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Singularly perturbed systems: Case of exchange of stability

Nikolai Nefedov[1] , Klaus Schneider[2]

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[1] Moscow State University
Faculty of Physics
Department of Mathematics
119899 Moscow
Russia

[2] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany

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Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

Singularly Perturbed Systems: Case of Exchange of Stability

N.N. Nefedov*
Moscow State University
Faculty of Physics
Department of Mathematics
119899 Moscow

K.R. Schneider†
Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstraße 39
10117 Berlin

Abstract. We consider systems of singularly perturbed ordinary differential equations and study the initial value problem on a finite interval. Our goal is to describe the asymptotic behavior of its solution with respect to ε in case of exchange of stability of a solution of the degenerate system considered as a steady state solution of the associated system. The obtained results extend the well-known fundamental theorems due to A.N. Tikhonov and A.B. Vasil'eva.

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1. Introduction

Modelling numerous processes in techniques and natural sciences leads to singularly perturbed systems of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,\varepsilon) \\ \varepsilon\,\frac{dy}{dt} &= g(x,y,t,\varepsilon) \end{aligned} \tag{1.1}$$

where $x \in R^k, y \in R^l$, and ε is a small positive parameter.
Basically we may distinguish two main approaches in the study of singularly perturbed systems: the longtime behavior of a sample of trajectories and the transition behavior of one trajectory on a finite time interval where in both cases the smallness of ε plays a crucial role.

*e–mail: nefedov@math384.phys.msu.su
†e–mail: schneide@WIAS-berlin.de

In what follows we consider the initial value problem to (1.1)

$$x(t_0, \varepsilon) = x^0, \; y(t_0, \varepsilon) = y^0, \quad t \in (t_0, t_e) =: T, \; t_e > t_0 \tag{1.2}$$

on a finite time interval and study the asymptotic behavior of its solution $(x(t,\varepsilon), y(t,\varepsilon))$ with respect to ε under conditions which are not included in the standard theory [3, 4, 10, 12, 13, 14, 15] and which can be viewed as an exchange of stability of an equilibrium solution of the associated system.

The motivation for considering such problems comes from the study of jumping behavior of the fast reaction rate in bimolecular reactions [9]. In order to be able to compare our results with the standard fundamental theorems due to A.N. Tikhonov [11, 12] and A.B. Vasil'eva [13] we will recall these theorems in the next section. In section 3 we introduce the basic assumptions and the key tool, the method of lower and upper solution. Section 4 contains our main results which say that an exchange of stability influences the asymptotic behavior of (1.1), (1.2) with respect to ε near the point of exchange of stability. In the case under consideration we have a change from an $O(\varepsilon)$-behavior to $O(\sqrt{\varepsilon})$. In the final section we illustrate our result by considering an example from the reaction kinetics.

2. Fundamental results of the standard theory

Let D_x and D_y be open bounded regions in R^k and R^l respectively, let J be the interval $J := \{\varepsilon \in R : 0 \leq \varepsilon < \varepsilon_* \ll 1\}$, let $D := D_x \times D_y \times T \times J$. Concerning the smoothness of f and g we suppose

(T_1). $f : D \to R^k$, $g : D \to R^l$ are continuous and continuously differentiable with respect to the first three variables.

It is obvious that the asymptotic behavior of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (1.1), (1.2) with respect to ε depends on the solution set of the equation

$$g(x, y, t, 0) = 0. \tag{2.1}$$

The first result in this direction is due to A.N. Tikhonov [11, 12]. To formulate his result we introduce the assumptions:

(T_2). Equation (2.1) has an isolated solution $y = \varphi(x,t)$ defined for $(x,t) \in D_x^0 \times \bar{T}$ where D_x^0 is a closed simply connected subset of D_x , and $\bar{T} := [t_0, t_e]$.

(T_3). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi(x,t), t, 0) \; , \; x(t_0) = x^0 \in D_x^0 \tag{2.2}$$

has a unique solution $\tilde{x}(t; t_0)$ defined on T.

(T_4). $y = \varphi(x,t)$ is an asymptotically stable equilibrium point of the associated system

$$\frac{dy}{d\tau} = g(x,y,t,0) \tag{2.3}$$

uniformly for $(x,t) \in D_x^0 \times \overline{T}$ (x and t are considered as parameters in (2.3)).

(T_5). The initial value problem

$$\frac{dy}{d\tau} = g(x^0, y, t_0, 0), \quad y(0) = y^0 \tag{2.4}$$

has a unique solution $\tilde{y}(\tau, y^0)$ which exists for $\tau \geq 0$ and tends to $\varphi(x^0, t_0)$ as $\tau \longrightarrow \infty$.

Hypothesis (T_5) says that y^0 is in the basin of attraction of the equilibrium point $\varphi(x^0, t_0)$ of (2.4).

