Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

A nonlocal quasilinear multi-phase system with nonconstant specific heat and heat conductivity

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submitted: December 16, 2010

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No. 1580
Berlin 2010

2010 *Mathematics Subject Classification.* 35K51, 35K59, 35K65, 45K05, 80A22.

Key words and phrases. Phase transitions, nonlocal models, quasilinear integrodifferential vectorial equation.

The authors gratefully acknowledge financial support by the MIUR-PRIN Grant 2008ZKHAHN “Phase transitions, hysteresis and multiscaling”, the GAČR Grant P201/10/2315, the program SMART-MATH of CNR/AVČR, the MIUR-PRIN Grant 20089PWTPS “Mathematical Analysis for inverse problems towards applications”, the DFG Research Center MATHEON in Berlin, and the IMATI of CNR in Pavia.

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Abstract

In this paper, we prove the existence and global boundedness from above for a solution to an integrodifferential model for nonisothermal multi-phase transitions under nonhomogeneous third type boundary conditions. The system couples a quasilinear internal energy balance ruling the evolution of the absolute temperature with a vectorial integro-differential inclusion governing the (vectorial) phase-parameter dynamics. The specific heat and the heat conductivity k are allowed to depend both on the order parameter χ and on the absolute temperature θ of the system, and the convex component of the free energy may or may not be singular. Uniqueness and continuous data dependence are also proved under additional assumptions.

1 Introduction

In this paper we consider a nonisothermal multi-phase transition process occurring in a bounded container $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, with Lipschitzian boundary $\partial\Omega$. The state variables describing the evolution of the system are the absolute temperature $\theta > 0$ and the vectorial order parameter $\chi \in \mathbb{R}^d$, $d \in \mathbb{N}$. Following the idea that was already described in the pioneering papers [27] and [5], but which has been only recently analyzed in a more systematic way (cf., e. g., [1]–[2], [6]–[7], [9]–[22], [25]), we take into account long range interactions between particles. Then the model equations resulting from the energy and entropy balance relations have the form

$$(e(\theta,\chi))_t + (\lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \text{div}\,(k(\theta,\chi)\nabla\theta) = 0 \quad \text{in } Q_\infty := \Omega \times (0,+\infty), \tag{1.1}$$

$$\mu(\theta)\chi_t + \lambda'(\chi) + b[\chi] + (\beta+\theta)\partial\varphi(\chi) + \theta\sigma'(\chi) + e_\chi(\theta,\chi) - \theta s_\chi(\theta,\chi) \ni 0 \quad \text{in } Q_\infty, \tag{1.2}$$

$$k(\theta,\chi)\nabla\theta\cdot\mathbf{n} + \gamma(\theta - \theta_\Gamma) = 0 \quad \text{on } \Sigma_\infty := \partial\Omega \times (0,\infty), \tag{1.3}$$

$$\theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega, \tag{1.4}$$

where $\mathbf{n}$ denotes the outward normal vector to $\partial\Omega$, and (1.2) has to be understood as an inclusion in $\mathbb{R}^d$, where $\partial\varphi$ is a possibly multivalued subdifferential of a general proper, convex, and lower semicontinuous function $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. The physical meaning of the functions e, s, λ, σ and of the positive constant β is explained in (1.16–1.19), while $b[\chi]$ (whose explicit form will be given) represents the nonlocal operator acting on χ. With an abuse of notation we have used the symbols λ', σ', e_χ, s_χ for gradient vectors in $\mathbb{R}^d$ and omitted the scalar product symbol between $\mathbb{R}^d$ vectors (like $b[\chi]$ and χ_t in (1.1)) in order not to overburden the presentation. The function μ in (1.2) represents the (bounded away from 0) mobility of the system, while γ denotes the heat transfer coefficient through the boundary $\partial\Omega$. The external temperature θ_Γ is a sufficiently regular boundary datum on Σ_∞, and θ_0, χ_0 are supposed to be two given initial configurations.

The main novelty here is to consider a multi-phase nonlocal phase field system in the case when the specific heat $c_V(\theta,\chi) = \partial_\theta e(\theta,\chi)$ and the heat conductivity $k(\theta,\chi)$ are not constant and depend on both the variables θ and χ. Suitable regularity and growth conditions will be specified in the following section.

Let us only note that many typical expressions for c_V in a two-phase system (i.e. in case $d=1$) can be included in our analysis. In the solid-liquid system mentioned above, for example, we may have different values $c_V^0(\theta)$ in the solid and $c_V^1(\theta)$ in the liquid phase, hence, we may define $c_V(\theta,\chi) = c_V^0(\theta) + \chi(c_V^1(\theta) - c_V^0(\theta))$ (cf. [26, Section IV.4]). The value of χ can be kept between 0 and 1 by setting $\varphi = I_{[0,1]}$ (the indicator function of $[0,1]$). The physically meaningful case in which the behaviour of c_V^0 and c_V^1 are powers of θ ($\sim \theta^\alpha$, $\alpha \geq 1$) near zero and bounded functions for large θ's can be covered by our analysis. Regarding the heat conductivity k, typical expressions of the type $k(\theta,\chi) = K_1(\theta)\chi + K_2(\theta)(1-\chi)$, in case of a two-phase system with $\chi \in [0,1]$, for quite general functions K_1 and K_2, are also allowed here.

The main goal of this paper is to study the global existence of solutions to system (1.1–1.4), coupling a suitable variational formulation of the semilinear parabolic partial differential equation (1.1) for θ to the integrodifferential inclusion (1.2) for χ. We also prove some uniform in time upper bound for the absolute temperature of the system (see Theorem 2.2 below). Uniqueness of solutions is obtained under additional assumptions, in particular, in case that the heat conductivity k in (1.1) depends only on θ and not on χ.

Before entering into the mathematical discussion of the problem, let us give a brief derivation of the system (1.1–1.4), emphasizing, in particular, the differences between local and nonlocal models.

We assume here that the multi-phase transition process can be completely described by the evolution of the state variables $\theta(x,t) > 0$, which represents the absolute temperature of the system, and the order parameter $\chi(x,t)$, which here is a vector in $\mathbb{R}^d$. We fix some constant reference temperature θ_c, which will be assumed to be equal to 1, for simplicity.

Inspired by the nonlocal Cahn-Hilliard model studied by Gajewski in [10], we consider the following nonlocal specific free energy

$$F[\theta,\chi] = f_0(\theta,\chi) + B[\chi],$$

where B is a potential that accounts for long range interaction between particles. More specifically, given a bounded, symmetric kernel $\kappa : \Omega \times \Omega \to \mathbb{R}$ and an even smooth function $G : \mathbb{R}^d \to \mathbb{R}$, we choose

$$B[\chi](x,t) := \int_\Omega \kappa(x,y)\, G(\chi(x,t) - \chi(y,t))\, \mathrm{d}y. \tag{1.5}$$

Note that the local potential $(\nu/2)|\nabla\chi|^2$ used often in the literature, see [26] and the references therein, can be obtained as a formal limit as $n \to \infty$ from the nonlocal one with the choice $G(\eta) = |\eta|^2/2,\ \kappa(x,y) = n^{N+2}\widetilde{\kappa}(|n(x-y)|^2)$, where $\widetilde{\kappa}$ is a nonnegative function with support in $[0,1]$ and $\nu = 1/N \int_{\mathbb{R}^N} \widetilde{\kappa}(|z|^2)|z|^2 \, \mathrm{d}z$. This follows from the formula

$$\int_\Omega n^{N+2}\widetilde{\kappa}(|n(x-y)|^2)\,|\chi(x)-\chi(y)|^2 \,\mathrm{d}y = \int_{\Omega_n(x)} \widetilde{\kappa}(|z|^2) \left| \frac{\chi\left(x+\frac{z}{n}\right)-\chi(x)}{\frac{1}{n}} \right|^2 \,\mathrm{d}z$$
$$\overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} \widetilde{\kappa}(|z|^2)\,\langle \nabla\chi(x), z\rangle^2 \,\mathrm{d}z = \nu|\nabla\chi(x)|^2$$

for a sufficiently regular χ, where we denote $\Omega_n(x) = n(\Omega - x)$. We have used the identity $\int_{\mathbb{R}^N} \widetilde{\kappa}(|z|^2)\,\langle v, z\rangle^2 \,\mathrm{d}z = 1/N \int_{\mathbb{R}^N} \widetilde{\kappa}(|z|^2)|z|^2 \,\mathrm{d}z$ for every unit vector $v \in \mathbb{R}^N$ (cf. the Introduction of [20] for further details on this topic).

Let E and S be the total energy and entropy densities, respectively. The process is governed by the internal energy and entropy balance relations over an arbitrary control volume $\Omega' \subset \Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega'} E(\theta,\chi)\,\mathrm{d}x + \int_{\partial\Omega'} \langle \mathbf{q}, \mathbf{n}\rangle \,\mathrm{d}A = \Psi(\Omega')\,, \tag{1.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega'} S(\theta,\chi)\,\mathrm{d}x + \int_{\partial\Omega'} \left\langle \frac{\mathbf{q}}{\theta}, \mathbf{n}\right\rangle \,\mathrm{d}A \geq 0\,, \tag{1.7}$$

where $\mathbf{q}$ is the heat flux vector, $\mathbf{n}$ is the unit outward normal to $\partial\Omega'$, and $\Psi(\Omega')$ is the energy exchange through the boundary of Ω' due to the nonlocal interactions. Since $B[\chi]$ is a potential field, it does not contribute to the entropy production in the Clausius-Duhem inequality (1.7).

The local form of the entropy balance reads

$$\theta S_t(\theta,\chi) + \operatorname{div}\mathbf{q} - \frac{\langle \mathbf{q}, \nabla\theta\rangle}{\theta} \geq 0,$$

and it has to be understood in the regularity context of Theorem 2.2 below. This is certainly satisfied if

$$\begin{aligned} &\langle \mathbf{q}, \nabla\theta\rangle \leq 0\,,\\ &\theta S_t(\theta,\chi) + \operatorname{div}\mathbf{q} \geq 0\,. \end{aligned}$$

Assuming $\theta > 0$ and a suitable regularity with respect to time (this will have to be justified in the next sections), we obtain from (1.6) that

$$\int_{\Omega'} (E_t - \theta S_t)\,\mathrm{d}x \leq \Psi(\Omega')\,. \tag{1.8}$$

Differentiating the identities $F = E - \theta S = f_0 + B[\chi]$ with respect to t, we obtain

$$F_t = E_t - \theta S_t - \theta_t S = \partial_\theta f_0 \theta_t + \partial_\chi f_0 \chi_t + B[\chi]_t\,, \tag{1.9}$$

where $\partial_\chi f_0$ stands for an element of Clarke's partial subdifferential of f_0 with respect to $\chi \in \mathbb{R}^d$, and $\partial_\theta f_0$ is the partial derivative of f_0 with respect to $\theta \in \mathbb{R}$. Consequently,

$$S = -\partial_\theta f_0 = s_0\,, \quad E = e_0 + B[\chi]\,, \quad f_0 = e_0 - \theta s_0\,, \tag{1.10}$$

and inequality (1.8) reads

$$\int_{\Omega'} \left(\partial_\chi f_0 \chi_t + B[\chi]_t\right)\, \mathrm{d}x \le \Psi(\Omega')\,. \tag{1.11}$$

The nonlocal interaction takes place only inside the domain Ω, hence $\Psi(\Omega) = 0$. A canonical way to satisfy these conditions independently of the evolution of χ consists in choosing the order parameter dynamics in the form

$$\mu(\theta)\chi_t \in -D_\chi \mathcal{F}[\theta, \chi] \tag{1.12}$$

with a factor $\mu(\theta) > 0$, where we denote

$$\mathcal{F}[\theta, \chi] \;=\; \int_\Omega F[\theta, \chi]\, \mathrm{d}x$$

and $D_\chi \mathcal{F}$ stands for the Clarke subdifferential of $\mathcal{F}$ with respect to the variable $\chi \in L^2(\Omega; \mathbb{R}^d)$. The inclusion sign in (1.12) accounts for the fact that $f_0(\theta, \chi)$ includes terms that are possibly not Fréchet differentiable. Condition (1.12) is based on the assumption that the system tends to move towards local minima of the free energy with a speed proportional to $1/\mu(\theta)$. Denoting

$$b[\chi](x,t) \;:=\; 2 \int_\Omega \kappa(x,y)\, G'\left(\chi(x,t) - \chi(y,t)\right)\, \mathrm{d}y\,, \tag{1.13}$$

where again, with an abuse of notation, G' stands for the d-component vector ∇G, we see that the inequality (1.11) holds without prescribing any relationship between $\mu(\theta)$ and $B[\chi]$, provided that we choose $\Psi(\Omega')$ in (1.6) as

$$\Psi(\Omega') = \int_{\Omega'} \left(-b[\chi]\chi_t + B[\chi]_t\right)\, \mathrm{d}x\,. \tag{1.14}$$

The differential form of the energy balance (1.6) then reads

$$E_t + \operatorname{div} \mathbf{q} \;=\; -b[\chi]\chi_t + B[\chi]_t\,. \tag{1.15}$$

The specific heat $c_V(\theta, \chi)$ is the only thermodynamic state function, which can be identified from the measurements, while the local internal energy and entropy densities are computed from the formulas

$$e_0(\theta,\chi) = e_0(0,\chi) + e(\theta,\chi), \quad e(\theta,\chi) = \int_0^\theta c_V(\tau,\chi)\, \mathrm{d}\tau\,, \tag{1.16}$$

$$s_0(\theta,\chi) = s_0(0,\chi) + s(\theta,\chi), \quad s(\theta,\chi) = \int_0^\theta \frac{c_V(\tau,\chi)}{\tau}\, \mathrm{d}\tau\,, \tag{1.17}$$

where $e_0(0,\chi), s_0(0,\chi)$ are in fact "integration constants", which we choose as

$$e_0(0,\chi) = \lambda(\chi) + \beta\varphi(\chi), \tag{1.18}$$

$$s_0(0,\chi) = -\sigma(\chi) - \varphi(\chi)\,, \tag{1.19}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous, the functions λ and σ are sufficiently regular on $\mathcal{D}(\varphi)$, and the parameter β is a positive constant.

Then, the free energy functional F has the form

$$F[\theta,\chi] \;=\; f(\theta,\chi) + \lambda(\chi) + B[\chi] + (\beta+\theta)\varphi(\chi) + \theta\sigma(\chi)\,, \tag{1.20}$$

where $f(\theta,\chi) = e(\theta,\chi) - \theta s(\theta,\chi)$.

