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The classical solvability of the contact angle problem for generalized equations of mean curvature type

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Abstract

Mean curvature equations of general quasilinear type in connection with contact-angle boundary conditions are considered in this paper. We investigate the existence, uniqueness and continuous dependence of the solution in classical function spaces. On the one hand, a survey of techniques and ideas developed in the seventies and eighties, mainly by Uraltseva, is presented. On the other hand, extensions of these results are also proposed: we formulate growth conditions for the general dependence of the potential on the $x_{N+1}-$ variable, and we extend the existence and uniqueness statements to this case. Moreover, the regularity assumptions on the right-hand side are relaxed, and alternative proofs for the higher-order estimates and the existence result are provided.

1 Introduction

We consider the problem to determine in a domain $\Omega \subset \mathbb{R}^{N+1}$ ($N \geq 2$ the space dimension) a $N-$dimensional hypersurface $S \subset \Omega$, obeying the relation

$$\mathrm{div}_S \, \sigma_q(x, \, \nu) + \sigma_x(x, \, \nu) = \Phi(x, \, \nu) \,, \tag{1}$$

where div_S is the surface divergence operator, and ν denotes a unit normal to S. The potential $\sigma : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$, $(x, q) \mapsto \sigma(x, q)$ is given and one-homogeneous in the $q-$variable. The right-hand side $\Phi : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ is a given function. In the case of isotropic data $\sigma(x, \, q) = \sigma(x) \, |q|$ and $\Phi(x, \, q) = \Phi(x)$, the equation (1) reduces to the problem of surfaces with prescribed mean curvature. We consider on the boundary $S \cap \partial\Omega$ the generalized contact-angle condition

$$\sigma_q(x) \cdot n(x) = \kappa(x) \,, \tag{2}$$

where n is the outward unit normal to $\partial\Omega$, and $\kappa : \partial\Omega \to \mathbb{R}$ is given.

More specifically, we are interested in graph solutions to the problem (1), (2). A graph solution can be defined (after a suitable change of coordinates) if $\Omega = G \times \mathbb{R}$ with a bounded domain $G \subset \mathbb{R}^N$, and if S is represented as the graph of a function $\psi : G \to \mathbb{R}$. The problem (1), (2) on the manifold S reduces to boundary value problem posed in the domain G. Define for $(\bar{x}, \, x_{N+1}) \in G \times \mathbb{R}$ and for $p \in \mathbb{R}^N$

$$\bar{\sigma}(\bar{x}, \, x_{N+1}, \, p) := \sigma(\bar{x}, \, x_{N+1}, \, -p, \, 1) \,. \tag{3}$$

and introduce a function $\bar{\bar{\Phi}} : G \times \mathbb{R} \times \mathbb{R}^N$ via

$$\bar{\bar{\Phi}}(\bar{x}, \, x_{N+1}, \, p) := \Phi(\bar{x}, \, x_{N+1}, \, \nu(p)), \quad \nu_i(p) := \begin{cases} -p_i/\sqrt{1 + |p|^2} & (i = 1, \ldots, N) \\ 1/\sqrt{1 + |p|^2} & i = N + 1 \end{cases} \,. \tag{4}$$

In the domain G, (1), (2) is equivalent to the contact angle problem

$$- \operatorname{div} \bar{\sigma}_p(\bar{x}, \, \psi, \, \nabla \psi) = \bar{\Phi}(\bar{x}, \, \psi, \, \nabla \psi) \quad \text{in } G \,, \tag{5}$$

$$-\bar{\sigma}_p(\bar{x}, \, \psi, \, \nabla \psi) \cdot n(\bar{x}) = \kappa(\bar{x}, \, \psi) \qquad \text{on } \partial G \,. \tag{6}$$

Physical applications of the model (1), (2) respectively (5), (6) are to find in thermodynamical contexts, where (1) is to interpret as the first variation of a surface free energy. The equation (1) is known as generalized Gibbs-Thomson relation: The surface S typically represents a phase transition, and σ is the surface tension on S; The right-hand side Φ in (1) may involve quantities such as chemical potential, temperature and mechanical stresses on S: see the book [Vis96], Ch. IV for models in crystallization, or [LM89], [GK10] for related mathematical developments. Technical applications for the model (1) are for instance processes in industrial crystal growth, where curvature effects on the crystallization interface are assumed to be responsible for the formation of defects (cf. [DDEN08]).

Equations of mean curvature type were thouroughly studied in the seventies, in connection both with the Dirichlet and the contact angle problem: see [Gia74], [Ger74], [Giu76] among others for the BV approach, see [Fin65], [Ser69], [Ura73], [Ura75], [Ura82], [SS76] a. o. for the classical approach, which is going to retain our attention in this paper.

The existence of graph solutions essentially relies on the gradient estimate for the function ψ. To our knowledge *local* estimates were obtained first in [Mir67], [BDM69] for the problem of minimal surfaces ($\Phi = 0, \sigma(q) = |q|$). The local boundedness of the gradient was proved for general quasilinear equations in [LU70] on the basis of profound results of geometric measure theory. Local estimates employing other methods were also derived early (cf. [Tru73]) by the authors of [GT01] (see Chapter 16). It is to note that the *a priori* estimate derived in these papers for C^2 solutions being local, they did not lead to the solvability of (5), (6).

The global estimate on the gradient for the contact angle problem (5), (6) was first obtained in the papers [Ura71, Ura73, Ura75] for general $\sigma = \sigma(q)$, mainly via extension of the methods of [LU70]. In [Ura71] the validity of these results was restricted to (strictly) convex $\mathcal{C}^{2,\alpha}-$domains G, a vanishing angle of contact. The theory for convex domains and a constant nonvanishing angle of contact κ was introduced in [Ura73]; The results were extended in [Ura75] to variable $\kappa = \kappa(\bar{x})$ and nonconvex $\mathcal{C}^3-$domains, but only for the case $\sigma = |q|$ (mean curvature equation). In these papers, it is assumed that $\Phi = \Phi(x)$. Other approaches to the results of [Ura75] for the mean curvature equation were discussed in the papers [SS76], Th. 3 or in [Ger79], that states the gradient estimate for (nonconvex) $\mathcal{C}^4$ domains. The boundedness result for gradient of solutions to the general quasilinear mean curvature equation with contact angle $\kappa = \kappa(\bar{x}, x_{N+1})$ was proved in [Ura82]. In the latest paper σ is allowed to depend on the $x_{N+1}-$variable, but only in a very particular way.

The arguments on existence, uniqueness and *a priori* estimates for the problem (5), (6) are spread in the literature (mostly in papers by Uraltseva). The paper [Ura82], where the general quasilinear case is treated, deals indeed only with the gradient estimate. In this paper, we aim at a complete overview on the classical solvability of the problem (5), (6) in smooth settings. We also propose two generalizations: A growth condition for the $x_{N+1}-$dependence of the function σ is formulated, and shown to yield well-posedness; The regularity assumptions for Φ are weakened.

2 Notations and statement of the main results

Let $N \geq 2$ denote the space dimension, and $G \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^{2,\alpha}$, $\alpha > 0$, $\Omega := G \times \mathbb{R}$. Throughout the paper, the function σ is assumed to satisfy

$$\sigma \in C^3(\overline{G} \times \mathbb{R} \times (\mathbb{R}^{N+1} \setminus \{0\})) \,. \tag{7}$$

We assume that there exist positive constants λ_j $(j = 0, 2)$ and μ_i $(i = 0, \ldots, 4)$ such that for all $(x, q) \in \Omega \times \mathbb{R}^{N+1}$

$$\lambda_0 \, |q| \leq \sigma(x, q) \leq \mu_0 \, |q| \tag{8a}$$

$$|\sigma_q(x, q)| \leq \mu_1 \tag{8b}$$

$$\frac{\lambda_2}{|q|} \, |\xi|^2 \leq \sum_{i,j=1}^{N+1} \sigma_{q_i, q_j}(x, q) \, \xi_i \, \xi_j \leq \frac{\mu_2}{|q|} \, |\xi|^2 \tag{8c}$$

$$\text{for all } \xi \in \mathbb{R}^{N+1} \text{ such that } \xi \cdot q = 0$$

$$\sum_{j=1}^{n+1} \sigma_{q_i, q_j}(x, q) \, q_j = 0 \quad \text{for } i = 1, \ldots, N+1 \tag{8d}$$

$$|\sigma_{q,x}(x, q)| \leq \mu_3 \,, \quad |\sigma_{q,x,x}(x, q)| \leq \mu_4 \,. \tag{8e}$$

The hypotheses (8a), (8b), (8c) and (8d) are well-knwon, and in particular satisfied if σ is convex and positively homogeneous of degree one in the q variable (cf. [LU70], [Ura71] for a proof). We need special assumptions on the $x_{N+1}-$derivatives of the function σ. We assume that

$$|\sigma_{x_{N+1}}(x, q)| + |\sigma_{x_{N+1}, x_{N+1}}(x, q)| \leq \mu_5 \, \frac{|q_{N+1}|^2}{|q|} \tag{9a}$$

$$|\sigma_{\bar{x}, x_{N+1}}| + |\sigma_{x_{N+1}, q}| + |\sigma_{\bar{x}, x_{N+1}, q}| + |\sigma_{x_{N+1}, x_{N+1}, q}| + |\sigma_{x_{N+1}, q, q}| \leq \mu_6 \, \frac{|q_{N+1}|}{|q|} \,. \tag{9b}$$

One purpose of the paper is also to relax the requirement of continuous differentiability of the right-hand side. We shall require that $\Phi \in V \subset W^{1,\infty}(\Omega \times \mathbb{R}^{N+1})$, where V is any closed linear subspace of $W^{1,\infty}$ that allows for $\nabla \Phi$ to have bounded traces on both sides of smooth submanifolds (for instance, $\nabla \Phi \in C_{pw}$ or even $\nabla \Phi \in W^{1,1}_{pw}$). We assume that

$$\Phi \in V(\Omega \times \mathbb{R}^{N+1}), \quad \kappa \in C^{1,\alpha}(\partial G \times \mathbb{R}) \, (\alpha > 0), \tag{10}$$

is required. Special assumptions are needed in connection with the $x_{N+1}-$derivatives of these functions:

$$\operatorname*{ess\,sup}_{\Omega \times \mathbb{R}^{N+1}} \Phi_{x_{N+1}} \leq -\gamma_0 < 0, \quad \kappa_{x_{N+1}} \geq 0 \,. \tag{11}$$

Choosing λ_0 as in (8a), there is a compatibility condition between the functions κ and σ:

$$\sup_{\partial G \times \mathbb{R}} |\kappa| < \lambda_0 \,, \quad \gamma_1 := \lambda_0 - \|\kappa\|_{L^\infty(\partial G \times \mathbb{R})} > 0 \,. \tag{12}$$

3

For the existence and uniqueness of the solution, we have to assume that the parameters γ_0, λ_2 and μ_5, μ_6 in the conditions (11), (8d) and (9) satisfy

$$\gamma_0 > \frac{(\mu_5 + \mu_6 + \|\Phi_q\|_{L^\infty(\Omega \times \mathbb{R}^{n+1})})^2}{4\,\lambda_2}\,. \tag{13}$$

The main result on existence, uniqueness and regularity for the problem (5), (6) is formulated in the following theorem.

