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Quasistatic small-strain plasticity in the limit of vanishing hardening and its numerical approximation

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Abstract. The quasistatic rate-independent evolution of the Prager-Ziegler-type model of linearized plasticity with hardening is shown to converge to the rate-independent evolution of the Prandtl-Reuss elastic/perfectly plastic model. Based on the concept of energetic solutions we study the convergence of the solutions in the limit for hardening coefficients converging to 0 by using the abstract method of Γ-convergence for rate-independent systems. An unconditionally convergent numerical scheme is devised and 2D and 3D numerical experiments are presented. A two-sided energy inequality is a posteriori verified to document experimental convergence rates.

1. Introduction, plasticity with hardening

There is an extensive engineering and mathematical literature addressing plasticity with hardening, e.g. [1, 2, 3, 18, 20, 21, 22, 34, 47, 49, 50, 51, 54]. The simplest example involving a quadratic stored energy is *linearized plasticity*. We will deal with the simplest variant of this linearized plasticity with isotropic hardening and possibly also kinematic hardening. The rate-independent plasticity without hardening, i.e. the *Prandtl-Reuss* elastic/perfectly plastic model, received attention already long time ago, see e.g. in [25, 28, 34, 46]. Since models without hardening are degenerate, they are mathematically much more challenging as the displacements lie in $BD(\Omega)$ only rather than in the much better space $H^1(\Omega)$. We refer to [11] for a recent analysis in the context of so-called energetic solutions. This work will be a crucial basic for our analysis.

The aim of this article is to show that starting from a model with hardening we have classical solutions that converge in the limit of vanishing hardening to solutions of the Prandtl-Reuss model. Beside justification of the Prandtl-Reuss model as such a limit, the motivation of considering and implementing a small hardening is also numerically justified, e.g. a-posteriori error estimates and convergence of iterative schemes can be proved [3, 19, 47]. One should nevertheless mention that there are algorithms allowing for a direct treating of the Prandtl-Reuss model without hardening, e.g. Newton's schemes work often very well for the standard models with zero hardening, but there is no convergence analysis for the time-dependent case (see [46, 47] for static and incremental problems). Thus, our paper can be seen as a natural synthesis of the rate-independent evolutionary approach from [11] and the numerical approximation with small hardening used in [46, 47].

Let us still remark that similar convergence analysis has been carried out for gradient-plasticity, using the gradient of the plastic strain both in stored energy and in dissipation potential. Namely, for vanishing coefficients in these gradient terms, in [17, Sect.9.2], this model was shown to converge to the Prandtl-Reuss model in terms of the energetic solutions. A similar justification has been given by vanishing viscosity in the plastic flow rule [53]. The case of vanishing inertia is treated in [36], but only for the case of fixed positive hardening. The peculiarity of the vanishing hardening presented here is the changing structure, because the limit Prandtl-Reuss problem has a different set of internal variables, since the isotropic-hardening parameter disappears. Moreover, the vanishing-hardening limit is especially adapted to analyze the simultaneous limit with temporal and finite-element discretizations, see our Section 5.

The plasticity problem is considered to be *rate independent*, i.e. no inertia is considered (which is usually called *quasistatic* and there are no internal time scales. Thus, the problem is invariant under time rescaling. This allows us to use the so-called *energetic formulation*, which in our convex case is fully equivalent to the

commonly used evolutionary variational inequalities, see [35]. However, the energetic formulation gives rise to a useful Γ-convergence theory, which will be employed in Sections 4 and 5.

The elastoplastic evolution is given in terms of a time-dependent *stored energy* functional $\mathscr{E}$ and a *dissipation potential* $\mathscr{R}$ being positively homogeneous of degree 1 (i.e. $\mathscr{R}(\lambda z) = \lambda \mathscr{R}(z)$ for $\lambda \geq 0$), which reflects the *rate-independence* of the process, namely the invariance under any monotone re-scaling of time. Both functionals are defined with respect to a suitable state space $Q := U \times Z$, where U and Z are Banach spaces. The triple $(Q, \mathscr{E}, \mathscr{R})$ will be called a *rate-independent system*. Thereby a state $q = (u, z) \in U \times Z = Q$ is given by the *displacement* field u and the *internal variable* z that describes here plastic strain and possibly hardening, in some other applications it may also be damage or some phase-transformation variables. We assume that $\mathscr{R}$ involves only z, which distinguishes it as a "slow" variable while u is a "fast" variable. We choose a further Banach space $X \supset Z$ on which $\mathscr{R}$ is coercive.

Formally, the evolution of a rate-independent system $(Q, \mathscr{E}, \mathscr{R})$ is given through solutions $q : [0, T] \to Q$ of the following system of *doubly nonlinear* degenerate *parabolic/elliptic variational inclusions*:

$$\partial_u \mathscr{E}(t, u(t), z(t)) \ni 0 \quad \text{and} \quad \partial \mathscr{R}\big(\dot{z}(t)\big) + \partial_z \mathscr{E}(t, u(t), z(t)) \ni 0 \tag{1.1}$$

for a.a. $t \in (0, T)$, where "∂" refers to a (partial) subdifferential, using that $\mathscr{R}(\cdot)$, $\mathscr{E}(t, \cdot, z)$, and $\mathscr{E}(t, u, \cdot)$ are always here convex functionals. The first relation is the balance of forces and the second is the plastic flow rule.

For our subsequent work it turns out that the notion of *energetic solutions* is better suited. The following precise definition is the basis for the rest of this paper. A function $q : [0, T] \to Q$ is called an energetic solution for $(Q, \mathscr{E}, \mathscr{R})$, if $t \mapsto \mathscr{E}(t, q(t))$ lies in $\mathrm{L}^\infty([0, T])$, if $t \mapsto \mathscr{E}'_t(t, q(t))$ lies in $\mathrm{L}^1([0, T])$, and if for all $t \in [0, T]$ the stability condition (S) and the energy balance (E) hold:

$$\text{(S)} \qquad \mathscr{E}(t, q(t)) \leq \mathscr{E}(t, \hat{q}) + \mathscr{R}(\hat{z} - z(t)) \quad \text{for all } \hat{q} = (\hat{u}, \hat{z}) \in Q, \tag{1.2a}$$

$$\text{(E)} \qquad \mathscr{E}(t, q(t)) + \mathrm{Diss}_{\mathscr{R}}(z, [0, t]) = \mathscr{E}(0, q_0) + \int_0^t \mathscr{E}'_s(s, q(s))\, \mathrm{d}s, \tag{1.2b}$$

with $\mathscr{E}'_t(t, q) = \frac{\partial}{\partial t} \mathscr{E}(t, q)$ and the dissipation functional $\mathrm{Diss}_{\mathscr{R}}(z, [0, t]) := \sup \sum_{j=1}^N \mathscr{R}(z(t_j) - z(t_{j-1}))$, where the supremum is taken over all partitions of $[0, t]$.

The major advantage of the concept of energetic solutions is that the formulation (1.2) is derivative free: it does not contain $\dot{q}$, $\partial \mathscr{R}$, nor $\mathscr{E}'_q$. Moreover, it shows immediately the basic energetic a priori estimates. If $\mathscr{E}$ is coercive in Z we immediately obtain

$$u \in \mathrm{B}([0, T]; U) \quad \text{and} \quad z \in \mathrm{B}([0, T]; Z) \cap \mathrm{BV}([0, T]; X),$$

where we used $\mathrm{Diss}_{\mathscr{R}}(z, [0, T]) < \infty$ and the assumed coercivity of $\mathscr{R}$ in X; here "$\mathrm{B}([0, T]; \cdot)$" and "$\mathrm{BV}([0, T]; \cdot)$" stands for bounded measurable and bounded variation functions on $[0, T]$, respectively. The equivalence of (1.1) and the energetic solution is standard for the case with hardening (see e.g. [35, Sect. 2]), while for the case without hardening the more subtle equivalence is established in [11].

We are now in a position to explain the main results and ideas of the paper. In Section 2 we describe the plasticity model with hardening. In that situation $\mathscr{E}$ is uniformly convex (implying a quadratic coercivity) and we choose as state space Q a closed subspaces of the Hilbert space $\mathrm{H}^1(\Omega; \mathbb{R}^d) \times \mathrm{L}^2(\Omega; \mathbb{R}^{d \times d}_{\mathrm{dev}}) \times \mathrm{L}^2(\Omega)$. The classical existence result for solutions (cf. e.g. [20, 24]) is stated in Proposition 2.1 for completeness.

In Section 3 we present the Prandtl-Reuss model for perfect elastoplasticity without hardening. We still have convexity, but the coercivity only holds with a linear lower bound. Hence, the underlying state space $Q_{\rm PR}$ becomes a closed subspace of $\mathrm{BD}(\Omega;\mathbb{R}^d) \times \mathscr{M}(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})$. One major problem in our limit passage is that Q for the hardening case is only weakly$*$ dense in $Q_{\rm PR}$ but not strongly. In Proposition 3.2 we provide a slight generalization of the existence result in [11] for energetic solutions for the Prandtl-Reuss model written a rate-independent system $(Q_{\rm PR}, \mathscr{E}_{\rm PR}, \mathscr{R}_{\rm PR})$. Using different techniques similar existence results were obtained much earlier in [25, 53]. For simplicity, we have restricted all our work to the case without volume and surface forces, and thus we do not need the classical *safe-load condition*. We restricted the loading to time-dependent Dirichlet boundary data $u_{\rm D}$, but everything presented here can be generalized to included volume and surface loading, when adding a suitable the safe-load condition.

In Section 4 we consider a family $(Q, \mathscr{E}_\varepsilon, \mathscr{R})$ of rate-independent systems and study the limit of its energetic solutions $q_\varepsilon : [0,T] \to Q$ for $\varepsilon \to 0$. To study the case of vanishing hardening, we assume that $\mathscr{E}_\varepsilon$ has the form

$$\mathscr{E}_\varepsilon(t,u,\pi,\eta) := \int_\Omega \frac{1}{2}\mathbb{C}\big(e(u{+}2u_{\rm D}(t)){-}\pi\big){:}\big(e(u){-}\pi\big) + \frac{\varepsilon}{2}\mathbb{H}\pi{:}\pi + \frac{\varepsilon}{2}b\eta^2\,\mathrm{d}x,$$

where $\varepsilon > 0$ measures the size of the hardening. Here $\pi(t,x) \in \mathbb{R}^{d\times d}_{\rm dev}$ is the plastic strain tensor and η the scalar isotropic hardening parameter. The dissipation potential (or equivalently on the elastic domain) is assumed to be independent of ε and to satisfy the quite general structure condition (4.1). Then, based on suitable a priori bounds and the abstract theory of Γ-convergence for energetic solutions of rate-independent systems (see [38]), Theorem 4.1 states that the solutions $q_\varepsilon = (u_\varepsilon, \pi_\varepsilon, \eta_\varepsilon)$ converge (in terms of subsequences of u- and π-components) to energetic solutions (u,π) of the Prandtl-Reuss system $(Q_{\rm PR}, \mathscr{E}_{\rm PR}, \mathscr{R}_{\rm PR})$.

In Section 5 we study the joint convergence of vanishing hardening and space-time discretization. Choosing the time step $\tau = T/N$ we define discrete solutions via the incremental problem

$$\text{Minimize } \ \mathscr{E}_\varepsilon(k\tau,u,\pi,\eta) + \mathscr{R}(\pi{-}\pi^{k-1}_{\varepsilon\tau h}, \eta{-}\eta^{k-1}_{\varepsilon\tau h}) \ \text{ subject to } \ (u,\pi,\eta) \in Q_h,$$

where h is the mesh parameter, i.e. the maximal mesh size in the finite-element discretization for the functions in Q (for conformal P1 elements for u and $P0$ elements for (π,η)). Again using the abstract Γ-convergence theory we establish convergence of (a subsequence of) the associated solutions $\bar q_{\varepsilon\tau h}$ (piecewise constant interpolant of the incremental solutions) to energetic solutions of the Prandtl-Reuss system $(Q_{\rm PR}, \mathscr{E}_{\rm PR}, \mathscr{R}_{\rm PR})$. We cannot establish convergence of the whole family $\bar q_{\varepsilon\tau h}$, as it is not known that the solutions of the Prandtl-Reuss system are unique. However, since it is shown in [11, Thm. 5.9] that the stresses

$$\sigma = \mathbb{C}\big(e(u{+}u_{\rm D}(t)){-}\pi\big) \tag{1.3}$$

are uniquely defined, and conclude that the stresses $\bar\sigma_{\varepsilon\tau h}$ converge without choosing a subsequence. This also explains why in [46] only convergence of the stresses (even with explicit error rates) could be shown.

Section 6 reports on numerical experiments were one two-dimensional and one three-dimensional situation are studied, which are set up in such a way that a shear band must form. We study the influence of varying the hardening parameter ε and the discretization parameters h and τ.