Using (1.20), we rewrite the phase dynamics (1.12) as

$$\mu(\theta)\chi_t + \lambda'(\chi) + b[\chi] + (\beta+\theta)\partial\varphi(\chi) + \theta\sigma'(\chi) + e_\chi(\theta,\chi) - \theta s_\chi(\theta,\chi) \ni 0\,, \tag{1.21}$$

while the internal energy balance (1.15) can be reformulated as

$$(e(\theta,\chi))_t + (\lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \mathrm{div}\,(k(\theta,\chi)\nabla\theta) = 0. \tag{1.22}$$

We now show that in the energy conserved case (that is, if we assume no-flux boundary conditions ($\gamma = 0$ in (1.3))), the model is compatible with the Öttinger-Grmela GENERIC formalism [17]. Set

$$\mathcal{E}[\theta,\chi](t) \;=\; \int_\Omega E(\theta,\chi)(x,t)\,\mathrm{d}x\,, \tag{1.23}$$

$$\mathcal{S}[\theta,\chi](t) \;=\; \int_\Omega S(\theta,\chi)(x,t)\,\mathrm{d}x\,, \tag{1.24}$$

$$\mathcal{B}[\chi](t) \;=\; \int_\Omega B[\chi](x,t)\,\mathrm{d}x\,. \tag{1.25}$$

We show that there exists a symmetric positive semidefinite matrix $\mathbf{M}[\theta,\chi]$ such that

$$\mathbf{M}[\theta,\chi]\begin{pmatrix} D_\theta\mathcal{E}[\theta,\chi] \\ D_\chi\mathcal{E}[\theta,\chi]\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \tag{1.26}$$

and such that the system (1.21–1.22) has the form

$$\frac{\partial}{\partial t}\begin{pmatrix} \theta \\ \chi \end{pmatrix} = \mathbf{M}[\theta,\chi]\begin{pmatrix} D_\theta\mathcal{S}[\theta,\chi] \\ D_\chi\mathcal{S}[\theta,\chi]\end{pmatrix}. \tag{1.27}$$

It suffices to choose (we omit the arguments of the state functions for simplicity)

$$\mathbf{M}[\theta,\chi] = \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}, \tag{1.28}$$

where M_0 is the differential operator

$$M_0[y] = -\frac{1}{c_V}\mathrm{div}\left(\theta^2 k(\theta,\chi)\nabla\frac{y}{c_V}\right)\,, \tag{1.29}$$

with homogeneous Neumann boundary condition, and m_{ij} are scalars given by the formulas

$$m_{11} = \frac{\theta}{\mu(\theta)\, c_V^2}(D_\chi \mathcal{E})^2\,, \tag{1.30}$$

$$m_{12} = -\frac{\theta}{\mu(\theta)\, c_V} D_\chi \mathcal{E}\,, \tag{1.31}$$

$$m_{22} = \frac{\theta}{\mu(\theta)}\,. \tag{1.32}$$

Note that $m_{12}^2 = m_{11} m_{22}$ and $m_{11} \geq 0, m_{22} \geq 0$; hence, $\mathbf{M}[\theta, \chi]$ is positive semidefinite. Furthermore, we have

$$\begin{pmatrix} D_\theta \mathcal{E} \\ D_\chi \mathcal{E} \end{pmatrix} = \begin{pmatrix} c_V \\ D_\chi \mathcal{E} \end{pmatrix} = \begin{pmatrix} c_V \\ \frac{\partial}{\partial \chi} e_0 + D_\chi \mathcal{B} \end{pmatrix}, \quad \begin{pmatrix} D_\theta \mathcal{S} \\ D_\chi \mathcal{S} \end{pmatrix} = \begin{pmatrix} c_V/\theta \\ \frac{\partial S}{\partial \chi} \end{pmatrix}. \tag{1.33}$$

We easily check that (1.26) holds, and (1.27) has the form

$$\begin{pmatrix} \theta_t \\ \chi_t \end{pmatrix} = \begin{pmatrix} \mathrm{div}\,(k(\theta, \chi)\nabla\theta)/c_V \\ 0 \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} c_V/\theta \\ \frac{\partial S}{\partial \chi} \end{pmatrix}. \tag{1.34}$$

In component form, we have

$$\theta_t = \frac{1}{c_V}\left(\mathrm{div}\,(k(\theta, \chi)\nabla\theta) + \frac{1}{\mu(\theta)} D_\chi \mathcal{E}\, D_\chi \mathcal{F}\right), \tag{1.35}$$

$$\chi_t = -\frac{1}{\mu(\theta)}\, D_\chi \mathcal{F}\,. \tag{1.36}$$

To see that (1.35)–(1.36) coincides with (1.21–1.22), it suffices to take into account the formula

$$\frac{\partial}{\partial t} e_0 = c_V\, \theta_t + \frac{\partial}{\partial \chi} e_0\, \chi_t = \mathrm{div}\,(k(\theta, \chi)\nabla\theta) + \frac{1}{\mu(\theta)} D_\chi \mathcal{B}\, D_\chi \mathcal{F} = \mathrm{div}\,(k(\theta, \chi)\nabla\theta) - \chi_t\, D_\chi \mathcal{B}\,.$$

This proves that the model is compatible both with the standard principles of thermodynamics and the generalized thermodynamic formalism introduced in [17].

We prove an existence result for a suitable variational formulation of system (1.1–1.4). Using a Moser technique, we also show that the temperature variable θ is globally bounded from above. The uniqueness result holds true for particular classes of potentials φ provided that the heat conductivity k in (1.1) does not depend on χ and c_V and μ satisfy a suitable growth condition around 0.

The paper is organized as follows. In Section 2, we state our assumptions on the data and our main results; in particular, global existence for a suitable variational formulation of (1.1–1.4). In Section 3, we prove some auxiliary results related to the Lipschitz continuity of solution operators to differential inclusions. The proof will be developed as follows: the problem is approximated by partial time discretization, regularization and cut-off procedure (cf. Subsection 4.1). Suitable a priori estimates (cf. Subsection 4.2) allow us to pass to the limit with respect to the time step and regularization parameters, while the cut-off is removed by proving an upper bound on the absolute temperature (which is independent of the truncation parameter) by means of Moser techniques (cf. Subsection 4.3). The uniqueness result is proved in Section 5.

2 Main results

In this section, we state our main results on solvability conditions for the system (1.1–1.4). We start by introducing a suitable variational formulation; to this end, we consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and for $t \in (0, \infty]$ we denote by $Q_t = \Omega \times (0, t)$ the open space-time cylinder and by Σ_t its lateral boundary $\partial\Omega \times (0, t)$. We use, for the sake of simplicity, the same symbol H for both $L^2(\Omega)$ and $L^2(\Omega\,;\,\mathbb{R}^N)$, while for arbitrary integer d, $\mathbf{H}$ denotes the space $L^2(\Omega; \mathbb{R}^d)$. H and $\mathbf{H}$ are both endowed with the standard scalar product which we denote by $(\cdot, \cdot)$. The symbol V stands for the space $H^1(\Omega)$, and V' for its dual space, while the symbol $\mathbf{V}$ denotes the space $H^1(\Omega; \mathbb{R}^d)$, $\langle \cdot, \cdot \rangle$ being the duality $V' - V$ and $\mathbf{V}' - \mathbf{V}$. Then, the following dense and continuous embeddings, where we identify H (and $\mathbf{H}$) with its dual space H' (and $\mathbf{H}'$), hold true: $V \hookrightarrow H \equiv H' \hookrightarrow V'$, and $\mathbf{V} \hookrightarrow \mathbf{H} \equiv \mathbf{H}' \hookrightarrow \mathbf{V}'$. Finally, we rewrite the system (1.1–1.4) in the following variational formulation:

$$\langle \partial_t(e(\theta, \chi)), z \rangle + \int_\Omega k(\theta, \chi) \nabla\theta \cdot \nabla z \,\mathsf{d}x + \int_{\partial\Omega} \gamma(\theta - \theta_\Gamma)\, z \,\mathsf{d}A \tag{2.1}$$

$$= - \int_\Omega \left(\lambda'(\chi)\partial_t\chi + \beta\,(\varphi(\chi))_t + b[\chi]\chi_t\right)\, z \,\mathsf{d}x \quad \forall z \in V, \quad \text{a.e. in } (0, \infty)\,,$$

$$\mu(\theta)\chi_t + \lambda'(\chi) + \theta\sigma'(\chi) + (\beta + \theta)\partial\varphi(\chi) + b[\chi] \tag{2.2}$$

$$+\, e_\chi(\theta, \chi) - \theta s_\chi(\theta, \chi) \ni 0 \quad \text{a.e. in } Q_\infty\,,$$

where (2.2) has to be understood as an inclusion in $\mathbb{R}^d$ with $b[\chi]$ defined by (1.13), and e and s are defined in (1.16–1.17). Letting (cf. (1.4)) $u_0 := e(\theta_0, \chi_0)$, we prescribe the initial conditions

$$e(\theta, \chi)(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega\,, \tag{2.3}$$

and suppose that the data fulfil the following assumptions.

Hypothesis 2.1. (Existence) *Let us fix positive constants* C_σ, C_λ, k_0, k_1, $\underline{c}$, $\bar{c}$, c_1, β, C_0, *and assume that*

(i) $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *is a proper, convex, and lower semicontinuous function,* $\mathcal{D}(\varphi)$ *is its domain;*

(ii) $\sigma,\, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi))$, $|\sigma'(r)| \leq C_\sigma$, $|\lambda'(r)| \leq C_\lambda$ *for all* $r \in \mathcal{D}(\varphi)$*;*

(iii) $\kappa \in W^{1,\infty}(\Omega \times \Omega)$, $\kappa(x, y) = \kappa(y, x)$ *a. e. in* $\Omega \times \Omega$, $G \in W^{2,\infty}(\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $G(z) = G(-z)$ *for all* $z \in (\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$ *;*

(iv) $k\ :\ \mathbb{R} \times \mathcal{D}(\varphi) \to (0 + \infty)$ *is a locally Lipschitz continuous function such that* $0 < k_0 \leq k(v, w) \leq k_1$ *for all* $v \in \mathbb{R}$ *and* $w \in \mathcal{D}(\varphi)$*;*

(v) *The function* μ *maps* $[0, \infty)$ *in* $(0, \infty)$ *and the function* $\theta \mapsto \frac{1+\theta}{\mu(\theta)}$ *is bounded and Lipschitz continuous on* $[0, \infty)$ *with Lipschitz constant* L_μ*;*

(vi) $c_V : [0,+\infty) \times \mathcal{D}(\varphi) \to [0,+\infty)$ *is a continuous function satisfying*

$$c_V(0,\chi) = 0, \quad 0 < c_V(\theta,\chi) \le \bar{c} \quad \forall \theta \in (0,+\infty), \quad \forall \chi \in \mathcal{D}(\varphi); \tag{2.4}$$
$$\underline{c} \le c_V(\theta,\chi) \quad \forall (\theta,\chi) \in [1,+\infty) \times \mathcal{D}(\varphi); \tag{2.5}$$
$$\text{the function } \theta \mapsto \frac{c_V(\theta,\chi)}{\theta} \quad \text{is integrable in } (0,1) \text{ for all } \chi \in \mathcal{D}(\varphi). \tag{2.6}$$

Moreover, for all $(\theta,\chi) \in [0,+\infty) \times \mathcal{D}(\varphi)$ *there exists the gradient* $(c_V)_\chi(\theta,\chi)$*, and it holds*

$$|(c_V)_\chi(\theta,\chi)| \le c_1 c_V(\theta,\chi), \quad |(c_V)_\chi(\theta,\chi_1) - (c_V)_\chi(\theta,\chi_2)| \le c_1|\chi_1 - \chi_2| \tag{2.7}$$
$$\text{for all } \theta \in [0,+\infty), \ \chi, \ \chi_1, \ \chi_2 \in \mathcal{D}(\varphi).$$

Let e *and* s *be defined by formulas (1.16–1.17) and suppose that*

$$0 < s(1,\chi) \le c_1 \quad \forall \chi \in \mathcal{D}(\varphi); \tag{2.8}$$
$$|s_\chi(\theta_1,\chi_1) - s_\chi(\theta_2,\chi_2)| \le c_1 \left(|\theta_1 - \theta_2| + |\chi_1 - \chi_2|\right)$$
$$\text{for all } \theta_1, \theta_2 \in [0,+\infty), \ \chi_1, \ \chi_2 \in \mathcal{D}(\varphi). \tag{2.9}$$

(vii) $\chi_0 \in \mathbf{V} \cap L^\infty(\Omega)^d$. *Moreover, for any* $C > 0$ *set*

$$\mathcal{D}_C(\varphi) = \{\chi \in \mathcal{D}(\varphi) : \exists \xi \in \partial\varphi(\chi) : |\xi| \le C\},$$

and assume that $\chi_0(x) \in \mathcal{D}_{C_0}(\varphi)$ *a. e. in* Ω*;*

(viii) $\theta_0, \ u_0 \in L^\infty(\Omega)$ *fulfil* $u_0 = e(\theta_0, \chi_0)$ *and* $\theta_0(x) > 0$ *a. e. in* Ω*;*

(ix) $\gamma \in L^\infty(\partial\Omega)$ *is a nonnegative function;*

(x) $\theta_\Gamma \in L^\infty(\Sigma_\infty)$ *is such that* $\theta_\Gamma(x,t) > 0$ *a.e. and* $\log(\theta_\Gamma) \in L^1(\Sigma_\infty)$*;*

(xi) *Let* $\zeta \in (W^{1,1}_{loc}(0,\infty))^d$ *be the solution to the differential inclusion*

$$\alpha(t)\zeta_t + \partial\varphi(\zeta) \ni g(t) \quad \text{a.e.}, \tag{2.10}$$

with the initial condition $\zeta(0) = \zeta_0$, $\zeta_0 \in \mathbb{R}^d$*, and given data* $g \in (L^\infty(0,\infty))^d$ *and* $\alpha \in L^\infty_{\mathrm{loc}}(0,\infty)$ *such that* $0 < \alpha_0 \le \alpha(t)$ *a.e. We assume that there exists a positive constant* $D > 0$ *such that for all* $C > 0$ *such that* $|g(t)| \le C$*, and* $\zeta_0 \in \mathcal{D}_C(\varphi)$*, we have*

$$|(g - \alpha\zeta_t)(t)| \le DC \quad \text{a.e.}. \tag{2.11}$$

We are now in position to state the existence theorem.

