^0) - b^*(r_h, u_h^0, r_h^0). \end{aligned} \tag{19}$$

Skew symmetry properties and decomposing r_h gives

$$\nu\|\nabla r_h^0\|^2 = -b^*(u_h, r_h', r_h^0) - b^*(r_h^0, u_h^0, r_h^0) - b^*(r_h', u_h^0, r_h^0). \tag{20}$$

Standard inequalities and (5) now provides

$$\nu\|\nabla r_h^0\|^2 \le C\|\nabla r_h'\|\|\nabla r_h^0\| + C_s\nu^{-1}\|f\|_{-1}\|\nabla r_h^0\|^2. \tag{21}$$

Using the small data condition, then dividing through by $\|\nabla r_h^0\|$ gives

$$\|\nabla r_h^0\| \le C\|\nabla r_h'\| \le \frac{C}{\gamma}. \tag{22}$$

The triangle inequality completes the proof, as

$$\|\nabla(u_h - u_h^0)\| = \|\nabla r_h\| \le \|\nabla r_h^0\| + \|\nabla r_h'\| \le \frac{C}{\gamma}. \tag{23}$$

□

Lemma 8. *If p_h is the Taylor-Hood pressure and p_h^0 is the Scott-Vogelius pressure then*

$$\|p_h^0 - (p_h - \gamma\nabla\cdot u_h)\| \le \tfrac{C}{\gamma}.$$

Proof. The Taylor-Hood and Scott-Vogelius solutions to (3)-(4) satisfy respectively

$$b^*(u_h, u_h, v_h) - (p_h, \nabla\cdot v_h) + \nu(\nabla u_h, \nabla v_h) + \gamma(\nabla\cdot u_h, \nabla\cdot v_h) = (f, v_h), \tag{24}$$
$$b^*(u_h^0, u_h^0, v_h) - (p_h^0, \nabla\cdot v_h) + \nu(\nabla u_h^0, \nabla v_h) = (f, v_h). \tag{25}$$

Subtracting (25) from (24) and rearranging gives

$$\begin{aligned} (p_h^0 - (p_h - \gamma\nabla\cdot u_h), \nabla\cdot v_h) = b^*(u_h^0, u_h^0 - u_h, v_h) &+ b^*(u_h^0 - u_h, u_h, v_h) \\ &+ \nu(\nabla(u_h^0 - u_h), v_h). \end{aligned} \tag{26}$$

From Lemma 2.1, Theorem 3.3 and bounds on solutions it follows that

$$(p_h^0 - (p_h - \gamma\nabla\cdot u_h), \nabla\cdot v_h) \le \frac{C}{\gamma}\|\nabla v_h\|. \tag{27}$$

Dividing (27) by $\|\nabla v_h\|$ and the LBB condition (of the Scott-Vogelius element) finishes the proof. □

3.2 The time-dependent case for the NSE

For the time-dependent case, we find an analogous result to the steady case. We consider the semi-discrete formulation, and extension to the usual temporal discretizations such as backward Euler and Crank-Nicolson is straight-forward, although technical. Thus we proceed to study the following problems: Given $u_h(0) \in V_h^0$, find $(u_h(t), p_h(t)) \in (X_h, P_h) \times (0, T]$ such that $\forall (v_h, q_h) \in (X_h, P_h)$, where $P_h = Q_h$ (Taylor-Hood) or $\widetilde{Q_h}$ (Scott-Vogelius),

$$\begin{aligned} ((u_h)_t, v_h) + b^*(u_h, u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) & \\ +\gamma(\nabla \cdot u_h, \nabla \cdot v_h) &= (f, v_h) \qquad (28) \\ (\nabla \cdot u_h, q_h) &= 0. \qquad (29) \end{aligned}$$

It is straight-forward to show (e.g. [9]) that this formulation admits unique solutions satisfying for $0 \le t \le T$,

$$\|u_h(t)\|^2 + \nu \int_0^t \|\nabla u_h(s)\|^2 \, ds + \gamma \int_0^t \|\nabla \cdot u_h(s)\|^2 \, ds \le C(data), \qquad (30)$$

$$\text{If } P_h = Q_h\text{: } \|p_h\| \le (1+\gamma) \cdot C(data), \qquad (31)$$

$$\text{If } P_h = \widetilde{Q_h}\text{: } \|p_h\| \le C(data). \qquad (32)$$

Remark 9. *For fully discrete case, there is a restriction that the time-step be small enough to get uniqueness, otherwise an analogous result holds.*

Remark 10. *With the following theorem, the bound* (31) *can be improved to be independent of* γ.

Theorem 11. *On a fixed mesh, the Taylor-Hood velocity solutions to* (28)-(29) *converge to the Scott-Vogelius solution with convergence order* γ^{-1} *in the energy norm, as* $\gamma \to \infty$*: if* u_h *is the Taylor-Hood solution and* u_h^0 *is the Scott-Vogelius solution, then*

$$\|u_h - u_h^0\|_{L^2(0,T;H^1(\Omega))} \le \frac{C}{\gamma}.$$

Remark 12. *The stability estimate* (30) *suggests the rate may be only* $\gamma^{-1/2}$ *since the Scott-Vogelius solution is pointwise divergence-free, but the theorem proves it is indeed faster.*

Proof. Our strategy for this proof is similar to that of the steady case. Let $(u_h^0, p_h^0) \in (V_h^0, \widetilde{Q_h}) \times [0, T]$ denote the solution of (28)-(29) using Scott-Vogelius elements, $(u_h, p_h) \in (V_h, Q_h) \times [0, T]$ for the Taylor-Hood solution, and the difference between them to be $r_h \in V_h \times [0, T]$, so that

$$u_h(t) = u_h^0(t) + r_h(t).$$

Again we orthogonally decompose $r_h(t) = r_h'(t) + r_h^0(t)$, where $r_h'(t) \in R_h$ and $r_h^0(t) \in V_h^0$; recall $V_h = V_h^0 \oplus R_h$ in the X_h inner product.

Consider (28) with an arbitrary test function $s_h \in R_h \subset V_h$. The Taylor-Hood and Scott-Vogelius solutions satisfy, respectively,

$$((u_h)_t, s_h) + b^*(u_h, u_h, s_h) + \nu(\nabla u_h, \nabla s_h) + \gamma(\nabla \cdot u_h, \nabla \cdot s_h) = (f, s_h) \qquad (33)$$

$$((u_h^0)_t, s_h) + b^*(u_h^0, u_h^0, s_h) - (p_h^0, \nabla \cdot s_h) + \nu(\nabla u_h^0, \nabla s_h) = (f, s_h). \qquad (34)$$

Subtracting and utilizing the following identities

$$\nabla \cdot u_h = \nabla \cdot r_h = \nabla \cdot r_h' \tag{35}$$
$$(\nabla r_h, \nabla s_h) = (\nabla r_h', \nabla s_h). \tag{36}$$

provides the equation

$$(r_{ht}, s_h) + \nu(\nabla r_h', \nabla s_h) + \gamma(\nabla \cdot r_h', \nabla \cdot s_h) = -b^*(r_h, u_h^0, s_h) - b^*(u_h, r_h, s_h) - (p_h^0, \nabla \cdot s_h).$$