Theorem 2.1. *Let all the assumptions of this section be satisfied for the domain G and the functions σ, Φ and κ. Then, the problem (5), (6) possesses a unique solution $\psi \in C^{2,\alpha}(\overline{G})$. Denoting S the graph of the function ψ, there is a constant c depending on all the data in their respective norm, such that $\|D^2\psi\|_{C^\alpha(\overline{G})} \leq c\,(\|\Phi\|_{C^\alpha(\overline{S})} + \|\kappa\|_{C^{1,\alpha}(\partial S)})$.*

A second result of the paper concerns the gradient estimate for solutions to (5), (6), which is the most essential step of the proof. In comparison to the result of [Ura82], we allow for a x_{N+1} dependence of σ, and we formulate the assumptions for the function Φ as integrability conditions.

Proposition 2.2. *Assumptions of Theorem 2.1 (the inequality (13) being not needed). Assume that $\psi \in C^2(\overline{G})$ is a solution to (5), (6). Let p and s be real numbers such that $p > N/2$ and $s > \max\{p, \frac{2Np}{2P-N}\}$. Then, there is a continuous (polynomial) function c such that*

$$\sup_G \sqrt{1 + |\nabla\psi|^2} \leq c(X,\, |\Phi\|_{L^s(S)},\, \|\Phi_x\|_{L^s(S)},\, \|\Phi_q\|_{L^{2s}(S)})\,,$$

where X depends on all the data in their respective norm, but not on Φ.

Preliminary propositions We terminate this section by stating explicitly a few elementary consequences of the hypotheses formulated in the preceding section (see [LU70] or [Ura73] for similar considerations). Due to (8a) and the Taylor formula, there is for all $(x, q) \in \Omega \times \mathbb{R}^{N+1} \setminus \{0\}$ a $\lambda \in\,]0, 1[$ such that[1]

$$0 = \sigma(x, 0) = \sigma(x, q) - \sigma_q(x, q)\,q + \frac{1}{2}\,\sigma_{q_i, q_j}(x,\,\lambda\,q)\,q_i\,q_j\,.$$

The properties (8d) and (8a) therefore implies for all $q \in \mathbb{R}^{N+1} \setminus \{0\}$ that

$$\sigma(x, q) = \sigma_q(x, q) \cdot q, \quad \sigma_q(x, q) \cdot q \geq \lambda_0\,|q|\,. \tag{14}$$

For $p \in \mathbb{R}^N$, $q := (-p, 1)$, it follows from (14) and the definition (3) that

$$\begin{aligned}
\bar{\sigma}_p(x, p) \cdot p &= \sigma_q(x, q) \cdot q - \sigma_{q_{N+1}}(x, q) \\
&= \sigma(x, q) - \sigma_{q_{N+1}}(x, q)\,.
\end{aligned}$$

Using (8a) and (8b), one therefore obtains from the previous assumptions on the growth of σ that

$$\bar{\sigma}_p(x, p) \cdot p \geq \lambda_0\,\sqrt{1 + |p|^2} - \mu_1 \quad \text{for all } (x, p) \in \Omega \times \mathbb{R}^N\,. \tag{15}$$

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[1] Whenever confusion is impossible, we use the convention that repeated indices imply summation.

Since $\sigma_q(x,\,q)\cdot q = \sigma(x,\,q)$, the assumption (8e) also implies that

$$|\sigma_x(x,\,q)| \le \mu_3 \,|q| \quad \text{for all } q \in \mathbb{R}^{N+1}\,. \tag{16}$$

For $\xi,\,p \in \mathbb{R}^N$, the relation (8c) elementarily implies that

$$\frac{\lambda_2\,|\xi^T|^2}{\sqrt{1+|p|^2}} \le \bar\sigma_{p_i,p_j}(x,\,p)\,\xi_i\,\xi_j \le \frac{\mu_2\,|\xi^T|^2}{\sqrt{1+|p|^2}}\,. \tag{17}$$

Here, $\xi^T = \xi^T(p) := \hat\xi - \frac{q}{|q|}\,(\hat\xi \cdot \frac{q}{|q|}) \in \mathbb{R}^{N+1}$, with $\hat\xi := (\xi_1,\dots,\xi_N,0)$ and $q := (-p,1)$.

We also need extensions into G of the data n and κ given on $\partial G \times \mathbb{R}$.

Remark 2.3 (Data extension). *Since G has a $\mathcal{C}^{2,\alpha}$ boundary, the unit normal has an extension $n :=$ $\nabla \mathrm{dist}(\cdot,\,\partial G)$ into G such that $n \in [C^{1,\alpha}(\overline{G})]^N$. Setting $n_{N+1} = 0$ and extending $n(\bar x)$ by a constant in the $N+1-$direction, we obtain that $n \in [C^{1,\alpha}(\overline{G} \times \mathbb{R})]^{N+1}$. Under the assumption (10), it is possible to assume that $\kappa \in C^{1,\alpha}(\overline{G} \times \mathbb{R})$. We can ensure that the inequality (12) is preserved.*

Finally, we recall some notations associated with the surface S. For $\psi \in C^2(\overline{G})$, the graph $S \subset \mathbb{R}^{N+1}$ of ψ is the set $S := \{(\bar x,\,x_{N+1}) \in \overline{G} \times \mathbb{R} \,:\, x_{N+1} = \psi(\bar x)\}$. A unit normal on the surface S is given by $\nu(\bar x,\,\psi(\bar x)) := \nu(\nabla\psi(\bar x))$ with $\nu(p)$ like in (4). The natural surface measure on the surface S is given by $dH_N := \sqrt{1+|\nabla\psi|^2}\,d\lambda_N$. For $f \in C^1(\mathbb{R}^{N+1})$, the differential operator

$$\delta\,f := \nabla f - (\nabla f \cdot \nu)\,\nu\,, \tag{18}$$

is identical on S with the surface gradient. Throughout the paper, we denote $\partial S := \{(\bar x,\,x_{N+1}) \in \partial G \times \mathbb{R} \,:\, x_{N+1} = \psi(\bar x)\}$. The tangential gradient of ψ on ∂G given by $\psi_t := \nabla\psi - (\nabla\psi \cdot n)\,n$ on ∂G. If α denotes the angle of contact between S and $\partial G \times \mathbb{R}$ (that is, $\cos\alpha := -\nabla\psi \cdot n/\sqrt{1+|\nabla\psi|^2}$ on ∂G), then

$$\sin\alpha = \left(\frac{1+|\psi_t|^2}{1+|\nabla\psi|^2}\right)^{1/2} \quad \text{on } \partial G\,. \tag{19}$$

Denote dH_{N-1} the standard surface measure on ∂G. Then, a natural surface measure on ∂S is defined by

$$ds = \sqrt{1+|\psi_t|^2}\,dH_{N-1} = \sin\alpha\,\sqrt{1+|\nabla\psi|^2}\,dH_{N-1}\,. \tag{20}$$

3 Global $L^\infty-$ estimate on $\nabla\psi$

In this section, we are concerned with *a priori* estimates satisfied by $\nabla\psi$ in $L^\infty(G)$ for a function $\psi \in C^2(\overline{G})$ satisfying (5), (6). The assumptions considered for the data are those of section 2. For local gradient estimates, we refer to the publications mentioned in the introduction. A gradient estimate up to the boundary of S was first proved in [Ura71], [Ura73] for convex domains $G \subset \mathbb{R}^N$ of class $\mathcal{C}^{2,\alpha}$, $\sigma = \sigma(q)$, $\Phi = \Phi(x)$, and $\kappa = $ const. The proof was extended in [Ura75] for $\sigma(q) = |q|$ to nonconvex $\mathcal{C}^3$ domains, $\kappa = \kappa(\bar x)$. For the later case, results are also to find in [SS76], [Ger79]. A worth-noticing

difference is the following: thanks to the Sobolev embedding theorem up to the boundary of S, Uraltseva allows for the limiting case $\gamma_0 = 0$ (cf. the condition (11)), while the proof in the last two papers can be carried out from more elementary considerations. Finally, Uraltseva extended her methods in the paper [Ura82] to general quasilinear mean curvature equations, $\kappa = \kappa(\bar{x}, x_{N+1})$, and G of class $\mathcal{C}^2$.

In this section, we present a proof of the gradient estimate *using Uraltseva's methods*. We slightly extend the result of [Ura82] allowing for a general x_{N+1} dependence of σ via the conditions (9), and tracking the dependence on the right-hand side in the gradient bound in terms of integrability conditions.

Throughout the section, $S \subset \mathbb{R}^{N+1}$ denotes a $N-$ dimensional submanifold that satisfies (1), (2). We abbreviate $\sigma = \sigma(x, \nu)$ and $\Phi = \Phi(x, \nu)$ on S. We start with a method to estimate integrals over ∂S which was the new ingredient for the advances in [Ura82] with respect to the former contributions [Ura71, Ura73].[2] In the following two lemmas, we recall the proof of this fundamental statement.

Lemma 3.1. *Let $S \subset \mathbb{R}^{N+1}$ denote a $N-$ dimensional manifold that satisfies* (1)*,* (2)*. Taking into account the assumptions* (8b) *and* (8e) *and the Remark 2.3, introduce the function $a_0 := |\Phi| + 2\,\mu_1\,|\nabla n| + \mu_3$. Then, for every nonnegative $f \in C^1(\mathbb{R}^{N+1})$*

$$\int_{\partial S} f\, \frac{ds}{\sin \alpha} \leq \gamma_1^{-1} \left(\int_S |\,\delta f\,|\, dH_N + \int_S a_0\, f\, dH_N \right),$$

where δ is defined by (18)*. The function $\sin \alpha$ and the measure ds are defined in* (19) *and* (20)*.*

Proof. On the surface S, define a vector field $T := -(n \cdot \nu)\,\sigma_q + (\sigma_q \cdot \nu)\,n$. Note that T is tangent on S. Denote moreover $\nu' := \sin \alpha^{-1}\,(n - (\nu \cdot n)\,\nu)$ the conormal on ∂S. We use the identity $\int_S T \cdot \delta f\, dH_N + \int_S \mathrm{div}_S T\, f\, dH_N = \int_{\partial S}(T \cdot \nu')\, f\, ds$. One easily verifies that

$$\begin{aligned} T \cdot \nu' &= \sin \alpha^{-1}\,(\sigma_q - \kappa\, n) \cdot \nu \\ &\geq (\lambda_0 - \|\kappa\|_{L^\infty(\partial S)})\,\sin \alpha^{-1} \quad \text{on } \partial S\,. \end{aligned} \tag{21}$$

We compute

$$\begin{aligned} \mathrm{div}_S T &= \delta_i((\sigma_q \cdot \nu)\, n_i - (\nu \cdot n)\,\sigma_{q_i}) \\ &= n_i\,\delta_i\,\sigma_q \cdot \nu + \sigma_q \cdot \nu\,\mathrm{div}_S n - \nu \cdot \delta_i\, n\,\sigma_{q_i} - (\nu \cdot n)\,\mathrm{div}_S \sigma_q \\ &\quad + n_i\,\delta_i\,\nu \cdot \sigma_q - n \cdot \delta_i\,\nu\,\sigma_{q_i}\,. \end{aligned} \tag{22}$$

Using the equation (1), it follows that $\mathrm{div}_S \sigma_q = \Phi - \sigma_x \cdot \nu$. Using the symmetry of the matrix $\{\delta_i\,\nu_j\}$, we show that $n_i\,\delta_i\,\nu \cdot \sigma_q - n \cdot \delta_i\,\nu\,\sigma_{q_i} = 0$. For $i \in \{1, \ldots, n\}$, the property (8d) and the identity (14) yield $\delta_i\,\sigma_q \cdot \nu = \sigma_{q_j, \delta_i}\,\nu_j + \sigma_{q_j, q_l}\,\delta_j\,\nu_l\,\nu_j = \sigma_{q_j, \delta_i}\,\nu_j = \sigma_{\delta_i}$, where $\sigma_{\delta_i} = \sigma_{x_i} - (\nu \cdot \sigma_x)\,\nu_i$. Thus

$$\mathrm{div}_S T = n_i\,\sigma_{\delta_i} + \sigma_q \cdot \nu\,\mathrm{div}_S n - \nu \cdot \delta_i\, n\,\sigma_{q_i} - (\nu \cdot n)\,(\Phi - \sigma_x \cdot \nu)\,, \tag{23}$$

and the estimate $|\,\mathrm{div}_S T\,| \leq a_0$ is an easy consequence of the contitions (8). The claim follows combining (21) and (23). $\qquad \square$

Note the following elementary precision concerning Lemma 3.1.