Theorem 2.2. (Existence) *Let Hypothesis 2.1 hold. Then there exists at least one pair* (θ, χ) *that solves system (2.1–2.3) and such that*

$$\theta \in L^\infty(Q_\infty) \cap L^2(0,\infty;V), \quad (e(\theta,\chi))_t \in L^2_{loc}(0,\infty;V'), \tag{2.12}$$

$$\theta(x,t) > 0 \quad \textit{a. e. in } Q_\infty, \tag{2.13}$$

$$\begin{aligned} &\chi \in L^\infty_{\mathsf{loc}}(Q_\infty)^d \cap L^\infty_{loc}(0,\infty;\mathbf{V}), \quad \chi_t \in L^\infty(Q_\infty)^d; \\ &\exists\, C > 0 \,:\, \chi(x,t) \in \mathcal{D}_C(\varphi) \quad \textit{a. e. in } Q_\infty. \end{aligned} \tag{2.14}$$

Moreover, there exists a positive constant $\overline{\theta}$ *independent of* t *such that the following uniform upper bound hold:*

$$\theta(x,t) < \overline{\theta} \quad \textit{for a. e. } (x,t) \in Q_\infty. \tag{2.15}$$

Hypothesis 2.3. (Uniqueness) *Assume that Hyp. 2.1 is satisfied and suppose moreover that*

(i) $k(\theta,\chi) = \bar{k}(\theta)$ *for all* $\theta \in \mathbb{R}$ *and* $\chi \in \mathbb{R}^d$*;*

(ii) *Fix* $T \in (0,\infty)$ *and suppose that there exists a positive constant* R*, depending only on* C*,* α_0*, and* T *such that the solutions* $\zeta_1, \zeta_2 \in W^{1,\infty}(0,T)$ *to* (2.10) *associated with data* $\zeta_{01}, \zeta_{02} \in \mathcal{D}_C(\varphi)$*,* $\alpha_1, \alpha_2 \in L^\infty(0,T)$*, and with* $g_1, g_2 \in L^\infty(0,T)$ *complying with the constraint*

$$|g_i(t)| \le C, \quad i = 1, 2, \quad \textit{a. e. in } (0,T), \tag{2.16}$$

satisfy for all $t \in (0,T)$ *the inequality*

$$\int_0^t |\dot\zeta_1 - \dot\zeta_2|(\tau)\, \mathsf{d}\tau + |\zeta_1 - \zeta_2|(t) \le R\Big(|\zeta_{01} - \zeta_{02}| + \int_0^t \left(\left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right|(\tau) + |g_1 - g_2|(\tau)\right) \mathsf{d}\tau\Big)\,; \tag{2.17}$$

(iii) *Define* $\tilde{c}(\theta) := \min\{c_V(v,\chi) : \chi \in \mathcal{D}(\varphi),\ v \ge \theta\}$ *and assume that* $\displaystyle\int_0^1 \frac{\tilde{c}(v)\mu(v)}{v^2}\, \mathsf{d}v = +\infty$*;*

(iv) *The function* $v \mapsto v^2/\mu(v)$ *is nondecreasing in* $(0,+\infty)$*;*

(v) *There exists* $\theta_* > 0$ *such that* $\theta_0(x) \ge \theta_*,\ \theta_\Gamma(x,t) \ge \theta_*$ *a.e.;*

(vi) *Assume that* $\tilde{c}(\theta) > 0$ *for every* $\theta \in (0,\infty)$*.*

Remark 2.4. Hyp. 2.1 allows for physically meaningful choices of c_V. We can choose, for example, a $(d+1)$-component model with $K = \{\chi_i \ge 0,\ \sum_{i=1}^d \chi_i \le 1\}$, $\chi_0 = 1 - \sum_{i=1}^d \chi_i$, $\varphi = I_K$ (indicator function of the set K), and $c_V(\theta,\chi) = \sum_{i=1}^d c_i(\theta)\chi_i$, where $c_i(\theta)$ behave asymptotically at 0 and ∞ like $\dfrac{\theta^\alpha}{1+\theta^\alpha}$. Further examples of potentials φ complying with Hyp. 2.1 (xi) and Hyp. 2.3 (ii) will be given in the following Section 3.

We state then our last result regarding uniqueness and continuous data dependence for (2.1–2.3).

Theorem 2.5. (Uniqueness) *Suppose that Hypothesis 2.3 is satisfied. Let $T \in (0,\infty)$ be fixed. Then, there exists a positive constant $\underline{\theta}(T)$ such that*

$$\theta(x,t) \geq \underline{\theta}(T) \quad \text{for a.e. } (x,t) \in Q_T\,. \tag{2.18}$$

Moreover, if (θ_1,χ_1), (θ_2,χ_2) are two solutions to (2.1–2.3) in the sense of Theorem 2.2 associated with initial data θ_{01},χ_{01} and θ_{02},χ_{02}, and boundary data $\theta_{\Gamma 1},\theta_{\Gamma 2}$, respectively, and $\hat\theta = \theta_1 - \theta_2$, $\hat\chi = \chi_1 - \chi_2$, $\hat\chi_0 = \chi_{01} - \chi_{02}$, $\hat\theta_0 = \theta_{01} - \theta_{02}$, $\hat\theta_\Gamma = \theta_{\Gamma 1} - \theta_{\Gamma 2}$, then, there exists a constant $C_T > 0$ such that

$$\begin{aligned}\int_0^T \int_\Omega |\hat\theta(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \max_{t\in[0,T]} \int_\Omega |\hat\chi(x,t)|^2\,\mathrm{d}x \qquad\qquad\qquad\qquad \\ \leq C_T \left(|\hat\theta_0|_H^2 + |\hat\chi_0|_H^2 + \int_0^T \int_{\partial\Omega} \gamma |\hat\theta_\Gamma(s,t)|^2 \,\mathrm{d}A\,\mathrm{d}t \right).\end{aligned} \tag{2.19}$$

Finally, beside Hypothesis 2.3, assume that

$$(\theta_\Gamma)_t \in L^2(0,T;L^2(\partial\Omega))\,. \tag{2.20}$$

Then, the θ-component of the solution (θ,χ) to (2.1–2.3) has the further regularity

$$\theta \in L^\infty(0,T;V), \quad \theta_t \in L^2(0,T;H)\,. \tag{2.21}$$

3 A differential inclusion

This section is devoted to the description of some properties of solutions to general differential inclusions of the form (2.10), which are used in the proof of Theorems 2.2, 2.5.

First we provide some examples of functions φ satisfying Hypothesis 2.1 (xi), and we prove some further properties for space and time dependent differential inclusions that follow exactly from this assumption. Finally, we give examples of functions φ satisfying Hypothesis 2.3 (ii).

Examples of functions complying with Hypothesis 2.1 (xi).

Proposition 3.1. *The function φ introduced in Hypothesis 2.1 (i) satisfies Hypothesis 2.1 (xi) in each of the following cases:*

(a) *if $d = 1$;*

(b) *if φ is the indicator function I_K associated with a closed, and convex set $K \subset \mathbb{R}^d$. In this case one has $D = 1$;*

(c) *if $\varphi(x) = f(M_K(x))$, where $f : [0,f_0) \to [0,+\infty)$ is an increasing and convex C^1 function such that $f(0) = f'(0) = 0$, $f_0 > 0$, and M_K is the Minkowski functional of K, a closed, convex set in $\mathbb{R}^d$ such that $B_r(0) \subset K \subset B_R(0)$, defined by the formula $M_K(x) = \inf\left\{s > 0\,;\, \frac{1}{s}x \in K\right\}$, for $x \in \mathbb{R}^d$. Then $D = R/r$.*

The proof of point (a) follows directly from [19, Prop. 3.4], (b) is obvious. Let us prove the point (c). In order to do that, we first need to prove the following auxiliary result.

Proposition 3.2. *Let $\varphi(x)$ be as in Prop. 3.1 (c). Then for every $C > 0$ there exists $C_1 > 0$ such that for every $x \in \mathcal{D}(\varphi)$ we have the following implications*

$$\varphi(x) \le C_1 \;\Rightarrow\; \sup\{|\eta| \,:\, \eta \in \partial\varphi(x)\} \le (R/r)C, \tag{3.1}$$

$$\varphi(x) \ge C_1 \;\Rightarrow\; \inf\{|\eta| \,:\, \eta \in \partial\varphi(x)\} \ge C. \tag{3.2}$$

Proof. We first prove the following equivalence

$$\eta \in \partial\varphi(x) \quad \Leftrightarrow \quad (\eta = cw, \quad w \in \partial M_K(x), \quad c = f'(M_K(x)))\ .$$

We clearly have $\partial\varphi(0) = \{0\}$. For $x \neq 0$, take $\gamma \in (0,1)$. Then, for $\eta \in \partial\varphi(x)$ and for all $y \in \mathcal{D}(\varphi)$, we have

$$\langle \eta, x - (x - \gamma(x-y))\rangle \ge \varphi(x) - \varphi(x - \gamma(x-y))\,,$$

hence

$$\langle \eta, x - y\rangle \ge \frac{1}{\gamma}\left(f(M_K(y)) - f(M_K(x)) - \gamma(M_K(x) - M_K(y))\right)\ .$$

Letting γ tend to 0, we obtain that $\dfrac{\eta}{f'(M_K(x))} \in \partial M_K(x)$. Conversely, for $w \in \partial M_K(x)$, we have

$$\langle f'(M_K(x))w, x - y\rangle \ge f'(M_K(x))(M_K(x) - M_K(y)) \ge f(M_K(x)) - f(M_K(y)),$$

which we wanted to prove.

Let now C be a given positive constant. For all $w \in \partial M_K(x)$ and $x \neq 0$ we have $(1/R) \le |w| \le (1/r)$. From this we deduce that if $f'(M_K(x))|w| < C$, then $f'(M_K(x)) < CR$, and so $M_K(x) < (f')^{-1}(CR)$ and $\varphi(x) < f((f')^{-1}(CR))$. We can choose $C_1 = f((f')^{-1}(CR))$, and (3.2) is proved. Suppose now that $f(M_K(x)) \le C_1$. Then we have $M_K(x) \le (f')^{-1}(CR)$, hence $f'(M_K(x)) \le CR$ and $f'(M_K(x))|w| \le (R/r)C$, and (3.1) is proved. ∎

We conclude the proof of Prop. 3.1 (c) by proving the following Proposition 3.3.

Proposition 3.3. *Let φ be as in Prop. 3.1 (c). Then Hypothesis 2.1 (xi) is satisfied with $D = R/r$.*

Proof. Consider some ζ satisfying inclusion (2.10) with initial datum ζ_0. Then, the following equality holds true for all $t \in (0,\infty)$:

$$|\alpha(t)\zeta_t|^2 + |g(t) - \alpha(t)\zeta_t|^2 + 2\alpha(t)\varphi(\zeta)_t = |g(t)|^2;$$

hence, we immediately deduce that, for all $t \in (0,\infty)$,

$$\varphi(\zeta)_t = \frac{1}{2\alpha(t)}\left(|g(t)|^2 - |g(t) - \alpha(t)\zeta_t|^2 - |\alpha(t)\zeta_t|^2\right). \tag{3.3}$$

By assumption, we have $|g(t)| \le C$ and $g(t) - \alpha(t)\zeta_t \in \partial\varphi(\zeta)$. In view of (3.2), there is a constant C_1 such that

$$\varphi(\zeta)_t \le \frac{1}{2\alpha(t)}(C^2 - C^2 - |\alpha(t)\zeta_t|^2) \le 0 \quad \forall t \in (0,\infty) \quad \text{if} \quad \varphi(\zeta) \ge C_1\,.$$

Hence,

$$\varphi(\zeta)_t(\varphi(\zeta) - C_1)^+ \le 0 \quad \forall t \in (0,\infty). \tag{3.4}$$

Integrating from 0 to t and using the assumption $\zeta_0 \in \mathcal{D}_C(\varphi)$, it follows that $\varphi(\zeta)(t) \le C_1$ for all $t \in (0,\infty)$. Then, using (3.1), together with the fact that $|g(t)| \le C$ for all $t \in (0,\infty)$, we finally obtain that

$$|g(t) - \alpha(t)\zeta_t| \le \frac{R}{r}C \quad \text{and} \quad |\alpha(t)\zeta_t| \le \left(\frac{R}{r}+1\right)C, \tag{3.5}$$

which concludes the proof. ■

Properties of the solution mapping under Hypothesis 2.1 (xi).

Proposition 3.4. *Let us consider the solutions $\zeta_1,\, \zeta_2 \in W^{1,\infty}(0,\infty)$ to* (2.10) *associated with data $\zeta_{01}, \zeta_{02} \in \mathcal{D}_C(\varphi)$, $\alpha_1, \alpha_2 \in L^\infty_{\mathsf{loc}}(0,\infty)$, and with $g_1, g_2 \in L^\infty(0,\infty)$ complying with the constraint*

$$|g_i(t)| \le C, \quad i = 1,\, 2, \quad \textit{a. e.},$$

and let Hypothesis 2.1 (xi) hold. Then there exists a constant L such that for every $t \in (0,\infty)$ we have

$$|\zeta_1 - \zeta_2|(t) \le |\zeta_{01} - \zeta_{02}| + L\int_0^t \left(\left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right| + |g_1 - g_2|\right) \mathsf{d}\tau\,. \tag{3.6}$$

Proof. Test the difference of the two inclusions (2.10) by $\zeta_1 - \zeta_2$ and divide the resulting inequality by α_1. Then we get for a.e. $t \in (0,\infty)$:

$$\langle \dot\zeta_1 - \dot\zeta_2, \zeta_1 - \zeta_2\rangle \le \left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right| |\langle \alpha_2\dot\zeta_2, \zeta_1 - \zeta_2\rangle| + |\langle g_1 - g_2, \zeta_1 - \zeta_2\rangle|\,.$$

Using the bound for $|\alpha_2\dot\zeta_2|$ (cf. Hypothesis 2.1 (xi), (2.11)), we get

$$\frac{\mathsf{d}}{\mathsf{d}t}|\zeta_1 - \zeta_2| \le L\left(\left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right| + |g_1 - g_2|\right)$$

from which (3.6) immediately follows by integrating over $(0,t)$. □

Proposition 3.5. *Let Hypothesis 2.1 (xi) hold, and let ζ_n and ζ be the solutions of* (2.10) *corresponding to the data (g_n, α_n, ζ_{0n}) and (g, α, ζ_0), respectively, with $|g_n(t)| \le C$, $\zeta_{0n} \in \mathcal{D}_C(\varphi)$. If $\{\zeta_{0n}\}$ converges to ζ_0 in $\mathbb{R}^d$, $\{g_n\}$ converges to g and $\{\alpha_n\}$ converges to α in $L^2(0,T)$ for some $T > 0$, then $\{\dot\zeta_n\}$ converges strongly to $\dot\zeta$ in $L^2(0,T)$.*

Proof. Test (2.10), written for ζ_n, by $\dot\zeta_n$ in order to obtain

$$\left(g_n - \alpha_n\dot\zeta_n\right)\dot\zeta_n = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\zeta_n)\,.$$

Now set $\eta_n = \dfrac{g_n}{\sqrt{\alpha_n}} - 2\sqrt{\alpha_n}\dot\zeta_n$. Then, by straightforward computations, we obtain that

$$\left|\frac{g_n}{\sqrt{\alpha_n}}\right|^2 - |\eta_n|^2 = 4\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\zeta_n)\,.$$

We know, by Prop. 3.4, that $\{\zeta_n\}$ converges uniformly to ζ and that φ is Lipschitz continuous on $\mathcal{D}_C(\varphi)$. Hence, integrating over $(0,t)$, we get that $|\eta_n|_{L^2(0,T)} \to |\eta|_{L^2(0,T)}$. Since we know that $\eta_n \to \eta$ weakly in $L^2(0,T)$ (since $\dot\zeta_n \to \dot\zeta$ weakly in $L^2(0,T)$), we get $\eta_n \to \eta$ strongly in $L^2(0,T)$, which is sufficient in order to conclude the desired convergence. ■

Examples of functions complying with Hypothesis 2.3 (ii).