A.N. Tikhonov has got essentially the result

Theorem 2.1 *Suppose the hypotheses $(T_1)-(T_5)$ hold. Then there exists a sufficiently small positive ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ the initial value problem (1.1), (1.2) has a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ satisfying*

$$\begin{aligned} \lim_{\varepsilon \to 0} x(t,\varepsilon) &= \tilde{x}(t, x^0) && \textit{for } t_0 \leq t \leq t_e, \\ \lim_{\varepsilon \to 0} y(t,\varepsilon) &= \varphi(\tilde{x}(t,x^0), t) && \textit{for } t_0 < t \leq t_e. \end{aligned}$$

In order to formulate the next theorem which is due to A.B. Vasil'eva [13], we introduce the concept of an asymptotic expansion of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (1.1), (1.2).

Definition 2.2 *An asymptotic expansion of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (1.1), (1.2) is a representation of $x(t,\varepsilon)$ and $y(t,\varepsilon)$ in the form*

$$z_a(t,\varepsilon) = Rz(t,\varepsilon) + \Pi z(\tau,\varepsilon) \tag{2.5}$$

where z is a placeholder for x and y respectively, $Rz(t,\varepsilon)$ is the regular part of the asymptotics, that is,

$$Rz(t,\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i R_i z(t), \tag{2.6}$$

and $\Pi z(\tau,\varepsilon)$ is the boundary layer correction near $t = t_0$,

$$\Pi z(\tau,\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i \Pi_i z(\tau) \tag{2.7}$$

where τ is the stretched variable $\tau = (t - t_0)/\varepsilon$. We denote by $Z_k(t,\varepsilon)$ the truncated part of (2.5)

$$Z_k(t,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i (R_i z(t) + \Pi_i z(\tau)).$$

Let F be some function defined on $R^k \times R \times J$. By means of the representation (2.5) we may rewrite $F(z_a(t,\varepsilon),t,\varepsilon)$ in the form

$$\begin{aligned} F(z_a(t,\varepsilon),t,\varepsilon) &= F(Rz(t,\varepsilon),t,\varepsilon) + F(z_a(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) \\ &\quad -F(Rz(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) =: RF + \Pi F \end{aligned} \tag{2.8}$$

where

$$RF := F(Rz(t,\varepsilon),t,\varepsilon),\ \Pi F := F(z_a(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) - F(Rz(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon). \tag{2.9}$$

In order to compute the coefficients $R_iz(t)$ and $\Pi_iz(\tau)$ we substitute (2.5) – (2.7) into (1.1), (1.2) and use the representation (2.8), (2.9). By equating expressions with the same power of ε (separately for t and τ) we obtain equations which let us determine the unknown coefficients of the asymptotic expansion. In particular, by assumption (T_2), $R_0x(t)$ and $R_0y(t)$ are uniquely determined by the degenerate system (2.1) and the initial value $x^0 : R_0x(t) = \tilde{x}(t,x^0), R_0y(t) = \varphi(R_0x(t),t)$. Note that $\Pi x_0(\tau)$ and $\Pi y_0(\tau)$ are determined by the initial value problems (see [13])

$$\begin{aligned} \frac{d\Pi_0 y}{d\tau} &= \Pi_0 g(R_0x(t_0)+\Pi_0x(\tau), R_0y(t_0)+\Pi_0y(\tau),t_0,0), \quad \Pi_0y(t_0) = y^0 - R_0y(t_0), \\ \frac{d\Pi_0 x}{d\tau} &= 0, \quad \Pi_0x(t_0) = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Pi_0 x(\tau) &\equiv 0 \\ \frac{d\Pi_0 y}{d\tau} &= \Pi_0 g(x^0, \varphi(x^0,t_0) + \Pi_0y(\tau), t_0, 0). \end{aligned} \tag{2.10}$$

Finally, we strengthen the assumptions (T_1) and (T_4) as follows.

$(\tilde{T}_1)$. *The functions f and g are $(k+2)$-times continuously differentiable with respect to all variables in the domain of interest.*

$(\tilde{T}_4)$. *All eigenvalues $\lambda_i(t)$ of the Jacobian $g_y(\tilde{x}(t,x^0),\varphi(\tilde{x}(t,x^0),t),t,0)$ satisfy*

$$\operatorname{Re}\lambda_i(t) < 0 \quad \text{for} \quad t \in T, \quad 1 \le i \le m.$$

Theorem 2.2 *We assume the hypotheses $(\tilde{T}_1),(T_2),(T_3),(\tilde{T}_4),(T_5)$ to hold. Let $(X_k(t,\varepsilon)$, $Y_k(t,\varepsilon))$ be the truncated parts of the asymptotic expansion of the solution of problem (1.1), (1.2) obtained by the method of boundary layer functions* (see [13], [14] for details). *Then there exists a sufficiently small ε_0 and a constant $c = c(\varepsilon_0)$ such that for*

$0 < \varepsilon \leq \varepsilon_0$ *the initial value problem* (1.1), (1.2) *has a unique solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *for* $t \in T$ *satisfying*

$$\begin{aligned} |x(t,\varepsilon) - X_k(t,\varepsilon)| &\leq c\,\varepsilon^{k+1}, \\ |y(t,\varepsilon) - Y_k(t,\varepsilon)| &\leq c\,\varepsilon^{k+1}. \end{aligned}$$

In particular, we have for $k = 0$:

$$x(t,\varepsilon) = R_0 x(t) + O(\varepsilon),\ y(t,\varepsilon) = R_0 y(t) + \Pi_0 y(\tau) + O(\varepsilon)$$

where $\Pi_0 y$ *is defined by (2.10).*

3. Exchange of Stability. Assumptions. Notation

In what follows we consider the case that if we replace in (2.3) x by $\tilde{x}(t;t_0)$ then the equilibrium point $\varphi(\tilde{x}(t;t_0),t)$ of (2.3) changes its stability when the parameter t passes some critical value $t_c, t_0 < t_c < t_e$. This situation can arise if (2.1) has two solutions intersecting for $t = t_c$.