Taking $s_h = r_h'$, then reducing with Lemmas 1 and 2, and (30) and (32) yields

$$\begin{aligned}
& (r_{ht}, r_h') + \nu\|\nabla r_h'\|^2 + \gamma\|\nabla \cdot r_h'\|^2 \\
= \; & -b^*(r_h, u_h^0, r_h') - b^*(u_h, r_h^0, r_h') - (p_h^0, \nabla \cdot r_h') \\
\leq \; & C_s\|\nabla r_h\|\|\nabla u_h^0\|\|\nabla r_h'\| + C_s\|\nabla u_h\|\|\nabla r_h^0\|\|\nabla r_h'\| + \|p_h^0\|\|\nabla \cdot r_h'\| \\
\leq \; & C_s\|\nabla r_h\|\|\nabla u_h^0\|M\|\nabla \cdot r_h'\| + C_s\|\nabla u_h\|\|\nabla r_h^0\|M\|\nabla \cdot r_h'\| + \|p_h^0\|\|\nabla \cdot r_h'\| \\
\leq \; & \left(C_s M\|\nabla r_h\|\|\nabla u_h^0\| + C_s M\|\nabla u_h\|\|\nabla r_h^0\| + \|p_h^0\|\right)\|\nabla \cdot r_h'\| \\
\leq \; & C\|\nabla \cdot r_h'\|.
\end{aligned} \tag{37}$$

We now step back, and proceed to bound r_h^0. Consider (28) with an arbitrary test function $w_h^0 \in V_h^0$. The Taylor-Hood and Scott-Vogelius solutions satisfy, respectively,

$$((u_h)_t, w_h^0) + b^*(u_h, u_h, w_h^0) + \nu(\nabla u_h, \nabla w_h^0) = (f, w_h^0), \tag{38}$$
$$((u_h^0)_t, w_h^0) + b^*(u_h^0, u_h^0, w_h^0) + \nu(\nabla u_h^0, \nabla w_h^0) = (f, w_h^0). \tag{39}$$

Subtracting gives

$$((r_h)_t, w_h^0) + \nu(\nabla r_h, \nabla w_h^0) = -b^*(u_h, u_h, w_h^0) + b^*(u_h^0, u_h^0, w_h^0), \tag{40}$$

which reduces to

$$((r_h)_t, w_h^0) + \nu(\nabla r_h^0, \nabla w_h^0) = -b^*(r_h, u_h, w_h^0) - b^*(u_h^0, r_h, w_h^0). \tag{41}$$

Taking $w_h^0 = r_h^0$ gives

$$((r_h)_t, r_h^0) + \nu\|\nabla r_h^0\|^2 = -b^*(r_h, u_h, r_h^0) - b^*(u_h^0, r_h, r_h^0) \tag{42}$$
$$= -b^*(r_h^0, u_h, r_h^0) - b^*(r_h', u_h, r_h^0) - b^*(u_h^0, r_h', r_h^0) \tag{43}$$

Using lemmas 1 and 2, and the uniform bound on solutions yields

$$\begin{aligned}
& ((r_h)_t, r_h^0) + \nu\|\nabla r_h^0\|^2 \\
& \leq C_s\|\nabla r_h^0\|^{3/2}\|\nabla u_h\|\|r_h^0\|^{1/2} + C_s\|\nabla r_h'\|\|\nabla u_h\|\|\nabla r_h^0\| + C_s\|\nabla r_h'\|\|\nabla u_h^0\|\|\nabla r_h^0\| \\
& \leq C_s\|\nabla r_h^0\|^{3/2}\|\nabla u_h\|\|r_h^0\|^{1/2} + C\|\nabla \cdot r_h'\|.
\end{aligned} \tag{44}$$

Adding (37) to (44) gives

$$\begin{aligned}
& ((r_h)_t, r_h^0) + ((r_h)_t, r_h') + \nu\|\nabla r_h^0\|^2 + \nu\|\nabla r_h'\|^2 + \gamma\|\nabla \cdot r_h'\|^2 \\
& \qquad \leq C_s\|\nabla r_h^0\|^{3/2}\|\nabla u_h\|\|r_h^0\|^{1/2} + (C + \|p_h^0\|)\|\nabla \cdot r_h'\|,
\end{aligned} \tag{45}$$

which reduces with orthogonality properties, the uniform bounds on solutions, then standard inequalities to

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|r_h\|^2 + \nu\|\nabla r_h\|^2 + \gamma\|\nabla \cdot r_h'\|^2 \\ \leq \quad & C_s\|\nabla r_h^0\|^{3/2}\|\nabla u_h\|\|r_h^0\|^{1/2} + C\|\nabla \cdot r_h'\| \\ \leq \quad & C\|r_h\|^2 + \nu\|\nabla r_h\|^2 + \frac{C}{2\gamma} + \frac{\gamma}{2}\|\nabla \cdot r_h'\|^2. \end{aligned} \quad (46)$$

This leaves

$$\frac{d}{dt}\|r_h\|^2 + \gamma\|\nabla \cdot r_h'\|^2 \leq C\|r_h\|^2 + \frac{C}{\gamma}. \quad (47)$$

The Gronwall inequality, $u_h(0) = u_h^0(0)$, and reducing gives us

$$\int_0^t \|\nabla \cdot r_h'\|^2 \, dt \leq \frac{C}{\gamma^2}, \quad (48)$$

which proves the theorem. □

4 Extension to turbulence models

Recent work on finite element methods for the 'α models' of fluid flow has proven their effectiveness at finding accurate solutions to flow problems on coarser spatial and temporal discretizations than are necessary for successful simulations of the Navier-Stokes equations [11, 12, 15, 1, 20, 14, 4, 3, 8]. We prove the convergence result for grad-div stabilized TH solutions to SV solutions of the Leray-α model; analogous results / proofs for the other α models follow similarly. Since a goal of the α-models is to find solutions on coarser meshes than would be used for the NSE, mass conservation of solutions can be very poor and thus heavy grad-div stabilization that preserves overall accuracy but improves the mass conservation will help to provide more physically relevant solutions.

The continuous Leray-α model formulation is: find $(u_h, p_h, w_h, \lambda_h) \in (X_h, P_h, X_h, P_h)$ such that $\forall (v_h, q_h, \chi_h, \psi_h) \in (X_h, P_h, X_h, P_h)$, where $P_h = Q_h$ (Taylor-Hood) or $\widetilde{Q_h}$ (Scott-Vogelius),

$$\begin{aligned} ((u_h)_t, v_h) + b^*(w_h, u_h, v_h) - (p_h, \nabla \cdot v_h) + \nu(\nabla u_h, \nabla v_h) & \\ +\gamma(\nabla \cdot u_h, \nabla \cdot v_h) &= (f, v_h), \quad (49) \\ (\nabla \cdot u_h, q_h) &= 0, \quad (50) \\ (w_h, \chi_h) + \alpha^2(\nabla w_h, \nabla \chi_h) + (\lambda_h, \nabla \cdot \chi_h) + \gamma(\nabla \cdot w_h, \nabla \cdot \chi_h) &= (u_h, \chi_h), \quad (51) \\ (\nabla \cdot w_h, \psi_h) &= 0. \quad (52) \end{aligned}$$

The equations (51)-(52) are the discretization of the α-filter, with discrete incompressiblity enforced. Advantages of using this discretization for the filter instead of the usual one are discussed in [1].

