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Extensions of the control variational method

Dedicated to Prof. Dr. Fredi Tröltzsch on the occasion of his 60th birthday

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Abstract

The control variational method is a development of the variational approach, based on optimal control theory. In this work, we give an application to a variational inequality arising in mechanics and involving unilateral conditions both in the domain and on the boundary, and we explore the extension of the method to time-dependent problems.

1 Introduction

The control variational method was introduced in Sprekels and Tiba [5], Arnăutu, Langmach, Sprekels, and Tiba [1]. A presentation of this new variational approach may be found in the monograph Neittaanmäki, Sprekels, and Tiba [4]. The minimization of the usual energy associated to a differential system is performed via optimal control theory, not via the calculus of variations as in the classical variational approach. This new general treatment of variational problems is very flexible and may offer several different solutions for the same problem as shown in Sprekels and Tiba [6], Sofonea and Tiba [7], including the case of nonconvex minimization problems. The new approach is relevant both from the theoretical and the numerical point of view. In Section 2, we give a new application of the control variational problem to a contact problem in mechanics. We quote Matei and Ciurcea [3] for the treatment of general contact problems via weak solvability and dual Lagrange multipliers.

The last section investigates the possibility to extend the control variational method to time-dependent problems. A parabolic equation is studied. Several interesting properties are put into evidence, and the possibility to reformulate the considered parabolic equation as an optimal control problem is analyzed.

2 A problem arising in mechanics

We consider a beam in $[0,1]$, clamped at the right end, that may make contact with two “distributed” obstacles described by functions $k_1(x) < k_2(x)$ in $[a,b] \subset (0,1)$. Moreover, at its left end the beam has to remain above a pointwise obstacle given by zero. We introduce the space

$$V = \{v \in H^2(0,1)\,;\; v(1) = v'(1) = 0\} \tag{2.1}$$

and the closed convex subset

$$K = \{v \in V\,;\; k_1(x) \le v(x) \le k_2(x)\,,\; x \in [a,b]\,,\; v(0) \ge 0\}\,. \tag{2.2}$$

We define the continuous bilinear form $\tilde{a} : V \times V \to \mathrm{I\!R}$,

$$\tilde{a}(u,v) \;=\; \int_0^1 a(x)\, u_{xx}(x)\, v_{xx}(x)\, dx\,,$$

where $a \in L^\infty(0,1)\,,\ M \geq a(x) \geq c > 0$ a.e. in $[0,1],$ is a given coefficient depending on the thickness.

The above contact problem is then described by the following variational inequality: find $y \in K$ such that

$$\tilde{a}(y, v-y) \geq \int_0^1 f(v-y)\, dx\,, \quad \forall\; v \in K\,, \tag{2.3}$$

where $f \in L^2(0,1)$ denotes the given vertical force acting on the beam and $y \in K$ is the corresponding vertical displacement. If K is nonvoid, then (2.3) has a unique solution $y \in K$, as it follows from standard variational inequalities theory; see Neitttaanmäki, Sprekels, and Tiba [4], p. 451.

We now define the auxiliary function $z \in H^2(0,1)$ by

$$z''(x) \;=\; f \quad \text{in } (0,1)\,, \quad z(0) \;=\; z'(0) \;=\; 0\,, \tag{2.4}$$

and the mapping $\ell \in L^\infty(0,1)$ by

$$\ell(x) \;:=\; a(x)^{-1}\,, \quad \text{for a.e. } x \in (0,1)\,.$$

We then introduce the following constrained optimal control problem:

$$\mathrm{Min} \left\{ \frac{1}{2} \int_0^1 \ell\, h^2\, dx \right\}\,, \tag{2.5}$$

subject to

$$y''(x) \;=\; \ell\, z \,+\, \ell\, h \quad \text{in } (0,1)\,, \tag{2.6}$$

$$y \;\in\; K\,. \tag{2.7}$$

Notice that (2.7) includes the Cauchy conditions (in $x = 1$) for the state equation given by (2.6), via (2.1), (2.2).