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[2]Some references on the original idea are also to find in [Ura82].

Lemma 3.2. *Assumptions of Lemma 3.1. Then $\sin \alpha > \gamma_1/\mu_1$ on ∂S.*

Proof. Denote $n' = \sin \alpha^{-1} (\nu - \cos \alpha \, n)$. It is easy to verify that $|n'| = 1$ on ∂S. From the conditions (8), it follows that $\mu_1 \geq \sigma_q \cdot n' = \sin \alpha^{-1} (\sigma - (\nu \cdot n) \, \kappa) \geq \sin \alpha^{-1}(\lambda_0 - \|\kappa\|_{L^\infty(\partial S)})$. $\qquad\square$

We now turn to the core of the proof of the gradient estimate. It was noticed for the first time in [Ura73] that under the condition (12), it is both convenient and sufficient to estimate the quantity

$$v(x) := \nu_{N+1}^{-1} \left(\sigma(x, \nu) - \kappa(x) \, (\nu \cdot n(x)) \right), \quad x \in S \tag{24}$$

since the conditions (8a) and (12) imply the inequalities

$$\gamma_1 \sqrt{1 + |\nabla \psi|^2} \leq v \leq \gamma_2 \sqrt{1 + |\nabla \psi|^2} \text{ on } S, \quad \gamma_2 := (\mu_0 + \|\kappa\|_{L^\infty(\partial G \times \mathbb{R})}). \tag{25}$$

The following Lemma provides the corner stone for the gradient estimate. We perform the computations for continuously differentiable Φ. In the case that $\Phi \in V$ (cf. (10)), the same is valid usign either the right or the left trace of $\nabla \Phi$ on S.

Lemma 3.3. *Let S be a $N-$ dimensional hypersurface that satisfies (1), (2), such that $\nu_{N+1} > 0$ on S. Let v be defined by (24) on S. Then, there are functions $a_1, a_2, b_1, \ldots, b_{N+1}$ such that for all $\eta \in C^1(\overline{S})$, the relation*

$$\int_S \nu_{n+1}^2 \, \sigma_{q_i, q_j} \, \delta_j \, v \, \delta_i \, \eta - \int_S \Phi_{x_{N+1}} \nu_{N+1} \, v \, \eta + \int_{\partial S} \kappa_{x_{N+1}} \nu_{N+1} \, v \, \eta \, \frac{ds}{\sin \alpha}$$
$$= \int_S \nu_{N+1} \left\{ a_1 \, \eta + b \cdot \delta \, \eta \right\} + \int_{\partial S} \nu_{N+1} \, a_2 \, \eta \, \frac{ds}{\sin \alpha}, \tag{26}$$

is valid. There are constants c_i, $i = 1, \ldots, 4$ depending only on the constants in the conditions (8), (9), on $\|\kappa_x\|_{L^\infty(S)}$ and on the domain G, such that

$$a_1 \leq \frac{-\lambda_2^2}{2} \, |\delta \, \nu|^2 + c_1 \, (1 + |\Phi_x| + |\Phi_q|) + c_2 \, |\Phi_q| \, \nu_{N+1} \, |\delta \, v|, \quad a_2 \leq c_3 \quad |b| \leq c_4. \tag{27}$$

Proof. Throughout the proof, $\sigma_q = \sigma_q(x, \nu)$ on S. Due to the assumption $\nu_{N+1} > 0$, S is the graph of a function $\psi \in C^2(\overline{G})$. For $k = 1, \ldots, N$, we denote d_k the tangential differential operator $d_k := \partial_{x_k} + \psi_{x_k} \, \partial_{x_{N+1}}$ on S. For $\eta \in C^1(\mathbb{R}^{n+1})$, we denote $\eta_{d_k} := \eta_x \cdot u^k$ with the tangent vector field $u_i^k := \delta_i^k$ for $i = 1, \ldots, N$, $u_{N+1}^k := \psi_{x_k}$.

For $k = 1, \ldots, N + 1$, we introduce $\xi_k := \sigma_{q_k} - \kappa \, n_k$, and $\zeta_k := \nu_{N+1}^{-1} \nu_k$, that is, $\zeta_k = -\psi_{x_k}$ for $k = 1, \ldots, N$, and $\zeta_{N+1} = 1$. The identity (14) yields

$$v = \sum_{k=1}^{N+1} (\sigma_{q_k} - \kappa \, \nu_k) \, \zeta_k = \xi \cdot \zeta \text{ on } S. \tag{28}$$

For $k \in \{1, \ldots, n\}$, we can differentiate the equation (5), multiply the result with $\eta \circ \psi = \eta(x, \psi)$ ($\eta \in C^1(\overline{S})$ arbitrary), and use integration by parts to obtain that

$$\int_G \frac{d\bar\sigma_{p_i}}{dx_k} \frac{d}{dx_i} \eta \circ \psi = \int_G \frac{d\bar\Phi}{dx_k} \eta \circ \psi + \int_{\partial G} n_i \frac{d\bar\sigma_{p_i}}{dx_k} \eta \circ \psi \, dH_{N-1},$$

which is nothing else but the identity

$$-\int_S \nu_{N+1}\, d_k\sigma_{q_i}\, d_i\eta = \int_S \nu_{N+1}\, d_k\Phi\, \eta - \int_{\partial S} \nu_{N+1}\, d_k\sigma_{q_i}\, n_i\, \eta\, \frac{ds}{\sin\alpha}\,, \tag{29}$$

with summation over $i = 1, \ldots, N$. Choosing $\xi_k\, \eta$ as testfunction in (29), it follows that

$$-\int_S \nu_{N+1}\, \xi_k\, d_k\sigma_{q_i}\, d_i\eta = \int_S \nu_{N+1}\, (d_k\sigma_{q_i}\, d_i\xi_k + \xi_k\, d_k\Phi)\, \eta - \int_{\partial S} \nu_{N+1}\, \xi_k\, d_k\sigma_{q_i}\, n_i\, \eta\, \frac{ds}{\sin\alpha}\,, \tag{30}$$

with summation over $i, k = 1, \ldots, N$. Using the symmetry of the matrix $\{\delta_j\, \nu_l\}$ and the fact that $\psi_{x_k}\, \nu_{N+1} = -\nu_k$, one verifies that

$$d_k\, \nu_j = \delta_k\, \nu_j + \psi_{x_k}\, \delta_{N+1}\, \nu_j = -\nu_{N+1}\, \delta_j\, \psi_{x_k}\,, \quad j = 1, \ldots, N+1\,. \tag{31}$$

For $i \in \{1, \ldots, N\}$, the latest yields

$$\begin{aligned}
\xi_k\, d_k\sigma_{q_i} &= \xi_k\, \sigma_{q_i,d_k} + \xi_k\, \sigma_{q_i,q_j}\, (\delta_k\, \nu_j + \psi_{x_k}\, \delta_{N+1}\, \nu_j) \\
&= \xi_k\, \sigma_{q_i,d_k} - \nu_{N+1}\, \sigma_{q_i,q_j}\, \delta_j\, \psi_{x_k}\, \xi_k
\end{aligned} \tag{32}$$

For $j \in 1, \ldots, N+1$, using that $\zeta_{N+1} = 1$ on S, we see that

$$\sum_{k=1}^{N} \delta_j\, \psi_{x_k}\, \xi_k = -\sum_{k=1}^{N+1} \delta_j\, \zeta_k\, \xi_k = -\delta_j(\zeta \cdot \xi) + \sum_{k=1}^{N+1} \zeta_k\, \delta_j\, \xi_k\,. \tag{33}$$

Using (8d), we compute that

$$\begin{aligned}
\nu_{N+1}\, \zeta_k\, \delta_j\, \xi_k &= \sigma_{q_k,q_l}\, \nu_k\, \delta_j\, \nu_l + \nu_k\, (\sigma_{q_k,\delta_j} - \delta_j(n_k\, \kappa)) \\
&= \nu_k\, (\sigma_{q_k,\delta_j} - \delta_j(n_k\, \kappa))\,,
\end{aligned} \tag{34}$$

with summation over $k = 1, \ldots, N+1$. Using (32), (33) and (34), we obtain for $i \in \{1, \ldots, N\}$ the identity

$$\begin{aligned}
\sum_{k=1}^{N} d_k\sigma_{q_i}\, \xi_k &= \sum_{k=1}^{N} \xi_k\, \sigma_{q_i,d_k} - \nu_{N+1}\, \sigma_{q_i,q_j}\, \left(-\delta_j\, v + \sum_{k=1}^{N+1} \zeta_k\, (\sigma_{q_k,\delta_j} - \delta_j(n_k\, \kappa))\right) \\
&= \sigma_{q_i,q_j}\, (\nu_{N+1}\, \delta_j\, v + \tilde{b}_j) + \sum_{k=1}^{N} \xi_k\, \sigma_{q_i,d_k}\,, \quad \tilde{b}_j := -\sum_{k=1}^{N+1} \nu_k\, (\sigma_{q_k,\delta_j} - \delta_j(n_k\, \kappa))\,.
\end{aligned} \tag{35}$$

Due to (8d), we easily see that $\sum_{i-1}^{N} \sigma_{q_i,q_j}\, d_i\eta = \sum_{i-1}^{N+1} \sigma_{q_i,q_j}\, \delta_i\, \eta$. Thus, we obtain that

$$\int_S \nu_{N+1}\, \xi_k\, d_k\sigma_{q_i}\, d_i\eta = \int_S \nu_{N+1}\sigma_{q_i,q_j}\, (\nu_{N+1}\, \delta_j\, v + \tilde{b}_j)\, \delta_i\, \eta + \int_S \nu_{N+1} \sum_{k=1}^{N} \xi_k\, \sigma_{q_i,d_k}\, d_i\eta\,. \tag{36}$$