The following lemma will be useful for the analysis in this section.

Lemma 13. *If* $(u_h, p_h, w_h, \lambda_h)$ *solves* (49)-(52) *then* $\|w_h\| \le \|u_h\|$.

Proof. The Lemma can be verified quickly by choosing $\chi_h = w_h$ in (51) and using the Cauchy-Schwarz inequality. □

Theorem 14. *On a fixed mesh the grad-div stabilized Taylor-Hood velocity solutions to* (49)-(52) *converge to the Scott-Vogelius velocity solution with convergence order* γ^{-1} *in the energy norm, as* $\gamma \to \infty$. *That is, if we denote the SV velocity solutions as* u_h^0 *and grad-div stabilized TH solution as* u_h *then*

$$\|\nabla(u_h - u_h^0)\| \le \frac{C}{\gamma}.$$

Proof. Let $(u_h^0, w_h^0, p_h^0, \lambda_h^0) \in (X_h, X_h, \tilde{Q}_h, \tilde{Q}_h) \times [0,T]$ denote the solution of (49)-(52) using Scott-Vogelius elements, $(u_h, w_h, p_h, \lambda_h) \in (X_h, X_h, Q_h, Q_h) \times [0,T]$ for the Taylor-Hood solution. Let the difference between u_h and u_h^0 be denoted by r_u and the difference between w_h and w_h^0 be denoted by r_w so that

$$\begin{aligned} u_h(t) &= u_h^0(t) + r_u(t), \ and \\ w_h(t) &= w_h^0(t) + r_w(t). \end{aligned}$$

Orthogonally decompose $r_u(t) = r_u'(t) + r_u^0(t)$, where $r_u'(t) \in R_h$ and $r_u^0(t) \in V_h^0$. Similarly, orthogonally decompose $r_w(t) = r_w'(t) + r_w^0(t)$ so that $r_w'(t) \in R_h$ and $r_w^0(t) \in V_h^0$.

Consider (49) and (51) with an arbitrary test function $s_h \in R_h \subset V_h$. The Taylor-Hood and Scott-Vogelius solutions satisfy, respectively,

$$((u_h)_t, s_h) + b^*(w_h, u_h, s_h) + \nu(\nabla u_h, \nabla s_h) + \gamma(\nabla \cdot u_h, \nabla \cdot s_h) = (f, s_h) \quad (53)$$

$$((u_h^0)_t, s_h) + b^*(w_h^0, u_h^0, s_h) - (p^0, \nabla \cdot s_h) + \nu(\nabla u_h^0, \nabla s_h) = (f, s_h). \quad (54)$$

Subtracting using previous identities gives

$$((r_u)_t, s_h) + \nu(\nabla r_u', \nabla s_h) + \gamma(\nabla \cdot r_u', \nabla \cdot s_h) = b^*(w_h^0, u_h^0, s_h) - b^*(w_h, u_h, s_h) - (p_h^0, s_h).$$

Taking $s_h = r_u'$, and reducing with Lemmas 1, 2 and 13, and uniqueness of solutions yields

$$\begin{aligned} & ((r_u)_t, r_u') + \nu\|\nabla r_u'\|^2 + \gamma\|\nabla \cdot r_u'\|^2 \\ = \ & b^*(w_h^0, u_h^0, r_u') - b^*(w_h, u_h, r_u') - (p_h^0, r_u') \\ \le \ & C_s(\|\nabla w_h^0\|\|\nabla u_h^0\|\|\nabla r_u'\| + \|\nabla w_h\|\|\nabla u_h\|\|\nabla r_u'\|) + \|p_h^0\|\|\nabla \cdot r_u'\| \\ \le \ & C\|\nabla \cdot r_u'\|. \end{aligned} \quad (55)$$

We now derive a similar bound for r_w'. Consider that the Taylor-Hood and Scott-Vogelius solutions satisfy the follow equations from (51)

$$(w_h, \chi_h) + \alpha^2(\nabla w_h, \nabla \chi_h) + (\lambda_h, \nabla \cdot \chi_h) + \gamma(\nabla \cdot w_h, \nabla \cdot \chi_h) = (u_h, \chi_h), \quad (56)$$

$$(w_h^0, \chi_h) + \alpha^2(\nabla w_h^0, \nabla \chi_h) + (\lambda_h^0, \nabla \cdot \chi_h) = (u_h^0, \chi_h). \quad (57)$$

Subtracting and choosing $\chi_h = r_w$ and rearranging gives

$$\|r_w\|^2 + \alpha^2\|\nabla r_w\|^2 + \gamma\|\nabla \cdot r'_w\|^2 = (r_u, r_w) - (\lambda_h^0, \nabla \cdot r'_w). \tag{58}$$

The Cauchy-Schwarz inequality and Lemma 2.2 yields

$$\|\nabla \cdot r'_w\| \leq \frac{C}{\gamma}. \tag{59}$$

Next we derive a bound for r_w^0. To do this we subtract (57) from (56) and choose $\chi_h = r_w^0$ which gives

$$(r_w, r_w^0) + \alpha^2\|\nabla r_w^0\|^2 = (r_u, r_w^0). \tag{60}$$

From here we rearrange by using Cauchy-Schwarz and equivalence of norms over finite dimensional Hilbert spaces which gives

$$\|\nabla r_w^0\| \leq C\left(\|\nabla r_u\| + \|\nabla r'_w\|\right). \tag{61}$$

We proceed similar to the time-dependent NSE case and bound r_u^0. Consider (49) with arbitrary test function $v_h^0 \in V_h^0$. The Taylor-Hood and Scott-Vogelius solutions saitsfy

$$((u_h)_t, v_h^0) + b^*(w_h, u_h, v_h^0) + \nu(\nabla u_h, \nabla v_h^0) = (f, v_h^0), \tag{62}$$
$$((u_h^0)_t, v_h^0) + b^*(w_h^0, u_h^0, v_h^0) + \nu(\nabla u_h^0, \nabla v_h^0) = (f, v_h^0). \tag{63}$$

Subtracting (63) from (62) rearranging and choosing $v_h = r_u^0$ gives

$$((r_u)_t, r_u^0) + \nu\|\nabla r_u^0\|^2 \leq |b^*(w_h^0, r_u, r_u^0)| + |b^*(r_w, u_h, r_u^0)|. \tag{64}$$

To majorize the first trilinear term in (64) use Lemmas 2.1 and 4.1, bounds on solutions and note that for orthogonal decompositions the triangle inequality is an equality. Lastly, using equivalence of norms gives

$$\begin{aligned} |b^*(w_h^0, r_u, r_u^0)| \leq C\|\nabla r_u\|\|\nabla r_u^0\| &\leq C\|\nabla r_u\|\|\nabla r_u^0\| + C\|\nabla r_u\|\|\nabla r'_u\| \\ &\leq C\|\nabla r_u\|^2 \\ &\leq C\|r_u\|^2. \end{aligned} \tag{65}$$

