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L^∞-estimates for divergence operators on bad domains

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ABSTRACT. In this paper we prove L^∞-estimates for solutions of divergence operators in case of mixed boundary conditions. In this very general setting the Dirichlet boundary part may be arbitrarily wild, i.e. no regularity conditions have to be imposed on it.

1. INTRODUCTION

It is well-known that L^∞-estimates for elliptic operators play an eminent role in the investigation of partial differential equations and systems. Such estimates are established since the work of De Giorgi, Nash and Moser in case of Dirichlet boundary conditions (compare [LU, Chapter III], [GT, Chapter 8] or [WYW]). In this paper, we are interested in mixed Dirichlet/Neumann boundary conditions. If the domain is Lipschitzian and the Neumann/Dirichlet boundary parts satisfy a certain compatibility condition, introduced by Gröger in [Grö], then global Hölder continuity of solutions is known ([GR], see also [Gri] and [HMRS]). We weaken these conditions considerably: first, the Dirichlet part may be an arbitrary closed subset of the boundary, and, secondly, only for points in the closure of the Neumann boundary part we require bi-Lipschitz charts. This includes domains which are not necessarily situated on one side of the boundary – as long as this boundary part carries a homogeneous Dirichlet condition. The reader may think, as an example, of a ball minus one half of its equatorial plane; another example is shown in Figure 1. The essential instruments are Gaussian estimates, derived in [AE], and a result of Duong and $\mathrm{M^c}$Intosh [DM] on Riesz transforms.

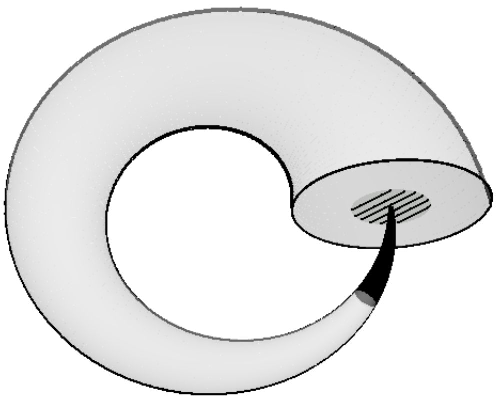

FIGURE 1. The black apex and the shaded circle carry the Dirichlet condition

2. PRELIMINARIES

All function spaces under consideration are real valued. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let Γ be an open subset of $\partial\Omega$. For all $q \in (1, \infty)$ we define $W^{1,q}_\Gamma(\Omega)$ as the closure of

$$C^\infty_\Gamma(\Omega) =: \{\psi|_\Omega : \psi \in C^\infty_c(\mathbb{R}^d),\ \operatorname{supp}\psi \cap (\partial\Omega \setminus \Gamma) = \emptyset\} \tag{2.1}$$

in the Sobolev space $W^{1,q}(\Omega)$. Of course, if $\Gamma = \emptyset$, then $W^{1,q}_\Gamma(\Omega) = W^{1,q}_0(\Omega)$. If q' is the dual exponent of q then we denote by $W^{-1,q'}_\Gamma(\Omega)$ the dual space of $W^{1,q}_\Gamma(\Omega)$. Throughout this paper we make the following assumption.

Assumption 2.1. *For all $x \in \overline{\Gamma}$ there is an open neighbourhood $\mathcal{V}_x$ of x and a bi-Lipschitz mapping F_x from $\mathcal{V}_x$ onto the open unit cube E, such that $F_x(x) = 0$ and $F_x(\Omega \cap \mathcal{V}_x)$ is equal to the lower open half cube $E_- = (-1,1)^{d-1} \times (-1,0)$ of E.*

Further, we suppose that μ is a bounded, Lebesgue measurable function on Ω, taking its values in the set of real $d \times d$-matrices, which, additionally, satisfies the usual ellipticity condition

$$\inf_{x\in\Omega} \inf_{|\xi|=1} \mu(x)\xi \cdot \xi > 0. \tag{2.2}$$

Define the (closed) form $\mathfrak{t}\colon W^{1,2}_\Gamma(\Omega) \times W^{1,2}_\Gamma(\Omega) \to \mathbb{R}$ by

$$\mathfrak{t}[u,v] := \int_\Omega \mu \nabla u \cdot \nabla v \, \mathrm{d}x. \tag{2.3}$$

We denote by A the operator associated with $\mathfrak{t}$ on $L^2(\Omega)$ and let S be the semigroup generated by $-A$.