We easily verify that

$$\int_S \nu_{N+1}\, \xi_k\, \sigma_{q_i,d_k}\, d_i\eta = -\int_S \nu_{N+1}\, d_i(\xi_k\, \sigma_{q_i,d_k})\, \eta + \int_{\partial S} \nu_{N+1}\, \xi_k\, \sigma_{q_i,d_k}\, n_i\, \frac{ds}{\sin\alpha}\,.$$

with summation over $k, i = 1, \dots, N$, and we can rewrite (30) as

$$
\int_S \nu_{N+1}\, \sigma_{q_i,q_j}\, (\nu_{N+1}\, \delta_j\, v + \tilde{b}_j)\, \delta_i\, \eta = \int_S \nu_{N+1}\, (d_i(\xi_k\, \sigma_{q_i,d_k}) - d_k\sigma_{q_i}\, d_i\xi_k - \xi_k\, d_k\Phi)\, \eta
$$
$$
+ \int_{\partial S} \nu_{N+1}\, (\xi_k\, d_k\sigma_{q_i} - \xi_k\, \sigma_{q_i,x_k})\, n_i\, \eta\, \frac{ds}{\sin\alpha}\,. \tag{37}
$$

Observe that $d_i(\xi_k\, \sigma_{q_i,d_k}) = d_i\xi_k\, \sigma_{q_i,d_k} + A_1$ with $A_1 := \xi_k\, d_i\sigma_{q_i,d_k}$. On the other hand, using the relation (31) $d_k\sigma_{q_i}\, d_i\xi_k = \sigma_{q_i,d_k}\, d_i\xi_k - A_2$

$$
A_2 := \nu_{N+1}\, \sigma_{q_i,q_j}\, \delta_j\, \psi_{x_k}\, (\sigma_{q_k,d_i} - d_i(\kappa\, n_k) - \nu_{N+1}\, \sigma_{q_k,q_j}\, \delta_j\, \psi_{x_i})\,. \tag{38}
$$

Due again to (31), we also have

$$
\xi_k\, d_k\Phi = \xi_k\, (\Phi_{d_k} + \Phi_{q_j}\, d_k\nu_j) = \xi_k\, (\Phi_{x_k} + \xi_k\, \psi_{x_k}\, \Phi_{x_{N+1}}) + \nu_{N+1}\, \Phi_{q_j}\, \xi_k\, \delta_j\, \zeta_k\,,
$$

with summation over $k = 1, \dots, N$ and $j = 1, \dots, N+1$. Observe that $\xi_k\, \psi_{x_k} = -v + \sigma_{q_{N+1}}$. It follows that $\xi_k\, d_k\Phi = -v\, \Phi_{x_{N+1}} + A_3$,

$$
A_3 := \sum_{k=1}^N \xi_k\, \Phi_{x_k} + \sigma_{q_{N+1}}\, \Phi_{x_{N+1}} + \nu_{N+1}\, \Phi_{q_j}\, \delta_j\, v - \Phi_{q_j}\, \sum_{k=1}^{N+1} \nu_k\, (\sigma_{q_k,\delta_j} - \delta_j(\kappa\, n_k))\,. \tag{39}
$$

Finally, observe that $\xi_k\, d_k\sigma_{q_i}\, n_i = \xi_k\, d_k(\sigma_q \cdot n) - \xi_k\, d_k n \cdot \sigma_q$. Using the fact that $\xi_k\, d_k$ is a tangential differential operator on ∂S, it follows from the boundary condition (2) that

$$
\xi_k\, d_k\sigma_{q_i}\, n_i = \xi_k\, d_k\kappa - \xi_k\, d_k n \cdot \sigma_q
$$
$$
= \xi_k\, \kappa_{x_k} + (\sigma_{q_{N+1}} - v)\, \kappa_{x_{N+1}} - \xi_k\, d_k n \cdot \sigma_q\,, \tag{40}
$$

with summation over $k = 1, \dots, N$. Thus $(\xi_k\, d_k\sigma_{q_i} - \xi_k\, \sigma_{q_i,x_k})\, n_i = -v\, \kappa_{x_{N+1}} + a_2$ with

$$
a_2 := \xi_k\, \kappa_{x_k} + \sigma_{q_{N+1}}\, \kappa_{x_{N+1}} - \xi_k\, d_k n \cdot \sigma_q - \xi_k\, (\sigma_{q_i,x_k} + \psi_{x_k}\, \sigma_{q_i,x_{N+1}})\, n_i\,. \tag{41}
$$

For $i = 1, \dots, n$, we define $b_i := \sigma_{q_i,q_j}\, \tilde{b}_j$, and we set $a_1 := A_1 + A_2 + A_3$. Then the representation (26) follows from (37). It remains to prove the estimates (27). Using the definition (35) of $\tilde{b}$ and the assumptions (8), (10), we easily prove that $|b| \leq \mu_2\, (|\sigma_{q,x}| + |\,\delta(\kappa\, n)|)$. We compute

$$
A_1 = \xi_k\, \{\sigma_{q_i,x_k,x_i} + \psi_{x_i}\, \sigma_{q_i,x_k,x_{N+1}} + \sigma_{x_k,q_i,q_j}\, d_i\nu_j + \sigma_{q_i,x_{N+1}}\, (-\delta_i\, \zeta_k + \zeta_i\, \delta_{N+1}\, \zeta_k)
$$
$$
- \zeta_k\, (\sigma_{x_i,x_{N+1},q_i} - \zeta_i\, \sigma_{q_i,x_{N+1},x_{N+1}} + \sigma_{x_{N+1},q_i,q_j}\, d_i\nu_j)\}\,,
$$

with summation over $i, k = 1, \dots, N$. We use the formula (cp. (14))

$$
\sum_{i=1}^n \sigma_{q_i,x_j}\, \psi_{x_i} = -\sum_{i=1}^{N+1} \sigma_{q_i,x_j}\, \zeta_i + \sigma_{q_{N+1},x_j} = -\nu_{N+1}^{-1}\, \sigma_{x_j} + \sigma_{q_{N+1},x_j}\,,
$$

and analogously, that $\sum_{i=1}^N \sigma_{q_i,x_j,x_l}\, \psi_{x_i} = -\nu_{N+1}^{-1}\, \sigma_{x_j,x_l} + \sigma_{q_{N+1},x_j,x_l}$, to prove with the help of (31) that

$$
|A_1| \leq |\xi|\, (|\sigma_{q,\bar{x},\bar{x}}| + |\sigma_{\bar{x},x_{N+1}}|\, \nu_{N+1}^{-1} + |\sigma_{q_{N+1},\bar{x}}|) + \nu_{N+1}\, |\sigma_{\bar{x},q,q}|\, |\,\delta\, \zeta|
$$
$$
+ |\sigma_{q,x_{N+1}}|\, |\,\delta\, \zeta| + (|\sigma_{q_{N+1},x_{N+1}}| + |\sigma_{x_{N+1}}|\, \nu_{N+1}^{-1})\, |\,\delta_{n+1}\, \zeta|
$$
$$
+ |\zeta|\, (\sigma_{\bar{x},q,x_{N+1}} - \nu_{N+1}^{-1}\, \sigma_{x_{N+1},x_{N+1}} + \sigma_{q_{N+1},x_{N+1},x_{N+1}} + \nu_{N+1}\, |\sigma_{x_{N+1},q,q}|\, |\,\delta\, \zeta|)\,.
$$

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Due to the assumptions (8) and (9), we therefore have $|A_1| \leq c_1 + c_2 \, \nu_{N+1} \, |\, \delta \, \zeta\,|$.

Using that $\sigma_{q_i, q_j} \, d_i = \sigma_{q_i, q_j} \, \delta_{x_i}$, we readily see that

$$
\begin{aligned}
A_2 &= \nu_{N+1} \, \sigma_{q_i, q_j} \, \delta_j \, \psi_{x_k} \, (\sigma_{q_k, x_i} - \delta_i(\kappa \, n_k) - \nu_{N+1} \, \sigma_{q_k, q_j} \, \delta_j \, \psi_{x_i}) \\
&\leq - \nu_{N+1}^2 \, \sigma_{q_i, q_j} \, \sigma_{q_k, q_j} \, \delta_j \, \zeta_i \, \delta_j \, \zeta_k + \nu_{N+1} \, (|\sigma_{q, \bar{x}}| + |\,\delta(\kappa \, n)|) \, |\, \delta \, \zeta\,| \, .
\end{aligned}
\tag{42}
$$

The condition (8c) implies that $\sigma_{q_i, q_j} \, \delta_j \, \zeta_k \, \sigma_{q_k, q_l} \, \delta_l \, \zeta_i \geq \gamma_2^2 \, |\, \delta \, \zeta\,|^2$, yielding $A_2 \leq -\lambda_2^2 \, \nu_{N+1}^2 \, |\, \delta \, \zeta\,|^2 + c \, \nu_{N+1} \, |\, \delta \, \zeta\,|$. Using the conditions (8), we easily see that

$$
|A_3| \leq \mu_1 \, |\Phi_x| + |\Phi_q| \, (|\sigma_{\bar{x}, q}| + |\,\delta(\kappa \, n)|) + \nu_{N+1} \, |\Phi_q| \, |\, \delta \, v\,| \, .
$$

Thus, for the function a_1 we have the following estimate:

$$
a_1 \leq -\lambda_2^2 \, |\, \delta \, \zeta\,|^2 \, \nu_{N+1}^2 + C \, \nu_{N+1} \, |\, \delta \, \zeta\,| + |\Phi_q| \, \nu_{N+1} \, |\, \delta \, v\,| + C \, (1 + |\Phi_x| + |\Phi_q|) \, ,
$$

where the constants depend on G, $\|\kappa_x\|_{L^\infty(S)}$ and the constants of the conditions (8), (9). It follows from Youngs inequality that

$$
a_1 \leq -\frac{\lambda_2^2}{2} \, |\, \delta \, \zeta\,|^2 \, \nu_{N+1}^2 + C \, (1 + |\Phi_x| + |\Phi_q|) + C \, |\Phi_q| \, \nu_{N+1} \, |\, \delta \, v\,| \, .
$$

Finally, we use the assumptions (9) to show that $|a_2| \leq \mu_1 \, (|\kappa_x| + |n_x| + |\sigma_{q, \bar{x}}|) + v \, \sigma_{q_i, x_{N+1}} \, n_i \leq C$. This concludes the proof. $\qquad\square$

From Lemma 3.3, there are several ways to finish the proof. Uraltseva's technique in [LU70, Ura71, Ura82] is based on estimating $w := \log v$. Assume that $\Phi_{x_{N+1}} \leq 0$ and $\kappa_{x_{N+1}} \geq 0$. Choosing $\eta \, v$ with η nonnegative as a testfunction in (26), and using (25), one easily deduces that

$$
\int_S \sigma_{q_i, q_j} \, \delta_j \, w \, \delta_i \, \eta + \int_S \nu_{N+1}^2 \, \sigma_{q_i, q_j} \, \delta_j \, v \, \delta_i \, v \, \eta \leq C_1 \int_S \{(1 + |\, \delta \, w\,|) \, \eta + |\, \delta \, \eta\,|\} + C_2 \int_{\partial S} \eta \, \frac{ds}{\sin \alpha} \, .
$$

Thus, Lemma 3.1 now yields $\int_S \sigma_{q_i, q_j} \, \delta_j \, w \, \delta_i \, \eta \leq C \, \int_S \{(1 + |\, \delta \, w\,|) \, \eta + |\, \delta \, \eta\,|\}$. It is possible to derive the boundedness of w like in the standard Stampacchia proof for second order elliptic equation with L^∞ coefficients, provided that a Sobolev embedding theorem is globally available on the manifold S (cf. [LU70], [MS73] for local embedding results, [Ura71], [Ura82] for the extension to global embedding). Here we rather show an elementary manner to finish the proof in the case that $\gamma_0 > 0$ in the condition (11). Under this strong monotonicity condition, the estimate on $\nabla \psi$ is only polynomial in the norm of the data.