We bound the second trilinear using Lemma 2.1 and uniform bound on solutions. Then we split the r_w term using the triangle inequality and use (61), which yields

$$\begin{aligned} |b^*(r_w, u_h, r_u^0)| &\leq C\|\nabla r_w\|\|\nabla r_u^0\| \\ &\leq C\|\nabla r'_w\|\|\nabla r_u^0\| + C\|\nabla r_w^0\|\|\nabla r_u^0\|. \end{aligned} \tag{66}$$

Adding $C\|\nabla r'_w\|\|\nabla r'_u\|$ and $C\|\nabla r_w^0\|\|\nabla r'_u\|$ to the right hand side of (66) and using orthogonality gives

$$|b^*(r_w, u_h, r_u^0)| \leq C\|\nabla r'_w\|\|\nabla r_u\| + C\|\nabla r_w^0\|\|\nabla r_u\| \tag{67}$$

We majorize the first right hand side term using Lemma 2.2, bounds on solutions and (59). Additionally, we majorize the second right hand side term using (61). After we combine like terms we are left with

$$|b^*(r_w, u_h, r_u^0)| \leq \frac{C}{\gamma} + C\|\nabla r_u\|^2. \tag{68}$$

From equivalence of norms we have that $\|\nabla r_u\| \leq C\|r_u\|$. Therefore,

$$((r_u)_t, r_u^0) + \nu\|\nabla r_u^0\|^2 \quad \leq \quad \frac{C}{\gamma} + C\|r_u\|^2. \tag{69}$$

Adding (69) and (55) gives

$$\frac{d}{dt}\|r_u\|^2 + 2\gamma\|\nabla \cdot r_u'\|^2 \leq C\|r_u\|^2 + \frac{C}{\gamma}. \tag{70}$$

Analogous to the time-dependent NSE proof, the Gronwall inequality, $u_h(0) = u_h^0(0)$ and reducing finishes the proof. □

4.1 Numerical Verification for the Leray-α model

To numerically verify the velocity convergence rate shown above we consider the benchmark 2D problem of channel flow over a forward-backward step. The domain Ω is a 40×10 rectangle with a 1×1 step 5 units into the channel at the bottom. The top and bottom of the channel as well as the step are prescribed with no-slip boundary conditions, and the sides are given the parabolic profile $(y(10-y)/25, 0)^T$. We use the initial condition $u_0 = (y(10-y)/25, 0)^T$ inside Ω, choose the viscosity $\nu = 1/600$ and run the test to T=10. The correct physical behavior is for an eddy to form behind the step (at larger T, the eddy will move down the channel and a new eddy will form).

A barycenter-refinement of a Delauney triangulation of Ω is used, which yields a total of 14,467 degrees of freedom for the (P_2, P_1^{disc}) SV computations and 9,427 for (P_2, P_1) TH. A Crank-Nicolson time discretization is chosen as the temporal discretization, with a timestep of $\Delta t = 0.01$. For the TH computations, we use grad-div stabilization parameters $\gamma = \{0,\ 1,\ 10,\ 100,\ 1,000,\ 10,000\}$.

Plots of the SV and TH solutions are shown in Figure 1, and the correct physical behavior is observed in both; in fact, these solutions are nearly indistinguishable. Plots of the TH solutions with $\gamma > 0$ are also nearly identical and so are omitted. Differences between the TH solutions with varying γ, and the SV solution are computed in the H^1 norm, and are shown (with rates) in Table 1; first order convergence is observed, in accordance with our theory. The divergence errors of the TH solutions are given in Table 1, which also display first order convergence. Also of particular interest is that the TH solution with $\gamma = 0$ has very poor mass conservation, even though its plot appears correct.

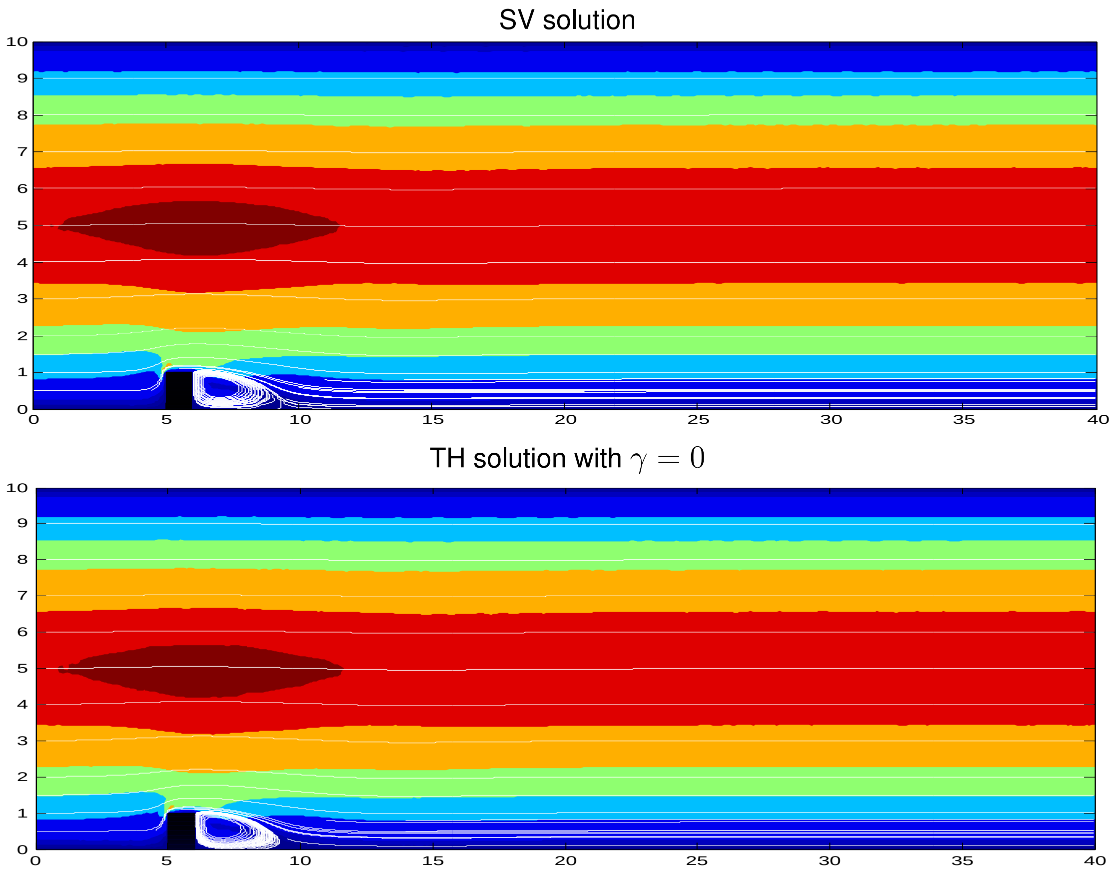

Figure 1: SV and TH solutions of the Leray-α model at $t = 10$.