2. Plasticity with hardening

Let us define

$$\mathbb{R}^{d\times d}_{\rm sym} := \Big\{ A\in\mathbb{R}^{d\times d} \,\Big|\, A^{\mathsf{T}} = A \Big\} \quad \text{and} \quad \mathbb{R}^{d\times d}_{\rm dev} := \Big\{ A\in\mathbb{R}^{d\times d}_{\rm sym} \,\Big|\, \operatorname{tr} A = 0 \Big\}. \tag{2.1}$$

We consider the classical formulation of linearized elastoplasticity with a trace-free plastic strain tensor $\pi \in \mathbb{R}^{d\times d}_{\rm dev}$ and a scalar, isotropic hardening parameter η. The system consists of the mechanical equilibrium $\operatorname{div}\sigma + f = 0$ for the elastic stress $\sigma = \mathbb{C}(e(u)-\pi)$, i.e.

$$-\operatorname{div}\big(\mathbb{C}(e(u)-\pi)\big) = f, \quad \text{where} \quad e(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^{\top}\Big) \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{2.2a}$$

and the evolution law (flow rule) for the internal, plastic variable $z = (\pi, \eta)$ in the form

$$\partial R\begin{pmatrix}\dot\pi\\ \dot\eta\end{pmatrix} + \begin{pmatrix}\operatorname{dev}\,\mathbb{C}\big(\pi - e(u)\big) + \mathbb{H}\pi\\ b\eta\end{pmatrix} \ni \begin{pmatrix}0\\0\end{pmatrix}. \tag{2.2b}$$

Here $\mathbb{C}$ is the symmetric positive semidefinite 4th-order elastic moduli tensor, i.e. $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$ for all $i,j,k,l = 1,\dots,d$. The *kinematic hardening tensor* $\mathbb{H}$ is a symmetric and positive semidefinite operator on $\mathbb{R}^{d\times d}_{\rm dev}$ (of Prager/Ziegler type [44, 55]), whereas the scalar parameter $b \geq 0$ determines the *isotropic hardening*.

The dissipation potential R is assumed to be lower semicontinuous, convex, and positively homogeneous of degree 1. The latter condition reflects rate independence, since the subdifferential $\partial R(\cdot)$ is homogeneous of degree 0. We define the elastic domain via $S = \partial R(0) \subset \mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R}$, which is closed and convex, and denote by δ_S the convex indicator function, i.e. $\delta_S(\varsigma,\xi) = 0$ for $(\varsigma,\xi) \in S$ and $+\infty$ otherwise. Then, we have $R = \delta_S^*$, where δ_S^* the Legendre-Fenchel conjugate functional to δ_S. Throughout we assume that R satisfies the coercivity estimate

$$\exists\, c > 0 \;\forall\, (\dot\pi,\dot\eta) \in \mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R}: \quad R(\dot\pi,\dot\eta) \geq c(|\dot\pi| + |\dot\eta|).$$

This condition is equivalent to assuming that S contains an open neighborhood of $(0,0) \in \mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R}$.

Then, by the classical duality relation of convex analysis we have

$$[\partial R]^{-1} = \partial R^* = \partial\delta_S^{**} = \partial\delta_S = N_S = \text{ the normal-cone mapping for } S.$$

The problem (2.2) can thus equivalently be written in a form which is more standard in the engineering literature, namely

$$\operatorname{div}\sigma + f = 0 \qquad \text{with} \quad \sigma = \mathbb{C}\big(e(u)-\pi\big), \tag{2.3a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\pi\\ \eta\end{pmatrix} \in N_S\begin{pmatrix}\operatorname{dev}\sigma - \mathbb{H}\pi\\ -b\eta\end{pmatrix}, \tag{2.3b}$$

which reveals that $\mathbb{H}\pi$ is in the position of the *back stress* to the *elastic stress* σ.

We assume that such elasto-plastic body occupies a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. The system is to be completed by boundary conditions. For this, we divide the boundary $\partial\Omega$ into two disjoint parts, $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$, and consider

$$u|_{\Gamma_{\rm D}} = u_{\rm D}(t) \;\text{ on } \Gamma_{\rm D}, \quad \big(\mathbb{C}\big(e(u)-\pi\big)\big)\nu = g \quad \text{on } \Gamma_{\rm N} = \partial\Omega\backslash\Gamma_{\rm D}. \tag{2.4}$$

Throughout this work we assume that $u_{\rm D}(t)$ may depend on time and is defined in all of Ω, and that

$$\Gamma_{\rm D} \;\text{ has a nonempty relative interior and} \tag{2.5a}$$

$$\text{a } (d-2)\text{-dimensional Lipschitz boundary.} \tag{2.5b}$$

Further, we make a transformation to the homogeneous Dirichlet boundary conditions, i.e. replace u by $u+u_{\rm D}$. Neglecting an (unimportant) additive constant thus created in the shifted functional $\mathscr{E}$, the function-space setting and the energetics can be defined by

$$U := \Big\{u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \;\Big|\; u|_{\Gamma_{\rm D}} = 0\Big\},\ Z := \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times\mathrm{L}^2(\Omega), \tag{2.6a}$$

$$Q := U\times Z, \quad X := \mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times\mathrm{L}^1(\Omega), \tag{2.6b}$$

$$\mathscr{E}(t,u,\pi,\eta) := \frac12\int_\Omega \mathbb{C}\big(e(u)-\pi\big):\big(e(u)-\pi\big) + \mathbb{H}\pi:\pi + b\eta^2\,\mathrm{d}x - \big\langle f_{\rm ext}(t),(u,\pi)\big\rangle, \tag{2.6c}$$

$$\mathscr{R}(\dot\pi,\dot\eta) := \int_\Omega \delta_S^*(\dot\pi,\dot\eta)\,\mathrm{d}x \tag{2.6d}$$

with the extended force $f_{\rm ext}(t) \in U^*\times Z^*$ in (2.6c) being defined by

$$\big\langle f_{\rm ext}(t),(u,\pi)\big\rangle := \int_\Omega f(t)\cdot u - \mathbb{C}e(u_{\rm D}(t)):(e(u)-\pi)\,\mathrm{d}x + \int_{\Gamma_{\rm N}} g(t)\cdot u\,\mathrm{d}\mathscr{H}^{d-1}. \tag{2.6e}$$

As the stability condition (S) in (1.2a) is an intrinsic part of the definition of energetic solutions, we introduce the set of stable states at time t via

$$\mathscr{S}(t) := \Big\{q\in Q \;\Big|\; \mathscr{E}(t,q)<\infty,\ \forall \hat q\in Q\colon \mathscr{E}(t,q)\le \mathscr{E}(t,\hat q)+\mathscr{R}(\hat q-q)\Big\}. \tag{2.7}$$

Then, assuming

$$u_{\rm D}\in \mathrm{W}^{1,1}(I;\mathrm{H}^1(\Omega;\mathbb{R}^d)),\quad f\in \mathrm{W}^{1,1}(I;L^{p_f}(\Omega;\mathbb{R}^d)),\quad g\in \mathrm{W}^{1,1}(I;L^{p_g}(\Gamma;\mathbb{R}^d)), \tag{2.8}$$

with $p_f = 2d/(d+2)$ and $p_g = 2-2/d$ if $d>2$, or $p_f>1$ and $p_g>1$ for $d=2$, or $p_f=1=p_g$ for $d=1$, we have $f_{\rm ext}\in \mathrm{W}^{1,1}(I;U^*\times Z^*)$ and the proof of the following proposition is standard, see [1, 20, 21, 22, 24, 35].

Proposition 2.1 (Existence of energetic solutions)**.** *Let $\mathbb{C}$, $\mathbb{H}$, $b\ge 0$, S, U, and Z be as described above, and (2.8) hold. Moreover assume that $q_0=(u_0,\pi_0,\eta_0)\in\mathscr{S}(0)$, i.e. q_0 is stable at time $t=0$. Furthermore, let the hardening form be coercive on $\mathrm{dom}(\delta_S^*)$, i.e.*

$$\inf\Big\{\frac{\mathbb{H}\pi:\pi+b\eta^2}{|\pi|^2+\eta^2}\;\Big|\; 0\neq(\pi,\eta)\in\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R},\ \delta_S^*(\pi,\eta)<\infty\Big\}>0. \tag{2.9}$$

Then, there is a unique energetic solution $q=(u,\pi,\eta)$ with $q(0)=q_0$ for the energetic rate-independent system $(U\times Z,\mathscr{E},\mathscr{R})$ defined in (2.6). Moreover, we have $q\in \mathrm{W}^{1,1}(I;U\times Z)$.

3. Plasticity without hardening

In the case without hardening, i.e. $\mathbb{H}=0$ and $b=0$, one obtains the *Prandtl-Reuss system* of perfect elastoplasticity. The peculiarity here is that the displacement no longer lives in the conventional Sobolev H^1-space but rather in the space of *functions with bounded deformations* introduced in [52], defined as

$$\mathrm{BD}(\Omega;\mathbb{R}^d) := \Big\{u\in \mathrm{L}^1(\Omega;\mathbb{R}^d)\;\Big|\; e(u)\in\mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\Big\}, \tag{3.1}$$

where $e(u)$ is the symmetric part of distributional gradient of u. For general bounded Borel sets $S\subset\mathbb{R}^d$ we set $\mathscr{M}(S,V)=C_0(S;V)^*$ denotes V-valued Radon measures on Ω (here $C_0(S;V)$ denotes continuous function with compact support on S). The

space $\mathrm{BD}(\Omega;\mathbb{R}^d)$ has a predual space, and thus a weak* topology. The weak* convergence means weak convergence in $\mathrm{L}^1(\Omega;\mathbb{R}^d)$ together with weak* convergence of $e(u)$ in $\mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. See [54, 11] for many details about BD-functions.

The rate-independent system $(Q_{\mathrm{PR}},\mathscr{E}_{\mathrm{PR}},\mathscr{R}_{\mathrm{PR}})$ for the Prandtl-Reuss system is given by

$$U_{\mathrm{PR}} := \mathrm{BD}(\Omega;\mathbb{R}^d), \quad Z_{\mathrm{PR}} = X_{\mathrm{PR}} := \mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}), \tag{3.2a}$$

$$Q_{\mathrm{PR}} := \big\{(u,\pi)\in U_{\mathrm{PR}}\times Z_{\mathrm{PR}} \mid e(u)-\pi|_\Omega \in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \\ u\circledS\nu\,\mathrm{d}\mathscr{H}^{d-1}+\pi|_{\Gamma_{\mathrm{D}}}=0 \text{ on } \Gamma_{\mathrm{D}}\big\}, \tag{3.2b}$$

$$\mathscr{E}_{\mathrm{PR}}(t,u,\pi) := \frac12\int_\Omega \mathbb{C}\big(e(u+2(u_{\mathrm{D}}(t))-\pi\big):\big(e(u)-\pi\big)\,\mathrm{d}x, \tag{3.2c}$$

$$\mathscr{R}_{\mathrm{PR}}(\dot\pi) := \int_{\Omega\cup\Gamma_{\mathrm{D}}} \delta_P^*(\cdot)\,\mathrm{d}\dot\pi(x) \quad \text{for } \dot\pi\in\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}), \tag{3.2d}$$

where $a\circledS b$ means the symmetrized tensorial product $\frac12(a\otimes b+b\otimes a)$ and where $P\subset\mathbb{R}^{d\times d}_{\mathrm{dev}}$ is the elasticity domain, which is assumed to be a bounded, closed convex neighborhood of 0. Equivalently, (3.2d) can also be written as $\mathscr{R}_{\mathrm{PR}}(\dot\pi)=\int_{\Omega\cup\Gamma_{\mathrm{D}}}\delta_P^*(\frac{\mathrm{d}\dot\pi}{\mathrm{d}|\dot\pi|})\,\mathrm{d}|\dot\pi|$ where $|\dot\pi|$ is the total variation of $\dot\pi$ and $\frac{\mathrm{d}\dot\pi}{\mathrm{d}|\dot\pi|}$ is the Radon-Nykodym derivative of $\mathrm{d}\dot\pi$ with respect to $|\dot\pi|$. See [11] for further details about functions on $\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$.

Functions from $\mathrm{BD}(\Omega;\mathbb{R}^d)$ have traces in $\mathrm{L}^1(\partial\Omega;\mathbb{R}^d)$. However, one has to be aware of jumps that can occur at the boundary, i.e. the measure $e(u)$ may concentrate on the boundary $\partial\Omega$. Thus, the classical boundary condition $u=0$ on Γ_{D} used in the definition of Q in (2.6b) is replaced by the more involved relation $u\circledS\nu\,\mathrm{d}\mathscr{H}^{d-1}+\pi|_{\Gamma_{\mathrm{D}}}=0$ on Γ_{D} for Q_{PR} in (3.2b). This relation has to be understood as an equality of measures on Γ_{D}, viz.

$$\forall\,\text{measurable } A\subset\Gamma_{\mathrm{D}}: \quad \int_A u\circledS\nu\,\mathrm{d}\mathscr{H}^{d-1}=\int_A \mathrm{d}\pi=\pi(A).$$

The relation simply means that any jump of u on the boundary with respect to the Dirichlet condition $u=0$ is due to a localized plastic deformation.

In fact, to define traces properly, one use suitable extensions of functions into a neighborhood of $\bar\Omega$. For $w\in\mathrm{H}^1(\mathbb{R}^d;\mathbb{R}^d)$ representing prescribed Dirichlet condition $w|_{\Gamma_{\mathrm{D}}}$, we define

$$A(w) := \big\{(u,\pi)\in\mathrm{BD}(\Omega;\mathbb{R}^d)\times\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}) \;\big|\; e(u)-\pi|_\Omega\in\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \\ \pi|_{\Gamma_{\mathrm{D}}}=(w-u)\circledS\nu\,\mathrm{d}\mathscr{H}^{d-1} \text{ on } \Gamma_{\mathrm{D}}\big\}. \tag{3.3}$$

Lemma 2.1 in [11] shows that $A: w\mapsto A(w)$ has a weak$\times$weak*-closed graph in $\mathrm{H}^1(\mathbb{R}^d;\mathbb{R}^d)\times(\mathrm{BD}(\Omega;\mathbb{R}^d)\times\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}))$. This shows that Q_{PR} from (3.2b), being equal to $A(0)$, is a closed subset of the Banach space of

$$\widetilde{Q}_{\mathrm{PR}} := \Big\{(u,\pi)\in\mathrm{BD}(\Omega;\mathbb{R}^d)\times\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}) \;\Big|\; e(u)-\pi|_\Omega\in\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\Big\}$$

which is equipped with the norm

$$\|(u,\pi)\|_{\widetilde{Q}_{\mathrm{PR}}} := \|u\|_{\mathrm{L}^1(\Omega)}+\|e(u)\|_{\mathscr{M}(\Omega)}+\|\pi\|_{\mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}})}+\|e(u)-\pi\|_{\mathrm{L}^2(\Omega)}.$$

Thus, Q_{PR} is itself a Banach space with the norm $\|\cdot\|_{\widetilde{Q}_{\mathrm{PR}}}$.