We study the singularly perturbed differential system (1.1) under the following assumptions

(S_1). $f : D \to R^k, g : D \to R^l$ are twice continuously differentiable with respect to all variables.

(S_2). Equation (2.1) has two different solutions $y = \varphi_1(x,t)$ and $y = \varphi_2(x,t)$ defined in $D_x^0 \times \overline{T}$ and with the same smoothness properties as f and g.

(S_3). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi_1(x,t), t, 0)\ ,\ x(t_0) = x^0$$

has a unique solution $\tilde{x}^1(t,x^0)$ defined on T. There is a point t_c in T such that

(i) For $t_0 \leq t < t_c$, the real parts of all eigenvalues of the Jacobian $G_1(t) := g_y(\tilde{x}^1(t,x^0), \psi_1(t), t, 0)$ are negative where $\psi_1(t)$ is defined by $\psi_1(t) := \varphi_1(\tilde{x}^1(t, x^0), t)$. That is $\psi_1(t)$ is an asymptotically stable equilibrium of the associated system

$$\frac{dy}{d\tau} = g(\tilde{x}^1(t,x^0), y, t, 0) \tag{3.1}$$

for $t \in [t_0, t_c)$.

(ii) For $t = t_c$ exactly one simple real eigenvalue $e_1(t)$ of $G_1(t)$ vanishes and crosses the imaginary axis transversally that is, $e_1(t_c) = 0,\ e_1'(t_c) > 0$.

(iii) For $t \in (t_c, t_e]$ exactly one eigenvalue of $G_1(t)$ has positive real part.

(S_4). The initial value problem

$$\frac{dy}{d\tau} = g(x^0, y, t_0, 0), \;\; y(0) = y^0$$

has a unique solution $\tilde{y}(\tau, y^0)$ which exists for $\tau \geq 0$ and tends to $\varphi_1(x^0, t_0)$ as $\tau \to \infty$.

Assumption (S_4) means that y^0 lies in the basin of attraction of the equilibrium point $\varphi_1(x^0, t_0)$ of (3.1).

From assumption $(S_1)-(S_3)$ it follows that $\psi_1(t)$ is a differentiable one-parameter family of equilibria of the associated system (3.1) which intersects for $t = t_c$ another differentiable one-parameter familiy of equilibria of system (3.1). The following assumption says that this family is related to the second root $\varphi_2(x, t)$ of (2.1).

(S_5). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi_2(x,t), t, 0) \;, \;\; x(t_c) = x^1 := \tilde{x}^1(t_c, x^0)$$

has a unique solution $\tilde{x}^2(t, x^1)$ defined on T such that

(i) For $t_c < t \leq t_e$ all eigenvalues of the Jacobian $G_2(t) := g_y(\tilde{x}^2(t,x^1), \psi_2(t), t, 0)$ are in the left half plane where $\psi_2(t)$ is defined by $\psi_2(t) := \varphi_2(\tilde{x}^2(t,x^1), t)$.

(ii) For $t = t_c$ exactly one simple real eigenvalue $e_2(t)$ of $G_2(t)$ vanishes and crosses the imaginary axis transversally, that is $e_2(t_c) = 0, \; e_2'(t_c) < 0$.

(iii) For $t \in [t_0, t_c)$ exactly one eigenvalue of $G_2(t)$ has positive real part.

Definition 3.1 *Under the assumption (S_3) and (S_5) the vector function $(\hat{x}(t), \hat{y}(t))$ defined by*

$$\hat{x}(t) := \begin{cases} \tilde{x}^1(t, x^0) & t_0 \leq t \leq t_c \\ \tilde{x}^2(t, x^1) & t_c \leq t \leq t_e \end{cases} , \; \hat{y}(t) := \begin{cases} \psi_1(t) & t_0 \leq t \leq t_c \\ \psi_2(t) & t_c \leq t \leq t_e \end{cases} \tag{3.2}$$

is referred to as the composed stable solution of (1.1) with respect to $\psi_1(t), \psi_2(t)$.