γ	$\|u_{TH}^{\gamma} - u_{SV}\|_{H^1}$	rate	$\|\nabla \cdot u_{TH}^{\gamma}\|$
0	2.0360	-	1.2466
1	0.1473	1.14	0.0085
10	0.0311	0.68	9.836E-4
10^2	0.0035	0.94	8.774E-5
10^3	3.616E-4	0.99	8.667E-6
10^4	3.622E-5	1.00	8.948E-7

Table 1: Convergence of the grad-div stabilized Taylor-Hood Leray-α solutions toward the Scott-Vogelius Leray-α solution, first order as $\gamma \to \infty$.

5 Extension to magnetohydrodynamic flows

To understand a fluid flow which is influenced by a magnetic field one must understand the mutual interaction of a magnetic field and a velocity field. The system of differential equations which describe the flow of an electrically conductive and nonmagnetic incompressible fluid (e.g. liquid sodium) are called magnetohydrodynamics (MHD). These equations are commonly used in metallurgical industries to heat, pump, stir and levitate liquid metals [5].

We consider the steady MHD in the form studied in, e.g., [6, 7], which is the Navier-Stokes equations coupled to the pre-Maxwell equations. For simplicity of the analysis, we restrict to homogeneous Dirichlet boundary conditions (or periodic) for both velocity and the magnetic field and consider a convex domain. The Galerkin finite element method that explicitly enforces incompressibility of both the velocity and magnetic fields and with grad-div stabilization of both velocity and magnetic fields is, $\forall (v_h,\ \chi_h, q_h, \psi_h) \in (X_h, X_h, Q_h, Q_h)$,

$$\begin{aligned} b^*(u_h, u_h, v_h) + \nu(\nabla u_h, \nabla v_h) - s b^*(B_h, B_h, v_h) & & \\ -(P_h, \nabla \cdot v_h) + \gamma(\nabla \cdot u_h, \nabla \cdot v_h) &= (f, v_h) & (71) \\ (\nabla \cdot u_h, q_h) &= 0 & (72) \\ \nu_m(\nabla B_h, \nabla\ \chi_h) - b^*(B_h, u_h,\ \chi_h) + b^*(u_h, B_h,\ \chi_h) & & \\ +(\lambda_h, \nabla \cdot\ \chi_h) + \gamma(\nabla \cdot B_h, \nabla \cdot\ \chi_h) &= (\nabla \times G,\ \chi_h) & (73) \\ (\nabla \cdot B_h, \psi_h) &= 0. & (74) \end{aligned}$$

The Lagrange multiplier is added in (73) so that the divergence of the magnetic field can be explicitly enforced via (74) without overdetermining the discrete system.

For the choice of (X_h, Q_h) to be Taylor Hood elements, both $\nabla \cdot u_h = 0$ and $\nabla \cdot B_h = 0$ are enforced weakly in (71)-(74), but if instead Scott-Vogelius elements are chosen then pointwise enforcement is recovered (choose $q_h = \nabla \cdot u_h$ and $\psi_h = \nabla \cdot B_h$). Similar to the NSE case, there is a ‘middle ground’ of improved mass conservation while using Taylor-Hood elements, if γ is chosen “large”. Note we consider the stabilization parameters to be equal only for simplicity since we consider their limiting behavior; in practice it may be necessary to choose them different for optimal accuracy.

Lemma 15. *Solutions to* (71) - (74) *exist and satisfy*

$$\|\nabla u_h\| \leq \nu^{-1}\|f\|_{-1} + s^{\frac{-1}{2}}\nu^{\frac{-1}{2}}\nu_m^{\frac{-1}{2}}\|G\|(=: M_1), \quad (75)$$

$$\|\nabla B_h\| \leq \nu^{\frac{-1}{2}}\nu_m^{\frac{-1}{2}}s^{\frac{-1}{2}}\|f\| + \nu_m^{-1}\|G\|(=: M_2). \quad (76)$$

If

$$\nu - C_s M_1 - 2sC_s M_2 > 0, and \quad (77)$$

$$\nu_m - C_s M_1 - 2C_s M_2 > 0 \quad (78)$$

then solutions are unique.

Proof. Existence of solutions is a straight forward application of the Leray-Schauder Theorem. To derive (75) and (76) we multiply (73) by s and add it to (71). Next we choose $v_h = u_h$ and $\chi_h = B_h$. Noting that $b^*(B_h, B_h, u_h) = -b^*(B_h, u_h, B_h)$ leaves

$$\nu\|\nabla u_h\|^2 + s\nu_m\|\nabla B_h\|^2 \leq (f, u_h) + s(\nabla \times G, B_h). \quad (79)$$

The bounds can be derived from (79) by using Young's inequality.

To derive sufficient conditions for uniqueness assume to get a contradiction that there are two solutions to (71)-(74), $(u_h^1, B_h^1, p_h^1, \lambda_h^1)$ and $(u_h^2, B_h^2, p_h^2, \lambda_h^2)$. Now let $D_u := u_h^1 - u_h^2$ and $D_B := B_h^1 - B_h^2$. Substituting u_h^1, u_h^2 into (71), and choosing $v_h = D_u$, subtracting and rearranging gives

$$\begin{aligned}\nu\|\nabla D_u\|^2 + \gamma\|\nabla \cdot D_u\|^2 &= b^*(u_h^2, u_h^2, D_u) - b^*(u_h^1, u_h^1, D_u) \\ &+ sb^*(B_h^1, B_h^1, D_u) - sb^*(B_h^2, B_h^2, D_u). \quad (80)\end{aligned}$$

Using standard inequalities and noting that $b^*(v, u, u) = 0$ we can rewrite (80) as

$$\begin{aligned}\nu\|\nabla D_u\|^2 + \gamma\|\nabla \cdot D_u\|^2 &= sb^*(B_h^1, D_B, D_u) + sb^*(D_B, B_h^2, D_u) \\ &- b^*(D_u, u_h^1, D_u). \quad (81)\end{aligned}$$

Scaling (73) by s and similar treatment gives

$$\begin{aligned}s\nu_m\|D_B\|^2 + s\gamma\|\nabla \cdot D_B\|^2 &= sb^*(B_h^1, D_u, D_B) + sb^*(D_B, u_h^2, D_B) \\ &- sb^*(D_u, B_h^1, D_B). \quad (82)\end{aligned}$$

Adding (81) and (82) and noting that $b^*(B_h^1, D_u, D_B) = -b^*(B_h^1, D_B, D_u)$, yields

$$\begin{aligned}\nu\|D_u\|^2 + s\nu_m\|D_B\|^2 &\leq sb^*(D_B, u_h^2, D_B) - sb^*(D_u, B_h^1, D_B) \\ &+ sb^*(D_B, B_h^2, D_u) - b^*(D_u, u_h^1, D_u). \quad (83)\end{aligned}$$

Utilizing Lemma 2.1 and Young's inequality we can now rewrite this as

$$\|\nabla D_u\|^2(\nu - C_s M_1 - 2sC_s M_2) + \|\nabla D_B\|^2 s(\nu_m - C_s M_1 - 2C_s M_2) \leq 0. \quad (84)$$

□

5.1 Convergence of velocity and magnetic field Taylor-Hood solutions to the Scott-Vogelius solution for steady MHD

We now extend the results above to the case of steady MHD, formulated by (71)-(74). Here there are two grad-div stabilization terms that arise in the analysis, but the main ideas of the proofs for the NSE carry through to this problem as well, although more technical details arise. An extension to time dependent MHD can be performed analogously to how the NSE was extended in Section 3.