Proposition 3.6. *The function φ, introduced in Hypothesis 2.1 (i) satisfies Hypothesis 2.3 (ii), in each of the following cases:*

(a) *if $d = 1$;*

(b) *if, for any $C > 0$, φ is a C^1-function with Lipschitz continuous derivative on $\mathcal{D}_C(\varphi)$;*

(c) *if $\varphi = I_K$, where K is either a polyhedron or a smooth convex set with nonempty interior.*

Proof. The proofs of (a) and (c) follow respectively from [19, Prop. 3.4] and [8, Thm. 7.1, p. 88]. We briefly show here how to proceed to prove case (b). Let us consider solutions $\zeta_1,\,\zeta_2 \in W^{1,\infty}(0,T)$ to (2.10) associated with the data $\zeta_{01},\zeta_{02} \in \mathcal{D}_C(\varphi)$, $\alpha_1,\alpha_2 \in L^\infty(0,T)$, and with $g_1, g_2 \in L^\infty(0,T)$ complying with the constraint

$$|g_i(t)| \le C, \quad i = 1,\,2, \quad \text{a.e. in } (0,T).$$

By Hypothesis 2.1 (xi), ζ_1, ζ_2 remain in $\mathcal{D}_{DC}(\varphi)$. Using the Lipschitz continuity of φ' on $\mathcal{D}_{DC}(\varphi)$, we obtain that there exists a positive constant Q such that the following inequality holds true a.e.:

$$\alpha_1|\dot\zeta_1 - \dot\zeta_2| \le |\alpha_1 - \alpha_2||\dot\zeta_2| + Q|\zeta_1 - \zeta_2| + |g_1 - g_2|.$$

Dividing by α_1, and using the bound for $|\alpha_2\dot\zeta_2|$, from Hypothesis 2.1 (xi) (cf. (2.11)), we get

$$\begin{aligned}|\dot\zeta_1 - \dot\zeta_2| &\le (C + DC)\left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right| + Q|\zeta_1 - \zeta_2| + |g_1 - g_2| \\ &\le M\left(\left|\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right| + |\zeta_1 - \zeta_2| + |g_1 - g_2|\right) \qquad (3.7)\end{aligned}$$

for some positive constant M (depending on C, D, Q). We can rewrite this inequality in the following convenient form, for $t \in (0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-Mt}|\zeta_1-\zeta_2|\right) \le \mathrm{e}^{-Mt}\left(\left|\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right| + |g_1-g_2|\right).$$

Integrating over $(0,t)$, and using the previous inequality (3.7), we get

$$|\dot\zeta_1-\dot\zeta_2|(t) \le L\left(|\zeta_{01}-\zeta_{02}| + \left|\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right| + |g_1-g_2| + \int_0^t\left(\left|\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right| + |g_1-g_2|\right)\mathrm{d}\tau\right)$$

for some positive constant L (depending on C, D, Q). Integrating once more in time we arrive at the desired inequality (2.17). ■

Proposition 3.7. *Let $f : [0, f_0) \to [0,+\infty)$ be an increasing, convex function with locally Lipschitz continuous derivative, $f(0) = f'(0) = 0$, and let K be a closed, convex set of class $C^{1,1}$ such that $B_r(0) \subset K \subset B_R(0)$. Then, $\varphi(x) = f(M_K(x))$ has a Lipschitz continuous derivative on $\mathcal{D}_C(\varphi)$ for any $C > 0$, i.e., Property (b) in Proposition 3.6 is satisfied.*

Proof. Let $C > 0$ be given. We denote $\mathcal{D}_C f = \{s \in (0, f_0) : f'(s) \le RC\}$, and let L_C be the Lipschitz constant of f' on $\mathcal{D}_C f$. For $x \in \mathcal{D}_C(\varphi)$ we have $|\varphi'(x)| \le C$, hence $f'(M_K(x)) \le RC$, that is, $M_K(x) \in \mathcal{D}_C f$. We now estimate the difference $|\varphi'(x)-\varphi'(y)|$ on $\mathcal{D}_C(\varphi)$. Assume first that $x \ne 0$, $y = 0$. Then

$$\begin{aligned}|\varphi'(x)-\varphi'(y)| &= |\varphi'(x)| = f'(M_K(x))|M_K'(x)| \le \frac{1}{r}f'(M_K(x))\\ &\le \frac{1}{r}L_C M_K(x) \le \frac{1}{r^2}L_C|x| = \frac{L_C}{r^2}|x-y|.\end{aligned}$$

Consider now the case $x \ne 0$, $y \ne 0$ and set $J_K(x) = M_K(x)M_K'(x)$, $J_K(y) = M_K(y)M_K'(y)$. The mapping J_K is Lipschitz continuous on $\mathbb{R}^d$ (with Lipschitz constant L_J) (see [8, Section 5.2]), and we have

$$\begin{aligned}|\varphi'(x)-\varphi'(y)| &= |f'(M_K(x))M_K'(x) - f'(M_K(y))M_K'(y)| \le \frac{f'(M_K(x))}{M_K(x)}|J_K(x)\\ &\quad - J_K(y)| + |M_K'(y)|\frac{f'(M_K(x))}{M_K(x)}|M_K(x)-M_K(y)|\\ &\quad + |M_K'(y)||f'(M_K(x)) - f'(M_K(y))|\\ &\le L_C\left(L_J + \frac{2}{r^2}\right)|x-y|,\end{aligned}$$

from which the assertion follows. ■

A relevant case for applications is, for example, $\varphi(x) = -\log(1-M_K^2(x))$, see [12].

4 Existence of solutions

This section is devoted to the proof of the existence result stated in Section 2. We use a technique based on approximations, a priori estimates, and passage to the limit.

Let us first write down our equations (2.1) and (2.2) as

$$\begin{aligned}&\langle (e(\theta,\chi))_t, z\rangle + \int_\Omega k(\theta,\chi)\nabla\theta\cdot\nabla z\,\mathrm{d}x + \int_{\partial\Omega}\gamma(\theta-\theta_\Gamma)\,z\,\mathrm{d}A\\ &= -\int_\Omega \left(\lambda'(\chi)\chi_t + \beta\left(\varphi(\chi)\right)_t + b[\chi]\chi_t\right)\,z\,\mathrm{d}x \quad \forall z\in V, \quad \text{a.e. in } (0,\infty), \qquad (4.1)\\ &\mu(\theta)\chi_t + (\beta+\theta)\partial\varphi(\chi) \ni -\lambda'(\chi) - \theta\sigma'(\chi) - b[\chi] - e_\chi(\theta,\chi) + \theta s_\chi(\theta,\chi)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } Q_\infty. \qquad (4.2)\end{aligned}$$

4.1 Approximation

Assuming Hypothesis 2.1 to hold, we proceed as follows: first we extend the domain of definition of $c_V(\theta,\chi)$ by putting $\widetilde{c}_V(\theta,\chi) = c_V(|\theta|,\chi)$ for $(\theta,\chi)\in\mathbb{R}\times\mathcal{D}(\varphi)$, and set

$$\widetilde{e}(\theta,\chi) = \int_0^\theta \widetilde{c}_V(\xi,\chi)\,\mathrm{d}\xi \quad \text{for } (\theta,\chi)\in\mathbb{R}\times\mathcal{D}(\varphi)\,.$$

$$\widetilde{\mu}_\varrho(\theta) = \begin{cases}\mu(|\theta|) & \text{if } |\theta|\le\varrho\\ \mu(\varrho)(|\theta|-\varrho) & \text{if } |\theta|\ge\varrho\end{cases},$$

We now fix a truncation parameter $\varrho\ge 1$, which will be determined below, and define

$$s_\chi^\varrho(\theta,\chi) = \begin{cases}\displaystyle\int_0^\theta \frac{(\widetilde{c}_V)_\chi(\xi,\chi)}{\xi}\,\mathrm{d}\xi & \text{if } |\theta|\le\varrho\\ \displaystyle\int_0^\varrho \frac{(\widetilde{c}_V)_\chi(\xi,\chi)}{\xi}\,\mathrm{d}\xi & \text{if } |\theta|\ge\varrho\end{cases}.$$

We fix an arbitrary $T>0$, and split the interval $[0,T]$ into an equidistant partition $0 = t_0, t_1,\dots,t_n$, $t_j = jT/n$ for $j=0,1,\dots,n$, $n\in\mathbb{N}$, with the intention to let n tend to ∞. We choose sequences $\{\theta_{0,n}\}_n\in V$ and $\{\theta_{\Gamma,n}\}_n\in W^{1,2}(0,T;L^2(\partial\Omega))$ of approximate data such that $\theta_{0,n}(x)\ge 1/n$ a.e., $\theta_{\Gamma,n}(x,t)\ge 1/n$ a.e., $\theta_{0,n}\to\theta_0$ strongly in H, and $\theta_{\Gamma,n}\to\theta_\Gamma$ strongly in $L^2(0,T;L^2(\partial\Omega))$.

An approximate solution (θ_n,χ_n) will be constructed successively in intervals $[t_{j-1},t_j]$ for $j=1,\dots,n$. Assuming that it is already known on $[0,t_{j-1}]$, we define

$$\bar{\chi}_n(x,t) = \chi_n(x,t_{j-1}) \quad x\in\Omega, \quad t\in(t_{j-1},t_j), \quad j=1,\dots,n\,, \qquad (4.3)$$

$$\bar{\theta}_n(x,t) = \begin{cases}\theta_{0,n} & \text{for } t\in[0,t_1)\\ \displaystyle\frac{n}{T}\int_{t_{j-2}}^{t_{j-1}}\theta_n(x,\tau)\,\mathrm{d}\tau & \text{for } t\in[t_{j-1},t_j), \quad j\ge 2\,.\end{cases}$$

With these notations, we then state the following approximating problem. We use only the index n for the variables here (omitting the ϱ dependence), for simplicity.

PROBLEM (P)$_{(n,\varrho)}$. Find two functions $\theta_n \in H^1(0,T;H) \cap L^\infty(0,T;V)$ and $\chi_n \in L^\infty(\Omega \times (0,T))^d$, $\partial_t \chi_n \in L^\infty(\Omega \times (0,T))^d$, such that $\theta_n \geq \varepsilon_n$ a.e. in Q_T for some $\varepsilon_n > 0$, $\chi_n \in \mathcal{D}_C(\varphi)$, and for all $t \in (0,T)$ and $z \in V$, we have

$$
\begin{aligned}
& \int_\Omega \partial_t \left(\frac{1}{n} \theta_n(t) + \widetilde{e}(\theta_n(t), \chi_n(t)) \right) z \, \mathrm{d}x + \int_\Omega k(\bar{\theta}_n(t), \bar{\chi}_n(t)) \nabla \theta_n(t) \cdot \nabla z \, \mathrm{d}x \\
& \qquad + \int_{\partial\Omega} \gamma(\theta_n(t) - \theta_{\Gamma,n}(t)) z \, \mathrm{d}A \\
= & \; - \int_\Omega \left((\lambda'(\chi_n)(t) + b[\chi_n](t)) \partial_t \chi_n(t) + \beta \partial_t \left(\varphi(\chi_n(t)) \right) \right) z \, \mathrm{d}x \, ,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
& \widetilde{\mu}_\varrho(\theta_n(t)) \partial_t \chi_n(t) + (\beta + |\theta_n(t)|) \partial \varphi(\chi_n)(t) \\
\ni \; & -\lambda'(\chi_n)(t) - |\theta_n(t)| \sigma'(\chi_n)(t) - b[\chi_n](t) \\
& -\widetilde{e}_\chi(\theta_n(t), \chi_n(t)) + |\theta_n(t)| s^\varrho_\chi(\theta_n(t), \chi_n(t)) \quad \text{a.e. in } \Omega \, ,
\end{aligned}
\tag{4.5}
$$

with initial conditions

$$
\theta_n(0) = \theta_{0,n}, \quad \chi_n(0) = \chi_0 \, . \tag{4.6}
$$

Lemma 4.1. *Under Hypothesis 2.1, for each $\varrho > 0$ and $n \in \mathbb{N}$ the* PROBLEM (P)$_{(n,\varrho)}$ *has a unique solution (θ_n, χ_n) with the required properties.*

Proof. On each interval (t_{j-1}, t_j), we can proceed as in the proof of [20, Thm. 2.2, p. 290]. We test a Galerkin approximation of (4.4) by the approximation of $\partial_t \theta_n$. The estimates are sufficient to pass to the limit in the Galerkin scheme and to obtain a solution on each interval (t_{j-1}, t_j). We only have to check that the initial conditions at $t_1, t_2, \ldots$ are well defined. Indeed, since on each interval (t_{j-1}, t_j) we have $\theta_n \in H^1(t_{j-1}, t_j; H) \cap L^\infty(t_{j-1}, t_j; V)$, we also obtain that $t \mapsto \theta_n(t, \cdot)$ is weakly continuous in (t_{j-1}, t_j) for every j with values in V. Moreover, χ_n is strongly continuous with values in $L^\infty(\Omega)^d$, and there exists a positive constant C (independent of n) such that $\chi_n(t, \cdot) \in \mathcal{D}_C(\varphi)$ on (t_{j-1}, t_j) for every $j = 1, \ldots, n$. Hence, we can define the initial conditions at $t = t_j$ by $\theta_n(t_j) = \theta_n(t_j -)$. ∎

4.2 A priori estimates

In this subsection, we perform suitable a priori estimates (independent of n) for the solution. In the following, we will denote by C any positive constant that depends only on the data of the problem but may vary from line to line. In particular, it will not depend on the truncation parameter ϱ and discretization parameter n. If such a dependence takes place, we use the symbol C_ϱ for a constant that depends on ϱ, but not on n. Again, the same symbols will denote constants that may differ from line to line.

Let us, for simplicity, in this subsection occasionally omit the indices n and write simply θ, χ instead of θ_n, χ_n if no confusion may arise. We denote (note that $\theta_n > 0$ and so $\tilde{e} = e$)

$$u_n(t) := \frac{1}{n}\theta_n(t) + e(\theta_n(t), \chi_n(t))) \quad \text{for } t \in (0, T) \,. \tag{4.7}$$

Estimate for χ_t. Equation (4.5) is of the form (2.10) with

$$\alpha(t) = \tilde{\alpha}(\theta) \ := \ \frac{\widetilde{\mu}_\varrho(\theta)}{\beta + \theta} \ \geq \ \frac{\mu_0(1+\theta)}{\beta+\theta} \ \geq \ \mu_0 \min\left\{1, \frac{1}{\beta}\right\} \,, \tag{4.8}$$

$$g(t) = \ell[\theta, \chi] := -\frac{1}{\beta+\theta}\left(\theta\sigma'(\chi) + \lambda'(\chi) + b[\chi] + e_\chi(\theta,\chi) - \theta s_\chi^\varrho(\theta,\chi)\right) . \tag{4.9}$$

First, let us note that, using (2.4), (2.7), and (2.8), we get

$$\begin{aligned} s_\chi^\varrho(\theta,\chi) &\leq \int_0^{\min\{\theta,\varrho\}} \frac{|(c_V)_\chi(\xi,\chi)|}{\xi}\,\mathrm{d}\xi \ \leq \ c_1 \int_0^1 \frac{c_V(\xi,\chi)}{\xi}\,\mathrm{d}\xi + c_1\int_1^\varrho \frac{\bar{c}}{\xi}\,\mathrm{d}\xi \\ &\leq c_1 s(1,\chi) + c_1\bar{c}\log\varrho \ \leq \ c_1^2 + c_1\bar{c}\log\varrho \,. \end{aligned} \tag{4.10}$$

Hence, owing to Hypothesis 2.1, we have

$$\begin{aligned} |\ell[\theta,\chi]| &\leq C_\sigma + \frac{1}{\beta}\left(C_\lambda + C_b\right) + \frac{\theta}{\beta+\theta}\sup_{0\leq\xi\leq\theta}|(c_V)_\chi(\xi,\chi)| + \frac{\theta}{\beta+\theta}|s_\chi^\varrho(\theta,\chi)| \\ &\leq C_\sigma + \frac{1}{\beta}\left(C_\lambda + C_b\right) + c_1 \sup_{0\leq\xi\leq\theta}|c_V(\xi,\chi)| + c_1^2 + c_1\bar{c}\log\varrho \\ &\leq C_\sigma + \frac{1}{\beta}\left(C_\lambda + C_b\right) + c_1\bar{c} + c_1^2 + c_1\bar{c}\log\varrho \,, \end{aligned}$$

where C_b denotes here the upper bound for the operator b defined in (1.13). Let us set

$$C_{\ell,\varrho} := C_\sigma + \frac{1}{\beta}\left(C_\lambda + C_b\right) + c_1\bar{c} + c_1^2 + c_1\bar{c}\log\varrho \,. \tag{4.11}$$

Then the conditions of Hypothesis 2.1 (xi) are satisfied with the choice $C = \max\{C_{\ell,\varrho},\, C_0\}$, where C_0 is defined in Hyp. 2.1 (vii) and $C_{\ell,\varrho}$ is defined in (4.11). Hence, we get the following estimates on χ:

$$|\chi_n|_{L^\infty(Q_T)} + |\partial_t\chi_n|_{L^\infty(Q_T)} + |\partial_t(\varphi(\chi_n))|_{L^\infty(Q_T)} \ \leq \ C(1+\log\varrho)^2 \,, \tag{4.12}$$

where now C is a constant independent of ϱ.