From this definition we obtain

$$\begin{aligned} \frac{d\hat{x}}{dt} &= f(\hat{x}(t), \hat{y}(t), t, 0) \\ 0 &= g(\hat{x}(t), \hat{y}(t), t, 0). \end{aligned} \tag{3.3}$$

In what follows let ν be any fixed small positive number. It is obvious that under the hypotheses $(S_1) - (S_5)$ Theorem 2.3 describes the asymptotic behavior of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of the initial value problem (1.1), (1.2) on the interval $(t_0, t_c - \nu)$. A similar approach is valid for $t \in (t_c + \nu, t_e)$ provided $y(t_c + \nu, \varepsilon) = y^\nu$ lies in the basin of attraction of the equilibrium point $\varphi_2(\tilde{x}^2(t_c + \nu, x^1), t_c + \nu)$ of the associated system

$$\frac{dy}{d\tau} = g(\tilde{x}^2(t, x^1), y, t, 0).$$

The critical interval is the interval $I_\nu := [t_c - \nu, t_c + \nu]$. To prove the existence of a solution of (1.1) defined on I_ν and to get an asymptotic approximation of it for small ε we will apply the method of upper and lower solutions. Following [1], [2], [7] we call $(\overline{x}(t,\varepsilon), \overline{y}(t,\varepsilon))$ and $(\underline{x}(t,\varepsilon), \underline{y}(t,\varepsilon))$ the upper and lower solution of (1.1), (1.2) respectively, provided they satisfy the following inequalities

$$\begin{aligned}
&\underline{x}(t,\varepsilon) \leq_+ \overline{x}(t,\varepsilon),\ \underline{y}(t,\varepsilon) \leq_+ \overline{y}(t,\varepsilon) \\
&\frac{d\underline{x}}{dt} - f(\underline{x}, y, t, \varepsilon) \leq_+ 0 \leq_+ \frac{d\overline{x}}{dt} - f(\overline{x}, y, t, \varepsilon) \\
&\varepsilon\,\frac{d\underline{y}}{dt} - g(x, \underline{y}, t, \varepsilon) \leq_+ 0 \leq_+ \varepsilon\,\frac{d\overline{y}}{dt} - g(x, \overline{y}, t, \varepsilon)
\end{aligned} \tag{3.4}$$

for $t \in T,\ x \in [\underline{x}(t,\varepsilon), \overline{x}(t,\varepsilon)],\ y \in [\underline{y}(t,\varepsilon), \overline{y}(t,\varepsilon)]$, and

$$\underline{x}(t_0,\varepsilon) \leq_+ x^0 \leq_+ \overline{x}(t_0,\varepsilon),\ \underline{y}(t_0,\varepsilon) \leq_+ y^0 \leq_+ \overline{y}(t_0,\varepsilon)$$

where $\leq_+$ means the partial ordering induced by the cone of vectors with nonnegative components. In order to be able to conclude that the existence of a lower and an upper solution of (1.1), (1.2) implies the existence of a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (1.1), (1.2) satisfying

$$\begin{aligned}
\underline{x}(t,\varepsilon) &\leq x(t,\varepsilon) \leq \overline{x}(t,\varepsilon), \\
\underline{y}(t,\varepsilon) &\leq y(t,\varepsilon) \leq \overline{y}(t,\varepsilon)
\end{aligned}$$

we introduce the following monotonicity assumption:

(S_6.) For $1 \leq i \leq k$, the functions $f_i(x_1, \ldots, x_{i-1}, \hat{x}_i(t), x_{i+1}, \ldots, x_k, y, t, \varepsilon)$ are nondecreasing in $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, y)$ and for $1 \leq j \leq l$ the functions $g_j(x, y_1, \ldots, y_{j-1}, \hat{y}_j(t), y_{j+1}, \ldots, y_l, t, \varepsilon)$ are nondecreasing in $(x, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_l, t, \varepsilon)$ for $\varepsilon \in J$ and $t \in I_\nu$.

Remark 3.2 *If we set $\eta := (x,y)$, $h := (f,g)$ then assumption (S_6) means that the function h is a socalled quasimonotone function of η. This property has been introduced by E. Kamke [6] and M. Müller [7] and plays now an important role in the theory of monotone systems [5, 8]. A consequence of a quasimonotone function is that its Jacobian is such that all off-diagonal elements are nonnegative that is, it maps the cone of nonnegative vectors into itself. Assumption (S_6) in particular implies $f_y \leq 0$ and $g_x \leq 0$.*

Let us introduce the matrix

$$G(t) := \begin{cases} G_1(t) & \text{for} \quad t_0 \leq t \leq t_c, \\ G_2(t) & \text{for} \quad t_c \leq t \leq t_e. \end{cases}$$

From the assumptions (S_3) and (S_5) it follows

$$G(t) = g_y((\hat{x}(t), \hat{y}(t), t, 0)$$

and that there is a sufficiently small $\nu > 0$ such that for $t \in I_\nu$ $G_i(t)$ has a unique simple eigenvalue $e_i(t)$ with a corresponding eigenvector $v_i(t)$ where e_i and v_i for $i = 1, 2$ are continuously differentiable and satisfy $e_i(t_c) = 0$ and $e'(t_c) \neq 0$. Thus, for $t \in I_\nu$ $G(t)$ has a simple eigenvalue $e(t)$ with an eigenvector $v(t)$ such that e and v are continuous in I_ν with $e(t_c) = 0$ and continuously diffentiable in $I_\nu \backslash \{t_c\}$.

Finally we need the assumptions

(S_7.) $v(t_c)$ is strictly positive.