If misunderstandings are not to be expected, we usually drop 'Ω' in the notation of spaces, i.e. we write $W^{1,q}_\Gamma$ instead of $W^{1,q}_\Gamma(\Omega)$ etc.

3. Main result

Our main results are as follow.

Theorem 3.1. *The semigroup S has a positive kernel satisfying Gaussian upper estimates, i.e.*

$$(e^{-tA} f)(x) = \int_\Omega K_t(x,y)\, f(y)\, dy, \quad x \in \Omega,\ f \in L^2(\Omega)$$

for some measurable function $K_t\colon \Omega \times \Omega \to [0,\infty)$ and there exist constants $b, c > 0$ and $\omega \in \mathbb{R}$ such that

$$0 \le K_t(x,y) \le \frac{c}{t^{d/2}}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}, \quad \text{for almost all } (x,y) \in \Omega \times \Omega \tag{3.1}$$

for all $t > 0$.

Remark 3.2. Note that the Gaussian upper bounds imply that for all $p \in [1,\infty]$ there exists a continuous semigroup $S^{(p)}$ on L^p which is consistent with S, compare [AE, Theorem 4.4]. If no confusion is possible then we denote by $-A$ the generator of $S^{(p)}$ and write S for $S^{(p)}$.

We emphasize that the coefficients do not have to be symmetric in Theorem 3.1.

Theorem 3.3. *Assume $\mu(x)$ is symmetric for all $x \in \Omega$. Let $q \in [2,\infty)$ and $\varepsilon > 0$. Then one has the following.*

i) *The space L^q, equipped with the $W_\Gamma^{-1,q}$-norm, is continuously mapped by $\big(A+\varepsilon\big)^{-\frac{1}{2}}$ into L^q.*

ii) *For every $\theta > \frac{1}{2}(1+\frac{d}{q})$, the operator $(A+\varepsilon)^{-\theta}$ admits an extension which continuously maps $W_\Gamma^{-1,q}$ into L^∞.*

For the proofs we need some prerequisites:

Lemma 3.4. *There are a continuous extension operator $\mathfrak{E}\colon W_\Gamma^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^d)$ and a $c > 0$ such that $\|\mathfrak{E}\psi\|_{L^1(\mathbb{R}^d)} \le c\,\|\psi\|_{L^1(\Omega)}$ for all $\psi \in W_\Gamma^{1,2}(\Omega) \cap L^1(\Omega)$.*

Proof. For all $x \in \Omega$ let B_x be a ball around x which does not intersect $\partial\Omega$. Further, for all $\partial\Omega \setminus \overline{\Gamma}$ let $\mathcal{U}_x$ be an open neighbourhood which does not intersect $\overline{\Gamma}$. Lastly, for all $x \in \overline{\Gamma}$, let $\mathcal{V}_x$ be an open neighbourhood which satisfies the condition in Assumption 2.1. Obviously, the union of the systems $\{B_x\}_{x\in\Omega}$, $\{\mathcal{U}_y\}_{y\in\partial\Omega\setminus\overline{\Gamma}}$ and $\{\mathcal{V}_z\}_{z\in\overline{\Gamma}}$ forms an open covering of $\overline{\Omega}$. Let $B_{x_1},\dots,B_{x_k},\mathcal{U}_{y_1},\dots,\mathcal{U}_{y_l},\mathcal{V}_{z_1},\dots,\mathcal{V}_{z_m}$ be a finite subcovering and let $\zeta_1^{(1)},\dots,\zeta_k^{(1)},\zeta_1^{(2)},\dots,\zeta_l^{(2)},\zeta_1^{(3)},\dots,\zeta_m^{(3)}$ be a smooth partition of unity over $\overline{\Omega}$, subordinated to this subcovering. Then $\psi = \sum_{r=1}^k \zeta_r^{(1)}\psi + \sum_{r=1}^l \zeta_r^{(2)}\psi + \sum_{r=1}^m \zeta_r^{(3)}\psi$ for all $\psi \in L^1(\Omega)$.