Estimate for θ. Taking $z = \theta_n$ in (4.4), we get

$$\begin{aligned} &\int_\Omega \partial_t\left(\frac{1}{n}\theta(t) + e(\theta(t),\chi(t))\right)\theta(t)\,\mathrm{d}x + \int_\Omega k(\bar{\theta}_n(t),\bar{\chi}_n(t))\nabla\theta(t)\cdot\nabla\theta(t)\,\mathrm{d}x \\ &+ \int_{\partial\Omega}\gamma(\theta(t) - \theta_{\Gamma,n}(t))\theta(t)\,\mathrm{d}A \\ &= -\int_\Omega \left(\lambda'(\chi)(t)\partial_t\chi(t) + \beta\partial_t\left(\varphi(\chi(t))\right) + b[\chi](t)\partial_t\chi(t)\right)\theta(t)\,\mathrm{d}x \,. \end{aligned} \tag{4.13}$$

Define

$$U(\theta,\chi) = \int_0^\theta c_V(v,\chi)v \,\mathsf{d}v \quad \text{for } (\theta,\chi) \in (0,\infty)\times\mathcal{D}(\varphi)\,.$$

We have $\partial_t U(\theta,\chi) = \theta\partial_t e(\theta,\chi) + (\partial_\chi U - \theta\partial_\chi e)\chi_t$. We integrate (4.13) from 0 to t and rewrite the first term as follows:

$$\begin{aligned}
&\int_0^t\int_\Omega \partial_t\left(\frac{1}{n}\theta(\tau) + e(\theta(\tau),\chi(\tau))\right)\theta(\tau)\,\mathsf{d}\tau\,\mathsf{d}x \\
&= \int_\Omega\left(\frac{1}{2n}\theta^2(t) + U(\theta(t),\chi(t)\right)\mathsf{d}x - \int_\Omega\left(\frac{1}{2n}\theta^2(0) + U(\theta(0),\chi(0)\right)\mathsf{d}x \\
&\quad - \int_0^t\int_\Omega(\partial_\chi U(\theta(\tau),\chi(\tau)) - \theta(\tau)\partial_\chi e(\theta(\tau),\chi(\tau)))\chi_t(\tau)\,\mathsf{d}\tau\,\mathsf{d}x\,.
\end{aligned}$$

There exist two constants C_1, C_2 such that $U(\theta,\chi) \geq C_1\theta^2 - C_2$. Hence, by (4.12) and the Gronwall's lemma, we obtain

$$\|\theta_n\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \leq C(1+\log\varrho)^2\,. \tag{4.14}$$

By comparison, we also deduce that

$$\|\partial_t u_n\|_{L^2(0,T;V')} \leq C(1+\log\varrho)^2. \tag{4.15}$$

Estimate for $\nabla\chi$. The function

$$\theta \mapsto \frac{\beta+\theta}{\widetilde{\mu}_\varrho(\theta)}$$

is Lipschitz continuous in $\mathbb{R}$ due to Hyp. 2.1 (v) and, with the help of the mean value theorem, it is straightforward to deduce that

$$\begin{aligned}
|\ell[\theta_1,\chi_1] - \ell[\theta_2,\chi_2]| &\leq |\sigma'(\chi_1) - \sigma'(\chi_2)| + \frac{C_\sigma}{\beta}|\theta_1-\theta_2| + \frac{1}{\beta}|\lambda'(\chi_1) - \lambda'(\chi_2)| \\
&\quad + \frac{C_\lambda}{\beta^2}|\theta_1-\theta_2| + \frac{1}{\beta}|b[\chi_1]-b[\chi_2]| + \frac{C_b}{\beta^2}|\theta_1-\theta_2| + \frac{1}{\beta+\theta_1}\int_0^{\theta_1} c_1|\chi_1-\chi_2|\,\mathsf{d}\xi \\
&\quad + \left|\frac{1}{\beta+\theta_1}\int_0^{\theta_1}(c_V)_\chi(\xi,\chi_2)\,\mathsf{d}\xi - \frac{1}{\beta+\theta_2}\int_0^{\theta_2}(c_V)_\chi(\xi,\chi_2)\,\mathsf{d}\xi\right| \\
&\quad + \frac{1}{\beta}\left|s^\varrho_\chi(\theta_1,\chi_1) - s^\varrho_\chi(\theta_2,\chi_2)\right| + \frac{|s^\varrho_\chi(\theta_2,\chi_2)|}{\beta^2}|\theta_1-\theta_2| \\
&\leq |\sigma'(\chi_1)-\sigma'(\chi_2)| + \frac{1}{\beta}\left(|\lambda'(\chi_1)-\lambda'(\chi_2)| + |b[\chi_1]-b[\chi_2]|\right) \\
&\quad + \frac{1}{\beta^2}\left(C_\sigma\beta + C_\lambda + C_b\right)|\theta_1-\theta_2| + \frac{c_1}{\beta}|\chi_1-\chi_2| + \frac{1}{\beta}\left(c_1\bar{c} + c_1\bar{c}\right)|\theta_1-\theta_2| \\
&\quad + \frac{c_1}{\beta}\left(|\theta_1-\theta_2| + |\chi_1-\chi_2|\right) + \frac{1}{\beta^2}\left(c_1^2 + c_1\log\varrho\right)|\theta_1-\theta_2|\,.
\end{aligned} \tag{4.16}$$

Hence, we can apply estimate (3.6) in Proposition 3.4 to (4.5) with (for $x, y \in \Omega$)

$$
\begin{aligned}
\alpha_1(t) &= \left(\frac{\widetilde{\mu}_\varrho(\theta)}{\beta+\theta}\right)(x,t),\\
\alpha_2(t) &= \left(\frac{\widetilde{\mu}_\varrho(\theta)}{\beta+\theta}\right)(y,t),\\
g_1(t) &= \ell[\theta(x,t),\chi(x,t)] = -\frac{1}{\beta+\theta}\Big(\theta\sigma'(\chi)+\lambda'(\chi)+b[\chi]+e_\chi(\theta,\chi)\\
&\qquad\qquad - \theta s^\varrho_\chi(\theta,\chi)\Big)(x,t),\\
g_2(t) &= \ell[\theta(y,t),\chi(y,t)] = -\frac{1}{\beta+\theta}\Big(\theta\sigma'(\chi)+\lambda'(\chi)+b[\chi]+e_\chi(\theta,\chi)\\
&\qquad\qquad - \theta s^\varrho_\chi(\theta,\chi)\Big)(y,t),
\end{aligned}
$$

and

$$
\zeta_1 = \chi(x,t), \quad \zeta_2 = \chi(y,t), \quad x, y \in \Omega.
$$

Hence, we get

$$
\begin{aligned}
|\chi(x,t)-\chi(y,t)| \le |\chi_0(x)-\chi_0(y)| + \hat{L}\Big(\int_0^t |\theta(x,s)-\theta(y,s)| \,\mathrm{d}s\\
+ \int_0^t \left(|b[\chi](x,s)-b[\chi](y,s)| + |\chi(x,s)-\chi(y,s)|\right)\,\mathrm{d}s\Big),
\end{aligned}
\tag{4.17}
$$

where $\hat{L}$ depends on L, L_μ, and the constants on the right hand side of (4.16). Now, recalling (1.13), we have, by (4.16), that

$$
\begin{aligned}
|\chi(x,t)-\chi(y,t)| &\le |\chi_0(x)-\chi_0(y)| + \hat{L}\int_0^t |\theta(x,s)-\theta(y,s)|\,\mathrm{d}s\\
&+ 2\hat{L}_\varrho \int_0^t\int_\Omega |\kappa(x,z)\left(G'(\chi(x,s)-\chi(z,s)) - G'(\chi(y,s)-\chi(z,s))\right)|\,\mathrm{d}z\,\mathrm{d}s\\
&+ 2\hat{L}\int_0^t\int_\Omega |G'(\chi(y,s)-\chi(z,s))\left(\kappa(x,z)-\kappa(y,z)\right)|\,\mathrm{d}z\,\mathrm{d}s\\
&+ \hat{L}\int_0^t |\chi(x,s)-\chi(y,s)|\,\mathrm{d}s\,.
\end{aligned}
\tag{4.18}
$$

Thus, in view of Hyp. 2.1 (iii), we obtain that

$$
\begin{aligned}
|\chi(x,t)-\chi(y,t)| &\le |\chi_0(x)-\chi_0(y)| + \hat{L}\int_0^t |\theta(x,s)-\theta(y,s)|\,\mathrm{d}s\\
&+ \hat{L}(2L_b+1)\int_0^t |\chi(x,s)-\chi(y,s)|\,\mathrm{d}s + 2\hat{L}L_b\int_\Omega |\kappa(x,z)-\kappa(y,z)|\,\mathrm{d}z\,,
\end{aligned}
\tag{4.19}
$$

where L_b is a constant depending on the Lipschitz constants of G and G', $\|\kappa\|_{L^\infty(\Omega\times\Omega)}$, $|\Omega|$, and T. From (4.19), using the assumptions $\chi_0 \in \mathbf{V}$ and $\kappa \in W^{1,\infty}(\Omega\times\Omega)$ (cf. Hyp. 2.1

(iii)), we immediately deduce that

$$|\nabla\chi| \leq C_\varrho \left(1 + \int_0^t \left(|\nabla\theta(\cdot,s)| + |\nabla\chi(\cdot,s)|\right) \,\mathrm{d}s\right) \quad \text{a.e. in } \Omega\,.$$

Now, with the help of Gronwall's lemma, we infer that

$$|\nabla\chi| \leq C_\varrho \left(1 + \int_0^t |\nabla\theta(\cdot,s)| \,\mathrm{d}s\right) \quad \text{a.e. in } \Omega\,. \tag{4.20}$$

Using finally (4.20) with (4.14), we get the desired estimate

$$\|\chi_n\|_{L^\infty(0,T;\mathbf{V})} \leq C_\varrho\,, \tag{4.21}$$

where C_ϱ denotes a positive constant depending increasingly on ϱ. From the definition (4.7) of u, it also follows that

$$\|u_n\|_{L^2(0,T;V)} \leq C_\varrho\,. \tag{4.22}$$

4.3 Lower and upper bounds on θ

In this subsection, we first prove a bound for $\log\theta$ entailing the strict positivity of the absolute temperature (in the limit when $n \to \infty$). Then, we prove a (time dependent) upper bound holding true for the solution component θ_n for n fixed, which enable us to proceed with the Moser iteration procedure in order to prove a uniform (independent of time, of n, and of ϱ) upper bound on θ. This permits us to remove the truncation parameter and to conclude the existence proof. Finally, we will prove a lower bound (independent of n) on θ holding true under the additional Hyp. 2.3 that we will use for the proof of uniqueness of solutions.

Estimate on $\log\theta$. Let us rewrite equation (4.4), by using (4.5), in the following form, for all $z \in V$,

$$\begin{aligned}\int_\Omega \partial_t\left(\frac{1}{n}\theta + e(\theta,\chi)\right) z \,\mathrm{d}x + \int_\Omega k(\bar\theta_n,\bar\chi_n)\nabla\theta\cdot\nabla z \,\mathrm{d}x + \int_{\partial\Omega} \gamma(\theta - \theta_{\Gamma,n})\, z \,\mathrm{d}A \\ = \int_\Omega \widetilde{\mu}_\varrho(\theta)\chi_t^2 + \theta\chi_t R(\theta,\chi) z \,\mathrm{d}x\,,\end{aligned} \tag{4.23}$$

where

$$R(\theta,\chi) := \sigma'(\chi) - s_\chi^\varrho(\theta,\chi) + \xi, \quad \xi \in \partial\varphi(\chi), \quad |\xi(x,t)| \leq C \text{ a.e.}\,,$$

where C is defined in Hypothesis 2.1 (xi). We prove now an estimate for $\log\theta$ in $L^2(0,T;V)$ by taking in (4.4) $z = T(\theta)$, where

$$T(\theta) := -\left(1 - \frac{1}{\theta}\right)^- = \begin{cases} 1 - \dfrac{1}{\theta} & \text{for } \theta \leq 1 \\ 0 & \text{for } \theta \geq 1 \end{cases}\,. \tag{4.24}$$

Notice that we are allowed to perform this estimate, with fixed n, because $\theta \geq \varepsilon_n > 0$ a.e. for all n (cf. Lemma 4.1). We get, using equation (4.5) and Hyp. 2.1 (iv),

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\theta,\chi) - \int_\Omega \chi_t \int_1^\theta (c_V)_\chi(\xi,\chi) T(\xi)\, \mathrm{d}\xi\, \mathrm{d}x + k_0 \|\nabla(\log\theta)^-\|_H^2$$
$$+ \int_{\partial\Omega} (\theta - \theta_{\Gamma,n})\, T(\theta)\, \mathrm{d}A \leq \int_\Omega \widetilde{\mu}_\varrho(\theta)|\chi_t|^2 T(\theta)\, \mathrm{d}x + \int_\Omega \theta T(\theta)\chi_t R(\theta,\chi)\, \mathrm{d}x\,, \tag{4.25}$$

where

$$E(\theta,\chi) = -\int_\Omega \int_1^\theta c_V(\xi,\chi) \left(1 - \frac{1}{\xi}\right)^- \mathrm{d}\xi \geq 0\,.$$

Note that the first term on the right-hand side of (4.25) is nonpositive, while the other term can be estimated using estimates (4.10), (4.12), and (4.14) on our solution (θ,χ). Regarding the second term on the left-hand side in (4.25), using (2.7) in Hyp. 2.1 (vi), we obtain that

$$\left| - \int_\Omega \chi_t \int_1^\theta (c_V)_\chi(\xi,\chi) T(\xi)\, \mathrm{d}\xi\, \mathrm{d}x \right| \leq c_1 E(\theta,\chi) \|\partial_t \chi\|_{L^\infty(\Omega)^d}.$$