(S_8.) $-g_{yy}(\hat{x}(t_c), \hat{y}(t_c), t_c, 0)v(t_c)v(t_c) >_+ r >_+ 0$.

(S_9). For $t_0 \leq t \leq t_c + \nu, \varepsilon \in J$ it holds

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &\leq_+ f(\hat{x}(t), \hat{y}(t), t, \varepsilon), \\ \varepsilon \frac{d\hat{y}(t)}{dt} &\leq_+ g(\hat{x}(t), \hat{y}(t), t, \varepsilon). \end{aligned}$$

Assumption (S_9) means that the composed stable solution is a lower solution of our initial value problem.

4. Asymptotic Behavior in Case of Exchange of Stability

The following theorem characterizes the influence of an exchange of stability of the family $\psi_1(t)$ of equilibria of (3.1) on the asymptotic behavior of the solution of the initial value problem (1.1),(1.2): near the critical point the usual $O(\varepsilon)$–behavior is replaced by an $O(\sqrt{\varepsilon})$– behavior. The proof of our main result is based on the application of the method of lower and upper solutions.

Theorem 4.1 . *Assume hypotheses* $(S_1) - (S_9)$ *to be valid. Then to any given small* $\nu > 0$ *there exists a sufficiently small* $\varepsilon_0 = \varepsilon_0(\nu)$ *such that for* $0 < \varepsilon \leq \varepsilon_0(\nu)$ *the initial value problem (1.1),(1.2) with* $x^0 \geq_+ \hat{x}(t_0), y^0 \geq_+ \hat{y}(t_0)$ *has a unique solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *satisfying*

$$\begin{aligned} \lim_{\varepsilon \to 0} x(t,\varepsilon) &= \hat{x}(t) \quad \textit{for } t \in \overline{T}, \\ \lim_{\varepsilon \to 0} y(t,\varepsilon) &= \hat{y}(t) \quad \textit{for } t_0 < t \leq t_e. \end{aligned}$$

Moreover we have

$$\begin{aligned} x(t,\varepsilon) &= \hat{x}(t) + 0(\varepsilon) & & \textit{for } \; t \in \overline{T} \\ y(t,\varepsilon) &= \begin{cases} \hat{y}(t) + \Pi_0 y(\tau) + 0(\varepsilon) & \textit{for } \; t_0 \leq t \leq t_c - \nu \\ \hat{y}(t) + 0(\varepsilon^{\frac{1}{2}}) & \textit{for } \; t \in I_\nu \\ \hat{y}(t) + 0(\varepsilon) & \textit{for } \; t_0 + \nu \leq t \leq t_e \end{cases} \end{aligned}$$

where $\Pi_0 y(\tau)$ *is the zeroth order boundary layer function.*

Proof. The proof proceeds in three steps. In the first step we consider the initial value problem (1.1), (1.2) on the interval $[t_0, t_c - \tilde{\nu}]$ where $\tilde{\nu}$ is any small positive number. It is obvious that under the hypotheses above Theorem 2.3 applies. Thus, to given $\tilde{\nu}$ there is an $\varepsilon = \varepsilon(\tilde{\nu})$ such that there exists a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (1.1), (1.2) on the interval $[t_0, t_c - \tilde{\nu}]$ with the asymptotic behavior as described above. Let

$$x_{\tilde{\nu}}^- := x(t_c - \tilde{\nu}, \varepsilon), \; y_{\tilde{\nu}}^- := y(t_c - \tilde{\nu}, \varepsilon). \tag{4.1}$$

Now we consider the initial value problem (1.1), (4.1) on the interval $I_{\tilde{\nu}}$. We prove the existence of a unique solution to this problem by applying the method of lower and upper solutions. First we note that according to hypothesis (S_9) $(\hat{x}(t), \hat{y}(t))$ is a lower solution to (1.1), (4.1) on the interval $I_{\tilde{\nu}}$. To construct an upper solution we use the functions

$$\overline{x}(t,\varepsilon) = \hat{x}(t) + \varepsilon \exp(\lambda t) v_f, \quad \overline{y}(t,\varepsilon) = \hat{y}(t) + \gamma\sqrt{\varepsilon} v(t) \tag{4.2}$$

where v_f is any strictly positive constant vector, the constants λ and γ will be choosen later in an appropriate way. It is clear that to given ε_1, $0 \le \varepsilon_1 \le \varepsilon(\tilde{\nu})$, there are positive numbers λ_1 and γ_1 such that for $0 \le \varepsilon \le \varepsilon_1$ and $\lambda \ge \lambda_1$, $\gamma \ge \gamma_1$ the following inequalities hold

$$\hat{x}(t_c - \tilde{\nu}) \le_+ x_{\tilde{\nu}}^- \le_+ \overline{x}(t_c - \tilde{\nu}, \varepsilon), \quad \hat{y}(t_c - \tilde{\nu}) \le_+ y_{\tilde{\nu}}^- \le_+ \overline{y}(t_c - \tilde{\nu}, \varepsilon).$$