We next define for all i and r a constant $c_{i,r} > 0$ and an operator $\mathfrak{E}_r^{(i)}$ from $C_\Gamma^\infty(\Omega)$ into $W^{1,2}(\mathbb{R}^d)$ such that $\mathbb{1}_\Omega\,\mathfrak{E}_r^{(i)}\psi = \zeta_r^{(i)}\,\psi$, $\|\mathfrak{E}_r^{(i)}\psi\|_{W^{1,2}(\mathbb{R}^d)} \le c_{i,r}\|\psi\|_{W^{1,2}(\Omega)}$ and $\|\mathfrak{E}_r^{(i)}\psi\|_{L^1(\mathbb{R}^d)} \le c_{i,r}\|\psi\|_{L^1(\Omega)}$ for all $\psi \in C_\Gamma^\infty(\Omega)$. First, for all $r \in \{1,\dots,k\}$ define the map $\mathfrak{E}_r^{(1)}\colon C_\Gamma^\infty(\Omega) \to W^{1,2}(\mathbb{R}^d)$ by $\mathfrak{E}_r^{(1)}\psi = \zeta_r^{(1)}\,\psi$. Then $\mathfrak{E}_r^{(1)}$ is continuous and $\|\mathfrak{E}_r^{(1)}\psi\|_{L^1(\mathbb{R}^d)} \le \|\psi\|_{L^1(\Omega)}$ for all $\psi \in C_\Gamma^\infty(\Omega)$. Secondly, let $r \in \{1,\dots,l\}$. Then $\operatorname{supp}\zeta_r^{(2)} \cap \overline{\Gamma} = \emptyset$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ and suppose that $\operatorname{supp}\psi \cap (\partial\Omega \setminus \Gamma) = \emptyset$. Then $\operatorname{supp}(\zeta_r^{(2)}\,\psi) \cap \partial\Omega = \emptyset$. Hence $\zeta_r^{(2)}\,\psi\,\mathbb{1}_\Omega \in C_c^\infty(\Omega) \subset W^{1,2}(\mathbb{R}^d)$. Define $\mathfrak{E}_r^{(2)}(\psi|_\Omega) = \zeta_r^{(2)}\,\psi\,\mathbb{1}_\Omega$. Clearly $\mathfrak{E}_r^{(2)}$ is well defined, and it satisfies the claimed requirements. Finally, let $r \in \{1,\dots,m\}$. Let $\widehat{\mathfrak{E}}$ be the even reflection from $W^{1,2}(E_-)$ into $W^{1,2}(E)$ (see [Giu, Lemma 3.4]). Then $\widehat{\mathfrak{E}}((\zeta_r^{(3)} \circ F_{z_r}^{-1})\mathbb{1}_{E_-})$ is continuous on E and $\operatorname{supp}\widehat{\mathfrak{E}}((\zeta_r^{(3)} \circ F_{z_r}^{-1})\mathbb{1}_{E_-}) \subset E$. Transforming back, it follows that

$$\operatorname{supp}\Big(\widehat{\mathfrak{E}}((((\zeta_r^{(3)}\,\psi) \circ F_{z_r}^{-1})\mathbb{1}_{E_-})\Big) \circ F_{z_r} \subset \mathcal{V}_{z_r}$$

for all $\psi \in C_\Gamma^\infty(\Omega)$. Hence we can define $\mathfrak{E}_r^{(3)}\colon C_\Gamma^\infty(\Omega) \to W^{1,2}(\mathbb{R}^d)$ by

$$(\mathfrak{E}_r^{(3)}\psi)(x) = \begin{cases} \Big(\widehat{\mathfrak{E}}((((\zeta_r^{(3)}\,\psi) \circ F_{z_r}^{-1})\mathbb{1}_{E_-})\Big)(F_{z_r}(x)) & \text{if } x \in \mathcal{V}_{z_r}, \\ 0 & \text{if } x \notin \mathcal{V}_{z_r}. \end{cases}$$

Clearly there exists a $c_{3,r} > 0$ such that $\|\mathfrak{E}_r^{(3)}\psi\|_{L^1(\mathbb{R}^d)} \le c_{3,r}\|\psi\|_{L^1(\Omega)}$ for all $\psi \in C_\Gamma^\infty(\Omega)$ and $\mathfrak{E}_r^{(3)}$ satisfies the claimed requirements.