Lemma 3.4. *Same assumptions as in Lemma 3.3. Then, there is a constant K depending on the constants in (8), (9), and on $\|\kappa_x\|_{L^\infty(S)}$ and G such that for all $1 \leq q < \infty$*

$$
\int_G v^{q-4} \, |\nabla v|^2 - \frac{1}{q} \int_G \Phi_{x_{N+1}} \, v^{q+1} \leq K \int_G \tilde{a} \, v^q \, , \quad \tilde{a} \leq 1 + |\Phi| + |\Phi_x| + |\Phi_q|^2 \, .
\tag{43}
$$

Proof. Choose in Lemma 3.3 $\eta = v^q$. We obtain that

$$
\begin{aligned}
q \int_S \nu_{N+1}^2 \, v^{q-1} \, \sigma_{q_i, q_j} \, \delta_j \, v \, \delta_i \, v &- \int_S \Phi_{x_{N+1}} \, \nu_{N+1} \, v^{q+1} + \int_{\partial S} \kappa_{x_{N+1}} \, \nu_{N+1} \, v^{q+1} \, \frac{ds}{\sin \alpha} \\
&= \int_S \nu_{N+1} \, \{a_1 \, v^q + q \, b \cdot \delta \, v \, v^{q-1}\} + \int_{\partial S} \nu_{N+1} \, a_2 \, v^q \, \frac{ds}{\sin \alpha} \, .
\end{aligned}
$$

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Due to (8b), $\sigma_{q_i,q_j} \, \delta_j \, v \, \delta_i \, v \geq \lambda_2 \, | \delta \, v|^2$. Applying Young's inequality, and using the bounds derived in Lemma 3.3 for the functions a_i, $|b|$ we can estimate

$$\nu_{N+1} \, a_1 \, v^q \leq c_1 \, (1 + |\Phi_x| + |\Phi_q|) \, \nu_{N+1} \, v^q + \frac{\lambda_2 \, q}{6} \, \nu_{N+1}^2 \, | \delta \, v|^2 \, v^{q-1} + \frac{3 \, c_2^2 \, |\Phi_q|^2}{2 \, \lambda_2 \, q} \, v^{q-1}$$

$$q \, \nu_{N+1} \, b \cdot \delta \, v \, v^{q-1} \leq \frac{\lambda_2 \, q}{6} \, \nu_{N+1}^2 \, | \delta \, v|^2 \, v^{q-1} + \frac{3 \, c_4^2 \, q}{2 \, \lambda_2} \, v^{q-1} \,.$$

Using the fact that $\kappa_{x_{N+1}} \geq 0$, and (25), we prove that

$$\frac{2 \, q \, \lambda_2}{3} \int_S \nu_{N+1}^2 \, v^{q-1} \, | \delta \, v|^2 - \int_S \Phi_{x_{N+1}} \, \nu_{N+1} \, v^{q+1}$$

$$\leq \int_S (\frac{3 \, c_2^2 \, |\Phi_q|^2}{2 \, \lambda_2 \, q} + \frac{3 \, c_4^2 \, q}{2 \, \lambda_2} + c_1 \, \gamma_2 \, (1 + |\nabla\Phi|)) \, v^{q-1} + \int_{\partial S} \nu_{N+1} \, |a_2| \, v^q \, \frac{ds}{\sin \alpha} \,.$$

We apply the estimates (25) and (27) and the Lemma 3.1 to estimate

$$\int_{\partial S} \nu_{N+1} \, |a_2| \, v^q \, \frac{ds}{\sin \alpha} \leq c_3 \, \gamma_2 \int_{\partial S} v^{q-1} \, \frac{ds}{\sin \alpha}$$

$$\leq c_3 \, \gamma_2 \, \gamma_1^{-1} \, ((q-1) \int_S v^{q-2} \, | \delta \, v| + \int_S |a_0| \, v^{q-1})$$

$$\leq \frac{\lambda_2 \, q}{6} \int_S \nu_{N+1}^2 \, | \delta \, v|^2 \, v^{q-1} + \int_S \{ \frac{3 c_3^2 \gamma_2^2 \, (q-1)^2}{2 \, \lambda_2 \, \gamma_1^2 \, q} \nu_{N+1}^{-2} \, v^{q-3} + c_3 \, \gamma_2 \, \gamma_1^{-1} \, |a_0| \, v^{q-1} \} \,.$$

Using (25) again and the bound derived in Lemma 3.1 for the function a_0, we derive the estimate

$$q \int_S \nu_{N+1} \, v^{q-2} \, | \delta \, v|^2 - \int_S \Phi_{x_{N+1}} \, v^q \quad \leq C \, q \int_S \tilde{a} \, v^{q-1}$$

$$\tilde{a} := \frac{1}{q} \, (q + |\Phi_x| + 1/q \, |\Phi_q|^2 + |\Phi|) \,, \tag{44}$$

where C depends on all the data but not on Φ. Note that $\int_S \nu_{N+1} \, v^{q-2} \, | \delta \, v|^2 = \int_G v^{q-2} \, | \delta \, v|^2 \geq \int_G \nu_{N+1}^2 \, v^{q-2} \, |\nabla v|^2$. The claim follows using again (25). $\qquad \square$

Proposition 3.5. *Same assumptions as in Lemma 3.4. Let $p > N/2$ and $\alpha_0 > \frac{2Np}{2p-N}$ arbitrary. Then, there exist a constant C depending on K, G, α_0 and p, and functions ζ_0, ζ_1 of α_0 and p such that* $\max_S v \leq C \, (1 + \|\Phi\|_{L^p(S)} + \|\Phi_x\|_{L^p(S)} + \|\Phi_q\|^2_{L^{2p}(S)})^{\zeta_1} \, \|v\|^{\zeta_0}_{L^{\alpha_0}(G)}.$

Proof. Due to the condition (11), Lemma 3.4 implies that

$$\int_G |\nabla v^{(q-2)/2}|^2 \leq \frac{K \, (q-2)^2}{4} \int_G \tilde{a} \, v^q \,. \tag{45}$$

We add $\|v^{(q-2)/2}\|^2_{L^2(G)}$ on both sides of (45). Thanks to Hölder's inequality, it follows that

$$\int_G \{|\nabla v^{(q-2)/2}|^2 + |v|^{q-2}\} \leq \frac{K \, (q-2)^2}{4} \int_G \tilde{a} \, v^q + \text{meas}(G)^{2/q} \, \|v\|^{q-2}_{L^q(G)} \,. \tag{46}$$

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Define $q_0 := \alpha_0/p'$, $p' = p/(p-1)$. The choice of α_0 garanties that $q_0 > N$. Define $\chi := \frac{q_0-2}{N-2} \frac{N}{q_0}$ if $N > 2$, and $\chi \in]p', +\infty[$ arbitrary if $N = 2$. The choice of α_0 implies that $\chi > p'$. We can also verify that $\frac{2\chi q}{q-2} \leq \frac{2N}{N-2}$ for $N \geq 3$, $\frac{2\chi q}{q-2} < \infty$ for $N = 2$, for all $q_0 \leq q < \infty$. It follows that the embedding $W^{1,2}(G) \hookrightarrow L^r(G)$ for $r := 2\chi q/(q-2)$ is continuous, and that the embedding constants are uniformly bounded. The relation (46) implies that

$$\|v\|_{L^{\chi q}(G)}^{q-2} = \|v^{(q-2)/2}\|_{L^r(G)}^2 \leq c\,((q-2)^2\, K\, \|\tilde{a}\|_{L^p(G)}\, \|v\|_{L^{p'q}(G)}^q + \operatorname{meas}(G)^{\frac{2}{q}+\frac{q-2}{qp}}\, \|v\|_{L^{p'q}(G)}^{q-2})$$

$$\leq c\,\max\{(q-2)^2\, K\, \|\tilde{a}\|_{L^p(G)},\ \operatorname{meas}(G)^{\frac{2}{q}+\frac{q-2}{qp}}\}\,\max\{\|v\|_{L^{p'q}(G)}^q,\ \|v\|_{L^{p'q}(G)}^{q-2}\}. \tag{47}$$

For $m \in \mathbb{N}$, set $\alpha_m := \frac{\chi}{p'}\alpha_{m-1}$, $A_m := \|v\|_{L^{\alpha_m}(G)}$. As a consequence of (47) with $q_m = \alpha_m/p'$, one finds the recursive inequalities $A_{m+1} \leq c_m^{1/(q_m-2)} A_m^{\xi_m}$ that imply

$$A_{m+1} \leq c_m^{1/(q_m-2)} \{\prod_{i=0}^{m-1} [c_i]^{\xi_{i+1}/(q_i-2)}\}\, A_0^{\prod_{i=0}^m \xi_i}\,,$$
$$\tag{48}$$
$$\xi_m := \begin{cases} \lambda_m := \frac{q_m}{q_m-2} & \text{if } A_m \geq 1 \\ 1 & \text{otherwise} \end{cases}\,,\quad c_m := c\,[(q_m-2)^2\, K\, \|\tilde{a}\|_{L^p(G)} + \operatorname{meas}(G)^{\frac{2}{q_m}+\frac{q_m-2}{q_m p}}]\,.$$

We now provide rough bounds for the products appearing in (48). We abbreviate $\tilde{\chi} := \chi/p' > 1$. Note first that

$$\log(\prod_{i=0}^m \xi_i) \leq \sum_{i=0}^m \log(q_i/(q_i-2)) \leq 2 \sum_{i=0}^m \frac{1}{q_i-2} \leq \frac{2}{q_0-2} \sum_{i=0}^\infty \tilde{\chi}^{-i}\,,$$

Thus $\zeta_0 := \prod_{i=0}^\infty \xi_i$ satisfies the estimate $\zeta_0 \leq \exp(2\tilde{\chi}/(q_0-2)(\tilde{\chi}-1))$. Observe also that

$$\log(\prod_{i=0}^{m-1} [c_i]^{\xi_{i+1}/(q_i-2)}) = \sum_{i=0}^{m-1} \frac{\xi_{i+1}}{q_i-2}\,\log(c_i)\,,$$
$$\tag{49}$$
$$\log(c_i) \leq \log c + \log(K\,\|\tilde{a}\|_{L^p(G)}) + 2\,\log(q_i-2) + (\frac{2}{q_i} + \frac{q_i-2}{q_i\, p})\,\log\operatorname{meas}(G)\,.$$

Using the estimate $\xi_{i+1} \leq q_0/(q_0-2)$ for $i \in \mathbb{N}$ we can bound

$$\sum_{i=0}^{m-1} \frac{\xi_{i+1}}{q_i-2}\,\log(q_i-2) \leq \frac{q_0}{q_0-2} \sum_{i=0}^{m-1} \frac{i\,\log\tilde{\chi} + \log q_0}{\tilde{\chi}^i\, q_0-2} \leq \frac{q_0\,\log q_0}{(q_0-2)^2}\,(\sum_{i=0}^{m-1} \frac{i+1}{\tilde{\chi}^i})\,,$$

and $\zeta_1 := \sum_{i=0}^\infty \frac{\xi_{i+1}}{q_i-2}$, $\zeta_2 := \sum_{i=0}^\infty \frac{\xi_{i+1}}{(q_i-2)}(\frac{2}{q_i} + \frac{q_i-2}{q_i})$ are obviously finite. Therefore, (49) implies that $\prod_{i=0}^{m-1} [c_i]^{\xi_{i+1}/(q_i-2)} \leq c_1(q_0)\,((K\|\tilde{a}\|_{L^p(G)})^{\zeta_1} + \operatorname{meas}(G)^{\zeta_2})$, and the claim follows from (48). $\qquad\square$

Everything is therefore reduced to estimating the L^{q_0}−norm of v for a $\alpha_0 > \frac{2Np}{2p-N}$. We directly obtain this bound, if we require the strong monotonicity condition (11). It trivially follows from (44) that for all $2 < t < \infty$

$$\|v\|_{L^t(G)} \leq \frac{C\,t}{\gamma_0}(1 + \|\Phi\|_{L^t(S)} + \|\Phi_x\|_{L^t(S)} + \|\Phi_q\|_{L^{2t}(S)})\,. \tag{50}$$

This achieves the proof of Theorem 2.2.