Now we prove that $\overline{x}(t,\varepsilon)$, $\overline{y}(t,\varepsilon)$ satisfy the differential inequalities (3.4) characterizing an upper solution of problem (1.1), (4.1) on the interval $I_{\tilde{\nu}}$. From (4.2), (S_6), (3.2), (S_1), and (3.3) we get

$$\begin{aligned}
\frac{d\overline{x}}{dt} - f(\overline{x}(t), \overline{y}(t), t, \varepsilon) \;&=\; \frac{d\hat{x}}{dt} + \varepsilon\lambda \exp(\lambda t) v_f \\
&-\; f(\hat{x}(t) + \varepsilon \exp(\lambda t) v_f, \hat{y}(t) + \gamma\sqrt{\varepsilon} v(t), t, \varepsilon) \\
&\ge_+\; \frac{d\hat{x}}{dt} + \varepsilon\lambda \exp(\lambda t) v_f \\
&-\; f(\hat{x}(t) + \varepsilon \exp(\lambda t) v_f, \hat{y}(t), t, \varepsilon) \\
&=\; \frac{d\hat{x}}{dt} + \varepsilon\lambda \exp(\lambda t) v_f \\
&-\; f(\hat{x}(t), \hat{y}(t), t, 0) - \varepsilon \exp(\lambda t) f_x(\hat{x}(t), \hat{y}(t), t, 0) v_f \\
&-\; \varepsilon f_\varepsilon(\hat{x}(t), \hat{y}(t), t, 0) + o(\varepsilon) \\
&=\; \varepsilon \exp(\lambda t)\Big[\lambda v_f - f_x(\hat{x}(t), \hat{y}(t), t, 0) v_f \\
&-\; \exp(-\lambda t) f_\varepsilon(\hat{x}(t), \hat{y}(t), t, \varepsilon)\Big] + o(\varepsilon).
\end{aligned}$$

Since $f_x(\hat{x}(t), \hat{y}(t), t, 0)$ and $f_\varepsilon(\hat{x}(t), \hat{y}(t), t, 0)$ are continuous in t and since v_f is strictly positive, to given ε_1 and $\tilde{\nu}$ there is a sufficiently large $\lambda_2 \ge \lambda_1$ such that for $t \in I_{\tilde{\nu}}$, $\lambda \ge \lambda_2$, and $0 < \varepsilon \le \varepsilon_1$

$$\lambda v_f - f_x(\hat{x}(t), \hat{y}(t), t, 0) - e^{-\lambda t} f_\varepsilon(\hat{x}(t), \hat{y}(t), t, \varepsilon) \ge_+ \kappa >_+ 0$$

where κ is some strictly positive vector. Consequently, there is a sufficiently small positive number ε_2, $0 < \varepsilon_2 \le \varepsilon_1$, such that for $t \in I_{\tilde{\nu}}$ and $0 < \varepsilon < \varepsilon_2$

$$\frac{d\overline{x}}{dt} - f(\overline{x}, \overline{y}, t, \varepsilon) \ge_+ 0.$$

Before we derive a similar inequality for $\varepsilon\ d\overline{y}/dt - g(\overline{x},\overline{y},t,\varepsilon)$ we note that according to hypothesis (S_8) and to the continuity of $g_{yy}(\hat{x}(t),\hat{y}(t),t,0)$ with respect to t there is a small positive number $\nu_2, \nu_2 \leq \tilde{\nu}$, and a strictly positive vector v_h such that for $t \in I_{\nu_2}$

$$-g_{yy}(\hat{x}(t),\hat{y}(t),t,0)v(t)v(t) \geq_+ v_h >_+ 0. \tag{4.3}$$

Using this inequality we get from (4.2), (S_6), (S_1), and (3.3)

$$\begin{array}{rl} & \varepsilon\,\dfrac{d\overline{y}}{dt} - g(\overline{x}(t),\overline{y}(t),t,\varepsilon) \\ \geq_+ & \varepsilon\,\dfrac{d\hat{y}}{dt} + \gamma\varepsilon^{3/2}\,\dfrac{dv}{dt} - g(\hat{x}(t),\hat{y}(t)+\sqrt{\varepsilon}\gamma v(t),t,\varepsilon) \\ = & -\sqrt{\varepsilon}\gamma g_y(\hat{x}(t),\hat{y}(t),t,0))v(t) + \varepsilon\dfrac{d\hat{y}}{dt} \\ - & g(\hat{x}(t),\hat{y}(t),t,0)) - \varepsilon g_\varepsilon(\hat{x}(t),\hat{y}(t),t,0)) \\ - & \varepsilon\dfrac{\gamma^2}{2}g_{yy}(\hat{x}(t),\hat{y}(t),t,0))v(t)v(t) + o(\varepsilon) \\ \geq_+ & \varepsilon\Big(\dfrac{\gamma^2}{2}v_h - g_\varepsilon(\hat{x}(t),\hat{y}(t),t,0) + \dfrac{d\hat{y}}{dt}\Big) + o(\varepsilon). \end{array}$$