In view of (3.2b), for any energetic solution $q\in\mathrm{L}^\infty(I;Q)$ the elastic stress $\sigma=\mathbb{C}(e(u+u_{\mathrm{D}})-\pi)$ lies in $\mathrm{L}^\infty(I;\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ even though its particular components

$\mathbb{C}e(u)$ and $\mathbb{C}\pi$ may exhibit spatial concentration. Note also that, comparing to (2.6e), we now consider $f = 0$ and $g = 0$ to prevent uncontrolled slip and thus blow-up of a-priori estimates. In fact, certain qualified nonvanishing $f = 0$ or $g = 0$ could be admitted if a suitable "safe load condition" is additionally imposed, cf. [11, Formulas (2.17)-(2.18)].

Existence of solution to the Prandtl-Reuss system has been proved in [25, 53] by using the method of vanishing viscosity (or Yosida regularization). In [14], the limit passage from the rate-independent incremental problem was done while using weaker notion of solutions. Only recently, in [11, Thm.4.5], Dal Maso et al. executed this limit procedure in the framework of energetic solutions. In [17, Sect.9], it was shown the Prandtl-Reuss limit to also be reached from a rate-independent model of gradient plasticity.

We give a slight variant of the existence proof from [11] which is based on quite different arguments based on abstract BV relaxations. This variant follows [48] but still relies on the technical results from [11]. However, we hope that it can be generalized more easily to other plasticity models. We introduce an intermediate notion of convergence in $Q_{\rm PR}$ denoted by "$\overset{\rm strict}{\longrightarrow}$" and defined via

$$(u_n,\pi_n)\overset{\rm strict}{\longrightarrow}(u,\pi)\ \iff\ \begin{cases}(u_n,\pi_n)\overset{*}{\rightharpoonup}(u,\pi) & \text{in } \mathrm{BV}(\Omega;\mathbb{R}^d)\times\mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\rm dev}),\\ |\pi_n|(\Omega\cup\Gamma_{\rm D})\to|\pi|(\Omega\cup\Gamma_{\rm D}) & \text{in } \mathbb{R},\\ e(u_n)-\pi_n|_\Omega\to e(u)-\pi|_\Omega & \text{in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}).\end{cases}\tag{3.4}$$

Moreover, we define the subspace

$$Q^0_{\rm PR}:=\big\{(u,\pi)\in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^d)\times\mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\ \big|\ u=0\text{ on }\Gamma_{\rm D},\ e(u)-\pi\in\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big\}.\tag{3.5}$$

Obviously, $Q^0_{\rm PR}$ is a strongly closed subspace of $Q_{\rm PR}$; in fact it is the strong closure of $\mathbb{P}\,Q$ for Q from (2.6b), where $\mathbb{P}(u,\pi,\eta)=(u,\pi)$ is the projection dropping the isotropic hardening parameter.

Our different approach is based on the following central conjecture, which states that a weakened version weak stability is sufficient to imply the desired stability. We believe that the implication (3.6) is true under quite general conditions. We will show that it holds at least in the case considered in [11], see Remark 3.3.

Conjecture 3.1. *For all $t\in[0,T]$ and all $(u,\pi)\in Q_{\rm PR}$ the following holds:*

$$\left.\begin{aligned}&\text{If } \mathscr{E}_{\rm PR}(t,u,\pi)\le\mathscr{E}_{\rm PR}(t,u{+}\hat u,\pi{+}\hat\pi)+\mathscr{R}_{\rm PR}(\hat\pi)\text{ for all }(\hat u,\hat\pi)\in Q^0_{\rm PR},\\ &\text{then } \mathscr{E}_{\rm PR}(t,u,\pi)\le\mathscr{E}_{\rm PR}(t,u{+}\tilde u,\pi{+}\tilde\pi)+\mathscr{R}_{\rm PR}(\tilde\pi)\text{ for all }(\tilde u,\tilde\pi)\in Q_{\rm PR}.\end{aligned}\right\}\tag{3.6}$$

The following arguments (from [48]) support the validity of (3.6). First one knows by [4, Thm. 2.38] that

$$\mathscr{J}:(\tilde u,\tilde\pi)\mapsto\mathscr{E}_{\rm PR}(t,u{+}\tilde u,\pi{+}\tilde\pi)+\mathscr{R}_{\rm PR}(\tilde\pi)$$

is sequentially lower semicontinuous on $Q_{\rm PR}$ with respect to the weak* convergence. Thus, the minimum over $(\tilde u,\tilde\pi)\in Q_{\rm PR}$ on the right-hand side of the lower line in (3.6) is achieved. Moreover, [4, Thm. 2.39] guarantees that $\mathscr{J}$ is even sequentially continuous with respect to the strict convergence (3.4) on $Q_{\rm PR}$. Under suitable conditions on $\partial\Omega$ and $\Gamma_{\rm D}$ it should be possible to show that $Q^0_{\rm PR}$ is dense in $Q_{\rm PR}$ with respect to the strict convergence, see [48]. Then, (3.6) follows by continuity and density.

As a side effect of the vanishing-hardening convergence below, we obtain the following existence result.

Proposition 3.2 (Energetic solution for the Prandtl-Reuss model)**.** *Let the rate-independent system* $(Q_{\rm PR}, \mathscr{E}_{\rm PR}, \mathscr{R}_{\rm PR})$ *be as described in (3.2) such that* (3.6) *holds. Moreover, assume* $u_{\rm D} \in \mathrm{W}^{1,1}(I;\mathrm{W}^{1,2}(\Omega;\mathbb{R}^d))$ *and that* $(u_0,\pi_0) \in Q_{\rm PR}$ *is stable at* $t=0$*. Then, there exists an energetic solution* (u,π) *with* $(u(0),\pi(0)) = (u_0,\pi_0)$*. Moreover,* $\mathbb{C}(e(u)-\pi)$ *and thus also the"true" elastic stress* σ *from* (1.3) *is determined uniquely.*

Proof. It suffices to merge Proposition 2.1 with Theorem 4.1 below for $\eta_0 \in \mathrm{L}^2(\Omega)$ such that $\delta_S(\pi_0,\eta_0) = \delta_P(\pi_0)$ and realize that (4.1) can always be satisfied for a suitable S, e.g. for that one from Example 4.2. □

Remark 3.3. Conjecture 3.1 holds in the case considered in [11]. The equivalence proved in [11, Prop. 3.5] was used for exactly the same purpose as we use (3.6), namely to show that the set of stable states is closed under the weak* convergence, see [11, Thm. 3.7]. There the following additional conditions were used:

$$\Omega \text{ has a } (d-1) \text{ dimensional } C^2 \text{ boundary,} \tag{3.7a}$$

$$\Gamma_{\rm D} \text{ has a } (d-2) \text{ dimensional } C^2 \text{ boundary,} \tag{3.7b}$$

$$\mathbb{C}e = \mathbb{C}_{\rm D}\big(e - \tfrac{\operatorname{tr} e}{d}\mathbb{I}\big) + \kappa(\operatorname{tr} e)\mathbb{I} \quad \text{with}$$
$$\mathbb{C}_{\rm D}{:}\mathbb{R}^{d\times d}_{\rm dev} \to \mathbb{R}^{d\times d}_{\rm dev} \text{ positive definite and } \kappa > 0. \tag{3.7c}$$

The proof of (3.6) for this case is obtained in [11, Sect. 3.2] as a result of a subtle regularity theory for the stress $\sigma = \mathbb{C}(e(u)-\pi) \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ (namely $\operatorname{dev}\sigma \in L^\infty(\Omega)$ and $\operatorname{div}\sigma = 0$ in Ω and $\sigma\nu = 0$ on $\Gamma_{\rm N} = \partial\Omega\backslash\Gamma_{\rm D}$) and a careful analysis of the stress-strain duality, see e.g. [11, Prop. 2.4].

In fact, in [11], the stronger hypotheses $f_{\rm ext} \in \mathrm{W}^{1,1}(I;\mathrm{L}^d(\Omega;\mathbb{R}^d))$ are assumed, which would lead here to the qualification $u_{\rm D} \in \mathrm{W}^{1,1}(I;\mathrm{W}^{2-1/d,d}(\Gamma_{\rm D};\mathbb{R}^d))$ that allows the extension of $u_{\rm D}$ in $\mathrm{W}^{1,1}(I;\mathrm{W}^{2,d}(\Omega;\mathbb{R}^d))$ so that $\operatorname{div}\mathbb{C}e(u_{\rm D})$ is in $\mathrm{W}^{1,1}(I;\mathrm{L}^d(\Omega;\mathbb{R}^d))$. Yet, one should realize a special character of the Dirichlet part of the load $f_{\rm ext}$ from (2.6e) that needs only $e(u_{\rm D}) \in \mathrm{W}^{1,1}(I;\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}))$ when tested by $e(u)-\pi \in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})$.

4. The limit of vanishing hardening

Now we present and prove the main result, namely that the time-dependent Prandtl-Reuss model can be obtained as a limit from the classical elastoplastic model with hardening, if the hardening approaches zero. Most previous existence results for the Prandtl-Reuss system are based on viscous regularizations of the flow rule, see e.g. [5, 14, 53]. The vanishing-hardening limit was investigated only in the context of the dynamical problem in [9, 10], where the inertial effects can be used to prevent instantaneous formation of shear bands. To the best of the authors knowledge, the limit of vanishing hardening in the strict rate-independent setting was not investigated so far. And it is exactly this limit, which can be combined easily with numerical space-time discretizations, see the following section.

We naturally need some moderate qualifications for the relation between the dissipative potentials δ_S^* and δ_P^* occurring in (2.6d) and (3.2d), respectively. We suppose

$$\forall \pi \in \mathbb{R}^{d\times d}_{\rm dev}: \qquad \delta_P^*(\pi) = \min_{\eta\in\mathbb{R}} \delta_S^*(\pi,\eta), \tag{4.1a}$$

$$\exists C{\in}\mathbb{R}\ \forall \pi{\in}\mathbb{R}^{d\times d}_{\rm dev}\ \exists \eta{\in}\mathbb{R}: \qquad \delta_S^*(\pi,\eta) = \delta_P^*(\pi) \text{ and } |\eta| \le C|\pi|. \tag{4.1b}$$

We show in Example 4.2 that these conditions hold in the standard cases used for modeling hardening.

Theorem 4.1 (Convergence of vanishing hardening). *Let all the assumptions of Proposition 2.1 and* (3.6) *hold. Define*

$$\mathscr{E}_\varepsilon(t,u,\pi,\eta) := \int_\Omega \frac{1}{2}\mathbb{C}\big(e(u{+}2u_{\rm D}(t)){-}\pi\big){:}\big(e(u){-}\pi\big) + \frac{\varepsilon}{2}\mathbb{H}\pi{:}\pi + \frac{\varepsilon}{2}b\eta^2\,{\rm d}x, \tag{4.2}$$

assume that $q_0=(u_0,\pi_0,\eta_0)\in Q$ is stable at $t=0$ for each $\varepsilon>0$, and denote by $q_\varepsilon=(u_\varepsilon,\pi_\varepsilon,\eta_\varepsilon)$ be the unique energetic solution to $(Q,\mathscr{E}_\varepsilon,\mathscr{R})$; cf. (2.6). *Then the following holds:*
(i) There is a constant C independent of ε giving the a-priori estimates

$$\big\|\pi_\varepsilon\big\|_{{\rm BV}(\bar I;{\rm L}^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C, \tag{4.3a}$$
$$\big\|e(u_\varepsilon){-}\pi_\varepsilon\big\|_{{\rm L}^\infty(I;{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C, \tag{4.3b}$$
$$\big\|e(u_\varepsilon)\big\|_{{\rm L}^\infty(I;{\rm L}^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C, \tag{4.3c}$$
$$\big\|\pi_\varepsilon\big\|_{{\rm L}^\infty(I;{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))}\le C/\sqrt{\varepsilon}, \tag{4.3d}$$
$$\big\|e(u_\varepsilon)\big\|_{{\rm L}^\infty(I;{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C/\sqrt{\varepsilon}, \tag{4.3e}$$
$$\big\|\eta_\varepsilon\big\|_{{\rm L}^\infty(I;{\rm L}^2(\Omega))}\le C/\sqrt{\varepsilon}. \tag{4.3f}$$

(ii) For $\varepsilon\to 0$, there is a subsequence of $\{(u_\varepsilon,\pi_\varepsilon)\}_{\varepsilon>0}$ converging weakly in ${\rm L}^\infty(I;{\rm BD}(\Omega;\mathbb{R}^d))\times{\rm BV}(\bar I;\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))$ to some (u,π) and also $\pi_\varepsilon(t)\overset{*}{\rightharpoonup}\pi(t)$ in $\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))$ for any $t\in\bar I$. Every such couple (u,π) is an energetic solution to $(Q_{\rm PR},\mathscr{E}_{\rm PR},\mathscr{R}_{\rm PR})$ with $(u(0),\pi(0))=(u_0,\pi_0)$; see* (1.2) *and* (3.2).

(iii) Moreover, the whole sequence of stresses $\{\mathbb{C}(e(u_\varepsilon){-}\pi_\varepsilon)\}_{\varepsilon>0}$ converges weakly in ${\rm L}^\infty(I;{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$ to $\mathbb{C}(e(u){-}\pi)$ and even strongly pointwise on $[0,T]$, i.e.*

$$\forall t\in[0,T]:\quad \mathbb{C}(e(u_\varepsilon(t)){-}\pi_\varepsilon(t))\to\mathbb{C}(e(u(t)){-}\pi(t)|_\Omega)\ \text{in}\ {\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \tag{4.4}$$

and analogous assertion holds for "true" stresses from (1.3).