Moreover, we treat the boundary integral in the following way:

$$\int_{\partial\Omega} (\theta - \theta_{\Gamma,n}) T(\theta)\, \mathrm{d}A \geq \int_{\partial\Omega} \Psi(\theta) - \Psi(\theta_{\Gamma,n})\, \mathrm{d}A, \tag{4.26}$$

where Ψ is defined as

$$\Psi(\theta) := \begin{cases} \theta - \log\theta & \text{for } \theta \leq 1 \\ 1 & \text{for } \theta \geq 1 \end{cases},$$

and is a convex function on $[0,+\infty)$ such that $\Psi'(\theta) = T(\theta)$ for all $\theta \in [0,+\infty)$. Using estimates (4.12) and (4.14), and Hyp. 2.1 (x), we deduce that

$$E(\theta(x,t),\chi(x,t)) + k_0 \int_0^t \|\nabla(\log\theta)^-\|_H^2\, \mathrm{d}\xi + \int_0^t \int_{\partial\Omega} \Psi(\theta)\, \mathrm{d}A\, \mathrm{d}\xi$$
$$\leq C_\varrho \left(1 + \int_0^t E(\theta,\chi)\right) + E(\theta_{0,n},\chi_0) + \int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n})\, \mathrm{d}A\, \mathrm{d}\xi \tag{4.27}$$

for a.e. $(x,t) \in Q_T$. Now, in view of (2.8), we have that

$$E(\theta_{0,n},\chi_0) = -\int_\Omega \int_1^{\theta_{0,n}} c_V(\xi,\chi_0) \left(1 - \frac{1}{\xi}\right)^- \mathrm{d}\xi\, \mathrm{d}x$$
$$\leq \int_\Omega \int_0^1 c_V(\xi,\chi_0) \left(\frac{1}{\xi} - 1\right) \mathrm{d}\xi\, \mathrm{d}x \leq \int_\Omega s(1,\chi_0) \leq c_1|\Omega|\,. \tag{4.28}$$

Moreover, we can estimate the term $\int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n})\, \mathrm{d}A\, \mathrm{d}\tau$ using Hyp. 2.1 (x) as follows:

$$\int_0^t \int_{\partial\Omega} \Psi(\theta_{\Gamma,n})\, \mathrm{d}A\, \mathrm{d}\xi \leq C \left(\|\theta_{\Gamma,n}\|_{L^\infty(\Sigma_t)} + \|\left(\log(\theta_{\Gamma,n})\right)^-\|_{L^1(\Sigma_t)}\right) \leq C\,. \tag{4.29}$$

Using a standard Gronwall's lemma in (4.27), together with the estimates (4.29) and (4.28), we obtain the desired bound

$$\|(\log\theta_n)^-\|_{L^2(0,T;V)} \leq C_\varrho,$$

which, together with estimate (4.14) for θ in $L^2(0,T;V)$, gives

$$\|\log\theta_n\|_{L^2(0,T;V)} \leq C_\varrho. \tag{4.30}$$

Lower bound for θ under Hypothesis 2.3 (iii)–(vi). Let us consider relation (4.23). Notice that, by the previous estimates, it follows that there exists a positive constant R_ϱ such that $|R(\theta,\chi)| \le R_\varrho$ a.e. in Q_T. Hence, for every $z \in V$ such that $z \ge 0$ a.e., we get the following inequality (notice that by virtue of Lemma 4.1, we have here $\theta_t \in L^2(0,T;H)$):

$$\int_\Omega \left(\frac{1}{n}\theta_t + c_V(\theta,\chi)\theta_t\right) z \,\mathrm{d}x + \int_\Omega k(\bar\theta_n,\bar\chi_n)\nabla\theta\cdot\nabla z \,\mathrm{d}x + \int_{\partial\Omega} \gamma(\theta-\theta_{\Gamma,n})\, z \,\mathrm{d}A$$
$$= \int_\Omega \tilde\mu_\varrho(\theta)\left(\chi_t + \frac{R(\theta,\chi)\theta}{2\tilde\mu_\varrho(\theta)}\right)^2 - \frac{R^2(\theta,\chi)\theta^2}{4\tilde\mu_\varrho(\theta)} z \,\mathrm{d}x$$
$$\ge -\int_\Omega \frac{R_\varrho^2\theta^2}{4\tilde\mu_\varrho(\theta)} z \,\mathrm{d}x\,. \tag{4.31}$$

We now compare this inequality with the following ODE:

$$\tilde c(w) w_t = -\frac{R_\varrho^2 w^2}{4\tilde\mu_\varrho(w)}, \quad w(0) = w_0\,, \tag{4.32}$$

where $\tilde c$ is defined in Hyp. (2.3) (iii) and $w_0 = \min_{x\in\Omega}\theta_0(x) \ge \theta_*$ (cf. Hyp. 2.3 (v)).

Notice that the solution w is decreasing and does not vanish in finite time, due to Hyp. 2.3 (iii). The function w does not depend on x, hence we may add to the ODE in (4.32) the 0 term $-\mathrm{div}\,(k(\bar\theta_n,\bar\chi_n)\nabla w)$. Using the fact that $w_t < 0$ and $\tilde c(w) \le \frac{1}{n} + c_V(w,\chi)$, we obtain, subtracting (4.31) from (4.32), the inequality

$$\int_\Omega \left(\left(\frac{1}{n} + c_V(w,\chi)\right) w_t - \left(\frac{1}{n} + c_V(\theta,\chi)\right)\theta_t\right) z \,\mathrm{d}x + \int_\Omega k(\bar\theta_n,\bar\chi_n)\nabla(w-\theta)\cdot\nabla z \,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \gamma(\theta-\theta_{\Gamma,n})\, z \,\mathrm{d}A \;\le\; \frac{R_\varrho^2}{4}\int_\Omega \left(\frac{\theta^2}{\tilde\mu_\varrho(\theta)} - \frac{w^2}{\tilde\mu_\varrho(w)}\right) z \,\mathrm{d}x\,.$$

We now take as test function $z = H_\varepsilon(w-\theta)$, where H_ε is the regularization of the Heaviside function H,

$$H_\varepsilon(v) = \begin{cases} 0 & \text{if } v \le 0 \\ v/\varepsilon & \text{if } v \in (0,\varepsilon) \\ 1 & \text{if } v \ge \varepsilon \end{cases}\,. \tag{4.33}$$

By virtue of Hyp. 2.3 (iv), (v), we get

$$\int_\Omega \left(\left(\frac{1}{n} + c_V(w,\chi)\right) w_t - \left(\frac{1}{n} + c_V(\theta,\chi)\right)\theta_t\right) H_\varepsilon(w-\theta) \,\mathrm{d}x \le 0,$$

and we can pass to the limit in this inequality for $\varepsilon \searrow 0$, getting

$$\int_\Omega \left(\left(\frac{1}{n} + c_V(w,\chi)\right) w_t - \left(\frac{1}{n} + c_V(\theta,\chi)\right)\theta_t\right) H(w-\theta) \,\mathrm{d}x \le 0\,,$$

that is,

$$\frac{\partial}{\partial t}\int_\Omega \left(\left(\frac{1}{n}w + e(w,\chi)\right) - \left(\frac{1}{n}\theta + e(\theta,\chi)\right)\right)^+ \mathrm{d}x$$
$$\le \int_\Omega \left(e_\chi(w,\chi) - e_\chi(\theta,\chi)\right)\chi_t H(w-\theta)\,\mathrm{d}x\,. \tag{4.34}$$

Notice now that, by Hyp. 2.1 (vi) (cf. (2.4) and (2.7)), we have

$$|e_\chi(w,\chi) - e_\chi(\theta,\chi)| \le \max_{\tau \le w_0}(c_V(\tau,\chi))|w-\theta| \le \bar{c}|w-\theta|\,.$$

Integrating (4.34) over $(0,t)$, and using the boundedness of χ_t in $L^\infty(Q_T)$, we get, by the choice of the initial data θ_0 and w_0, that

$$\int_\Omega \left(\left(\frac{1}{n} w + e(w,\chi) \right) - \left(\frac{1}{n}\theta + e(\theta,\chi) \right) \right)^+ (t)\,\mathsf{d}x \ \le\ \bar{c} \int_\Omega (w-\theta)^+ \,\mathsf{d}x\,. \tag{4.35}$$

For $w > \theta$, we have

$$\frac{e(w,\chi) - e(\theta,\chi)}{w-\theta} = \frac{\int_\theta^w c_V(\tau,\chi)\,\mathsf{d}\tau}{w-\theta} \ge \frac{\int_\theta^w \tilde{c}(\tau)\,\mathsf{d}\tau}{w-\theta} \ge \frac{1}{w}\int_0^w \tilde{c}(\tau)\,\mathsf{d}\tau =: \tilde{C}(w)\,.$$

The function $\tilde{C}$ is nondecreasing, $\tilde{C}(w) > 0$ for $w > 0$. Hence,

$$\left(\frac{1}{n} w + e(w,\chi) \right) - \left(\frac{1}{n}\theta + e(\theta,\chi) \right) \ge \tilde{C}(w)(w-\theta) \quad \text{for } w \ge \theta\,. \tag{4.36}$$

Inequality (4.35) then yields

$$\tilde{C}(w(t)) \int_\Omega (w-\theta)^+(t)\,\mathsf{d}x \le C \int_0^t \int_\Omega (w-\theta)^+ \,\mathsf{d}x\,\mathsf{d}s$$

for every $t \in (0,T)$. From Gronwall's lemma we conclude that

$$\theta_n(x,t) \ge w(t) \quad \text{a.e. in } Q_T\,. \tag{4.37}$$

Upper bound for θ. Let us denote the right-hand side in (4.4) by

$$M(\theta,\chi) := ((\lambda'(\chi)(t) + b[\chi](t))\partial_t\chi(t) + \beta\partial_t\left(\varphi(\chi(t))\right))$$

which, due to the previous estimates is bounded by a positive constant, say, $\tilde{M}_\varrho$. Then, we compare the inequality, for all $z \in V$, $z \ge 0$ a.e.,

$$\begin{aligned} &\int_\Omega \partial_t \left(\frac{1}{n}\theta(t) + e(\theta(t),\chi(t)) \right)\, z\,\mathsf{d}x + \int_\Omega k(\bar{\theta}_n(t),\bar{\chi}_n(t))\nabla\theta(t)\cdot\nabla z\,\mathsf{d}x \\ &+ \int_{\partial\Omega} \gamma(\theta(t) - \theta_{\Gamma,n}(t)) z\,\mathsf{d}A \le \int_\Omega \tilde{M}_\varrho\, z\,\mathsf{d}x, \end{aligned} \tag{4.38}$$

with the following ODE:

$$\frac{1}{n}\,\dot{v}_n = \tilde{M}_\varrho, \quad v(0) = v_0\,, \tag{4.39}$$

where $v_0 = \max\{\sup\theta_0, \sup\theta_\Gamma\}$ (cf. Hyp. 2.1 (viii), (x)).

Then we have that

$$v_n(t) = v_0 + \tilde{M}_\varrho n t\,.$$

We proceed as above, adding to the ODE the 0 term $-\mathrm{div}\,(k(\bar{\theta}_n,\bar{\chi}_n)\nabla v)$, and subtracting (4.39) from (4.38). We test the resulting inequality by $H_\varepsilon(\theta-v)$ (cf. (4.33)). Using the fact that $(1/n + c_V(w,\chi))\,\dot{v}_n \geq 0$ and that $\theta \in H^1(0,T;H)$, we can let ε tend to 0, getting

$$\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{1}{n}\theta + e(\theta,\chi)\right) - \left(\frac{1}{n}v + e(v,\chi)\right)\right)^+ \mathrm{d}x \\ - \int_\Omega \left(e_\chi(\theta,\chi) - e_\chi(v,\chi)\right)\chi_t H(\theta-v)\,\mathrm{d}x \leq 0\,.$$

Using now the Lipschitz continuity of e_χ (cf. Hyp. 2.1 (vi)) and the boundedness of χ_t, we obtain that

$$\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{1}{n}\theta + e(\theta,\chi)\right) - \left(\frac{1}{n}v + e(v,\chi)\right)\right)^+ \mathrm{d}x \leq C_\varrho \int_\Omega (\theta-v)^+\,\mathrm{d}x\,.$$

Integrating over $(0,t)$ and using the choice of the initial condition v_0, we get

$$\int_\Omega (\theta-v)^+(t)\,\mathrm{d}x \leq C_\varrho \int_0^t \int_\Omega (\theta-v)^+\,\mathrm{d}x\,\mathrm{d}\tau,$$

and, applying Gronwall's lemma, we get the desired upper bound

$$\theta_n(x,t) \leq v_n(t) \quad \text{a.e. in } Q_T\,. \tag{4.40}$$

Moser estimate. In order to conclude the proof of existence of solutions to (2.1–2.3), it remains only to prove that the θ component of the solution (θ,χ) is bounded from above independently of T, ϱ and n.

This will enable us to choose ϱ sufficiently large in such a way that in this range of values of θ we have $s_\chi = s_\chi^\varrho$. To this end, we perform the following Moser estimate.

We will make repeated use of the well-known interpolation inequality (cf. [3])

$$\|v\|_H \leq A\left(\eta\|\nabla v\|_H + \eta^{-N/2}\|v\|_{L^1(\Omega)}\right), \tag{4.41}$$

which holds for every $v \in V$ and every $\eta \in (0,1)$, with a positive constant A independent of v and η.

Following the ideas already exploited in [20, Prop. 3.10, p. 296], for $j \in \mathbb{N}$, we choose in (4.4) $z = ((\theta_n - \theta_R)^+)^{2^j-1} \in L^2(0,T;H^1(\Omega))$, with $\theta_R = \max\{\Theta_\Gamma - 1, \Theta, 1\}$, where

$$|\theta_{\Gamma,n}(x,t)| \leq \Theta_\Gamma \quad \text{a.e. in } \Sigma_T, \quad |\theta_{0,n}(x)| \leq \Theta \quad \text{a.e. in } \Omega.$$

We know that $z \in L^2(0,T;H^1(\Omega))$, due to the upper bound on θ_n proved in (4.40).

Here below, we denote by $C_i,\ i=1,2,\ldots$ some positive constants that may depend on the data of the problem, but not on j, T, ϱ, and n. We omit again the indices n in θ_n, χ_n, for simplicity.