4 Higher-order estimates

The gradient bound is crucial in the problem (5), (6). Higher-order estimates can be derived whenever a $L^\infty-$bound on the derivatives of ψ has been proved, since the equation (5) is then a *uniformely elliptic equation of quasilinear type*, due to (cp. (17))

$$\sum_{i,j=1}^{N} \bar{\sigma}_{p_i,p_j} \xi_i \xi_j \geq \frac{\lambda_2 |\xi^T|^2}{\sqrt{1+|\nabla\psi|^2}} \geq \frac{\lambda_2 |\xi|^2}{(1+|\nabla\psi|^2)^{3/2}} \quad \text{for all } \xi \in \mathbb{R}^N.$$

Define $c_6 := \sup_G \sqrt{1+|\nabla\psi|^2}$. The following Lemma states the Hoelder continuity estimate.

Lemma 4.1. *Assume that G is a domain of class $\mathcal{C}^2$. Let $\psi \in \mathcal{C}^2(\overline{G})$ be a solution to* (5), (6). *Then, for all $\beta \in [0,1[$, there is $c = c(G, c_6, \beta)$ such that*

$$\|\nabla\psi\|_{C^{0,\beta}(\overline{G})} \leq c\left(1 + \|\nabla(n\,\kappa)\|_{L^\infty(S)} + \|\Phi\|_{L^\infty(S)}\right),$$

Proof. Due to Remark 2.3 and Gauss's divergence theorem, ψ satisfies

$$\int_G (\bar{\sigma}_p + \kappa\,n) \cdot \nabla\xi = \int_G (\bar{\Phi} - \operatorname{div}(\kappa\,n))\,\xi \quad \forall \xi \in W^{1,1}(G). \tag{51}$$

Here and throughout the proof, the functions $\bar{\sigma}$ and $\bar{\Phi}$ are evaluated at $(\bar{x}, \psi, \nabla\psi)$. In order to simplify the discussion, we prove the regularity in a smooth open domain $G_0 \subset G$, assuming that $\Gamma_0 := \partial G \cap \overline{G_0}$ is flat and such that the $N-1$ first basis vectors are tangent on Γ_0 and $n = e_N$ on Γ_0. In the general case, it is possible to use the definition of a domain of class $\mathcal{C}^2$ to locally map a neighbourhood of $\bar{x} \in \partial G$ onto the model configuration.

For $l = 1, \ldots, N-1$, we insert the testfunction $\partial_{x_l}\xi$ for $\xi \in C_c^1(G_0 \cup \Gamma_0)$ in (51). Using integration by parts, it follows that

$$-\int_G \{\bar{\sigma}_{p_i,p_j}\,\partial^2_{x_j,x_l}\psi + \bar{\sigma}_{p_i,x_l} + \bar{\sigma}_{p_i,x_{N+1}}\,\partial_{x_l}\psi + \partial_{x_l}(\kappa\,n_i)\}\,\partial_{x_i}\xi$$
$$+\int_{\partial G}(\bar{\sigma}_{p_i} + \kappa\,n_i)\,\partial_{x_i}\xi\,n_l = \int_G (\bar{\Phi} - \operatorname{div}(\kappa\,n))\,\partial_{x_l}\xi. \tag{52}$$

Since $n_l = 0$ on Γ_0, the choice of ξ yields vanishing of the surface integral. Equivalently

$$\int_G \bar{\sigma}_{p_i,p_j}\,\partial_{x_j}\psi_{x_l}\,\partial_{x_i}\xi = \int_G V \cdot \nabla\xi, \tag{53}$$
$$V_i := -\bar{\sigma}_{p_i,x_l} - \bar{\sigma}_{p_i,x_{N+1}}\,\partial_{x_l}\psi - \partial_{x_l}(n_i\,\kappa) - (\bar{\Phi} - \operatorname{div}(\kappa\,n))\,\delta_l^i \quad \text{for } i = 1, \ldots, N. \tag{54}$$

Using in particular the growth assumptions (8), and (9b), it follows that $|V| \leq \mu_2 + \mu_1 + \mu_6 + \|\nabla(\kappa\,n)\|_{L^\infty(S)} + \|\Phi\|_{L^\infty(S)}$. According to classical linear regularity theory (cf. for instance the Theorem 3.16 in [Tro87]), there is for $0 \leq \beta < 1$ arbitrary a constant c depending only on β, G_0, the ellipticity constant of the matrix $\{\bar{\sigma}_{p_i,p_j}\}$ and its norm in L^∞ such that

$$\|\psi_{x_l}\|_{C^{0,\beta}_{\text{loc}}(G_0 \cup \Gamma_0)} \leq c\,\|V\|_{[L^\infty(G_0)]^n}. \tag{55}$$

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It follows that $\psi_t := \nabla\psi - (n \cdot \nabla\psi)\, n \in C^{0,\beta}_{\text{loc}}(G_0 \cup \Gamma_0)$. Using an open covering of ∂G, and applying the reasoning locally to each section, we obtain that $\psi_t \in C^{0,\beta}(\partial G)$ with corresponding norm estimate. We show show that also $\psi_n := n \cdot \nabla\psi$ satisfies a Hölder condition on ∂G. For $\bar{x} \in \overline{G}$, $y \in \mathbb{R}$, define

$$H(\bar{x},\, y) := \bar{\sigma}_p(\bar{x},\, \psi(\bar{x}),\, \psi_t(\bar{x}) + n(\bar{x})\, y) \cdot n(\bar{x}) + \kappa(\bar{x},\, \psi(\bar{x}))\,.$$

Using the growth condition (8b), $|H(\bar{x},\, y)| \leq \mu_1 + \|\kappa\|_{L^\infty(S)}$ for all $(\bar{x}, y) \in \overline{G} \times \mathbb{R}$. Moreover, for $\bar{x}_1,\, \bar{x}_2 \in \overline{G},\, y \in \mathbb{R}$

$$
\begin{aligned}
|H(\bar{x}_1,\, y) - H(\bar{x}_2,\, y)| \leq & \|\bar{\sigma}_{p,\bar{x}}\|_{L^\infty}\, |\bar{x}_1 - \bar{x}_2| + \|\bar{\sigma}_{p,x_{N+1}}\|_{L^\infty}\, |\psi(\bar{x}_1) - \psi(\bar{x}_2)| \\
& + \|\bar{\sigma}_{p,p}\|\, (|\psi_t(\bar{x}_1) - \psi_t(\bar{x}_2)| + |y|\, |n(\bar{x}_1) - n(\bar{x}_2)|) \\
& + \|\kappa_{\bar{x}}\|_{L^\infty}\, |\bar{x}_1 - \bar{x}_2| + \|\kappa_{x_{N+1}}\|_{L^\infty}\, |\psi(\bar{x}_1) - \psi(\bar{x}_2)|\,,
\end{aligned}
$$

so that the following estimate holds:

$$\frac{|H(\bar{x}_1,\, y) - H(\bar{x}_2,\, y)|}{|\bar{x}_1 - \bar{x}_2|^\beta} \leq c\, (1 + \|\psi_t\|_{C^{0,\beta}(\partial G)} + |y|\, \|n\|_{C^{0,\beta}(\overline{G})})\,. \tag{56}$$

By virtue of the condition (8c), note that

$$\partial_y H(\bar{x},\, y) = \bar{\sigma}_{p_i,p_j}\, n_j(\bar{x})\, n_i(\bar{x}) \geq \lambda_2\, (1 + |\nabla\psi|^2)^{-1/2}\, (1 - (\nu \cdot n)^2) \geq \lambda_2\, c_6^{-3}\,. \tag{57}$$

On the other hand, the boundary condition (6) implies that $H(x,\, \psi_n(x)) = 0$ on ∂G. For $\bar{x},\, \bar{x}' \in \partial G$ arbitrary, it follows that

$$
\begin{aligned}
\lambda_2\, c_6^{-3}\, (\psi_n(\bar{x}) - \psi_n(\bar{x}')) &\leq \int_{\psi_n(\bar{x})}^{\psi_n(\bar{x}')} \partial_y H(\bar{x},\, s)\, ds \\
&= H(\bar{x},\, \psi_n(\bar{x}')) - H(\bar{x},\, \psi_n(\bar{x})) = H(\bar{x},\, \psi_n(\bar{x}')) - H(\bar{x}',\, \psi_n(\bar{x}'))\,.
\end{aligned}
$$

The latest yields

$$\frac{|\psi_n(\bar{x}) - \psi_n(\bar{x}')|}{|\bar{x} - \bar{x}'|^\beta} \leq c\, \frac{|H(\bar{x},\, \psi_n(\bar{x}')) - H(\bar{x}',\, \psi_n(\bar{x}'))|}{|\bar{x} - \bar{x}'|^\beta}\,. \tag{58}$$

Therefore, taking (56) into account

$$\|\psi_n\|_{C^{0,\beta}(\partial G)} \leq c\, (1 + \|\psi_t\|_{C^{0,\beta}(\overline{G})} + c_6\, \|n\|_{C^{0,\beta}(\overline{G})}) \tag{59}$$

which finally implies that $\nabla\psi \in C^{0,\beta}(\partial G)$. Return to (53) for $\xi \in C^1_c(G_0)$. With the help of regularity results for linear equations (cf. for instance the Theorem 3.16 in [Tro87]), it now follow that $\partial_{x_l}\psi \in C^{0,\beta}_{\text{loc}}(G_0 \cup \Gamma_0)$ for $l = 1, \ldots N - 1$ with corresponding norm estimate. Since the same relation is valid for $l = N$ if the testfunction ξ vanishes on ∂G (note: the operator $(\bar{\sigma}_p + \kappa\, n) \cdot \nabla$ is tangent on ∂G), we can argue the same for ψ_n in view of (59) $\qquad \square$

The estimate in $C^{2,\alpha}$ is obtained with similar ideas.

Lemma 4.2. *Same assumptions as in Lemma 4.1. Then,* $\|D^2\psi\|_{C^\alpha(\overline{G})} \leq C\, (\|\Phi\|_{C^\alpha(\overline{S})} + \|\kappa\|_{C^{1,\alpha}(\partial S)})$, *where C depends on the constants in the conditions* (8)*,* (9)*,* (12) *and on c_6.*

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Proof. Consider the relation (53). Lemma 4.1 implies that $\bar{\sigma}_{p_i,p_j} \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$. Analogously, $\bar{\Phi} \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$ (cf. (4) and (7)).