Let us note that (4.3c) cannot yield any uniform ${\rm L}^\infty(I;{\rm W}^{1,1}(\Omega;\mathbb{R}^d))$-estimate for u_ε because Korn's inequality does not hold for this limit case, cf. [43].

An example for stable initial conditions that are independent of ε is $\pi_0=0$ and $\eta_0=0$ and u_0 minimizing $\mathscr{E}_\varepsilon(t,\cdot,0,0)$ which obviously does not depend on ε; note that $\delta_S(0,0)=\delta_P(0)$. In principle, one may weaken the requirement of stability of initial conditions independently of ε and another approximation would allow for $u_0\in{\rm W}^{1,1}(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ and $\pi_0\in{\rm L}^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$. For this we would need to approximate both u_0 and π_0 by some $u_{0\varepsilon}$ and $\pi_{0\varepsilon}\in{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ so that $e(u_{0\varepsilon}){-}\pi_{0\varepsilon}\to e(u_0){-}\pi_0$ in ${\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ and $\|\pi_{0\varepsilon}\|_{{\rm L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})}=o(\varepsilon^{-1/2})$ so that $\int_\Omega\varepsilon\mathbb{H}\pi_{0\varepsilon}:\pi_{0\varepsilon}\,{\rm d}x\to 0$.

Proof of Theorem 4.1. We divide the proof to the six steps.

Step 1 (a-priori estimates): The coercivity of $\mathscr{R}$ and of the "$\mathbb{C}$-part" of $\mathscr{E}_\varepsilon$ then gives the estimates (4.3a,b). By the obvious inequality $\|e(u)\|_{{\rm L}^1(\Omega;\mathbb{R}^{d\times d})}\le({\rm meas}^d(\Omega))^{1/2}\|e(u){-}\pi\|_{{\rm L}^2(\Omega;\mathbb{R}^{d\times d})}+\|\pi\|_{{\rm L}^1(\Omega;\mathbb{R}^{d\times d})}$, we obtain also (4.3c). From the "$\mathbb{H}$-part" of $\mathscr{E}_\varepsilon$ and from $\frac12\varepsilon\eta^2$, we further obtain (4.3d) and (4.3f), respectively. Then, by the triangle inequality $\|e(u)\|_{{\rm L}^2(\Omega;\mathbb{R}^{d\times d})}\le\|e(u){-}\pi\|_{{\rm L}^2(\Omega;\mathbb{R}^{d\times d})}+\|\pi\|_{{\rm L}^2(\Omega;\mathbb{R}^{d\times d})}$, we eventually obtain (4.3e).

Step 2 (selection of convergent subsequences): Let us take the energetic solution $(u_\varepsilon, \pi_\varepsilon, \eta_\varepsilon)$ to the problem $(U{\times}Z, \mathscr{E}_\varepsilon, \mathscr{R}, u_0, \pi_0, \eta_0)$, which exists due to Proposition 2.1. For the convergence we use the estimates (4.3a-c) and select the subsequence by Banach's selection principle. Moreover, by Helly's selection principle and metrizability of the weak* topology on the balls in $\mathscr{M}(\Omega{\cup}\Gamma_{\rm D})$, we can also claim $\pi_\varepsilon(t) \overset{*}{\rightharpoonup} \pi(t)$ in $\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})$ for all $t\in\bar I$. Then also

$$\sigma_\varepsilon(t) = \mathbb{C}\big(e(u_\varepsilon(t)){-}\pi_\varepsilon(t)\big) \rightharpoonup \sigma(t) = \mathbb{C}\big(e(u(t)){-}\pi(t)|_\Omega\big) \quad \text{ in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \tag{4.5}$$

for all $t\in\bar I$; it follows from the a-priori estimate (4.3b) and the uniqueness of the stress, cf. the arguments in [34, Sect.4.2.3] or [11, Thm.5.9]. Also, for any $t\in\bar I$, $u_\varepsilon(t)$ converges weakly* in $\mathrm{BD}(\Omega;\mathbb{R}^d)$ to a limit which can be uniquely identified with $u(t)$ because its strain $e(u(t)) = \mathbb{C}^{-1}\sigma(t) + \pi(t)|_\Omega \in \mathscr{M}(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ has already been determined uniquely when choosing $\pi_\varepsilon(t) \overset{*}{\rightharpoonup} \pi(t)$ in $\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})$ and because of the prescribed Dirichlet boundary conditions.

Step 3 (stability): The convergence in the stability condition (1.2a) requires essentially an explicit construction of *mutual recovery sequences* in the spirit of [38]. Here, because of the disappearance of the isotropic hardening in the limit, we must consider a modification of this concept. For each t the solutions $q_\varepsilon(t) = (u_\varepsilon(t), \pi_\varepsilon(t), \eta_\varepsilon(t))$ form a stable sequence, i.e. $\sup_{\varepsilon>0}\mathscr{E}_\varepsilon(t,q_\varepsilon(t)) < \infty$ and $q_\varepsilon(t)\in\mathscr{S}_\varepsilon(t)$ with $\mathscr{S}_\varepsilon(t)$ from (2.7) with $\mathscr{E}_\varepsilon$ in place of $\mathscr{E}$, such that $(u_\varepsilon,\pi_\varepsilon) \overset{*}{\rightharpoonup} (u(t),\pi(t)) \in Q_{\rm PR}$, i.e.

$$u_\varepsilon(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{in } \mathrm{BD}(\Omega;\mathbb{R}^d), \tag{4.6a}$$

$$\pi_\varepsilon(t) \overset{*}{\rightharpoonup} \pi(t) \qquad \text{in } \mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}), \tag{4.6b}$$

$$e(u_\varepsilon(t)){-}\pi_\varepsilon(t) \rightharpoonup e(u(t)){-}\pi(t) \qquad \text{in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}). \tag{4.6c}$$

In particular, this implies $\|\pi_\varepsilon(t)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} = \mathscr{O}(\varepsilon^{-1/2})$ and $\|\eta_\varepsilon(t)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} = \mathscr{O}(\varepsilon^{-1/2})$.

Now, for fixed t and a given test function $(\tilde u,\tilde\pi)\in Q_{\rm PR}$ we have to construct a so-called mutual recovery sequence $(\tilde q_\varepsilon)_{\varepsilon>0}$ in the spirit of [38] such that

$$\begin{aligned}\limsup_{\varepsilon\to0}\ &\mathscr{E}_\varepsilon(t,\tilde q_\varepsilon) + \mathscr{R}(\tilde q_\varepsilon{-}q_\varepsilon(t)) - \mathscr{E}_\varepsilon(t,q_\varepsilon(t))\\ &\le \mathscr{E}_{\rm PR}(\tilde u,\tilde\pi) + \mathscr{R}_{\rm PR}(\tilde\pi{-}\pi(t)) - \mathscr{E}_{\rm PR}(t,u,\pi(t)).\end{aligned} \tag{4.7}$$

The main point is that we first restrict ourselves to special test functions $(\tilde u,\tilde\pi)\in Q_{\rm PR}$, namely those given in the form (where we temporarily drop the time dependence)

$$\begin{aligned}&(\tilde u,\tilde\pi,\tilde\eta) = (u,\pi,0) + (\hat u,\hat\pi,\hat\eta)\\ &\text{with}\quad (\hat u,\hat\pi)\in Q^1_{\rm PR} := \mathrm{H}^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d){\times}\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}),\\ &\text{and}\quad \hat\eta\in\mathrm{L}^2(\Omega)\ \text{ such that }\ \delta^*_S(\hat\pi,\hat\eta) = \delta^*_P(\hat\pi),\end{aligned} \tag{4.8}$$

where $\mathrm{H}^1_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d) := U$ from (2.6a). Then we put

$$\tilde q_\varepsilon = (\tilde u_\varepsilon,\tilde\pi_\varepsilon,\tilde\eta_\varepsilon) = (u_\varepsilon{+}\tilde u{-}u, \pi_\varepsilon{+}\tilde\pi{-}\pi, \eta_\varepsilon{+}\tilde\eta{-}0); \tag{4.9}$$

note that the difference $\tilde q_\varepsilon - q_\varepsilon = (\tilde u{-}u,\tilde\pi{-}\pi,\tilde\eta) = \hat q$ is independent of ε. Hence,

$$\begin{aligned}\mathscr{R}(\tilde\pi_\varepsilon{-}\pi_\varepsilon,\tilde\eta_\varepsilon{-}\eta_\varepsilon) &= \int_\Omega \delta^*_S(\tilde\pi_\varepsilon{-}\pi_\varepsilon,\tilde\eta_\varepsilon{-}\eta_\varepsilon)\,\mathrm{d}x = \int_\Omega \delta^*_S(\tilde\pi{-}\pi,\tilde\eta)\,\mathrm{d}x\\ &= \int_\Omega \delta^*_S(\hat\pi,\hat\eta)\,\mathrm{d}x = \int_\Omega \delta^*_P(\hat\pi)\,\mathrm{d}x = \mathscr{R}_{\rm PR}(\tilde\pi{-}\pi).\end{aligned} \tag{4.10}$$

Thus, we have the middle term in the limsup in (4.7) under control.

To control the limit of the two energy terms in the limsup (4.7) we use a cancellation property of the quadratic functionals $\mathscr{E}_\varepsilon$ and $\mathscr{E}_{\rm PR}$: because of the quadratic structure of both $\mathscr{E}_\varepsilon$ and $\mathscr{E}_{\rm PR}$, for $f_{\rm ext}$ defined again by (2.6e) now with $f=0$ and $g=0$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathscr{E}_\varepsilon(t,\tilde u_\varepsilon,\tilde\pi_\varepsilon,\tilde\eta_\varepsilon)-\mathscr{E}_\varepsilon(t,u_\varepsilon,\pi_\varepsilon,\eta_\varepsilon)\\
&=\lim_{\varepsilon\to 0}\int_\Omega \frac12\mathbb{C}\big(e(\tilde u_\varepsilon{-}u_\varepsilon)-\tilde\pi_\varepsilon{+}\pi_\varepsilon\big){:}\big(e(\tilde u_\varepsilon{+}u_\varepsilon)-\tilde\pi_\varepsilon{-}\pi_\varepsilon\big)\\
&\qquad+\frac{\varepsilon}{2}\mathbb{H}(\tilde\pi_\varepsilon{-}\pi_\varepsilon){:}(\tilde\pi_\varepsilon{+}\pi_\varepsilon)+\frac{\varepsilon}{2}(\tilde\eta_\varepsilon{-}\eta_\varepsilon)(\tilde\eta_\varepsilon{+}\eta_\varepsilon)\,{\rm d}x-\big\langle f_{\rm ext}(t),(\tilde u_\varepsilon{-}u_\varepsilon,\tilde\pi_\varepsilon{-}\pi_\varepsilon)\big\rangle\\
&=\lim_{\varepsilon\to 0}\int_\Omega \frac12\mathbb{C}\big(e(\hat u)-\hat\pi\big){:}\big(e(2u_\varepsilon{+}\hat u)-2\pi_\varepsilon{-}\hat\pi\big)+\frac{\varepsilon}{2}\mathbb{H}\hat\pi{:}(2\pi_\varepsilon{+}\hat\pi)\\
&\qquad+\frac{\varepsilon}{2}\tilde\eta(\tilde\eta_\varepsilon{+}\eta_\varepsilon)\,{\rm d}x-\big\langle f_{\rm ext}(t),(\hat u,\hat\pi)\big\rangle\\
&=\int_\Omega \frac12\mathbb{C}\big(e(\hat u)-\hat\pi\big){:}\big(e(2u{+}\hat u)-2\pi{-}\hat\pi\big)+{\rm d}x-\big\langle f_{\rm ext}(t),(\hat u,\hat\pi)\big\rangle\\
&=\int_\Omega \frac12\mathbb{C}\big(e(\tilde u{-}u)-\tilde\pi{+}\pi\big){:}\big(e(\tilde u{+}u)-\tilde\pi{-}\pi\big)\,{\rm d}x-\big\langle f_{\rm ext}(t),(\tilde u{-}u,\tilde\pi{-}\pi)\big\rangle\\
&=\mathscr{E}_{\rm PR}(t,\tilde u,\tilde\pi)-\mathscr{E}_{\rm PR}(t,u,\pi),
\end{aligned}\tag{4.11}$$

where we used (4.6c)

$$\begin{aligned}
&\Big|\int_\Omega\frac{\varepsilon}{2}\mathbb{H}\hat\pi{:}(2\pi_\varepsilon{+}\hat\pi)\,{\rm d}x\Big|\le\frac{\varepsilon}{2}|\mathbb{H}|\,\big\|\hat\pi\big\|_{\mathrm{L}^2(\Omega)}\big\|2\pi_\varepsilon{+}\hat\pi\big\|_{\mathrm{L}^2(\Omega)}=\mathscr{O}\big(\sqrt{\varepsilon}\big)\to 0\quad\text{and}\\
&\Big|\int_\Omega\frac{\varepsilon}{2}b\tilde\eta(\tilde\eta_\varepsilon{+}\eta_\varepsilon)\,{\rm d}x\Big|\le\frac{\varepsilon}{2}b\big\|\hat\eta\big\|_{\mathrm{L}^2(\Omega)}\big\|\tilde\eta_\varepsilon{+}\eta_\varepsilon\big\|_{\mathrm{L}^2(\Omega)}=\mathscr{O}\big(\sqrt{\varepsilon}\big)\to 0.
\end{aligned}\tag{4.12}$$

This allows for the limit passage in the mutual recovery sequence condition (4.7). Using the stability of q_ε we know that each term in the limsup is non-negative. Hence we conclude that the right-hand side is non-negative as well. Recalling the special choice of $(\tilde u,\tilde\pi)$ in (4.8) we have derived the following stability condition for the limit $(u(t),\pi(t))\in Q_{\rm PR}$:

$$\forall(\hat u,\hat\pi)\in Q^1_{\rm PR}:\quad \mathscr{E}_{\rm PR}(t,u(t),\pi(t))\le\mathscr{E}_{\rm PR}(t,u(t){+}\hat u,\pi(t){+}\hat\pi)+\mathscr{R}_{\rm PR}(\hat\pi),\tag{4.13}$$

where $Q^1_{\rm PR}$ is defined in (4.8). Since $Q^0_{\rm PR}$ is the (strong) closure of $Q^1_{\rm PR}$ in $Q_{\rm PR}$ and $\mathscr{E}_{\rm PR}$ and $\mathscr{R}_{\rm PR}$ are continuous, we have shown that $(u(t),\pi(t))$ satisfies the upper condition in the assumed property (3.6). Thus, the lower condition holds as well and the desired stability (S) in (1.2a) is established.