An important observation in the control variational method establishes the connection between the problems (2.3) and (2.5)–(2.7).

Theorem 2.1 *The control problem* (2.5)–(2.7) *has a unique optimal pair* $[y^*, h^*] \in K \times L^2(0,1),$ *and* y^* *is the unique solution of* (2.3).

Proof. Since $\ell \geq M^{-1} > 0$, the cost functional (2.5) is coercive. The optimal control problem (2.5)–(2.7) is strictly convex, which shows the existence and uniqueness of an optimal pair

$[y^*, h^*] \in K \times L^2(0,1)$ by standard arguments; see Neittaanmäki, Sprekels, and Tiba [4], Ch. 2.1.

We perform admissible variations of the form $y^* + \lambda(v - y^*)$, $h^* + \lambda(k - h^*)$, for any $\lambda \in [0,1]$, $k \in L^2(0,1)$, and $v \in K$ satisfying

$$v'' = \ell z + \ell k \quad \text{in } (0,1)\,. \tag{2.8}$$

By the optimality of $[y^*, h^*]$, we get

$$\frac{1}{2}\int_0^1 \ell\,(h^*)^2\,dx \le \frac{1}{2}\int_0^1 \ell\,(h^* + \lambda(k-h^*))^2\,dx\,.$$

Dividing by $\lambda > 0$ and letting $\lambda \searrow 0$, we obtain that

$$0 \le \int_0^1 \ell\,h^*(k - h^*)\,dx \tag{2.9}$$

for any $[k, v] \in L^2(0,1) \times K$ satisfying (2.8).

Replacing in (2.9) k and h^* by (2.8), (2.6), respectively, we find that

$$\begin{aligned} 0 &\le \int_0^1 ((y^*)'' - \ell z)(v'' - (y^*)'')\,\ell^{-1}\,dx \\ &= \int_0^1 a(x)(y^*)''(v'' - (y^*)'')\,dx - \int_0^1 f(x)(v - y^*)\,dx\,, \quad \forall\, v \in K\,, \end{aligned} \tag{2.10}$$

using integration by parts. Relation (2.10) gives the desired conclusion. ∎

Remark: We can rewrite (2.5) as

$$\begin{aligned} \frac{1}{2}\int_0^1 \ell\,h^2\,dx &= \frac{1}{2}\int_0^1 a(x)(y'' - \ell z)^2\,dx \\ &= \frac{1}{2}\int_0^1 a(x)(y'')^2\,dx - \int_0^1 f\,y\,dx + \frac{1}{2}\int_0^1 \ell\,z^2\,dx\,, \end{aligned}$$

which is the usual energy functional up to some constant (the last term). This shows that the control variational method is equivalent to the classical variational method. The use of control theory in the minimization process allows for a big flexibility and for the development of new approximation approaches. Notice as well that the solution of (2.5)–(2.7) may be obtained by using the simplest piecewise linear continuous finite elements although the original problem (2.3) is of order four.

We indicate now such an approximation method:

$$\operatorname*{Min}_{[y,h]\in K\times L^2(0,1)} \left\{ \frac{1}{2}\int_0^1 \ell\, h^2\, dx + \frac{1}{2\varepsilon}\int_0^1 (y'' - \ell\, h - \ell\, z)^2\, dx \right\}, \quad \varepsilon > 0\,. \tag{2.11}$$

By standard variational arguments, for any $\varepsilon > 0$ there is a unique minimizer $[y_\varepsilon, h_\varepsilon] \in K \times L^2(0,1)$, and we have $y_\varepsilon \to y^*$ strongly in V, $h_\varepsilon \to h^*$ strongly in $L^2(0,1)$, for $\varepsilon \searrow 0$.

Theorem 2.2 *Under the above assumptions, the pair $[y^*, h^*] \in K \times L^2(0,1)$ is optimal for the problem (2.5)–(2.7) if and only if there is some $r^* \in L^2(0,1)$ such that*

$$0 \le \int_0^1 \ell\, h^*(k - h^*)\, dx + \int_0^1 r^*(v'' - \ell\, k - \ell\, z)\, dx \tag{2.12}$$

for all $k \in L^2(0,1)$, $v \in K$.