Thus, there are positive numbers $\varepsilon_3,\ \varepsilon_3 \leq \varepsilon_2$, and a sufficiently large γ_0, such that for $0 < \varepsilon \leq \varepsilon_3, \gamma > \gamma_0$ and $t \in I_{\nu_2}$

$$\varepsilon\frac{d\overline{y}}{dt} - g(\overline{x}(t),\overline{y}(t),t,\varepsilon) \geq_+ 0.$$

Consequently, we have proved the existence of a lower and an upper solution of (1.1), (4.1) on I_{ν_2} which imply under the hypothesis (S_6) the existence of a unique solution of (1.1), (4.1) on I_{ν_2} satisfying the estimate of Theorem 4.1.
Let

$$x^+_{\nu^*} := \overline{x}(t_c + \nu_2/2,\varepsilon),\ y^+_{\nu^*} := \overline{y}(t_c + \nu_2/2,\varepsilon). \tag{4.4}$$

In the last step we apply Theorem 2.3 to the initial value problem (1.1), (4.4) on the interval $[t_0 + \nu_2/2, t_e]$ for $0 < \varepsilon < \varepsilon_0(\nu_2)$ where we assume that $\varepsilon_0(\nu_2)$ is so small such that $x^+_{\nu^*}$ is in the domain of attraction of the stable root φ_2 and that the corresponding boundary layer is contained in the interval $(t_c + \nu_2/2, t_c + \nu_2)$ for $0 < \varepsilon \leq \varepsilon_0(\nu_2)$. This completes the proof of the theorem. $\square$

Now we consider the initial value problem (1.1) (1.2) with $x^0 < \hat{x}(t_0),\ y^0 < \hat{y}(t_0)$. To this end we replace hypotheses (S_6) and (S_9) as follows:

$(\tilde{S}_6)$. For $i = 1,\dots k,\ \varepsilon \in J$ and $t \in I_\nu$ the functions $f_i(x_1,\dots,x_{i-1},\tilde{x}^2_i(t,x^1),\ x_{i+1},\dots,x_k,y,t,\varepsilon)$ are non-decreasing in $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k,y)$ and for $i = 1,\dots,l$, $\varepsilon \in J,\ t \in I_\nu$ the functions $g_i(x,y_1,\dots,y_{i-1},\ \psi_{2i}(t),\ y_{i+1},\dots,y_l,t,\varepsilon)$ are non-decreasing in $(x,y_1,\dots,\ y_{i-1},y_{i+1},\dots,y_l)$ in the region defined by $x_j \in [\tilde{x}^2_j(t,x^1), \overline{x}_j(t,\varepsilon)], j = 1,\dots,k,\ y_i \in [\psi_{2j}(t),\overline{y}_j(t,\varepsilon)]\ ,\ j = 1,\dots,l.$

$(\tilde{S}_9)$. For $t \in I_\nu$ where ν is any given small positive number we have

$$\begin{aligned} \frac{d\tilde{x}^2}{dt} &\leq f(\tilde{x}^2(t,x^1), \psi_2(t), t, \varepsilon) \\ \varepsilon\, \frac{d\psi_2}{dt} &\leq g(\tilde{x}_2(t,x^1), \psi_2(t), t, \varepsilon). \end{aligned}$$

Then the following theorem is valid.

Theorem 4.2 *Assume the hypotheses $(S_1)-(S_5), (\tilde{S}_6), (S_7), (S_8), (\tilde{S}_9)$ to be valid. Then to any small $\nu > 0$ there exists a sufficiently small $\varepsilon_0 = \varepsilon_0(\nu)$ such that for $0 < \varepsilon \leq \varepsilon_0(\nu)$ the initial value problem (1.1), (1.2) with $x^0 < \hat{x}(0), y^0 < \hat{y}(0)$ has a unique solution $(x(t,\varepsilon),\ y(t,\varepsilon))$ satisfying*

$$x(t,\varepsilon) = \hat{x}(t) + 0(\varepsilon) \text{ for } t \in \overline{T}\backslash I_\nu^-$$

$$y(t,\varepsilon) = \begin{cases} \hat{y}(t) + \Pi_0 y(\tau) + 0(\varepsilon) & \text{for} \quad t_0 < t \leq t_c - \nu \\ \hat{y}(t) + 0(\varepsilon^{\frac{1}{2}}) & \text{for} \quad t_c \leq t < t_c + \nu \\ \hat{y}(t) + 0(\varepsilon) & \text{for} \quad t_c + \nu \leq t \leq T. \end{cases}$$

For $t \in I_\nu^- := \{t \in R : t_c - \nu \leq t \leq t_c\}$ we have

$$\begin{aligned} \tilde{x}^2(t,x^1) &\leq x(t,\varepsilon) \leq \hat{x}(t) + \varepsilon \exp(\lambda t) v_f, \\ \varphi_2(\tilde{x}^2(t,x^1),t) &\leq y(t,\varepsilon) \leq \hat{y}(t) + \gamma\sqrt{\varepsilon} v(t). \end{aligned}$$

Proof. The proof of this theorem proceeds essentially in the same line as the proof of Theorem 4.1. The upper solution on I_ν is exactly the same, the lower solution differs from that one in Theorem 4.1 on the interval I_ν^- and implies a different estimate on this interval. □