Step 4 (upper energy balance): Using the energy balance (1.2b) for q_ε, for $f_{\rm ext}$ defined again by (2.6e) now with $f=0$ and $g=0$, we have

$$\begin{aligned}
&\int_\Omega\frac12\mathbb{C}\big(e(u_\varepsilon(t)){-}\pi_\varepsilon(t)\big){:}\big(e(u_\varepsilon(t)){-}\pi_\varepsilon(t)\big)+\frac{\varepsilon}{2}\mathbb{H}\pi_\varepsilon(t){:}\pi_\varepsilon(t)+\frac{\varepsilon}{2}b\eta_\varepsilon(t)^2\,{\rm d}x\\
&\quad+{\rm Diss}_{\mathscr{R}}(\pi_\varepsilon,\eta_\varepsilon;[0,t])-\big\langle f_{\rm ext}(t),(u_\varepsilon(t),\pi_\varepsilon(t))\big\rangle\le\int_0^t\big\langle\dot f_{\rm ext},(u_\varepsilon,\pi_\varepsilon)\big\rangle{\rm d}t\\
&\quad+\int_\Omega\frac12\mathbb{C}\big(e(u_0){-}\pi_0\big){:}\big(e(u_0){-}\pi_0\big)+\frac{\varepsilon}{2}\mathbb{H}\pi_0{:}\pi_0+\frac{\varepsilon}{2}b\eta_0^2\,{\rm d}x-\big\langle f_{\rm ext}(0),(u_0,\pi_0)\big\rangle.
\end{aligned}\tag{4.14}$$

Note that $e(u_\varepsilon(t)){-}\pi_\varepsilon(t)$ is well defined because of the bound (4.3a) and because $u_\varepsilon(t)$ is then determined uniquely by minimizing $\mathscr{E}_\varepsilon(t,\cdot,\pi_\varepsilon(t),\eta_\varepsilon(t))$. We can pass

to the limit in (4.14) by using weak lower semicontinuity of $\mathscr{E}_{\rm PR}(t,\cdot,\cdot)$ and that the already mentioned convergence $e(u_\varepsilon(t))-\pi_\varepsilon(t) \rightharpoonup e(u(t))-\pi(t)$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})$, and also by using

$$\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathscr{R}}(\pi_\varepsilon,\eta_\varepsilon;[0,t]) \geq \liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathscr{R}_{\rm PR}}(\pi_\varepsilon;[0,t]) \geq \mathrm{Diss}_{\mathscr{R}_{\rm PR}}(\pi;[0,t]). \quad (4.15)$$

The first inequality in (4.15) is just by $\delta_S^*(\pi,\eta) \geq \delta_P^*(\pi)$, while the second one is by weak lower-semi continuity of $\mathrm{Diss}_{\mathscr{R}_{\rm PR}}(\,\cdot\,;[0,t])$. Eventually, we can forget the terms $\int_\Omega \frac{\varepsilon}{2}\mathbb{H}\pi_\varepsilon(t){:}\pi_\varepsilon(t) + \frac{\varepsilon}{2}\eta_\varepsilon(t)^2\,\mathrm{d}x$ while $\int_\Omega \frac{\varepsilon}{2}\mathbb{H}\pi_0{:}\pi_0\,\mathrm{d}x$ converges to 0 because we have assumed $\pi_0 \in \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and also $\int_\Omega \frac{\varepsilon}{2} b\eta_0^2\,\mathrm{d}x \to 0$ by $\eta_0 \in \mathrm{L}^2(\Omega)$. The limit passage in (4.14) results to the upper energy estimate

$$\mathscr{E}_{\rm PR}(t,u(t),\pi(t)) + \mathrm{Diss}_{\mathscr{R}_{\rm PR}}(\pi;[0,t]) \leq \mathscr{E}_{\rm PR}(0,u_0,\pi_0) + \int_0^t \big\langle \dot f_{\rm ext},(u,\pi)\big\rangle \mathrm{d}t. \quad (4.16)$$

Step 5 (lower energy balance): Recalling the upper energy estimate (4.16) we obtain the desired energy balance (E) in (1.2b), by using the standard argument that stability of $q(t)$ for all $t\in\bar I$ implies the lower energy inequality, cf. [33, 42].

Step 6 (improved convergence): The convergence of the whole sequence $(\mathbb{C}e(u_\varepsilon)-\pi_\varepsilon)_{\varepsilon>0}$ is due to the mentioned uniqueness of the elastic stresses. Having the pointwise stability, by abstract arguments from theory of rate-independent processes, the stored energy converges pointwise, i.e. $\mathscr{E}_\varepsilon(t,u_\varepsilon(t),\pi_\varepsilon(t),\eta_\varepsilon(t)) \to \mathscr{E}_{\rm PR}(t,u(t),\pi(t))$ for any $t\in[0,T]$. In view of (4.2) and (3.2c) and using also (4.5), we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_\Omega \frac12 \mathbb{C}\big(e(u_\varepsilon(t))-\pi_\varepsilon(t)\big){:}\big(e(u_\varepsilon(t))-\pi_\varepsilon(t)\big)\,\mathrm{d}x\\
&\quad\le \lim_{\varepsilon\to 0} \int_\Omega \frac12 \mathbb{C}\big(e(u_\varepsilon(t))-\pi_\varepsilon(t)\big){:}\big(e(u_\varepsilon(t))-\pi_\varepsilon(t)\big) + \frac{\varepsilon}{2}\mathbb{H}\pi_\varepsilon(t){:}\pi_\varepsilon(t) + \frac{\varepsilon}{2} b\eta_\varepsilon(t)^2\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0} \mathscr{E}_\varepsilon(t,u_\varepsilon(t),\pi_\varepsilon(t),\eta_\varepsilon(t)) - \int_\Omega \sigma_\varepsilon(t){:}e(u_{\rm D}(t))\,\mathrm{d}x\\
&\quad= \mathscr{E}_{\rm PR}(t,u(t),\pi(t)) - \int_\Omega \sigma(t){:}e(u_{\rm D}(t))\,\mathrm{d}x\\
&\quad= \int_\Omega \frac12 \mathbb{C}\big(e(u(t))-\pi(t)\big){:}\big(e(u(t))-\pi(t)\big). \qquad (4.17)
\end{aligned}$$

Thus, by positive definiteness of $\mathbb{C}$ and using an appropriate equivalent norm on the Hilbert space $\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})$ induced by $\mathbb{C}$, we obtain $\liminf_{\varepsilon\to 0}\|e(u_\varepsilon(t))-\pi_\varepsilon(t)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})} \le \|e(u(t))-\pi(t)\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})}$. From this and from (4.5), the strong convergence (4.4) follows by the uniform convexity of the Hilbert space $\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d})$. □

Example 4.2 (Dissipation potential)**.** An example for the dissipation potential, considered e.g. in [21, 22, 35, 51] (often with for $q_{\rm H}=1$), is

$$\delta_S^*(\dot\pi,\dot\eta) := \delta_P^*(\dot\pi) + \delta_K(\dot\pi,\dot\eta) \quad\text{with}\quad K := \Big\{(\dot\pi,\dot\eta)\in\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R}\;\Big|\;\dot\eta \le -q_{\rm H}\delta_P^*(\dot\pi)\Big\} \quad (4.18)$$

for $P\subset\mathbb{R}^{d\times d}_{\rm dev}$ a convex, bounded, and closed neighbourhood of the origin and $q_{\rm H}>0$. We want to identify S via the definition

$$S = \partial\delta_S^*(0) = \Big\{(\varsigma,\xi)\in\mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R}\;\Big|\;\forall(\dot\pi,\dot\eta):\ \dot\eta\le -q_{\rm H}\delta_P^*(\dot\pi)\ \Rightarrow\ \delta_P^*(\dot\pi)\ge \dot\pi{:}\varsigma+\dot\eta\xi\Big\}.$$

As $\delta_P^*(0)$ is finite, any $(\varsigma,\xi)\in S$ satisfies implies $\xi\ge 0$. Then the maximal value of $\dot\pi{:}\varsigma+\dot\eta\xi$ is attained at $\dot\eta = -q_{\rm H}\delta_P^*(\dot\pi)$ and thus equals to $\dot\pi{:}\varsigma - q_{\rm H}\delta_P^*(\dot\pi)\xi$. The

inequality $\delta_P^*(\dot\pi) \geq \dot\pi{:}\varsigma + \dot\eta\xi$ then reduces to $\dot\pi{:}\varsigma \leq (1{+}q_{\rm H}\xi)\delta_P^*(\dot\pi)$ for all $\dot\pi \in \mathbb{R}^{d\times d}_{\rm dev}$. By the 1-homogeneity of δ_P^*, this gives $\varsigma \in (1{+}q_{\rm H}\xi)\partial\delta_P^*(0)$. Using $\partial\delta_P^*(0) = P$, we obtain the characterization

$$S = \Big\{(\varsigma,\xi) \in \mathbb{R}^{d\times d}_{\rm dev}\times\mathbb{R} \,\Big|\, \xi \geq 0 \ \text{ and } \ \varsigma \in (1{+}q_{\rm H}\xi)P\Big\}. \tag{4.19}$$

Note that the constraint $\dot\eta \leq -q_{\rm H}\delta_P^*(\dot\pi) \leq 0$ in (4.18) makes the evolution of the hardening η unidirectional, namely the hardening variable never can increase while ξ and thus also the set of ς satisfying $\delta_{P_0}^*(\varsigma) \leq 1 + q_{\rm H}\xi$ in (4.19) cannot only decrease. Also (4.1a) obviously holds just with P from (4.18), and $C = q_{\rm H}$ in (4.1b) with $L = |P|$ =Lipschitz constant of δ_P^*, and $\eta = -\delta_P^*(\pi)$ in (4.1b).

Remark 4.3. The isotropic hardening η can be omitted if $b = 0$ because then there is no driving force for the isotropic hardening and, if $\eta_0 = 0$, then identically $\eta \equiv 0$. Then the dissipation potential (2.6d) reduces to $\delta_P^*(\dot\pi)$, cf. (3.2d) above.

5. Numerical approximation and its convergence

Numerical convergence results for the problem with hardening are classical, cf. [21, 49, 50, 54] for a survey. However, most of the numerical results are based on the assumption of (often not justified) higher regularity, which then allows us to derive convergence rates, cf. the above references and e.g., [3]. Actually, regularity for perfect plasticity up to the boundary is a difficult problem, requiring high smoothness of the boundary, as documented in [7, 13, 16, 29]. Besides, the hardening parameters are considered fixed. Moreover, there are some numerical studies directly for Prandtl-Reuss plasticity, e.g. [26, 28, 45, 46]. This suggests a chance for an unconditional convergence jointly with both hardening parameters and numerical discretization parameters approaching 0. This is indeed what we will prove, cf. Theorem 5.1 below.

In the context of rate-independent processes, it is natural to discretize in time just by the fully implicit 1st-order Euler scheme. We consider the constant time-step τ, and abbreviate $t_\tau^k := k\tau$.

We further consider spatial approximation, using C^0-conforming P1-elements for the approximation of u and P0-elements for the approximation of π and η. We will further make a spatial discretization. For this, we assume that we are given a sequence of triangulations $\{\mathscr{T}_h\}_{h>0}$ of the polyhedral domain Ω without hanging nodes but otherwise entirely general. We suppose that $h > 0$ range over countable sets of positive real numbers with accumulation points at 0, and that $\max_{E\in\mathscr{T}_h}\mathrm{diam}(E) \leq h$. The finite-dimensional subspaces of $\mathrm{L}^2(\Omega)$ and $\mathrm{W}^{1,2}(\Omega)$ related to P0- and P1-elements and subordinate to the triangulation $\mathscr{T}_h$ respectively by $V_{0,h}$ and $V_{1,h}$.