Now define $\mathfrak{E}\colon C_\Gamma^\infty(\Omega) \to W^{1,2}(\mathbb{R}^d)$ by

$$\mathfrak{E} = \sum_{r=1}^k \mathfrak{E}_r^{(1)} + \sum_{r=1}^l \mathfrak{E}_r^{(2)} + \sum_{r=1}^m \mathfrak{E}_r^{(3)}.$$

Then $\mathfrak{E}$ is continuous and $\|\mathfrak{E}\psi\|_{L^1(\mathbb{R}^d)} \le (k + l + \sum_{r=1}^m c_{3,r})\|\psi\|_{L^1(\Omega)}$ for all $\psi \in C_\Gamma^\infty(\Omega)$. By construction $\mathfrak{E}\psi = \psi$ for all $\psi \in C_\Gamma^\infty(\Omega)$. Therefore $\mathfrak{E}$ extends to a continuous extension operator from $W_\Gamma^{1,2}(\Omega)$ into $W^{1,2}(\mathbb{R}^d)$ and from $L^1(\Omega)$ into $L^1(\mathbb{R}^d)$. □

Remark 3.5. In case of Lipschitz domains Ω it is well known that the full space $W^{1,2}(\Omega)$ admits an extension operator to $W^{1,2}(\mathbb{R}^d)$, which simultaniously also extends L^1, see [GT, Theorem 7.25] or [Giu, Theorem 3.10]. In contrast, we exploited in Lemma 3.4 the detailed structure of the space $W_\Gamma^{1,2}(\Omega)$.

Proof of Theorem 3.1. By Theorem 4.4 in [AE] it suffices to show that the form domain $W_\Gamma^{1,2}(\Omega)$ of $\mathfrak{t}$ satsifies the following four conditions:

a) $W_0^{1,2}(\Omega) \subseteq W_\Gamma^{1,2}(\Omega)$,
b) there is a linear, continuous extension operator $\mathfrak{E}\colon W_\Gamma^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^d)$ which maps $W_\Gamma^{1,2}(\Omega) \cap L^1(\Omega)$ continuously into $L^1(\mathbb{R}^d)$,
c) $u \in W_\Gamma^{1,2}(\Omega)$ implies $|u|,\ |u| \wedge \mathbb{1} \in W_\Gamma^{1,2}(\Omega)$,
d) $u \in W_\Gamma^{1,2}(\Omega)$, $v \in W^{1,2}(\Omega)$, and $|v| \le u$ implies $v \in W_\Gamma^{1,2}(\Omega)$.

Condition a) is obvious. Condition b) is shown in Lemma 3.4. We next show Condition c). Assume $u \in W_\Gamma^{1,2}$. Then, by definition, there is a sequence $\{u_n\}_n$ in $C_c^\infty(\mathbb{R}^d)$, such that $\operatorname{supp} u_n \cap (\partial\Omega \setminus \Gamma) = \emptyset$ and $\lim \|u_n|_\Omega - u\|_{W^{1,2}(\Omega)} = 0$. Let $n \in \mathbb{N}$. It is well-known that $u_n^+, u_n^- \in W^{1,2}(\mathbb{R}^d)$. Moreover, the supports of u_n^+, u_n^- also have a positive distance to $\partial\Omega \setminus \Gamma$. There exists a $\theta_n \in C_c^\infty(\mathbb{R}^d)$ such that the supports of $\theta_n * u_n^+$ and $\theta_n * u_n^-$ also have positve distance to $\partial\Omega \setminus \Gamma$, and, secondly,

$$\|u_n^+ - \theta_n * u_n^+\|_{W^{1,2}(\mathbb{R}^d)} \le \tfrac{1}{n} \quad \text{and} \quad \|u_n^- - \theta_n * u_n^-\|_{W^{1,2}(\mathbb{R}^d)} \le \tfrac{1}{n}.$$

Clearly, $(\theta_n * u_n^+)|_\Omega \in C_\Gamma^\infty$, and we may estimate

$$\begin{aligned}\|(\theta * u_n^+)|_\Omega - u^+\|_{W^{1,2}(\Omega)} &\le \|(\theta_n * u_n^+)|_\Omega - u_n^+|_\Omega\|_{W^{1,2}(\Omega)} + \|u_n^+|_\Omega - u^+\|_{W^{1,2}(\Omega)} \\ &\le \|\theta_n * u_n^+ - u_n^+\|_{W^{1,2}(\mathbb{R}^d)} + \|u_n^+|_\Omega - u^+\|_{W^{1,2}(\Omega)} \\ &\le \tfrac{1}{n} + \|u_n^+|_\Omega - u^+\|_{W^{1,2}(\Omega)}.\end{aligned}$$

But the second term approaches 0 for $n \to \infty$, since the mapping $v \mapsto v^+$ from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega)$ is continuous for arbitrary domains Ω, see [MM]. The same way one proves that $\lim(\theta_n * u_n^-)|_\Omega = u^-$ in $W^{1,2}(\Omega)$. Hence $|u| = u^+ + u^- \in W_\Gamma^{1,2}$.