The definition (54) together with Lemma 4.1 now implies that $V \in [C^{0,\alpha}(\overline{G})]^N$ (cp. (54)). Thus, the linear regularity theory (cf. Theorem 3.17 in [Tro87]) now yields for $l = 1, \ldots, N-1$

$$\|\psi_{x_l}\|_{[C^{1,\alpha}(\overline{G})]^N} \leq c \, \|V\|_{[C^{0,\alpha}(\overline{G})]^n} \, .$$

We are now allowed to differentiate the relation $H(\bar{x}, \psi_n(\bar{x})) = 0$ in any tangential direction τ over ∂G, which yields $\partial_y H(\bar{x}, \psi_n(\bar{x})) \, (\tau \cdot \nabla \psi_n) = \tau \cdot \nabla_{\bar{x}} H(\bar{x}, \psi_n(\bar{x}))$ for $\bar{x} \in \partial G$. Due to commutation rules, the mixed-derivatives $\psi_{t,n}$ belongs to $C^{0,\alpha}(\overline{G})$, with corresponding continuity estimates. In order to show that also $\psi_{n,n} \in C^{0,\alpha}(\overline{G})$, we use the previous results in connection with equation (5) yielding on Γ_0

$$\bar{\sigma}_{p_N,p_N} \, \psi_{n,n} = \bar{\Phi} - \sum_{i,j=1}^{N-1} \bar{\sigma}_{p_i,p_j} \, \psi_{x_i,x_j} - 2 \sum_{i=1}^{N-1} \bar{\sigma}_{p_i,p_N} \, \psi_{x_i,x_N} - \bar{\sigma}_{p_i,x_i} - \bar{\sigma}_{p_i,x_{N+1}} \, \psi_{x_i} \in C^{0,\alpha}(\overline{G}) \, .$$

Since $n_i \, \bar{\sigma}_{p_i,p_j} \, n_j \geq \lambda_2 \, c_6^{-3}$, the function $(n_i \, \bar{\sigma}_{p_i,p_j} \, n_j)^{-1}$ belongs also to $C^{0,\alpha}(\partial G)$. We finally can conclude that $\psi_{n,n} \in C^{0,\alpha}(\Gamma_0)$, and that $\psi_{n,n} \in C^{0,\alpha}(\partial G)$ due to localization arguments. Thus $D^2 \psi \in C^{0,\alpha}(\partial G)$, and the claim follows (Theorem 3.17 in [Tro87]). $\qquad \square$

5 *A priori* estimates on ψ in L^∞

The natural $W^{1,1}$ estimate, and the global boundedness of weak solutions to (5), (6) have been discussed in different papers. In the case that σ and κ do not depend on the x_{N+1} variable, and that $\Phi = \Phi(\bar{x}, x_{N+1})$ the inequality $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$ and the condition (11) is known to be sufficient to obtain these bounds. The arguments easily carry over to the general case.

Lemma 5.1. *Assume that $\psi \in W^{1,1}(G)$ is a weak solution to* (5), (6). *Assume that* (11) *is valid. Assume that $p > 2N$. Then, there is a constant depending on $2N - p$, on G, on the constants μ_1, γ_1 and γ_0, and on $\|\kappa_{\bar{x}}(\cdot, 0)\|_{L^\infty(G)}$ such that*

$$\|\psi\|_{L^\infty(G)} \leq c \, (1 + \|\bar{\Phi}(\bar{x}, 0, \nabla \psi)\|_{L^p(G)}^2) \, .$$

Proof. Multiply the equation with $\xi \in W^{1,1}(G)$ and integrate by parts. We add the zero $\int_G \mathrm{div}(\kappa(\bar{x}, 0) \, n \, \xi) - \int_{\partial G} \kappa(\bar{x}, 0) \, \xi$, to obtain the identity

$$\int_G (\bar{\sigma}_q(\bar{x}, \psi, \nabla \psi) + \kappa(\bar{x}, 0) \, n) \cdot \nabla \xi + \int_{\partial G} (\kappa(\bar{x}, \psi) - \kappa(\bar{x}, 0)) \, \xi$$
$$= \int_G (\bar{\Phi}(\bar{x}, \psi, \nabla \psi) - \bar{\Phi}(\bar{x}, 0, \nabla \psi)) \, \xi + \int_G (\bar{\Phi}(\bar{x}, 0, \nabla \psi) - \mathrm{div}(\kappa(\bar{x}, 0) \, n)) \, \xi \, .$$

Choose $\xi = (\psi - k)^+$, $k \in \mathbb{R}^+$. Due to (11), $(\kappa(\bar{x}, \psi) - \kappa(\bar{x}, 0)) \, (\psi - k)^+ \geq 0$, and $(\bar{\Phi}(\bar{x}, \psi, \nabla \psi) - \bar{\Phi}(\bar{x}, 0, \nabla \psi)) \, (\psi - k)^+ \leq -\gamma_0 \, \psi \, (\psi - k)^+$. Using (15), and the constant γ_1 from (12), we can prove that

$$\gamma_1 \int_G |\nabla (\psi - k)^+| - \mu_1 \, \mathrm{meas}(A_k) + \gamma_0 \int_G \psi \, (\psi - k)^+$$
$$\leq \int_G (|\bar{\Phi}(\bar{x}, 0, \nabla \psi)| + |\mathrm{div} \, \kappa(\bar{x}, 0) \, n)|) \, (\psi - k)^+ \, ,$$

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where $A_k := \operatorname{supp}(\psi - k)^+$. Using Young's and Hölder's inequalties, we can prove that

$$\gamma_1 \,\|(\psi - k)^+\|_{W^{1,1}(G)} \leq (\mu_1 + \frac{\gamma_1^2}{2\gamma_0})\, \operatorname{meas}(A_k) + \frac{1}{2\gamma_0} \int_{A_k} (|\bar{\Phi}(\bar{x},\, 0,\, \nabla\psi)| + |\operatorname{div}\kappa(\bar{x},\, 0)\, n)|)^2$$

$$\leq (\mu_1 + \frac{\gamma_1^2}{2\gamma_0})\, \operatorname{meas}(A_k) + \frac{1}{2\gamma_0}\, (\|\bar{\Phi}(\bar{x},\, 0,\, \nabla\psi)\|_{L^p(G)}^2 + \|\operatorname{div}(\kappa(\bar{x},\, 0)\, n)\|_{L^p(G)}^2)\, \operatorname{meas}(A_k)^{\frac{N-1}{N}+\epsilon},$$

with $\epsilon := 1/N - 2/p$. It follows that $\psi \leq c\,(1 + \|\bar{\Phi}(\bar{x},\, 0,\, \nabla\psi)\|_{L^p(G)}^2)$ (Stampacchia's Lemma, cf. [Tro87], Lemma 2.9). We prove analogously a lower bound, and the claim follows. $\qquad\square$

6 Existence

It was shown for the first time in [Ura71] that *a priori* estimates on the gradient of C^2 solutions to (5), (6) joined to the Hoelder estimate of Lemma 4.1 leads to an existence theorem via continuation methods in Banach-spaces exposed in [LU68], Ch. 10. Here, existence is obtained via the implicit function theorem.[3] Note that we need somewhat weaker hypotheses on Φ than usually in the literature. Moreover, the condition (13) seems not to be yet known in the present context. At first, we formulate a simple continuation Lemma.

Proposition 6.1. *Let X, Y, Z be Banach spaces such that $Y \hookrightarrow X$ with compact embedding. For $a, b \in \mathbb{R}$, $a < b$, let $\mathcal{G} : X \times]a,\, b[\to Z$ be a Fréchet differentiable mapping, such that the derivative $\partial_x \mathcal{G}(x^*,\, \lambda^*) \in \mathcal{L}(X,\, Z)$ is an isomorphism for all $(x^*,\, \lambda^*) \in X \times]a,\, b[$. Assume that there is $K > 0$ such that for all $\lambda \in]a,\, b[$, all solutions $x \in X$ to the equation $\mathcal{G}(x,\, \lambda) = 0$ belong to $B_K(0;\, Y)$. If there is $(x_0,\, \lambda_0) \in X \times]a, b[$ such that $\mathcal{G}(x_0,\, \lambda_0) = 0$, then the equation $\mathcal{G}(x,\, \lambda) = 0$ has a unique solution in $B_K(0;\, Y)$ for all $\lambda \in [a, b]$.*

Proof. Define $M := \{\lambda \in [a,\, b] : \exists x \in X,\, \mathcal{G}(x,\, \lambda) = 0\}$. The set M is nonvoid since $\mathcal{G}(x_0, \lambda_0) = 0$. Moreover $\lambda^* := \sup M$ belongs to M. To see this, choose $\{\lambda_k\}_{k \in \mathbb{N}} \subseteq M$, $\lambda_k \to \lambda^*$. By definition, there is $x_k \in X$ such that $\mathcal{G}(x_k,\, \lambda_k) = 0$. By assumption $x_k \in B_K(0;\, Y)$ for all $k \in \mathbb{N}$, and therefore, there is a subsequence x_{k_j} that strongly converges in X to some x^*. Obviously, $\mathcal{G}(x^*,\, \lambda^*) = 0$, implying $\lambda^* \in M$.

Seeking a contradiction, assume that $\lambda^* < b$. Then, due to the implicit function theorem (see [GT01], Th. 17.6), there is an open neighborhood $]\lambda^* - \epsilon,\, \lambda^* + \epsilon[$ in $]a, b[$ such that the equation $\mathcal{G}(x,\, \lambda) = 0$ defines a unique implicit vector-valued function $\lambda \mapsto x(\lambda) \in X$. Therefore $\lambda^* \neq \sup M$, the contradiction. Analogously, one shows that $\inf M = a$. This proves the existence.

If x_1, $x_2 \in X$ both solve $\mathcal{G}(x,\, \lambda) = 0$, then $\partial_x \mathcal{G}(x^*,\, \lambda)\,(x_1 - x_2) = 0$ for some $x^* \in [x_1,\, x_2]$. Due to the assumption that $\partial_x \mathcal{G}$ is an isomorphism, the uniqueness follows. $\qquad\square$

Theorem 6.2. *Assumptions of the Theorem 2.1. Then, there is a unique $\psi \in C^{2,\alpha}(\overline{G})$ that solves (5), (6).*

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[3]We thank the referree for the indication that a similar simplification of the existence proof was already achieved in the Second Edition (1972) of the book [LU68], which unfortunately was not translated into English.