The approximate solution $q_{\varepsilon\tau h}^k = (u_{\varepsilon\tau h}^k, \pi_{\varepsilon\tau h}^k, \eta_{\varepsilon\tau h}^k)$ is then obtained recursively for $k = 1, ..., T/\tau$ as the solution to the incremental problem

$$\left.\begin{array}{ll} \text{Minimize} & \mathscr{E}_\varepsilon(t_k,u,\pi,\eta) + \mathscr{R}(\pi{-}\pi_{\varepsilon\tau h}^{k-1}, \eta{-}\eta_{\varepsilon\tau h}^{k-1}) \\ \text{subject to} & u \in V_{1,h}^d, \quad \pi \in V_{0,h}^{d\times d}, \quad \eta \in V_{0,h}, \end{array}\right\} \tag{5.1}$$

with $\mathscr{E}_\varepsilon$ from (4.2) and $\mathscr{R}$ from (4.18), when starting from $q_{\varepsilon\tau h}^0 = q_0$. It is notable that (5.1) has a unique solution $q_{\varepsilon\tau h}^k$, $k = 1, ..., T/\tau$, which satisfies the two-sided

energy inequality

$$L^k_{\varepsilon\tau h} := \int_{t_{k-1}}^{t_k} \langle \dot f_{\rm ext}(t), q^k_{\varepsilon\tau h}\rangle \,{\rm d}t \\ \le \mathscr{E}_\varepsilon\big(t^k_\tau, q^k_{\varepsilon\tau h}\big) - \mathscr{E}_\varepsilon\big(t^{k-1}_\tau, q^{k-1}_{\varepsilon\tau h}\big) + {\rm Diss}_{\mathscr{R}}\big(q_{\varepsilon\tau h};[t_{k-1},t_k]\big) \\ \le \int_{t_{k-1}}^{t_k} \langle \dot f_{\rm ext}(t), q^{k-1}_{\varepsilon\tau h}\rangle \,{\rm d}t =: U^k_{\varepsilon\tau h}, \quad (5.2)$$

and the discrete stability

$$\forall \tilde q{=}(\tilde u,\tilde\pi,\tilde\eta){\in}V^d_{1,h}{\times}V^{d\times d}_{0,h}{\times}V_{0,h}\colon\quad \mathscr{E}_\varepsilon\big(t_k,q^k_{\varepsilon\tau h}\big)\le \mathscr{E}_\varepsilon\big(t_k,\tilde q\big)+\mathscr{R}(\tilde\pi{-}\pi^{k-1}_{\varepsilon\tau h},\tilde\eta{-}\eta^{k-1}_{\varepsilon\tau h}), \quad (5.3)$$

see [40, 42].

Let us define the piecewise affine interpolant $(u_{\varepsilon\tau h},\pi_{\varepsilon\tau h},\eta_{\varepsilon\tau h})$ by

$$\big[u_{\varepsilon\tau h},\pi_{\varepsilon\tau h},\eta_{\varepsilon\tau h}\big](t) := \frac{t{-}t^{k-1}_\tau}{\tau}\big(u^k_{\varepsilon\tau h},\pi^k_{\varepsilon\tau h},\eta^k_{\varepsilon\tau h}\big) + \frac{t^k_\tau{-}t}{\tau}\big(u^{k-1}_{\varepsilon\tau h},\pi^{k-1}_{\varepsilon\tau h},\eta^{k-1}_{\varepsilon\tau h}\big) \\ \text{for}\ \ t\in[t^{k-1}_\tau,t^k_\tau]\ \ \text{with}\ \ k=0,...,K_\tau:=T/\tau. \quad (5.4)$$

Besides, we define also the back-ward piecewise constant interpolant

$$\big[\bar u_{\varepsilon\tau h},\bar\pi_{\varepsilon\tau h},\bar\eta_{\varepsilon\tau h}\big](t) := \big(u^k_{\varepsilon\tau h},\pi^k_{\varepsilon\tau h},\eta^k_{\varepsilon\tau h}\big) \qquad \text{for } t\in(t^{k-1}_\tau,t^k_\tau],\ \ k=1,...,K_\tau. \quad (5.5)$$

For fixed $\varepsilon>0$, taking into account the uniqueness of the energetic solution $q_\varepsilon=(u_\varepsilon,\pi_\varepsilon,\eta_\varepsilon)$ as stated in Proposition 2.1, one obtains the convergence of the whole sequence $\tau\to0$ and $h\to0$, cf. [37, Sect.4.2] or also [23]. The strategy is now to merge this convergence with the limit $\varepsilon\to0$ as stated in Theorem 4.1. Clearly we have the successive convergence of the type

$$\lim_{\varepsilon\to0}\ \lim_{\substack{\tau\to0\\ h\to0}}\ q_{\varepsilon\tau h}=q. \quad (5.6)$$

By arguments as in [31] or [6, Cor.4.8(ii)], one can formulate this result by means of an implicit "stability criterion", involving functions $E_1,E_2:\mathbb{R}^+\to\mathbb{R}^+$, such that all cluster points in $\{(u_{\varepsilon\tau h},\pi_{\varepsilon\tau h},\eta_{\varepsilon\tau h})\}_{\varepsilon>0,\tau>0,h>0,\tau\le E_1(\varepsilon),h\le E_2(\varepsilon)}$ yield energetic solutions, i.e. one gets the conditional convergence of the type:

$$\lim_{\substack{\varepsilon\to0,\ \tau\to0,\ h\to0\\ \tau\le E_1(\varepsilon),\ h\le E_2(\varepsilon)}}\ q_{\varepsilon\tau h}=q. \quad (5.7)$$

By a careful inspection of the proof of Theorem 4.1 and a sophisticated construction of the mutual recovery sequence, we can improve (5.7) to the following unconditional convergence result.

Theorem 5.1 (Unconditional convergence of FEM-discretisation)**.** *Let the assumptions for Theorem 4.1 hold. Then every sequence* $\{(u_{\varepsilon\tau h},\pi_{\varepsilon\tau h},\eta_{\varepsilon\tau h})\}_{\varepsilon,\tau,h>0}$ *of the approximate solutions obtained by (5.1) contains a subsequence which converges for* $(\varepsilon,\tau,h)\to(0,0,0)$ *weakly* in the topologies indicated in (4.3a-c), i.e.*

$$\lim_{(\varepsilon,\tau,h)\to(0,0,0)}\bar u_{\varepsilon\tau h}=u \qquad \textit{weakly* in}\ \ {\rm L}^\infty(I;{\rm BD}(\Omega;\mathbb{R}^d)), \quad (5.8a)$$

$$\lim_{(\varepsilon,\tau,h)\to(0,0,0)}\bar\pi_{\varepsilon\tau h}=\pi \qquad \textit{weakly* in}\ \ {\rm BV}(\bar I;\mathscr{M}(\Omega{\cup}\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})), \quad (5.8b)$$

$$\lim_{(\varepsilon,\tau,h)\to(0,0,0)}e(u_{\varepsilon\tau h}){-}\pi_{\varepsilon\tau h}=e(u){-}\pi \quad \textit{weakly* in}\ \ {\rm L}^\infty(I;{\rm L}^2(\Omega;\mathbb{R}^{d\times d})). \quad (5.8c)$$

Any limit (u,π) *obtained by this way is an energetic solution in the sense of (1.2) to the Prandtl-Reuss model* $(Q_{\rm PR},\mathscr{E}_{\rm PR},\mathscr{R}_{\rm PR},u_0,\pi_0)$ *as defined in (3.2). Moreover, the whole sequence (not only selected subsequences) of stresses* $\{\mathbb{C}\big(e(u_{\varepsilon\tau h}){-}\pi_{\varepsilon\tau h}\big)\}_{\varepsilon,\tau,h>0}$

converges weakly in* $\mathrm{L}^\infty(I;\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}))$*, and even strongly pointwise, i.e. for any* $t\in[0,T]$*, it holds*

$$\lim_{(\varepsilon,\tau,h)\to(0,0,0)} \mathbb{C}(e(u_{\varepsilon\tau h}(t))-\pi_{\varepsilon\tau h}(t)) = \mathbb{C}(e(u(t))-\pi(t)|_\Omega) \quad \textit{strongly in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}). \tag{5.9}$$

Proof. We highlight only the differences to the proof of Theorem 4.1. It essentially concerns only Step 3, namely a construction and usage of a suitable mutual recovery sequence.

Let us abbreviate $\mathscr{E}_{\varepsilon h}(t,\cdot,\cdot,\cdot) := \mathscr{E}_\varepsilon(t,\cdot,\cdot,\cdot)+\delta_{V_{1,h}^d\times V_{0,h}^{d\times d}\times V_{0,h}}(\cdot,\cdot,\cdot)$. Again consider $(u,\pi)\in Q_{\mathrm{PR}}$ and a stable sequence $(u_{\varepsilon h},\pi_{\varepsilon h}) \overset{*}{\rightharpoonup} (u,\pi)$ in Q_{PR}, i.e.

$$u_{\varepsilon h} \overset{*}{\rightharpoonup} u \qquad \text{in } \mathrm{BD}(\Omega;\mathbb{R}^d), \tag{5.10a}$$

$$\pi_{\varepsilon h} \overset{*}{\rightharpoonup} \pi \qquad \text{in } \mathscr{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}), \tag{5.10b}$$

$$e(u_{\varepsilon h})-\pi_{\varepsilon h} \rightharpoonup e(u)-\pi \qquad \text{in } \mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{5.10c}$$

with $(u_{\varepsilon h},\pi_{\varepsilon h})\in V_{1,h}^d\times V_{0,h}^{d\times d}$ and an arbitrary $\eta_{\varepsilon h}\in V_{0,h}$. To facilitate also the time discretization, a floating time $t_\tau\to t$ has to be considered instead of the fixed time considered before in (4.11). Here stability of the sequence $\{(t_\tau,u_{\varepsilon h},\pi_{\varepsilon h},\eta_{\varepsilon h})\}_{\varepsilon,h>0}$ for the collection of functionals $\{(\mathscr{E}_{\varepsilon h},\mathscr{R})\}_{\varepsilon,h>0}$ means $\sup_{\varepsilon,h>0}\mathscr{E}_{\varepsilon h}(t_\varepsilon,u_{\varepsilon h},\pi_{\varepsilon h},\eta_{\varepsilon h})<\infty$ and $(u_{\varepsilon h},\pi_{\varepsilon h},\eta_{\varepsilon h})\in\mathscr{S}_{\varepsilon h}(t_\tau)$ with $\mathscr{S}_{\varepsilon h}$ from (2.7) but with $\mathscr{E}$ replaced by $\mathscr{E}_{\varepsilon h}$. In particular, $\|\pi_{\varepsilon h}\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(\varepsilon^{-1/2})$ and $\|\eta_{\varepsilon h}\|_{\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(\varepsilon^{-1/2})$.

We consider projectors $\Pi_{h,1}$ and $\Pi_{h,0}$ onto the P1- and P0-finite element spaces, respectively, which are assumed to satisfy

$$\Pi_{h,0}f\to f \text{ in } \mathrm{L}^2(\Omega) \quad\text{and}\quad \Pi_{h,1}g\to g \text{ in } \mathrm{H}^1(\Omega) \tag{5.11}$$

for all $f\in\mathrm{L}^2(\Omega)$ and all $g\in\mathrm{H}^1(\Omega)$. Again consider arbitrary $\hat u\in H^1(\Omega;\mathbb{R}^d)$ with $\hat u|_{\Gamma_{\mathrm{D}}}=0$ and $\hat\pi\in\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))$ as in (4.8). Then it is always possible to take $\hat\eta_h\in V_{0,h}$ such that $\delta_S^*(\Pi_{h,0}\hat\pi,\hat\eta_h)=\delta_P^*(\Pi_{h,0}\hat\pi)$. Here we used that S and P contain 0, are convex, and do not depend on x and hence the element-wise averaging involved in $\Pi_{h,0}$ allows for a point-wise construction on each element. Then, likewise in (4.9), put

$$\tilde u_{\varepsilon h}=u_{\varepsilon h}+\Pi_{h,1}\hat u\,,\qquad \tilde\pi_{\varepsilon h}=\pi_{\varepsilon h}+\Pi_{h,0}\hat\pi \quad\text{and}\quad \tilde\eta_{\varepsilon h}=\eta_{\varepsilon h}+\hat\eta_h. \tag{5.12}$$

Then one has an analog of (4.10):

$$\begin{aligned}\mathscr{R}(\tilde\pi_{\varepsilon h}-\pi_{\varepsilon h},\tilde\eta_{\varepsilon h}-\eta_{\varepsilon h}) &= \int_\Omega \delta_S^*(\tilde\pi_{\varepsilon h}-\pi_{\varepsilon h},\tilde\eta_{\varepsilon h}-\eta_{\varepsilon h})\,\mathrm{d}x = \int_\Omega \delta_S^*(\Pi_{h,0}(\tilde\pi-\pi),\tilde\eta_h)\,\mathrm{d}x\\ &= \int_\Omega \delta_S^*(\Pi_{h,0}\hat\pi,\hat\eta_h)\,\mathrm{d}x = \int_\Omega \delta_P^*(\Pi_{h,0}\hat\pi)\,\mathrm{d}x\\ &\to \int_\Omega \delta_P^*(\hat\pi)\,\mathrm{d}x = \mathscr{R}_{\mathrm{PR}}(\hat\pi),\end{aligned} \tag{5.13}$$

where the last convergence uses the $\mathrm{L}^2(\Omega)$ convergence assumed in (5.11).