The property $|u| \wedge \mathbb{1} \in W_\Gamma^{1,2}$ is obtained similarily, this time using the continuity of the maps $u \mapsto |u|$ and then $v \mapsto v \wedge \mathbb{1}$ from $W^{1,2}$ into $W^{1,2}$. See also [MM]. Finally, Condisition d) can essentially be proved the same way, this time using the continuity result for the map $u \mapsto u \wedge v$ from [MM]. □

We next turn to the proof of Theorem 3.3.

Theorem 3.6. *Assume $\mu(x)$ is symmetric for all $x \in \Omega$. For all $q \in (1,2]$ and $\varepsilon > 0$ the operator $(A+\varepsilon)^{-1/2}$ admits an extension as a continuous operator from L^q into $W^{1,q}_\Gamma$.*

For the proof we need the following result (see [DM, Theorem 2], see also [Ouh, Chapter 7.7]).

Proposition 3.7. *Let B be a positive, selfadjoint operator on L^2 with form domain W. Suppose there exists a $c > 0$ such that $\|\nabla\psi\|_{L^2} \le c\|B^{1/2}\psi\|_{L^2}$ for all $\psi \in W$. Assume that W is invariant under multiplication by bounded functions with bounded, continuous first derivatives. Moreover, assume that there exist $\beta > d/2$ and $C > 0$ such that the kernel L of the semigroup $(\mathrm{e}^{-tB})_{t>0}$ satisfies bounds*

$$|L_t(x,y)| \le \frac{C}{t^{d/2}}\left(1 + \frac{|x-y|^2}{t}\right)^{-\beta} \tag{3.2}$$

for all $t > 0$ and $x, y \in \Omega$. Then, for all $j \in \{1,\dots,d\}$, the operator $\frac{\partial}{\partial x_j}B^{-1/2}$ is of weak type $(1,1)$, and, thus can be extended from L^2 to a bounded operator on L^q for all $q \in (1,2]$.

Proof of Theorem 3.6. It follows from Theorem 3.1 that the kernel of the semigroup S satisfies the estimate (3.1). Define $\nu := \max(\varepsilon,\omega)$, where ω is as in (3.1). Note that A_q generates a contraction semigroup on L^q by [Ouh, Chapter 4.6]. Hence $(A+\nu)^{-1/2}$ extends to the continuous operator $(A_q+\nu)^{-1/2}$ on L^q.

Next we show that for all $j \in \{1,\dots,d\}$ the operator $\frac{\partial}{\partial x_j}(A+\nu)^{-1/2}$ extends to a bounded operator from L^q into itself.

We wish to apply Proposition 3.7 to the operator $B := A+\nu$ and $W := W^{1,2}_\Gamma$. By a classical result on forms the space $W^{1,2}_\Gamma$ is the domain of $(A+\nu)^{1/2}$. Hence there exists a $c > 0$ such that $\|\nabla\psi\|_{L^2} \le c\|(A+\nu)^{1/2}\psi\|_{L^2}$ for all $\psi \in W$. The invariance property of W under multiplication is concluded by straight forward arguments, see [HR, Proposition 3.8]. The semigroup kernel for $A+\nu$ satisfies again (3.1), but without the factor $\mathrm{e}^{\omega t}$ due to the definition of ν. Moreover, it is easy to see that the resulting Gaussian bounds from Theorem 3.1 are even much stronger than the bounds required in (3.2), since the function $r \mapsto (1+r)^\beta\,\mathrm{e}^{-br}$ from $[0,\infty)$ into $\mathbb{R}$ is bounded for every $\beta > 0$.