Now set $u = (\theta - \theta_R)$ and take $z = (u^+)^{2^j-1}$ in (4.4) to obtain that

$$\left\langle \frac{1}{n}\theta_t(t) + (e(\theta(t),\chi(t)))_t, (u^+)^{2^j-1} \right\rangle + \int_\Omega k(\bar\theta_n(t),\bar\chi_n(t))\nabla\theta(t)\cdot\nabla(u^+)^{2^j-1}\,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \gamma(\theta(t)-\theta_{\Gamma,n})\,(u^+)^{2^j-1}\,\mathrm{d}A$$
$$= -\int_\Omega \left(\lambda'(\chi)(t)\chi_t(t) + \beta\left(\varphi(\chi(t))_t\right) - b[\chi](t)\chi_t(t)\right)\,(u^+)^{2^j-1}\,\mathrm{d}x. \tag{4.42}$$

Our aim is to prove that there exists a positive constant C^* (independent of T, ϱ, and n) such that

$$\|\theta(t)\|_{L^\infty(\Omega)} \le C^*\,(1+\log\varrho)^{4+2N} \quad \text{for a.e. } t\in(0,T). \tag{4.43}$$

The first term on the left-hand side can be rewritten as

$$\left\langle \frac{1}{n}\theta_t(t) + (e(\theta(t),\chi(t)))_t, (u^+)^{2^j-1} \right\rangle = \frac{1}{n}\langle\theta_t,(u^+)^{2^j-1}\rangle + \langle\theta_t, c_V(\theta,\chi)(u^+)^{2^j-1}\rangle$$
$$+ \langle e_\chi(\theta,\chi)\chi_t,(u^+)^{2^j-1}\rangle. \tag{4.44}$$

Let us deal with the second term on the right-hand side in (4.44), using Hyp. 2.1 (vi) (notice that by (2.5), we have $c_V \ge \underline{c}$ in the set where $\theta\ge\theta_R\ge 1$) as follows:

$$\langle\theta_t, c_V(\theta,\chi)(u^+)^{2^j-1}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}E_j(u,\chi) - \int_\Omega\left(\int_0^u (c_V)_\chi(\xi+\theta_R,\chi)(\xi^+)^{2^j-1}\,\mathrm{d}\xi\right)\chi_t\,\mathrm{d}x, \tag{4.45}$$

where

$$\underline{c}\,2^{-j}\int_\Omega (u^+)^{2^j}\,\mathrm{d}x \le E_j(u,\chi) := \int_\Omega\int_0^u c_V(\xi+\theta_R,\chi)(\xi^+)^{2^j-1}\,\mathrm{d}\xi\,\mathrm{d}x \le \bar c\,2^{-j}\int_\Omega (u^+)^{2^j}\,\mathrm{d}x. \tag{4.46}$$

Then, using (4.12) and Hyp. 2.1 (vi), we infer that

$$\int_\Omega\left(\int_0^u (c_V)_\chi(\xi+\theta_R,\chi)(\xi^+)^{2^j-1}\,\mathrm{d}\xi\right)\chi_t\,\mathrm{d}x \le C_1(1+\log\varrho)^2E_j(u,\chi). \tag{4.47}$$

Moreover, by Hyp. 2.1, we get the inequality

$$\int_\Omega k(\bar\theta_n,\bar\chi_n)\nabla u\nabla\left((u^+)^{2^j-1}\right)\,\mathrm{d}x + \int_{\partial\Omega}\gamma\,(u-\theta_{\Gamma,n}+\theta_R)\,(u^+)^{2^j-1}\,\mathrm{d}A$$
$$\ge k_0\frac{2^j-1}{2^{2j-2}}\int_\Omega\left|\nabla\left((u^+)^{2^{j-1}}\right)\right|^2\,\mathrm{d}x + \int_{\partial\Omega}\gamma\,\left((u^+)^{2^j} - (u^+)^{2^j-1}\right)\,\mathrm{d}A\,.$$

Now, set $\Phi_j = (u^+)^{2^{j-1}}$. Regarding the terms on the right-hand side in (4.42) and the last term in (4.44), using (4.12) and Hyp. 2.1 (vi), we realize that

$$-\int_\Omega\left(\lambda'(\chi)(t)\chi_t(t) + \beta\left(\varphi(\chi(t))\right)_t - b[\chi](t)\chi_t(t) - e_\chi(\theta,\chi)\chi_t\right)\,(u^+)^{2^j-1}\,\mathrm{d}x$$
$$\le C_2(1+\log\varrho)^2\left(\int_\Omega|\Phi_j|^2\,\mathrm{d}x + 1\right)\,.$$

Let us now set $E_j^n = E_j + \dfrac{2^{-j}}{n} \displaystyle\int_\Omega |\Phi_j|^2 \, \mathrm{d}x$. Then, with the help of Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} E_j^n(u,\chi) + \frac{k_0(2^j-1)}{2^{2j-2}} \int_\Omega |\nabla \Phi_j|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma \, |\Phi_j|^2 \, \mathrm{d}A \\
&\quad \le \left(1 - 2^{-j}\right) \int_{\partial\Omega} \gamma \, |\Phi_j|^2 \, \mathrm{d}A + 2^{-j} \int_{\partial\Omega} \gamma \, \mathrm{d}A + C_1 (1 + \log \varrho)^2 E_j(u,\chi) \\
&\qquad + C_2 (1+\log\varrho)^2 \int_\Omega \left(|\Phi_j|^2 + 1\right) \, \mathrm{d}x \, .
\end{aligned}$$

Multiplying the above inequality by 2^j, in view of the upper bound for $E_j(u,\chi)$ in (4.46), we find out that

$$\begin{aligned}
&2^j \frac{\mathrm{d}}{\mathrm{d}t} E_j^n(u,\chi) + 2k_0 \int_\Omega |\nabla \Phi_j|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma |\Phi_j|^2 \, \mathrm{d}A \\
&\quad \le \; 2^j C_3 (1+\log\varrho)^2 \left(1 + \int_\Omega |\Phi_j|^2 \, \mathrm{d}x\right) . \qquad (4.48)
\end{aligned}$$

We now use the interpolation inequality (4.41) and note that, thanks to estimate (4.14), we have

$$\|\Phi_1\|^2_{L^1(\Omega)} = \left(\int_\Omega u^+ \, \mathrm{d}x\right)^2 \le C_4 (1+\log\varrho)^4, \quad \|\Phi_j\|^2_{L^1(\Omega)} = \|\Phi_{j-1}\|_H^4 \, .$$

Thus, we derive the inequalities

$$\int_\Omega |\Phi_1|^2 \, \mathrm{d}x \le \; 2A^2 \left(\eta^2 \|\nabla \Phi_1\|_H^2 + \eta^{-N} C_4 (1+\log\varrho)^4\right) , \qquad (4.49)$$

$$\int_\Omega |\Phi_j|^2 \, \mathrm{d}x \le \; 2A^2 \left(\eta^2 \|\nabla \Phi_j\|_H^2 + \eta^{-N} \|\Phi_{j-1}\|_H^4\right) \quad \text{for } j > 1 \, . \qquad (4.50)$$

For $j = 1$, we infer from (4.48) and (4.49) that

$$\begin{aligned}
&2 \frac{\mathrm{d}}{\mathrm{d}t} E_1^n(u,\chi) + 2k_0 \int_\Omega |\nabla \Phi_1|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma |\Phi_1|^2 \, \mathrm{d}A \\
&\quad \le \; C_5 (1+\log\varrho)^2 \left(1 + \eta^2 \int_\Omega |\nabla \Phi_1|^2 \, \mathrm{d}x + \eta^{-N} (1+\log\varrho)^4\right) .
\end{aligned}$$

Choosing $\eta = \sqrt{k_0}/(\sqrt{C_5(1+\log\varrho)^2})$, we find that

$$2 \frac{\mathrm{d}}{\mathrm{d}t} E_1^n(u,\chi) + k_0 \int_\Omega |\nabla \Phi_1|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma |\Phi_1|^2 \, \mathrm{d}A \le C_6 \left(1+\log\varrho\right)^{6+N} . \qquad (4.51)$$

For $j > 1$, we get

$$\begin{aligned}
&2^j \frac{\mathrm{d}}{\mathrm{d}t} E_j^n(u,\chi) + 2k_0 \int_\Omega |\nabla \Phi_j|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma |\Phi_j|^2 \, \mathrm{d}A \\
&\quad \le \; 2^j C_7 (1+\log\varrho)^2 \left(1 + \eta^2 \int_\Omega |\nabla \Phi_j|^2 \, \mathrm{d}x + \eta^{-N} \|\Phi_{j-1}\|_H^4\right) .
\end{aligned}$$

Choosing $\eta = \sqrt{k_0}/(\sqrt{2^j(C_7(1+\log\varrho)^2)})$, we conclude from (4.48) and (4.50) that

$$
\begin{aligned}
&2^j \frac{\mathrm{d}}{\mathrm{d}t} E_j^n(u,\chi) + k_0 \|\nabla\Phi_j\|_H^2 + \int_{\partial\Omega} \gamma |\Phi_j|^2 \, \mathrm{d}A \\
&\quad \le 2^{j\left(\frac{N}{2}+1\right)} C_8 (1+\log\varrho)^2 \left(1 + (1+\log\varrho)^N \|\Phi_{j-1}\|_H^4\right) . \qquad (4.52)
\end{aligned}
$$

By assumption, we have $\|\Phi_j(0)\|_H^2 = 0$. Hence, integrating (4.51) and (4.52) with respect to time and using the lower bound in (4.46), we obtain that

$$
\begin{aligned}
&\|\Phi_1(t)\|_H^2 \le C_9 (1+\log\varrho)^{6+N} , \\
&\|\Phi_j(t)\|_H^2 \le C_{10} 2^{j\left(\frac{N}{2}+1\right)} (1+\log\varrho)^2 \left(1 + (1+\log\varrho)^N \max_{0\le\tau\le t} \|\Phi_{j-1}(\tau)\|_H^4\right) .
\end{aligned}
$$

Define now

$$
z_j(t) = \max_{0\le\tau\le t} \sqrt{\|u(\tau)\|_{L^{2^j}(\Omega)}} = \max_{0\le\tau\le t} \|\Phi_j(\tau)\|_H^{2^{-j}} .
$$

Then we have

$$
\begin{aligned}
&z_1(t) \le C_{11} \, (1+\log\varrho)^{(6+N)/4} , \\
&z_j(t) \le C_{10}^{2^{-j-1}} 2^{j(\frac{N}{2}+1)2^{-j-1}} (1+\log\varrho)^{(N+2)2^{-j-1}} \max\{1, z_{j-1}(t)\} .
\end{aligned}
$$

In particular, putting $y_j(t) = \max\{1, z_j(t)\}$, we get

$$
y_1(t) \le C_{11} (1+\log\varrho)^{(6+N)/4}, \qquad (4.53)
$$

$$
y_j(t) \le \left(C_{12} (1+\log\varrho)^{(\frac{N}{2}+1)}\right)^{2^{-j}} 2^{\frac{j}{2}\left(\frac{N}{2}+1\right)2^{-j}} \, y_{j-1}(t) , \quad \text{for } j \ge 2. \qquad (4.54)
$$

Hence, passing to the logarithm in the inequality (4.54) and summing up the result from 2 to j, we obtain

$$
\begin{aligned}
\log y_j(t) &\le \sum_{i=2}^{j} 2^{-i} \Big(\log\left(C_{12}(1+\log\varrho)^{\frac{N}{2}+1}\right) + \frac{i}{2} \left(\frac{N}{2}+1\right) \log 2 \Big) + \log(y_1(t)) \\
&= \log\left(C_{12}(1+\log\varrho)^{\frac{N}{2}+1}\right) \sum_{i=2}^{j} 2^{-i} + \left(\frac{N}{2}+1\right) \log 2 \sum_{i=2}^{j} i 2^{-i} \\
&\quad + \log\left(C_{11} \, (1+\log\varrho)^{\frac{(6+N)}{4}}\right) \\
&\le \log\left(2^{\left(\frac{N}{2}+1\right)C_{13}}\right) + \log\left(C_{14} \, (1+\log\varrho)^{N+2}\right) ,
\end{aligned}
$$

independently of j and $t > 0$. Hence, we get

$$
y_j(t) \le 2^{\left(\frac{N}{2}+1\right)C_{13}} \left(C_{14} \, (1+\log\varrho)^{N+2}\right) ,
$$

independently of j and $t > 0$. Choosing a proper $\tilde{C}$, which is independent of ϱ, we can conclude that

$$
\sup_{t\ge 0,\, j\in\mathbb{N}} \sqrt{\|u(t)\|_{L^{2^j}(\Omega)}} \le \tilde{C}(1+\log\varrho)^{2+N} .
$$

Formula (4.43) now immediately follows.

4.4 Passage to the limit as $n \to \infty$

Our aim now is to pass to the limit in (4.4–4.6) as $n \to \infty$.

From (4.12), (4.14), (4.15), (4.21), (4.22), it follows that, up to the extraction of some subsequence of n as $n \to \infty$, there exist three functions $u,\, \theta : (0,T) \to H,\ \chi : (0,T) \to \mathbf{H}$, such that we have (as a consequence of the generalized Ascoli theorem, see, e.g., [24, Cor. 8, p. 90])

$$
\begin{aligned}
u_n \to u \quad & \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V) \\
& \text{and strongly in } C^0([0,T];V') \cap L^2(0,T;H), && (4.55)\\
\theta_n \to \theta \quad & \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), && (4.56)\\
\chi_n \to \chi \quad & \text{weakly star in } L^\infty(0,T;\mathbf{V}), && (4.57)\\
\partial_t \chi_n \to \partial_t \chi \quad & \text{weakly star in } L^\infty(\Omega \times (0,T))^d, && (4.58)\\
\chi_n \to \chi \quad & \text{strongly in } C^0([0,T];\mathbf{H}) . && (4.59)
\end{aligned}
$$

Moreover, as χ_n are uniformly bounded, it is easy to see that

$$
\bar{\chi}_n \to \chi \quad \text{weakly star in } L^\infty(0,T;\mathbf{V}) \quad \text{and strongly in } L^q(Q_T)^d, \tag{4.60}
$$

for every $q \in (1,\infty)$, as $n \to \infty$. Note that (cf. (4.7)) $u \geq 0$ and $\theta \geq 0$ a.e. Then it turns out that, at least for a subsequence, $u_n \to u$ a.e. Now, we denote by $\psi(\,\cdot\,,\chi)$ the inverse function of e with respect to the first variable, that is, $e(\psi(w,\chi)) = w$ for all $(w,\chi) \in [0,\infty) \times \mathcal{D}(\varphi)$. Since e is continuous, increasing in θ, and such that $e(\theta,\chi) \geq e_1\theta - e_2$ for some constants $e_1, e_2 > 0$, we infer that ψ is continuous with linear growth in $[0,\infty)\times\mathcal{D}(\varphi)$ and $\psi(0,\chi) = 0$. The Nemytskii operator is therefore continuous in $L^2(Q_T)$, and so this function is continuous and increasing with linear growth. Hence, we have that

$$
\theta_n = \psi\left(u_n - \theta_n/n, \chi_n\right) \to \psi(u,\chi) \quad \text{strongly in } L^2(0,T;H) .
$$

Hence, $\theta = \psi(u,\chi)$, or, equivalently, $u = e(\theta,\chi)$. Finally, we check that also $\bar{\theta}_n$ converge strongly to θ in $L^2(0,T;H)$, at least for a subsequence of $n \to \infty$. Indeed, from the definition of $\bar{\theta}_n$, we get

$$
\int_0^T \int_\Omega |\bar{\theta}_n(x,t) - \theta_n(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\bar{\theta}_n(x,t) - \theta_n(x,t)|^2 \,\mathrm{d}t\,\mathrm{d}x \tag{4.61}
$$

$$
= \int_0^{T/n} \|\theta_n(t) - \theta_{0,n}\|_H^2 \,\mathrm{d}t + \left(\frac{n}{T}\right)^2 \sum_{j=2}^n \int_\Omega \int_{t_{j-1}}^{t_j} \left| \int_{t_{j-2}}^{t_{j-1}} \left(\theta_n(x,t) - \theta_n(x,\tau)\right) \mathrm{d}\tau \right|^2 \mathrm{d}t\,\mathrm{d}x.
$$

By (4.56), the first integral on the right hand side of (4.61) can be estimated as follows:

$$
\int_0^{T/n} \|\theta_n(t) - \theta_{0,n}\|_H^2 \,\mathrm{d}t \leq \frac{C}{n} .
$$

Regarding the second term in (4.61), we proceed as follows:

$$
\begin{aligned}
&\left(\frac{n}{T}\right)^2 \sum_{j=2}^{n} \int_\Omega \int_{t_{j-1}}^{t_j} \left| \int_{t_{j-2}}^{t_{j-1}} (\theta_n(x,t) - \theta_n(x,\tau))\, \mathrm{d}\tau \right|^2 \mathrm{d}t\, \mathrm{d}x \\
&\le \frac{n}{T} \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-2}}^{t_{j-1}} \|\theta_n(t) - \theta_n(\tau)\|_H^2\, \mathrm{d}\tau\, \mathrm{d}t \\
&\le \frac{n}{T} \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} \int_{t-t_{j-1}}^{t-t_{j-2}} \|\theta_n(t) - \theta_n(t-h)\|_H^2\, \mathrm{d}h\, \mathrm{d}t \\
&\le \frac{n}{T} \int_0^{2T/n} \left(\int_h^T \|\theta_n(t) - \theta_n(t-h)\|_H^2\, \mathrm{d}t \right) \mathrm{d}h\,,
\end{aligned}
$$

where we have used the new variable $h = t - \tau$. The last integral tends to 0, because θ_n converge strongly to θ, which is mean continuous in $L^2(Q_T)$. This implies that

$$
\bar\theta_n \to \theta \quad \text{strongly in } L^2(0,T;H),
$$

and this allows us to pass to the limit in (4.4–4.6) as $n \to \infty$. Notice moreover that from estimate (4.30) we also deduce that there exists a function $\zeta : (0,T) \to V$ such that

$$
\log\theta_n \to \zeta \quad \text{weakly in } L^2(0,T;V).
$$

Using the strong convergence of θ_n and the maximal monotonicity of the extended $\log$ graph, we also deduce that $\zeta = \log\theta \in L^2(0,T;V)$, hence $\theta > 0$ a.e.