Proof. In the first step, we prove the existence claim assuming that $\Phi \in C^{1,\alpha}(\overline{G} \times \mathbb{R} \times \mathbb{R}^{N+1})$. Let $0 < \beta < \alpha$. Define two mappings $\mathcal{G}_1 : C^{2,\beta}(\overline{G}) \times\,]-1,\, 1[\to C^\beta(\overline{G})$ and $\mathcal{G}_2(w,\, \lambda) : C^{2,\beta}(\overline{G}) \times\,]-1,\, 1[\to C^{1,\beta}(\partial G)$ via

$$\mathcal{G}_1(w,\, \lambda) := -\frac{d}{dx_i}\bar{\sigma}_{p_i}(\bar{x},\, w,\, \nabla w) - \bar{\Phi}(\bar{x},\, w,\, \nabla w) + (1-\lambda)\left(\partial_{x_i}\bar{\sigma}_{p_i}(\bar{x},\, 0,\, 0) + \bar{\Phi}(\bar{x},\, 0,\, 0)\right)$$

$$\mathcal{G}_2(w,\, \lambda) := -\bar{\sigma}_{p_i}(\bar{x},\, w,\, \nabla w)\, n_i(\bar{x}) - \kappa(\bar{x},\, w) + (1-\lambda)\left(\bar{\sigma}_{p_i}(\bar{x},\, 0,\, 0)\, n_i(\bar{x}) + \kappa(\bar{x},\, 0)\right).$$

We define a mapping $\mathcal{G}(w,\, \lambda) := (\mathcal{G}_1(w,\, \lambda),\, \mathcal{G}_2(w,\, \lambda))$. Obviously, $\mathcal{G}(0,0) = 0$. Moreover, due to the regularity assumptions on σ, Φ and κ, the mapping $\mathcal{G}$ is clearly Fréchet-differentiable. The derivative $\partial_w\mathcal{G}(w^*,\, \lambda^*)$ at an arbitrary point $(w^*,\, \lambda^*) \in C^{2,\beta}(\overline{G}) \times\,]-1,\, 1[$ in the direction $w \in C^{2,\beta}(\overline{G})$ has the expression

$$\partial_w\mathcal{G}(w^*,\, \lambda^*)\, w = \begin{cases} -\frac{d}{dx_i}(\bar{\sigma}^*_{p_i,p_j}\, \partial_{x_j}w + \bar{\sigma}^*_{p_i,x_{N+1}}\, w) - \bar{\Phi}^*_{x_{N+1}}\, w - \bar{\Phi}^*_{p_i}\, w_{x_i} & \in C^\beta(\overline{G}) \\ -(\bar{\sigma}^*_{p_i,p_j}\, \partial_{x_j}w + \bar{\sigma}^*_{p_i,x_{N+1}}\, w)\, n_i(\bar{x}) - \kappa^*_{x_{N+1}}\, w & \in C^{1,\beta}(\partial G). \end{cases}$$

where the indice $*$ means that the value is taken at $(\bar{x},\, w^*(\bar{x}),\, \nabla w^*(\bar{x}))$. In the Lemma 6.3 below, we show that for every $f \in C^\beta(\overline{G}) \times C^{1,\beta}(\partial G)$, the equation $\partial_w\mathcal{G}(w^*,\, \lambda^*)\, w = f$ has a unique solution in $w \in C^{2,\beta}(\overline{G})$, that is nothing else but the invertibility of the Fréchet derivative $\partial_w\mathcal{G}(w^*,\, \lambda^*)$.

Moreover, any function $w \in C^{2,\beta}(\overline{G}) := X$ satisfying $\mathcal{G}(w,\, \lambda) = 0$ solves the problem (1), (2) with right-hand $\widetilde{\Phi}(x,\, q) := \Phi(x,\, q) + (1-\lambda)\, (\sigma(\bar{x},\, 0,\, 0) - \Phi(\bar{x},0,0))$, and with contact angle $\widetilde{\kappa}(x) := \kappa(x) + (1-\lambda)\, (\sigma_q(\bar{x},\, 0,\, 0) \cdot n(\bar{x}) - \kappa(\bar{x},0))$. Due to the results of the preceding sections 3, 4 and 5, all solutions to the equation $\mathcal{G}(w,\, \lambda) = 0$ lay therefore in a bounded set of $C^{2,\alpha}(\overline{G}) =: Y$.

The assumptions of the Lemma 6.1 are satisfied, and we obtain in particular the existence of a unique $\psi \in C^{2,\alpha}(\overline{G})$ such that $\mathcal{G}(w,\, 1) = 0$, that is the claim.

In order to obtain the Fréchet differentiability of $\mathcal{G}$, we had to assume in the first step of the proof that $\Phi \in C^{1,\alpha}$. In the second step, we have to show that this assumption can be removed. Let $\Phi \in V$ (cf. (10)). At first, we apply the Sobolev extension operator outside of G in the $\bar{x}-$ variable, to obtain for arbitrary fixed $q \in \mathbb{R}^{N+1}$ that $\Phi(\cdot,\, q)$ is in $W^{1,\infty}(\mathbb{R}^{N+1})$. Obviously, $\sup_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}} \Phi_{x_{N+1}} = \sup_{\Omega\times\mathbb{R}^{N+1}} \Phi_{x_{N+1}} \leq -\gamma_0$. We choose $\Phi_\epsilon(x,\, q) := \int_{\mathbb{R}^{N+1}} w_\epsilon(x-y)\, \Phi(y,q)\, dy$, where w_ϵ is a smooth nonnegative mollifier. Then, the sequence $\{\Phi_\epsilon(q)\} \subset C^\infty(\mathbb{R}^{N+1})$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}^{N+1})$, and $\Phi_{\epsilon,x_{N+1}} \leq -\gamma_0$. Moreover $\Phi_\epsilon(q) \to \Phi(q)$ in $W^{1,p}(\Omega)$ for all $1 \leq p < \infty$.

For $\epsilon > 0$, let $\psi_\epsilon \in C^{2,\alpha}(\overline{G})$ denote the unique solution to (5), (6) with right-hand Φ_ϵ. This solution exists according to the first step. Moreover, the sequence $\{\psi_\epsilon\}$ is uniformly bounded in $C^{2,\alpha}(\overline{G})$, since the bounds obtained in the sections 3, 4 and 5 only depend on the $W^{1,\infty}$ norm of Φ_ϵ and on γ_0. The claim follows letting $\epsilon \to 0$. □

Lemma 6.3. *Assumptions of Theorem 6.2. For every $w^* \in C^{2,\beta}(\overline{G})$ and $f \in C^\beta(\overline{G}) \times C^{1,\beta}(\partial G)$, the equation $\partial_w\mathcal{G}(w^*,\, \lambda^*)\, w = f$ has a unique solution in $w \in C^{2,\beta}(\overline{G})$.*

Proof. Existence is clear and follows from standard linear theory (cf. for instance the Theorem 3.28 in [Tro87]). For the uniqueness, we assume that $w_i \in C^{2,\beta}(\overline{G})$ is a solution for $i = 1, 2$. Then, the difference $\tilde{w}$ satisfies $\partial_w\mathcal{G}(w^*,\, \lambda^*)\, \tilde{w} = 0$. We abbreviate $\xi := \nabla\tilde{w}$. We moreover define $q^* :=$

$(\nabla w^*, -1) \in \mathbb{R}^{N+1}$, $\hat{\xi} = (\xi, 0) \in \mathbb{R}^{N+1}$, and the orthogonal part to q^* via $\xi^T := \hat{\xi} - (\hat{\xi} \cdot \frac{q^*}{|q^*|}) \frac{q^*}{|q^*|}$.
Using also (14), we obtain that

$$
\begin{aligned}
\bar{\sigma}^*_{p,x_{N+1}} \cdot \xi &= -\sigma_{q,x_{N+1}}(\bar{x}, w^*, q^*) \cdot \hat{\xi} \\
&= -\sigma^*_{q,x_{N+1}} \cdot \xi^T - \sigma^*_{q,x_{N+1}} \cdot \frac{q^*}{|q^*|} \frac{q^*}{|q^*|} \cdot \hat{\xi} = -\sigma^*_{q,x_{N+1}} \cdot \xi^T + [q^*_{N+1}]^{-1} \sigma^*_{x_{N+1}} \xi^T_{N+1} \,.
\end{aligned}
$$
(60)

where $\sigma^* =$ value at $(\bar{x}, w^*, q^*)$. Using the assumptions (9), it follows that $|\bar{\sigma}^*_{p,x_{N+1}} \cdot \xi| \le (\mu_5 + \mu_6) |\xi^T|/|q^*|$. On the other hand, it follows from (8d) that $\bar{\sigma}^*_{p_i,p_j} \xi_i \xi_j \ge \lambda_2 |\xi^T|^2/|q^*|$. Thus, employing Young's inequality, we obtain the inequality

$$
\bar{\sigma}^*_{p_i,p_j} \xi_i \xi_j + \bar{\sigma}^*_{p_i,x_{N+1}} \xi_i \tilde{w} \ge (1 - \delta_1) \lambda_2 \frac{|\xi^T|^2}{\sqrt{1 + |\nabla w^*|^2}} - \frac{(\mu_5 + \mu_6)^2}{4\delta_1 \lambda_2} \tilde{w}^2 \,,
$$
(61)

with $\delta_1 \in]0, 1[$ arbitrary. On the other hand, using the definition (4) of $\nu_j(p)$ for $j = 1, \ldots, N+1$, we compute for $i = 1, \ldots, N$ the derivative

$$
\partial_{p_i} \nu_j(p) = \begin{cases} \frac{-1}{\sqrt{1+|p|^2}} \left(\delta^j_i - \frac{p_i p_j}{1+|p|^2} \right) & \text{for } j \in \{1, \ldots, N\} \\ \frac{-p_i}{(\sqrt{1+|p|^2})^{3/2}} & \text{if } j = N+1 \,. \end{cases}
$$

We have for $k = 1, \ldots, N$ that $\partial_{p_j} \nu_k(\nabla w^*) \xi_j = \xi^T_k/|q^*|$. Since $\bar{\Phi}_{p_j} = \Phi_{q_k} \partial_{p_j} \nu_k(p)$, we easily see that $|\bar{\Phi}^*_{p_j} \xi_j| \le |\Phi_q| |\xi^T|/|q^*|$. It follows for $\delta_2 \in]0, 1[$ arbitrary that

$$
|\bar{\Phi}^*_{q_j} \xi_j \tilde{w}| \le \delta_2 \lambda_2 \frac{|\xi^T|^2}{\sqrt{1 + |\nabla w^*|^2}} + \frac{|\Phi_q|^2}{4\delta_2 \lambda_2} \tilde{w}^2 \,.
$$
(62)

Summarizing, (61) and (62) imply for $\delta_1 + \delta_2 \le 1$ the inequality

$$
\bar{\sigma}^*_{p_i,p_j} \xi_i \xi_j + \bar{\sigma}^*_{p_i,x_{N+1}} \xi_i \tilde{w} - \Phi^*_{x_{N+1}} \tilde{w}^2 - \bar{\Phi}^*_{p_j} \xi_j \tilde{w} \ge \left(\gamma_0 - \frac{(\mu_5 + \mu_6)^2}{4\delta_1 \lambda_2} + \frac{|\Phi_q|^2}{4\delta_2 \lambda_2} \right) \tilde{w}^2 \,.
$$
(63)

Due to the equation $\partial_w \mathcal{G}(w^*, \lambda^*) \tilde{w} = 0$, we have the identity

$$
\int_G \{\bar{\sigma}^*_{p_i,p_j} \partial_i \tilde{w} \partial_j \tilde{w} + \bar{\sigma}^*_{p_i,x_{N+1}} \partial_i \tilde{w} \tilde{w} - \Phi^*_{x_{N+1}} \tilde{w}^2 - \bar{\Phi}^*_{p_j} \partial_j \tilde{w} \tilde{w}\} + \int_{\partial G} \kappa^*_{x_{N+1}} \tilde{w}^2 = 0 \,.
$$

Since $\kappa_{x_{N+1}} \ge 0$, we can use (63) and the assumption to show that $\tilde{w} = 0$. $\qquad\square$

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