Taking into account the quadratic structure of both $\mathscr{E}_{\varepsilon h}$ (if restricted on the finite-dimensional FE subspaces) and (5.13), we have

$$\begin{aligned}
&\mathscr{E}_{\varepsilon h}(t_\tau,\tilde u_{\varepsilon h},\tilde\pi_{\varepsilon h},\tilde\eta_{\varepsilon h})-\mathscr{E}_{\varepsilon h}(t_\tau,u_{\varepsilon h},\pi_{\varepsilon h},\eta_{\varepsilon h})\\
&=\int_\Omega\frac12\mathbb{C}\big(e(\tilde u_{\varepsilon h}{-}u_{\varepsilon h})-\tilde\pi_{\varepsilon h}{+}\pi_{\varepsilon h}\big){:}\big(e(\tilde u_{\varepsilon h}{+}u_{\varepsilon h})-\tilde\pi_{\varepsilon h}{-}\pi_{\varepsilon h}\big)\\
&\qquad+\frac{\varepsilon}{2}\mathbb{H}(\tilde\pi_{\varepsilon h}{-}\pi_{\varepsilon h}){:}(\tilde\pi_{\varepsilon h}{+}\pi_{\varepsilon h})+\frac{\varepsilon}{2}(\tilde\eta_{\varepsilon h}{-}\eta_{\varepsilon h})(\tilde\eta_{\varepsilon h}{+}\eta_{\varepsilon h})\,\mathrm{d}x\\
&\qquad-\big\langle f_{\rm ext}(t_\tau),(\tilde u_{\varepsilon h}{-}u_{\varepsilon h},\tilde\pi_{\varepsilon h}{-}\pi_{\varepsilon h})\big\rangle\\
&=\int_\Omega\frac12\mathbb{C}\big(e(\Pi_{h,1}\hat u)-\Pi_{h,0}\hat\pi\big){:}\big(e(2u_{\varepsilon h}{+}\Pi_{h,0}\hat u)-2\pi_{\varepsilon h}{-}\Pi_{h,0}\hat\pi\big)\\
&\qquad+\frac{\varepsilon}{2}\mathbb{H}\Pi_{h,0}\hat\pi{:}(2\pi_{\varepsilon h}{+}\Pi_{h,0}\hat\pi)+\frac{\varepsilon}{2}\hat\eta_h(\tilde\eta_{\varepsilon h}{+}\eta_{\varepsilon h})\,\mathrm{d}x-\big\langle f_{\rm ext}(t_\tau),(\Pi_{h,1}\hat u,\Pi_{h,0}\hat\pi)\big\rangle.
\end{aligned}\tag{5.14}$$

From this, by using the convergence (5.10), (5.13), and the quadratic structure of $\mathscr{E}_{\rm PR}$, one can see that (4.11) modifies as follows:

$$\begin{aligned}
&\lim_{(\varepsilon,\tau,h)\to(0,0,0)}\mathscr{E}_{\varepsilon h}(t_\tau,\tilde u_{\varepsilon h},\tilde\pi_{\varepsilon h},\tilde\eta_{\varepsilon h})-\mathscr{E}_{\varepsilon h}(t_\tau,u_{\varepsilon h},\pi_{\varepsilon h},\eta_{\varepsilon h})\\
&\quad=\int_\Omega\frac12\mathbb{C}\big(e(\hat u)-\hat\pi\big){:}\big(e(2u{+}\hat u)-2\pi{-}\hat\pi\big)\,\mathrm{d}x-\big\langle f_{\rm ext}(t),(\hat u,\hat\pi)\big\rangle\\
&\quad=\int_\Omega\frac12\mathbb{C}\big(e(\tilde u{-}u)-\tilde\pi{+}\pi\big){:}\big(e(\tilde u{+}u)-\tilde\pi{-}\pi\big)\,\mathrm{d}x-\big\langle f_{\rm ext}(t),(\tilde u{-}u,\tilde\pi{-}\pi)\big\rangle\\
&\quad=\mathscr{E}_{\rm PR}(t,\tilde u,\tilde\pi)-\mathscr{E}_{\rm PR}(t,u,\pi),
\end{aligned}\tag{5.15}$$

where we used (5.10c) and also the estimates analogous to (4.12). This allows for the limit passage in the stability condition for the discretised problem with hardening, i.e. in

$$\bar{\mathscr{E}}_{\varepsilon\tau h}(t,\bar u_{\varepsilon\tau h}(t),\bar\pi_{\varepsilon\tau h}(t),\bar\eta_{\varepsilon\tau h}(t))\le\bar{\mathscr{E}}_{\varepsilon\tau h}(t,\tilde u,\tilde\pi,\tilde\eta)+\mathscr{R}(\tilde\pi{-}\bar\pi_{\varepsilon\tau h}(t),\tilde\eta{-}\bar\eta_{\varepsilon\tau h}(t))\tag{5.16}$$

for any $t\in[0,T]$ and any $(\tilde u,\tilde\pi,\tilde\eta)\in V^d_{1,h}\times V^{d\times d}_{0,h}\times V_{0,h}$, where we used the notation $\bar{\mathscr{E}}_{\varepsilon\tau h}(t,q):=\mathscr{E}_{\varepsilon h}(t^k_\tau,q)$ for $t\in(t^{k-1}_\tau,t^k_\tau]$, cf. (5.3) and the definition (5.5). In fact, one is to use (5.15) with the fact that, for all $t\in[0,T]$, one has $\bar{\mathscr{E}}_{\varepsilon\tau h}(t,q)=\mathscr{E}_{\varepsilon h}(t_\tau,q)$ with $t_\tau:=\min\{t^k_\tau\ge t;\ k=0,...,T/\tau\}\to t$ for $\tau\to0$. Thus we can prove the weaker stability condition (4.13), apply (3.6) to obtain the full stability, and finish as in the proof of Theorem 4.1. □

6. Numerical experiments

We present computational experiments on 2D and 3D Lipschitz domains, so, theoretically, only Theorem 4.1(i-ii) but not (iii) is at our disposal. The two-dimensional setting allows us to use finer discretizations and thereby to consider smaller values of the hardening parameter ε. Throughout this section we employ an isotropic material with isotropic hardening and without kinematic hardening. More specifically, we consider the ansatz (4.2) with $\mathbb{C}e=\lambda(\mathrm{tr}\,e)\mathbb{I}+2\mu\,e$ and with $\mathbb{H}=0$. Having in mind materials like steel, we consider the Lamé constants $\lambda=79\,\mathrm{GPa}$ and $\mu=52.7\,\mathrm{GPa}$, which correspond to Young's modulus 137 GPa and Poisson's ratio 0.3. Moreover, we employ the hardening parameter $b=\mu$ in (4.2) and study different choices of ε. We further consider the dissipation determined by S in (4.19) with $P=\{\varsigma\in\mathbb{R}^{d\times d}_{\rm dev}\,|\,|\varsigma|\le\sigma_{\rm y}\}$ with $\sigma_{\rm y}:=450\mathrm{MPa}$ and $q_{\rm H}=1/\sigma_{\rm y}$. The coercivity condition (2.9) holds even if $\mathbb{H}=0$ because $|\pi|\to\infty$ makes here also $\eta\to-\infty$; more

specifically, $\eta \le -q_{\mathrm{H}}\sigma_{\mathrm{y}}|\pi|$ and thus the infimum in (2.9) is exactly $bq_{\mathrm{H}}^2\sigma_{\mathrm{y}}^2/(1+q_{\mathrm{H}}^2\sigma_{\mathrm{y}}^2)$. The nonlinear systems of equations arising in each time step were solved by a Newton method using ideas from [8].

6.1. **Two-dimensional experiment.** We consider a rectangular specimen with slightly generalized Dirichlet boundary conditions:

$$\Omega = (-5\tfrac{L}{2}, 5\tfrac{L}{2}) \times (-\tfrac{L}{2}, \tfrac{L}{2}), \tag{6.1a}$$

$$\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{D}}^{\mathrm{top}} \cup \Gamma_{\mathrm{D}}^{\mathrm{bottom}}, \qquad \Gamma_{\mathrm{D}}^{\mathrm{top}} = (-5\tfrac{L}{2}, \tfrac{L}{2}) \times \{\tfrac{L}{2}\}, \tag{6.1b}$$

$$\Gamma_{\mathrm{D}}^{\mathrm{bottom}} = (-\tfrac{L}{2}, 5\tfrac{L}{2}) \times \{-\tfrac{L}{2}\}, \tag{6.1c}$$

$$u_{\mathrm{D}}(t,x) = 0 \qquad \text{for } x \in \Gamma_{\mathrm{D}}^{\mathrm{bottom}}, \text{ and} \tag{6.1d}$$

$$(u_{\mathrm{D}})_2(t,x) = -t\,2\cdot 10^{-3}\mathrm{m/s} \qquad \text{for } x \in \Gamma_{\mathrm{D}}^{\mathrm{top}}, \tag{6.1e}$$

i.e., only the normal displacement on $\Gamma_{\mathrm{D}}^{\mathrm{top}}$ is prescribed (and gradually increasing in time). Allowing the top side gliding facilitates the development of a shear band better than if it would be fixed, although this requires a modification of the arguments presented in Sections 2-5; in particular, instead of $u \circledS \nu \,\mathrm{d}\mathscr{H}^{d-1} + \pi = 0$ in (3.2b), one should prescribe $u\cdot\nu\,\mathrm{d}\mathscr{H}^{d-1} + \nu\cdot\pi\cdot\nu = 0$ on this "gliding" boundary. Moreover, we set $T = 1\,\mathrm{s}$ but note that, in fact, this physical unit is only related to (6.1e), otherwise it is irrelevant in the considered rate-independent case.

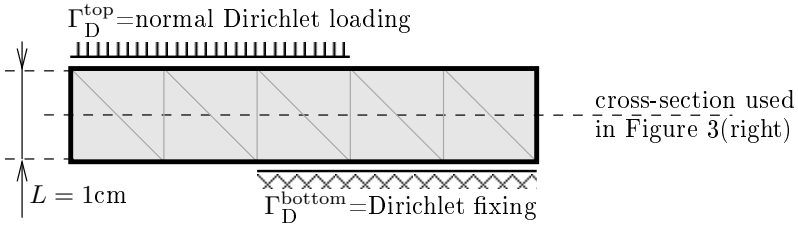

FIGURE 1. Geometry for the 2D example in Sect. 6.1

The employed triangulations are so-called red-refinements of mesh-width $h_\ell = 2^{-\ell}\cdot 10^{-2}\mathrm{m}$ of the coarse triangulation with $h_0 = L = 10^{-2}\mathrm{m}$ and 10 triangles displayed in Figure 1. We emphasize that the triangulations do not match the expected 45°-oriented slip band in this example. We always employ the time step

$$\tau = \tau_\ell = h_\ell/v_0, \quad \text{with} \quad v_0 = 0.04\,\mathrm{m/s}. \tag{6.2}$$

Snapshots of the discrete evolution with $\ell = 7$, i.e., a triangulation with $4^7\cdot 10 = 163\,840$ triangles, and $\varepsilon = 1/64$ for $t = 0.25\mathrm{s}$, $0.5\mathrm{s}$, $0.75\mathrm{s}$, and $1\mathrm{s}$ are shown in Figure 2, where we plotted the hardening variable $\eta_h(\cdot,t)$ together with the magnified discrete displacement field $u_h(\cdot,t)$. We observe that a plastic region develops which connects those points on the upper and lower boundary of Ω at which the type of the boundary conditions change.

To visualize the dependency of η_h on ε, we displayed for $\ell = 7$ and $\varepsilon = 4^{-m}$, $m = 0,1,2,3$, in the left and right plot of Figure 3 the functions

$$t \mapsto \|\eta_h(\cdot,t)\|_{\mathrm{L}^2(\Omega)}, \qquad x_K^{\mathrm{cs}} \mapsto -\eta_h\big(x_K^{\mathrm{cs}}, T\big), \tag{6.3}$$

where x_K^{cs} ranges over a neighbourhood of the cross-section indicated in Figure 1; more specifically, x_K^{cs} are those midpoints of elements whose distance from this cross-section is less than $10^{-4}\mathrm{m}$.

The entire functions $\eta_h(\cdot,T)$ at the end of the evolution are displayed for $\varepsilon = 4^{-m}$, $m = 0,1,2,3$ and $\ell = 7$ in Figure 4. We observe a diffuse plastic region for large hardening parameters εb and, on the other hand, a narrower region with sharper interfaces (= a slip band) for the smallest value of εb.

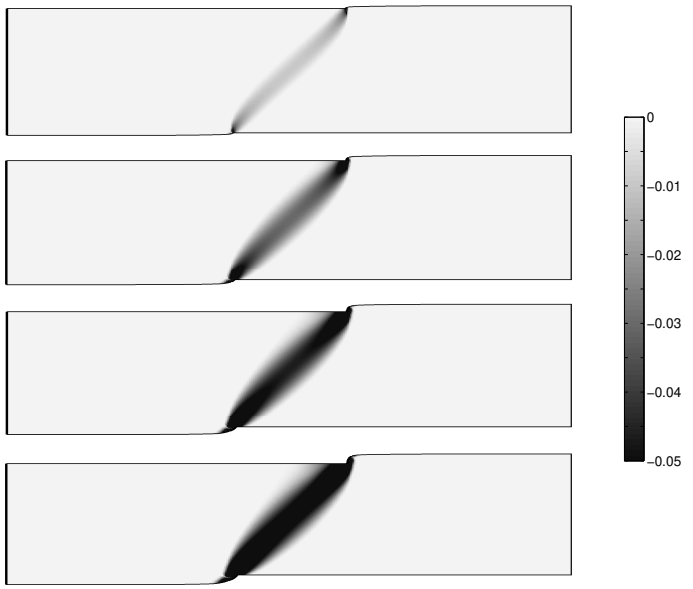

FIGURE 2. Snapshots of the time evolution of one slip-band depicted via the hardening variable η_h with $\varepsilon = 1/64$ in Example 6.1; $t = 0.25$s, 0.5s, 0.75s, and 1s. The displacement is magnified by a factor 4.

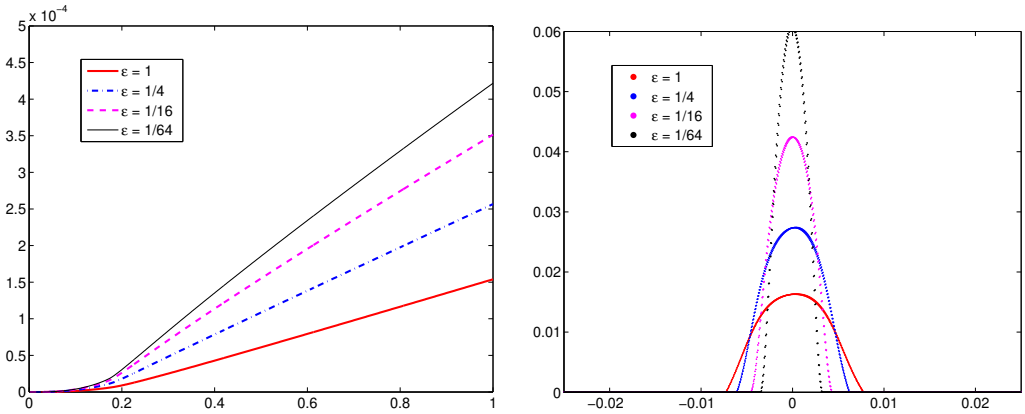

FIGURE 3. Illustration of development of a shear band when $\varepsilon \downarrow 0$: the $L^2(\Omega)$-norm of $\eta_h(\cdot, t)$ for time $0 \le t \le T = 1$ showing the blow-up tendency $\mathcal{O}(1/\sqrt{\varepsilon})$ predicted in (4.3f) (left) and a spatial profile of $-\eta_h(\cdot, T)$ in Example 6.1 along the middle cross-section indicated in Fig. 1 at the terminal time $t = T$ (right).