Thus the operator $(A+\nu)^{-1/2}$ maps L^q continuously into $W^{1,q}$ for all $q \in (1,2]$. Since $D((A+\nu)^{-1/2}) = D((A+\varepsilon)^{-1/2})$, with equivalent graph norm, it follows that the operator $(A+\varepsilon)^{-1/2}$ maps L^q continuously into $W^{1,q}$. It remains to verify the correct boundary behavior of the images. If $f \in L^2 \subset L^q$, then $(A+\varepsilon)^{-1/2}f \in W^{1,2}_\Gamma \subset W^{1,q}_\Gamma$. Then the assertion follows from the continuity of $(A+\varepsilon)^{-1/2}$, the closedness of $W^{1,q}_\Gamma$ in $W^{1,q}$ and the density of L^2 in L^q. □

Proof of Theorem 3.3. Statement i) follows by duality from Theorem 3.6.

'ii).' Let b, c, ω be as in (3.1). Set $\delta := \omega + 1$. Then $\|e^{-t(A+\delta)}\|_{1\to\infty} \le c\,t^{-\frac{d}{2}}$ for all $t > 0$. One also has the contraction property $\|e^{-t(A+\delta)}\|_{\infty\to\infty} \le 1$ for all $t > 0$ for the semigroup operators, see [Ouh, Theorem 4.9]. Then by interpolation $\|e^{-t(A+\delta)}\|_{q\to\infty} \le c^{\frac{1}{q}}\,t^{-\frac{d}{2q}}$ for all $t > 0$ and $q \in [1, \infty)$. Let $\tau \in (\frac{d}{2q}, \infty)$. Then

$$(A+\delta)^{-\tau} = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau-1} e^{-t(A+\delta)}\,dt,$$

by [Paz, Section 2.6]. Hence the operator $(A+\delta)^{-\tau}$ is bounded from L_q into L_∞. Therefore if $\theta > \frac{1}{2}(1 + \frac{d}{q})$ then Statement i) implies that L^q, endowed with the $W_\Gamma^{-1,q}$-norm, is mapped continuously by $(A+\delta)^{-\theta}$ into L^∞. Then ii) follows by the density of L^q in $W_\Gamma^{-1,q}$. □

4. Concluding remarks

- Everything can be carried over to complex spaces without difficulties. The form $\mathfrak{t}$ has then to be defined by taking the conjugate of v in (2.3), and $W_\Gamma^{-1,q}$ has to be defined as the space of continuous antilinear forms on $W_\Gamma^{1,q'}$.
- Quite similar as in the proof of Theorem 3.3, upper Gaussian estimates imply that a finite resolvent power maps L^2 into L^∞. The point is here, however, that the $W^{-1,q}$-calculus allows for jumps in the conormal derivative of solutions across internal interfaces. Thus, one can deal on the right hand side of the elliptic equation with distributions which are concentrated on internal interfaces. In electrostatics, for instance, a charge density on an interface causes a jump in the normal component of the dielectric displacement, see for instance [Tam, Chapter 1]. Secondly, it is essential that the resolvent power may be taken smaller than 1 for $q > d$.
- It can be shown that the index $\frac{1}{2}(1 + \frac{d}{q})$ in Theorem 3.3 is optimal: in smooth situations and Dirichlet boundary conditions, $A^{-s/2}$ provides a topological isomorphism between L^q and $W_0^{s,q}$. But $W_0^{s,q}$ embeds into L^∞ only if $s > \frac{d}{q}$.
- We do not know whether Hölder continuity for the semigroup kernel holds for the solution in our setting, but feel that it cannot be expected.
- It is not hard to see that the continuous extension of $(A+\varepsilon)^{-\theta}$ to $W_\Gamma^{-1,q}$ is nothing else as the restriction of the operator $(\widehat{A}+\varepsilon)^{-\theta}$ to $W_\Gamma^{-1,q}$, where $\widehat{A}$ is the operator which extrapolates A to $W_\Gamma^{-1,2}$. Compare [Ouh, Subsection 1.4.2].
- It seems an interesting question whether the operator $\breve{A} := \widehat{A}|_{W_\Gamma^{-1,q}}$, satisfies a resolvent estimate like $\|(\breve{A}+\lambda)^{-1}\| \le \frac{M}{\lambda}$ uniformly for all $\lambda > 0$ and, thus, enables a selfconsistent definition of fractional powers for $\breve{A} + \varepsilon$ on $W_\Gamma^{-1,q}$. Unfortunately, the attempts to prove this within the context of this paper have failed, compare [HR, Lemma 5.7].

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