Theorem 16. *On a fixed mesh the grad-div stabilized Taylor-Hood velocity and magnetic field solutions to* (71)-(74) *converge to the Scott-Vogelius velocity and magnetic field solutions with convergence order* γ^{-1} *in the energy norm, as* $\gamma \to \infty$*: if* (u_h, B_h) *is the Taylor-Hood solution and* (u_h^0, B_h^0) *is the Scott-Vogelius solution, then*

$$\|\nabla(u_h - u_h^0) + \|\nabla(B_h - B_h^0)\| \le \frac{C}{\gamma}.$$

Proof. Let $(u_h^0, p_h^0, B_h^0, \lambda_h^0) \in (V_h^0, \tilde{Q}_h, V_h^0, \tilde{Q}_h)$ denote the solution of (71)-(74) using Scott-Vogelius elements, $(u_h, p_h, B_h, \lambda_h) \in (V_h, Q_h, V_h, Q_h)$ for the Taylor-Hood solution. Additionally, denote the difference between the velocity solutions and the magnetic field solutions by $r_u \in V_h$ and $r_B \in V_h$, so that

$$\begin{aligned} u_h &= u_h^0 + r_u, \\ B_h &= B_h^0 + r_B. \end{aligned}$$

Plugging in the Taylor-Hood and Scott-Vogelius solutions into (71) gives the following equations: $\forall v_h \in V_h$,

$$b^*(u_h, u_h, v_h) + \nu(\nabla u_h, \nabla v_h) - sb^*(B_h, B_h, v_h) + \gamma(\nabla \cdot u_h, \nabla \cdot v_h) = (f, v_h), \quad (85)$$

$$b^*(u_h^0, u_h^0, v_h) + \nu(\nabla u_h^0, \nabla v_h) - sb^*(B_h^0, B_h^0, v_h) - (p_h^0, \nabla \cdot v_h) = (f, v_h). \quad (86)$$

Subtracting (86) from (85) gives

$$\begin{aligned} \nu(\nabla r_u, \nabla v_h) + \gamma(\nabla \cdot u_h, v_h) &= -b^*(u_h^0, r_u, v_h) - b^*(r_u, u_h, v_h) \\ &\quad + sb^*(B_h, r_B, v_h) + sb^*(r_b, B_h^0, v_h) - (p_h^0, \nabla \cdot v_h). \end{aligned} \quad (87)$$

Similarly, plugging in the Taylor-Hood and Scott-Vogelius solutions into (73) gives the following two equations: $\forall \chi_h \in V_h$,

$$\begin{aligned} \nu_m(\nabla B_h, \nabla\ \chi_h) - b^*(B_h, u_h,\ \chi_h) + b^*(u_h, B_h,\ \chi_h) & \\ + \gamma(\nabla \cdot B_h, \nabla \cdot\ \chi_h) &= (\nabla \times G,\ \chi_h), \end{aligned} \quad (88)$$

$$\begin{aligned} \nu_m(\nabla B_h^0, \nabla\ \chi_h) - b^*(B_h^0, u_h^0,\ \chi_h) + b^*(u_h^0, B_h^0,\ \chi_h) & \\ + (\lambda_h^0, \nabla \cdot\ \chi_h) &= (\nabla \times G,\ \chi_h). \end{aligned} \quad (89)$$

Subtracting (89) from (88) results in the following equality,

$$\begin{aligned} \nu_m(\nabla r_B, \nabla\ \chi_h) + \gamma(\nabla \cdot B_h, \nabla \cdot\ \chi_h) &= b^*(B_h, r_u,\ \chi_h) + b^*(r_B, u_h^0,\ \chi_h) \\ &\quad - b^*(u_h^0, r_B,\ \chi_h) - b^*(r_u, B_h,\ \chi_h) + (\lambda_h^0, \nabla \cdot\ \chi_h). \end{aligned} \quad (90)$$

Orthogonally decompose $r_u =: r_u^0 + r_u'$ and $r_B =: r_B^0 + r_B'$ where $r_u^0, r_B^0 \in V_h^0$ and $r_u', r_B' \in R_h$ and choosing $v_h = r_u'$ in (87), $\chi_h = r_B'$ in (90) and adding the two resulting equations yields

$$\begin{aligned}\nu\|\nabla r_u'\|^2 + \gamma\|\nabla\cdot r_u'\|^2 + \nu_m\|\nabla r_B'\|^2 + \gamma\|\nabla\cdot r_B'^2\|^2 = -b^*(u_h^0, r_u^0, r_u') \\ -b^*(r_u, u_h, r_u') + sb^*(B_h, r_B, r_u') + sb^*(r_b, B_h^0, r_u') - (p_h^0, \nabla\cdot r_u') \\ +b^*(B_h, r_u, r_B') + b^*(r_B, u_h^0, r_B') - b^*(u_h^0, r_B^0, r_B') \\ -b^*(r_u, B_h, r_B') + (\lambda_h^0, \nabla\cdot r_B')\end{aligned} \quad (91)$$

From (75), (76) and Lemmas 2.1 and 2, we can transform (91) to

$$\begin{aligned}\gamma\frac{\|\nabla\cdot r_u'\|^2 + \|\nabla\cdot r_B'\|^2}{\|\nabla\cdot r_u\| + \|\nabla\cdot r_B\|} \le C_sM_1M\|\nabla r_u^0\| + C_sM_1M\|\nabla r_u\| \\ +sC_sM_2M\|\nabla r_B\| + sC_sM_2M\|\nabla r_B\| + \|p^0\| + C_sM_2M\|\nabla r_u\| \\ +C_sM_1M\|\nabla r_B\| + C_sM_1M\|\nabla r_B^0\| + C_sM_2M\|\nabla r_B\| + M\|\lambda_h^0\|\end{aligned} \quad (92)$$

Since, u_h, u_h^0, B_h and B_h^0 are all bounded by data that implies that r_u, r_u^0, r_B and r_B^0 are as well. Therefore,

$$\|\nabla\cdot r_u'\| + \|\nabla\cdot r_B'\| \le \frac{C}{\gamma} \quad (93)$$

It remains to bound $\|r_u^0\|$ and $\|r_B^0\|$. We will majorize the terms individually and then combine the results. First, setting $v_h = r_u^0$ in (85) and (86), and rearranging gives the following

$$\nu(\nabla u_h, \nabla r_u^0) = -b^*(u_h, u_h, r_u^0) + sb^*(B_h, B_h, r_u^0) + (f, r_u^0), \quad (94)$$
$$\nu(\nabla u_h^0, \nabla r_u^0) = -b^*(u_h^0, u_h^0, r_u^0) + sb^*(B_h^0, B_h^0, r_u^0) + (f, r_u^0). \quad (95)$$