We experimentally studied the validity of the two-sided energy inequality discussed in Section 5. Figure 5 displays the discrete upper and lower bounds for discretizations defined through $\ell = 4, 5, 6, 7$ uniform refinements with corresponding time-step sizes and for fixed $\varepsilon = 1/16$. We see in the left plot of Figure 5 that the discrete upper bound dominates the lower bound for $\ell = 4$. This is predicted by the theoretical estimates (5.2) which assume an exact solution of the nonlinear systems of equations. Let us remark that Figure 5(left) refers to the problem with transformed Dirichlet condition and the displayed curves depend on the specific choice of u_D and do not represent the real physical power; more specifically, our choice of u_D was just to be the (approximate) elasto-plastic solution of the original problem, which makes the "transformed" power zero until the plasticification starts. To compare the upper and lower bound for different discretization parameters we scaled

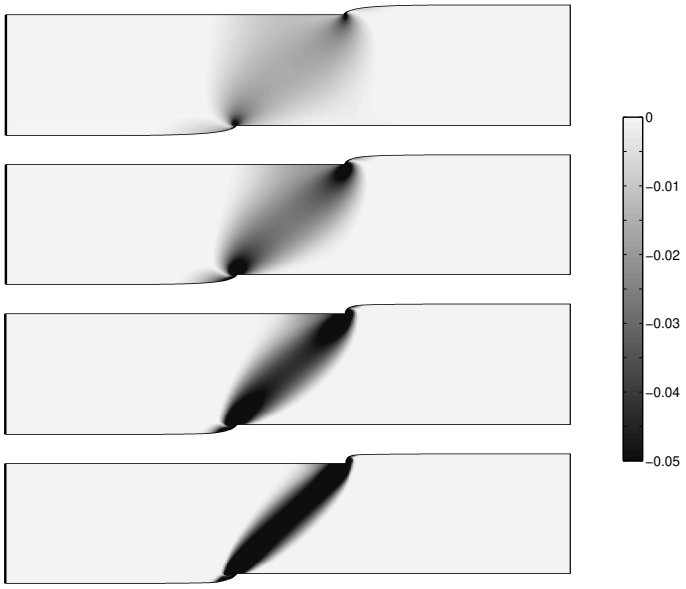

FIGURE 4. Displacement u_h and hardening variable η_h ($\sim$ |plastic strain|) for $t = T$ and decreasing hardening parameter $\varepsilon = 1$, $1/4$, $1/16$, and $1/64$, showing how a diagonal slip-band becomes more and more pronounced. The underlying triangulation has 163 840 triangles, the discrete displacement is displayed magnified by a factor 4.

them by $1/\tau$ and in the right plot of Figure 5 we plotted their difference with a logarithmic scale on the y-axis. We see that the difference decays linearly to 0 with the spatial and temporal discretization parameters.

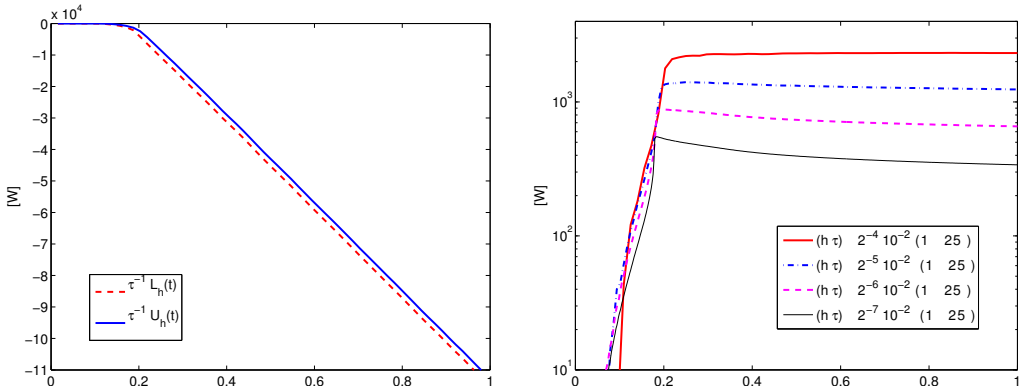

FIGURE 5. Rates (=power) of the upper and the lower bounds in (5.2) for $h = 2^{-4}$cm and $\tau \doteq 16 \cdot 10^{-3}$s (left) and difference between them in a semi-logarithmic scaling for different discretization parameters, indicating convergence like $\mathcal{O}(\tau)$ (right); $\varepsilon = 1/16$.

6.2. **Three-dimensional experiment.** Although we know that approximations converge unconditionally to the unique solution, the qualitative behaviour of the discrete solutions may depend on the employed triangulations in a preasymptotic range. In our three-dimensional experiment the underlying triangulations match the expected geometry of the slip band.

We let

$$\Omega = (-2L, 2L) \times (-\tfrac{L}{2}, \tfrac{L}{2})^2, \tag{6.4a}$$

$$\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{D}}^{\mathrm{top}} \cup \Gamma_{\mathrm{D}}^{\mathrm{bottom}}, \qquad \Gamma_{\mathrm{D}}^{\mathrm{top}} = (0, \tfrac{L}{2}) \times (-\tfrac{L}{2}, \tfrac{L}{2}) \times \{\tfrac{L}{2}\}, \tag{6.4b}$$

$$\Gamma_{\mathrm{D}}^{\mathrm{bottom}} = (-2L, 0) \times (-\tfrac{L}{2}, \tfrac{L}{2}) \times \{-\tfrac{L}{2}\}, \tag{6.4c}$$

$$u_{\mathrm{D}}(t,x) = 0 \qquad \text{for } x \in \Gamma_{\mathrm{D}}^{\mathrm{bottom}}, \text{ and} \tag{6.4d}$$

$$(u_{\mathrm{D}})_3(t,x) = -t\, 2 \cdot 10^{-3}\mathrm{m/s} \quad \text{for } x \in \Gamma_{\mathrm{D}}^{\mathrm{top}}, \tag{6.4e}$$

for $L = 1 \cdot 10^{-2}$m, i.e., again only the normal displacement is prescribed on the top part of Γ_{D}, and $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}}$

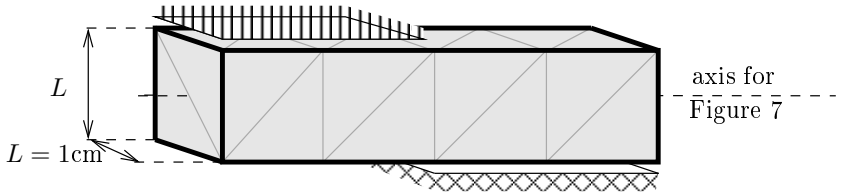

FIGURE 6. Geometry for the 3D example in Sect. 6.2

The employed triangulations are red-refinements of mesh-width $h_\ell = 2^{-\ell} \cdot 10^{-2}$m with $8^\ell \cdot 24$ tetrahedra of the coarse triangulation with $h_0 = 10^{-2}$m displayed in Figure 6. We always used the time-step size $\tau = 0.025 \cdot 2^{-\ell}$s.

For $\ell = 4$ and $\varepsilon = 4^{-m}$, $m = 0, 1, 2$, we displayed in the left and right plot of Figure 7 the functions

$$t \mapsto \|\eta_h(\cdot, t)\|_{\mathrm{L}^2(\Omega)}, \qquad x_K^{\mathrm{ax}} \mapsto -\eta_h\big(x_K^{\mathrm{ax}}, T\big),$$

where, like in (6.3), x_K^{ax} ranges over a neighbourhood of the horizontal axis in Figure 6; more specifically, x_K^{ax} are those midpoints of elements whose distance from this axis is less than 10^{-4}m.

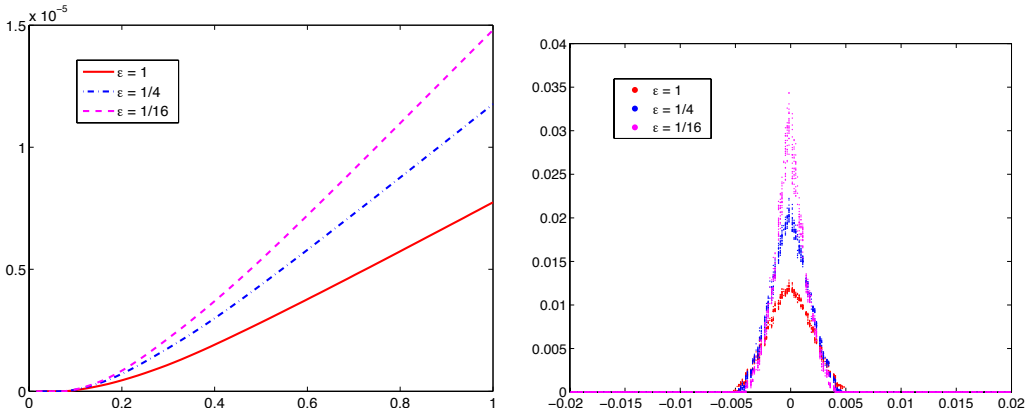

FIGURE 7. The $\mathrm{L}^2(\Omega)$-norm of $\eta_h(\cdot, t)$ for time $0 \le t \le T = 1$ (left) and a spatial profile of $\eta_h(T, \cdot)$ along the horizontal axis of Ω in Example 6.2

The functions $\eta_h(T, \cdot)$ are displayed for $\ell = 7$ and $\varepsilon = 4^{-m}$, $m = 0, 1, 2, 3$ in Figure 8. We observe that for ε tending to zero, the developed plastic region becomes narrower and the interfaces less diffuse.

Remark 6.1 (Regularity, rate of convergence)**.** If Ω is smooth and the loading has additional regularity in time, it has been shown in [29] that the energetic solution

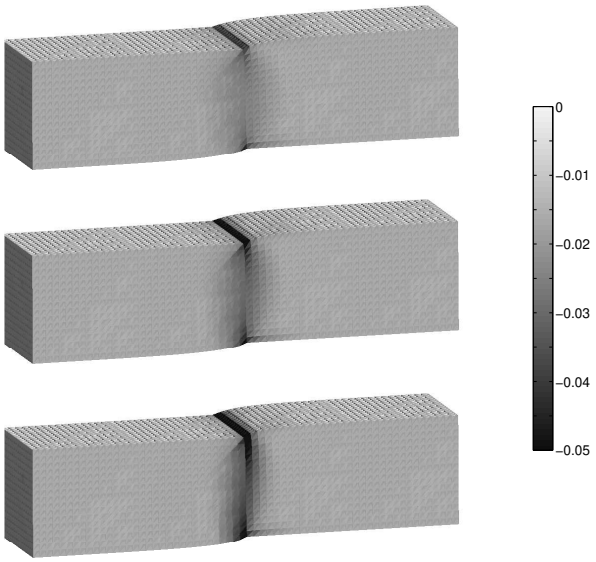

FIGURE 8. Displacement u_h and hardening variable η_h ($\sim$ |plastic strain|) for $t = T$ and $\varepsilon = 1$, $1/4$, and $1/16$, showing how a vertical slip-band becomes more and more pronounced. The underlying triangulation has 98 304 tetrahedra, the displacement is magnified by a factor 4.

$(u_\varepsilon, z_\varepsilon)$ admits additional regularity properties, namely

$$u_\varepsilon \in \mathrm{L}^\infty(I;\mathrm{W}^{3/2-\alpha}(\Omega;\mathbb{R}^d)), \qquad z \in \mathrm{L}^\infty(I;\mathrm{W}^{1/2-\alpha}(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R})), \quad \text{for some } \alpha>0. \tag{6.5}$$

This can further be used, cf. [30], to derive the rate of convergence:

$$\big\|u_\varepsilon - \bar u_{\varepsilon\tau h}\big\|_{\mathrm{L}^\infty(I;\mathrm{H}^1(\Omega;\mathbb{R}^d))} + \big\|z_\varepsilon - \bar z_{\varepsilon\tau h}\big\|_{\mathrm{L}^\infty(I;\mathrm{L}^2(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R}))} \le C(\alpha,\varepsilon)\big(\sqrt{\tau} + \sqrt[4-\alpha]{h}\big). \tag{6.6}$$

The constant $C(\alpha,\varepsilon)$ naturally depends on the positive-definiteness constant ε of $\mathscr{E}(t,\cdot,\cdot)$ and one can identify from [30] the explicit upper bound in the form $C(\alpha,\varepsilon) = C_\alpha\varepsilon^{-2}\mathrm{e}^{c/\varepsilon}$ with some $c, C \in \mathbb{R}$. Obviously, $C(\alpha,\varepsilon)$ grows essentially like $\mathrm{e}^{c/\varepsilon}$. In smooth cases, one thus gets an explicit hint how to converge simultaneously with hardening and discretisation, namely

$$\tau = o\big(\mathrm{e}^{-2c/\varepsilon}\big), \qquad h = o\big(\mathrm{e}^{-(4-\alpha)c/\varepsilon}\big). \tag{6.7}$$

This extreme upper bound cannot be used practically, but it has the advantage that it is mathematically rigorous. Nevertheless we hope that it is too pessimistic and that it will be possible to improve the bounds considerably, at least in particular cases.

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