5. Example: Fast bimolecular rection with monomolecular slow reaction

In this section we apply our results to the following differential system which describes a fast bimolecular reaction including slow monomolecular reactions (see [9] and references therein)

$$\begin{aligned} \varepsilon\, \frac{dy}{dt} &= \varepsilon\,(I_a(t) - g_1(y)) - r(y,z), \\ \varepsilon\, \frac{dz}{dt} &= \varepsilon\,(I_b(t) - g_2(z)) - r(y,z). \end{aligned} \tag{5.1}$$

To (5.1) we consider the initial value problem

$$y(0,\varepsilon) = y^0,\ z(0,\varepsilon) = z^0, \qquad 0 < t < t_e. \tag{5.2}$$

Concerning the inputs I_a and I_b we assume that they are nonnegative and twice continuously differentiable for $t > 0$ (this assumption can be relaxed!), for g_1, g_2, and r we consider the special case

$$g_1(y) \equiv y,\ \ g_2(z) \equiv z,\ \ r(y,z) \equiv yz. \tag{5.3}$$

By means of the coordinate transformation $y = y, z = y - x$ we get from (5.1) and (5.3) the singularly perturbed system

$$\begin{aligned} \varepsilon\, \frac{dy}{dt} &= \varepsilon\,(I_a(t) -\ y) - y\,(y - x) \equiv g(x,y,t,\varepsilon), \\ \frac{dx}{dt} &= I_a(t) - I_b(t) - x \equiv f(x,y,t,\varepsilon) \end{aligned} \tag{5.4}$$

and the initial condition

$$y(0,\varepsilon) = y^0,\ \ x(0,\varepsilon) = x^0 = y^0 - z^0. \tag{5.5}$$

The last equation in (5.4) can be integrated. Taking into account (5.5) we obtain

$$\tilde{x}(t,x^0) = e^{-t}\Big(x^0 + \int_0^t e^s(I_a(s) - I_b(s))ds\Big) \tag{5.6}$$

such that (5.4) and (5.5) are equivalent to

$$\varepsilon\,\frac{dy}{dt} = \varepsilon(I_a(t) - y) - y(y - \tilde{x}(t,x^0)), \quad y(0,\varepsilon) = y^0. \tag{5.7}$$

The corresponding degenerate equation has the solutions $y = y^{(1)}(t) \equiv 0,\ y = y^{(2)}(t) \equiv \tilde{x}(t,x^0)$. Consequently, if $\tilde{x}(t,x^0)$ does not change its sign in $[0,t_e]$ then Theorem 2.1 can be applied. It can be easily checked that $y^1(t) \equiv 0$ is an asymptotically stable equilibrium of the associated system

$$\frac{dy}{d\tau} = -y(y - \tilde{x}(t,x^0))$$

if $\tilde{x}(t,x^0)$ is negative for $t \in [0,t_e]$. If $\tilde{x}(t,x^0)$ changes its sign at $t = t_c \in (0,t_e)$ then we have the case of exchange of stability which was considered in Theorem 4.1. To obtain an explicit expression for the corresponding composed stable solution we consider the special case

$$I_a(t) \equiv 1\ ,\ I_b(t) \equiv 1 + \cos t.$$

Then (5.6) reads

$$\tilde{x}(t,x^0) \ = \ \left(x^0 + \frac{1}{2}\right)e^{-t} - \frac{\cos t + \sin t}{2}. \tag{5.8}$$

For $0 < x^0 < \frac{1}{2}\left(\sqrt{2}\, e^{\pi/4} - 1\right)$ and $t_e = \frac{\pi}{4}$ the equation

$$\left(x^0 + \frac{1}{2}\right) e^{-t} - \frac{\cos t + \sin t}{2} = 0 \tag{5.9}$$

has a unique solution $t = t_c$ in $\left(0, \frac{\pi}{4}\right)$. It can be easily shown that $y^2(t) \equiv \tilde{x}(t, x^0)$ is stable for $[0, t_c)$ and $y^1(t) \equiv 0$ is stable for $(t_c, \frac{\pi}{4}]$. Consequently, the composed stable solution reads

$$\hat{y}(t) = \begin{cases} \left(\frac{1}{2} + x^0\right) e^{-t} - \frac{\sin t + \cos t}{2} & \text{for} \quad 0 \le t \le t_c, \\ 0 & \text{for} \quad t_c \le t \le \frac{\pi}{4}. \end{cases}$$

Now we check the hypotheses of Theorem 4.1. In our case it is easy to see that the hypotheses $(S_1) - (S_5)$ are satisfied. From (5.4) we get that g is nondecreasing in x if we replace y by any nonnegative function $\tilde{y}(t)$, additionally we have $-g_{yy} = 2$. Thus, the assumptions (S_6) and (S_8) are valid. $v(t)$ can be set identically one. Since the derivative of $\hat{y}$ is strictly negative for $0 \le t \le t_c$ and I_a is nonnegative it can be easily verified that $\hat{y}$ fulfills assumption (S_9). Consequently, Theorem 4.1 can be applied to the initial value problem (5.1), (5.2) or equivalently to (5.7).

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