Subtracting (95) from (94), rewriting the nonlinear terms with standard identities and reducing with orthogonality properties gives

$$\begin{aligned}\nu\|\nabla r_u^0\|^2 &\le |b^*(u_h^0, r_u', r_u^0)| + |b^*(r_u, u_h, r_u^0)| \\ &+ |sb^*(B_h, r_B, r_u^0)| + |sb^*(r_B, B_h^0, r_u^0)|.\end{aligned} \quad (96)$$

Choosing $\chi_h = r_B^0$ in (88) and (89), and rearranging gives the following equalities

$$\nu_m(\nabla B_h, \nabla r_B^0) = b^*(B_h, u_h, r_B^0) - b^*(u_h, B_h, r_B^0) + (\nabla\times G, r_B^0), \quad (97)$$
$$\nu_m(\nabla B_h^0, \nabla r_B^0) = b^*(B_h^0, u_h^0, r_B^0) - b^*(u_h^0, B_h^0, r_B^0) + (\nabla\times G, r_B^0). \quad (98)$$

Subtracting (98) from (97), rewriting the nonlinear terms and reducing with orthogonality properties gives

$$\begin{aligned}\nu_m\|\nabla r_B^0\|^2 &\le |b^*(B_h, r_u, r_B^0)| + |b^*(r_B, u_h^0, r_B^0)| \\ &+ |b^*(u_h^0, r_B', r_B^0)| + |b^*(r_u, B_h, r_B^0)|.\end{aligned} \quad (99)$$

Adding (96) and (99) gives the following upper bound

$$\begin{aligned}\nu\|\nabla r_u^0\|^2+\nu_m\|\nabla r_B^0\|^2 &\leq |b^*(u_h^0,r_u',r_u^0)|+|b^*(r_u,u_h,r_u^0)|+|sb^*(B_h,r_B,r_u^0)|\\ &+ |sb^*(r_B,B_h^0,r_u^0)|+|b^*(B_h,r_u,r_B^0)|+|b^*(r_B,u_h^0,r_B^0)|\\ &+ |b^*(u_h^0,r_B',r_B^0)|+|b^*(r_u,B_h,r_B^0)| \qquad (100)\end{aligned}$$

Now using Lemma 2.1, (75), (76) and the triangle inequality yields

$$\begin{aligned}\nu\|\nabla r_u^0\|^2+\nu_m\|\nabla r_B^0\|^2 &\leq C_s(M_1\|\nabla r_u^0\|^2+M_1\|\nabla r_B^0\|^2\\ &+ 2sM_2\|\nabla r_B'\|\|\nabla r_u^0\|+2M_1\|\nabla r_u'\|\|\nabla r_u^0\|\\ &+ 2M_2\|\nabla r_u'\|\|\nabla r_B^0\|++2M_1\|\nabla r_B'\|\|\nabla r_B^0\|\\ &+ 2sM_2\|\nabla r_B^0\|\|\nabla r_u^0\|+2M_2\|\nabla r_u^0\|\|\nabla r_B^0\|) \qquad (101)\end{aligned}$$

The first 2 terms may be subtracted from both sides of (101) immediately. The subsequent terms may be handled using Young's inequality to yield

$$\begin{aligned}(\frac{\nu}{2}-C_sM_1-2sC_sM_2-2C_s\quad M\quad {}_2)\|\nabla r_u^0\|^2+(\frac{\nu_m}{2}-C_sM_1-2sC_sM_2-2C_sM_2)\|\nabla r_B^0\|^2\\ \leq 16\nu^{-1}s^2C_s^2M_2^2\|\nabla r_B'\|^2+16\nu^{-1}C_2^2M_1^2\|\nabla r_u'\|^2\\ + 16\nu_m^{-1}C_s^2M_2^2\|\nabla r_u'\|^2+16\nu_m^{-1}C_s^2M_1^2\|\nabla r_B'\|^2 \qquad (102)\end{aligned}$$

Provided that

$$\begin{aligned}\tfrac{\nu}{2} &\quad -C_sM_1-2sC_sM_2-2C_sM_2>0,\ and\\ \tfrac{\nu_m}{2} &\quad -C_sM_1-2sC_sM_2-2C_sM_2>0 \qquad (103)\end{aligned}$$

it follows from the triangle inequality that

$$\|\nabla(u_h-u_h^0)\|+\|\nabla(B_h-B_h^0)\|\leq\frac{C}{\gamma}. \qquad (104)$$

□

5.2 Numerical verification for steady MHD

To numerically verify the MHD convergence theory, we select the test problem with solution

$$u=<\cos(y),\sin(x)>^T,\quad P=\sin(x+y),\quad B=<x,-y>^T,$$

on the unit square with $\nu=\nu_m=1$, $s=1$ and f and g calculated from this information.

The mesh used was a barycenter-refined uniform triangulation of Ω, which provided a total of $4,324$ degrees of freedom for the (P_2,P_1) TH computations and $6,600$ for (P_2,P_1^{disc}) SV. The results are shown in Table 2, and first order convergence in the H^1 norm is observed for both velocity and the magnetic field.

γ	$\|u_{TH}^{\gamma}-u_{SV}\|_{H^1}$	rate	$\|\nabla\cdot u_{TH}^{\gamma}\|$	$\|B_{TH}^{\gamma}-B_{SV}\|_{H^1}$	rate	$\|\nabla\cdot B_{TH}^{\gamma}\|$
0	7.052E-4	-	5.45E-4	4.293E-6	-	1.74E-6
1	4.740E-4	-	3.19E-4	2.923E-6	-	8.93E-7
10	1.729E-4	0.41	8.44E-5	1.138E-6	0.41	2.96E-7
10^2	2.688E-5	0.81	1.16E-5	1.813E-7	0.80	4.66E-8
10^3	2.860E-6	0.97	1.22E-6	1.936E-8	0.97	4.97E-9
10^4	2.879E-7	1.00	1.23E-7	1.947E-9	1.00	5.00E-10

Table 2: Convergence of the grad-div stabilized Taylor-Hood steady MHD solutions toward the Scott-Vogelius steady MHD solution, first order as $\gamma \to \infty$.

6 Extrapolating to approximate the $\gamma = \infty$ solution

The previous sections verified that provided the SV element is stable the grad-div stabilized TH solutions to Stokes type problems converge to the SV solution as $\gamma \to \infty$. However, in practice there are limitations on how large γ may be chosen, because as γ increases the resulting linear system becomes ill-conditioned. In this section we consider linearly and quadratically extrapolating from grad-div stabilized TH velocity solutions found with smaller γ to approximate the SV solution in an effort to improve mass conservation.

Let the true solutions to (3) $-$ (4) be given by

$$u = \begin{bmatrix} (x^4-2x^3+x^2)(4y^3-6y^2+2y) \\ -(y^4-2y^3+y^2)(4x^3-6x^2+2x) \end{bmatrix}, \quad (105)$$

$$P = x+y+\frac{1}{2}(\cos(y)^2+\sin(x)^2), \quad (106)$$

on the unit square with $\nu = \frac{1}{100}$.