4.5 Conclusion of the existence proof

Let us now introduce the limit problem obtained by passing to the limit as $n \to \infty$ in (4.4–4.6) in the previous subsection.

PROBLEM (P)$_\varrho$. For fixed $T > 0$, and $\varrho > 0$, find two functions $\theta \in L^2(0,T;V)$ and $\chi \in L^\infty(\Omega \times (0,T))^d$, $\chi_t \in L^\infty(\Omega \times (0,T))^d$ such that, for $t \in (0,T)$, we have

$$
\begin{aligned}
&\langle (e(\theta(t),\chi(t)))_t, z\rangle + \int_\Omega k(\theta(t),\chi(t))\nabla\theta(t)\cdot\nabla z\, \mathrm{d}x + \int_{\partial\Omega} \gamma(\theta(t) - \theta_\Gamma)\, z\, \mathrm{d}A \\
&= - \int_\Omega \left(\lambda'(\chi)(t)\chi_t(t) + \beta\left(\varphi(\chi(t))\right)_t - b[\chi](t)\chi_t(t)\right) z\, \mathrm{d}x \quad \forall z \in V, \qquad (4.62)\\
&\tilde\mu_\varrho(\theta(t))\chi_t(t) + (\beta + \theta(t))\partial\varphi(\chi)(t) \ni -\lambda'(\chi)(t) - \theta(t)\sigma'(\chi)(t) \\
&- b[\chi](t) - e_\chi(\theta(t),\chi(t)) + \theta(t) s_\chi^\varrho(\theta(t),\chi(t)) \quad \text{a.e. in } \Omega\,, \qquad (4.63)
\end{aligned}
$$

and for $t = 0$ the functions $e(\theta,\chi)$ and χ satisfy the initial conditions

$$
e(\theta,\chi)(0) = u_0, \quad \chi(0) = \chi_0\,. \qquad (4.64)
$$

We thus have proved the following result.

Proposition 4.2. *Let Hyp. 2.1 hold true and let $\varrho > 0$, $\tilde{\mu}_\varrho$ and s^ϱ_χ be defined as at the beginning of Section 4.1. Then* PROBLEM (P)$_\varrho$ *has a solution in* $(0, T)$.

In order to conclude the proof of Theorem 2.2, we only need to remove the truncation ϱ. This can be done using the Moser estimate (4.43).

This bound for θ, indeed, allows us, for a suitable $\varrho \geq 1$ satisfying

$$C^*(1 + \log \varrho)^{4+2N} \leq \varrho/2,$$

with, e.g., $C^* = \tilde{C}^2$, to remove the truncation ϱ and to conclude that the solution to PROBLEM (P$_\varrho$) is in fact a solution to PROBLEM (P).

Finally, let us note that we have now found a solution on $(0, T)$ and we get that χ is weakly continuous with values in $\mathbf{V}$, $e(\theta, \chi) \in C^0([0, T]; H)$, and θ is bounded uniformly in time in $L^\infty(\Omega)$ due to the Moser estimate. Hence, we can continue the solution starting from time T and extend it on the whole time interval $(0, \infty)$.

This concludes the proof of Theorem 2.2. ■

5 Uniqueness

In this Section, we prove Theorem 2.5. Hence, we assume Hypothesis 2.3. First let us note that the lower bound for θ (2.18) directly follows by passing to the limit as n tends to ∞ in (4.37). Now we are ready to proceed in the proof of uniqueness and finally we will prove the regularity result (2.21) under the further assumption (2.20).

Uniqueness. In what follows, we denote by $R_0, R_1, R_2, \ldots$ suitable constants that possibly depend on T, but not on the solutions. We start with the proof of uniqueness of solutions. Equation (2.2) is for (almost) all $x \in \Omega$ of the form (2.10), with

$$\alpha(\theta) = \frac{\mu(\theta)}{\beta + \theta},$$
$$g = \ell[\theta, \chi] = -\frac{1}{\beta + \theta}\left(\lambda'(\chi) + \theta\sigma'(\chi) + b[\chi] + e_\chi(\theta, \chi) - \theta s_\chi(\theta, \chi)\right).$$

Within the range $0 < \theta < \overline{\theta}$ and $\chi \in \mathcal{D}_C(\varphi)$, $|\chi_t| \leq C$, of admissible values for the solutions, and, thanks to Hyp. 2.1 (ii) (iii) (v) (vi), all nonlinearities in (2.1–2.2) are Lipschitz continuous. Using the notation from Theorem 2.5, we obtain, as a consequence of (2.17), for a. e. $(x, t) \in Q_\infty$ the estimate

$$\int_0^t |\hat{\chi}_t(x, \tau)| \,\mathrm{d}\tau + |\hat{\chi}(x, t)|$$
$$\leq R_0(T)\left(|\hat{\chi}_0(x)| + \int_0^t \left(|\hat{\theta}(x, \tau)| + |\hat{\chi}(x, \tau)| + \int_\Omega |\hat{\chi}(y, \tau)| \,\mathrm{d}y\right) \mathrm{d}\tau\right), \tag{5.1}$$

with some constant $R_0(T)$. Integrating over Ω, and by Gronwall's argument, we obtain that

$$\int_\Omega |\hat{\chi}(y,t)| \,\mathrm{d}y \le R_1 \left(\int_\Omega |\hat{\chi}_0(y)| \,\mathrm{d}y + \int_0^t \int_\Omega |\hat{\theta}(y,\tau)| \,\mathrm{d}y \,\mathrm{d}\tau \right) , \tag{5.2}$$

and hence, we get

$$\int_0^t |\hat{\chi}_t(x,\tau)| \,\mathrm{d}\tau + |\hat{\chi}(x,t)| \le R_2 \left(|\hat{\chi}_0(x)| + \int_0^t |\hat{\theta}(x,\tau)| \,\mathrm{d}\tau + \int_0^t \int_\Omega |\hat{\theta}(y,\tau)| \,\mathrm{d}y \,\mathrm{d}\tau \right) \tag{5.3}$$

for a. e. $x \in \Omega$ and every $t \in [0,T]$. In particular,

$$\int_0^t \int_\Omega |\hat{\chi}_t(x,\tau)| \,\mathrm{d}x \,\mathrm{d}\tau \le R_3 \left(\int_\Omega |\hat{\chi}_0(x)| \,\mathrm{d}x + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)| \,\mathrm{d}x \,\mathrm{d}\tau \right) . \tag{5.4}$$

We now multiply (5.3) by $|\hat{\chi}(x,t)|$ and integrate over Ω to get for all $t \in [0,T]$ that

$$\int_\Omega |\hat{\chi}(x,t)|^2 \,\mathrm{d}x \le R_4 \left(\|\hat{\chi}_0\|^2_{\mathbf{H}} + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \right) . \tag{5.5}$$

The crucial point is to exploit Eq. (2.1) properly. Notice first that we have

$$b[\chi]\chi_t(x,t) = 2B[\chi]_t(x,t) + 2\int_\Omega \kappa(x,y)\, G'(\chi(x,t) - \chi(y,t))\, \chi_t(y,t) \,\mathrm{d}y . \tag{5.6}$$

We integrate the difference of the two equations (2.1), written for (θ_1, χ_1) and (θ_2, χ_2), from 0 to t, rewriting the terms $b[\chi_i](\chi_i)_t$ according to (5.6). Take $z = K(\theta_1) - K(\theta_2)$ in the resulting equation, where $K(u) = \int_0^u \bar{k}(s) \,\mathrm{d}s$, $u \in \mathbb{R}$, and integrate it again over $(0,t)$. Using the lower bound for θ, the Lipschitz continuity of all nonlinearities (φ is Lipschitz continuous on $\mathcal{D}_C(\varphi)$ with constant C), the properties of K (cf. Hyp. 2.1 (iv)), and denoting

$$\hat{\Theta}(x,t) = \int_0^t \hat{\theta}(x,\tau)\,\mathrm{d}\tau, \quad \hat{\Theta}_\Gamma(x,t) = \int_0^t \hat{\theta}_\Gamma(x,\tau)\,\mathrm{d}\tau ,$$

using (4.37), we obtain for each $t \in (0,T)$ that

$$\begin{aligned} k_0 c_V(w) \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{k_0}{2} \int_\Omega |\nabla \hat{\Theta}(x,t)|^2 \,\mathrm{d}x + \frac{k_0}{2} \int_{\partial\Omega} \gamma \hat{\Theta}^2_\Gamma(s,t) \,\mathrm{d}A \\ \le R_5 \Big(\int_0^t \int_{\partial\Omega} \gamma \hat{\theta}^2_\Gamma(s,\tau) \,\mathrm{d}A \,\mathrm{d}\tau + \|\hat{\theta}_0\|^2_H + \|\hat{\chi}_0\|^2_{\mathbf{H}} + \int_0^t \int_\Omega |\hat{\chi}(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \\ + \int_0^t \int_0^\xi \int_\Omega \int_\Omega \kappa(x,y) |\hat{\chi}_t(y,\tau)| |\hat{\theta}(x,t)| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\tau \,\mathrm{d}\xi \Big) . \end{aligned} \tag{5.7}$$

The last term on the right-hand side of the above inequality can be estimated, using (5.4), by

$$\begin{aligned} \int_0^t \int_\Omega \int_\Omega \kappa(x,y) |\hat{\chi}_t(y,\tau)| |\hat{\theta}(x,t)| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}\tau \le R_6 \int_\Omega |\hat{\theta}(x,t)| \,\mathrm{d}x \int_0^t \int_\Omega |\hat{\chi}_t(y,\tau)| \,\mathrm{d}y \,\mathrm{d}\tau \\ \le R_7 \left(\int_\Omega |\hat{\theta}(x,t)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |\hat{\chi}_0(x)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \right)^{1/2} . \end{aligned}$$

Combining the last two inequalities again with the Gronwall's lemma, we obtain for each $t \in [0,T]$ the estimate

$$
\begin{aligned}
&\int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2 \, \mathrm{d}x \ \mathrm{d}\tau + \int_\Omega |\nabla \hat{\Theta}(x,t)|^2 \, \mathrm{d}x + \int_{\partial\Omega} \gamma \hat{\Theta}^2(s,t) \, \mathrm{d}A \\
&\quad \leq R_8 \left(\|\hat{\theta}_0\|_H^2 + \|\hat{\chi}_0\|_{\mathbf{H}}^2 + \int_0^t \int_{\partial\Omega} \gamma \hat{\theta}_\Gamma^2(s,\tau) \, \mathrm{d}A \ \mathrm{d}\tau + \int_0^t \int_\Omega |\hat{\chi}(x,\tau)|^2 \ \mathrm{d}x \ \mathrm{d}\tau \right) .
\end{aligned}
\tag{5.8}
$$

We now multiply (5.8) by $2R_4$, add the result to (5.5), and see that Gronwall's argument can be applied again to arrive at the final estimate

$$
\begin{aligned}
& \int_\Omega |\hat{\chi}(x,t)|^2 \, \mathrm{d}x + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau \\
\leq \quad & R_8 \left(\|\hat{\theta}_0\|_H^2 + \|\hat{\chi}_0\|_{\mathbf{H}}^2 + \int_0^t \int_{\partial\Omega} \gamma \hat{\theta}_\Gamma^2(s,\tau) \, \mathrm{d}A \, \mathrm{d}\tau \right) .
\end{aligned}
\tag{5.9}
$$

Regularity. We prove now the regularity (2.21) for θ under the further assumption (2.20). In order to do that, let us consider, instead of (4.4), the following approximated equation

$$
\begin{aligned}
& \int_\Omega \partial_t \left(\frac{1}{n} \theta_n(t) + e(\theta_n(t), \chi_n(t)) \right) z \ \mathrm{d}x + \int_\Omega \bar{k}(\theta_n(t)) \nabla \theta_n(t) \cdot \nabla z \, \mathrm{d}x \\
& \quad + \int_{\partial\Omega} \gamma(\theta_n(t) - \theta_{\Gamma,n}(t)) z \, \mathrm{d}A \\
= \quad & - \int_\Omega \left((\lambda'(\chi_n)(t) + b[\chi_n](t)) \partial_t \chi_n(t) + \beta \partial_t \left(\varphi(\chi_n(t)) \right) \right) z \, \mathrm{d}x .
\end{aligned}
\tag{5.10}
$$

Since (cf. Hypothesis 2.3 (i)) the heat conductivity $\bar{k}$ is independent of χ, we can now test (5.10) by $K(\theta_n)_t$, where $K(\theta_n) = \int_0^{\theta_n} \bar{k}(s) \ \mathrm{d}s$ and, integrating over $(0,t)$, we obtain, using (4.37), the estimate

$$
\begin{aligned}
& \left(\frac{1}{n} + k_0 c_V(w) \right) \int_0^t \int_\Omega (\theta_n)_t \, \mathrm{d}x \, \mathrm{d}\tau + \frac{1}{2} \|\nabla (K(\theta_n))(t)\|_H^2 + k_0 k_1 \int_{\partial\Omega} \gamma K(\theta_n)^2(t) \, \mathrm{d}A \\
& \leq \frac{1}{2} \|\nabla (K(\theta_{0n}))\|_H^2 + \int_0^t \int_{\partial\Omega} |\gamma| |(\theta_{\Gamma,n})_t| K(\theta_n) \, \mathrm{d}A \, \mathrm{d}\tau + C k_1 \int_0^t \int_\Omega |(\theta_n)_t| \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned}
$$

Here C is a bound in $L^\infty(Q_T)$ for the term $(\lambda'(\chi_n) + b[\chi_n]) \partial_t \chi_n + \beta \partial_t \left(\varphi(\chi_n) \right)$ that we already obtained in Theorem 2.2, and k_0, k_1 are the positive constants introduced in Hypothesis 2.1 (iv). Using now assumption (2.20) and Hypothesis 2.1 (viii), (ix), together with a standard Gronwall lemma, we obtain the (independent of n) bound

$$
\|(\theta_n)_t\|_{L^2(0,T;H)} + \|K(\theta_n)\|_{L^\infty(0,T;V)} \leq C ,
$$

which leads immediately (due to Hypothesis 2.1 (iv)) to the desired estimate (2.21).

With this, Theorem 2.5 is proved. ■

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