Let γ_k ($k = 1, 2$ or 3) denote a distinct stabilization parameter and let $(u_h^{\gamma_k}, p_h^{\gamma_k})$ denote Taylor-Hood solutions of (3)-(4) with stabilization parameters γ_k. Additionally, let (u_{Ex}, p_{Ex}) denote the extrapolated solution and (u_h^0, p_h^0) denote the Scott-Vogelius solution to (3)-(4).

Computations were done on a barycenter-refined uniform triangulation of Ω, which provided 2162 degrees of freedom for the (P_2, P_1) TH elements and 3300 degrees of freedom for the (P_2, P_1^{disc}) SV element.

The results in Table 3 are for linear extrapolated solutions, and and Table 4 summarizes the results for quadratic extrapolated solutions. Little improvement is seen in linear extrapolation, but a dramatic improvement is observed for quadratic.

References

[1] A. Bowers and L. Rebholz. Increasing accuracy and efficiency in FE computations of the Leray-deconvolution model. *Numerical Methods for Partial Differential Equations*, to

γ_1	γ_2	$\|\nabla \cdot u_h^{\gamma_1}\|$	$\|\nabla \cdot u_h^{\gamma_2}\|$	$\|\nabla \cdot u_{Ex}\|$	$\|u_{Ex} - u_h^0\|_{H^1}$
1	10	2.1946e-4	2.2585e-5	2.9595e-6	6.3507e-6
1	100	2.1964e-4	2.2653e-6	1.1318e-7	2.9811e-7
10	50	2.2585e-5	4.5292e-6	4.1681e-6	7.6739e-6
10	100	2.2585e-5	2.2653e-6	2.0621e-6	3.7978e-6
50	100	4.5292e-6	2.2653e-6	2.2427e-6	4.1293e-6

Table 3: Improved mass conservation using linear extrapolation.

γ_1	γ_2	γ_3	$\|\nabla \cdot u_{Ex}\|_{H^1}$	$\|u_{EX} - u_h^0\|_{H^1}$
1	10	100	7.30832e-10	2.070203e-9
1	50	100	1.47103e-10	4.167215e-10

Table 4: Improved mass conservation using quadratic extrapolation

appear, 2010.

[2] M. Case, V. Ervin, A. Linke, and L. Rebholz. Improving mass conservation in FE approximations of the Navier Stokes equations using C^0 velocity fields: A connection between grad-div stabilization and Scott-Vogelius elements. *Submitted (available as tech report at http://www.wias-berlin.de/preprint/1510/wias_preprints_1510.pdf)*, 2010.

[3] S. Chen, C. Foias, D. Holm, E. Olson, E. Titi, and S. Wynne. The Camassa-Holm equations as a closure model for turbulent channel and pipe flow. *Phys. Rev. Lett.*, 81:5338–5341, 1998.

[4] S. Chen, D. Holm, L. Margolin, and R. Zhang. Direct numerical simulations of the Navier-Stokes alpha model. *Physica D*, 133:66–83, 1999.

[5] P.A. Davidson. *An introduction to magnetohydrodynamics*. Cambridge, 2001.

[6] M. Gunzburger and C. Trenchea. Analysis and discretization of an optimal control problem for the time-periodic mhd equations. *J. Math Anal. Appl.*, 308(2):440–466, 2005.

[7] M. Gunzburger and C. Trenchea. Analysis of optimal control problem for three-dimensional coupled modified navier-stokes and maxwell equations. *J. Math Anal. Appl.*, 333:295–310, 2007.

[8] D. Holm and B. Guerts. Leray and lans-α modelling of turbulent mixing. *J. of Turbulence*, 7(10), 2006.

[9] W. Layton. *An introduction to the numerical analysis of viscous incompressible flows*. SIAM, 2008.

[10] W. Layton, C. Manica, M. Neda, M.A. Olshanskii, and L. Rebholz. On the accuracy of the rotation form in simulations of the Navier-Stokes equations. *J. Comput. Phys.*, 228(5):3433–3447, 2009.

[11] W. Layton, C. Manica, M. Neda, and L. Rebholz. Numerical analysis and computational testing of a high-accuracy Leray-deconvolution model of turbulence. *Numerical Methods for Partial Differential Equations*, 24(2):555–582, 2008.

[12] W. Layton, C. Manica, M. Neda, and L. Rebholz. Numerical analysis and computational comparisons of the NS-omega and NS-alpha regularizations. *Comput. Methods Appl. Mech. Engrg.*, 199:916–931, 2010.

[13] A. Linke. Collision in a cross-shaped domain — A steady 2d Navier-Stokes example demonstrating the importance of mass conservation in CFD. *Comp. Meth. Appl. Mech. Eng.*, 198(41–44):3278–3286, 2009.

[14] E. Lunasin, S. Kurien, M. Taylor, and E.S. Titi. A study of the Navier-Stokes-alpha model for two-dimensional turbulence. *Journal of Turbulence*, 8:751–778, 2007.

[15] C. Manica, M. Neda, M.A. Olshanskii, and L. Rebholz. Enabling accuracy of Navier-Stokes-alpha through deconvolution and enhanced stability. *M2AN: Mathematical Modelling and Numerical Analysis*, to appear, 2010.

[16] M. Olshanskii, G. Lube, T. Heister, and J. Löwe. Grad-div stabilization and subgrid pressure models for the incompressible Navier-Stokes equations. *Comp. Meth. Appl. Mech. Eng.*, 198:3975–3988, 2009.

[17] M.A. Olshanskii. A low order Galerkin finite element method for the Navier-Stokes equations of steady incompressible flow: A stabilization issue and iterative methods. *Comp. Meth. Appl. Mech. Eng.*, 191:5515–5536, 2002.

[18] M.A. Olshanskii and A. Reusken. Grad-Div stabilization for the Stokes equations. *Math. Comp.*, 73:1699–1718, 2004.

[19] J. Qin. *On the convergence of some low order mixed finite elements for incompressible fluids*. PhD thesis, Pennsylvania State University, 1994.

[20] L. Rebholz and M. Sussman. On the high accuracy NS-α-deconvolution model of turbulence. *Mathematical Models and Methods in Applied Sciences*, 20:611–633, 2010.

[21] L.R. Scott and M. Vogelius. Conforming finite element methods for incompressible and nearly incompressible continua. In *Large-scale computations in fluid mechanics, Part 2*, volume 22-2 of *Lectures in Applied Mathematics*, pages 221–244. Amer. Math. Soc., 1985.

[22] L.R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *Mathematical Modelling and Numerical Analysis*, 19(1):111–143, 1985.

[23] S. Zhang. A new family of stable mixed finite elements for the 3d Stokes equations. *Math. Comp.*, 74(250):543–554, 2005.

[24] S. Zhang. Divergence-free finite elements on tetrahedral grids for $k \geq 6$. *Submitted*, 2010.

[25] S. Zhang. Quadratic divergence-free finite elements on powell-sabin tetrahedral grids. *Submitted*, 2010.