Proof. We take admissible variations in (2.11), of the form $y_\varepsilon + \lambda(v - y_\varepsilon)$, $h_\varepsilon + \lambda(k - h_\varepsilon)$, $\lambda \in [0,1]$, $v \in K$, $k \in L^2(0,1)$. We use the optimality of $[y_\varepsilon, h_\varepsilon]$, divide by $\lambda > 0$, and take $\lambda \searrow 0$ (ε is fixed now) to obtain

$$0 \le \int_0^1 \ell\, h_\varepsilon(k - h_\varepsilon)\, dx + \int_0^1 r_\varepsilon(v'' - y_\varepsilon'' - \ell\, k + \ell\, h_\varepsilon)\, dx\,, \tag{2.13}$$

where $r_\varepsilon = \frac{1}{\varepsilon}(y_\varepsilon'' - \ell\, h_\varepsilon - \ell\, z) \in L^2(0,1)$, and we use that, consequently, $-y_\varepsilon'' + \ell\, h_\varepsilon = -\ell\, z - \varepsilon\, r_\varepsilon$.

By (2.13), and using the inequality $-\varepsilon |r_\varepsilon|^2 \le 0$, we get that

$$0 \le \int_0^1 \ell\, h_\varepsilon(k - h_\varepsilon)\, dx + \int_0^1 r_\varepsilon(v'' - \ell\, k - \ell\, z)\, dx \tag{2.14}$$

for any $v \in K$, $k \in L^2(0,1)$.

We may assume that $0 \in K$ and fix in (2.14) $v = 0$, $k = -z + w$, $w \in L^2(0,1)$ arbitrary with $|w|_{L^2(0,1)} \le 1$. We infer that

$$0 \le \int_0^1 \ell\, h_\varepsilon(w - z - h_\varepsilon)\, dx - \int_0^1 r_\varepsilon\, \ell\, w\, dx\,.$$

Since the first integral is bounded by $h_\varepsilon \to h^*$ in $L^2(0,1)$, we obtain that $\{r_\varepsilon\}$ is bounded in $L^2(0,1)$, since w is arbitrary in the unit ball of $L^2(0,1)$. Hence, there exists some $r^* \in L^2(0,1)$ such that $r_\varepsilon \to r^*$ weakly in $L^2(0,1)$, on a subsequence. One may now pass to the limit as $\varepsilon \searrow 0$ in (2.14) to prove (2.12). This shows the necessity of (2.12). The sufficiency of this condition follows by choosing an admissible pair $[v,k]$ for the optimal control problem (2.5)–(2.7). Then, we have

$$0 \le \int_0^1 \ell\, h^*(k-h^*)\, dx \le \frac{1}{2}\int_0^1 \ell(h^*)^2\, dx + \frac{1}{2}\int_0^1 \ell\, k^2\, dx - \int_0^1 \ell(h^*)^2\, dx\,,$$

which concludes the proof. ■

Remark: Relation (2.12) is the first-order optimality condition for the control problem (2.5)–(2.7). Relation (2.14) plays the same role for the approximating problem (2.11). The functions r^*, respectively r_ε, are the Lagrange multipliers associated with the state equation. Similar arguments work in the evolution case; see Bergounioux and Tiba [2]. The information provided by (2.12), respectively (2.14), is related to the fact that the gradient of the cost functional with respect to the control vanishes (or has a certain orientation with respect to the constraints) at the optimal point. This information may be exploited in applying gradient algorithms in the numerical solution of (2.5)–(2.7).

3 Applications to parabolic equations

We discuss the model problem $(\Omega \subset \mathbb{R}^d\,,\ T>0)$:

$$y_t - \Delta y = f \quad \text{in}\ \Omega\times]0,T[\,, \tag{3.1}$$

$$y = 0 \quad \text{on}\ \partial\Omega \times [0,T]\,, \tag{3.2}$$

$$y(x,0) = y_0(x) \quad \text{in}\ \Omega\,. \tag{3.3}$$

The standard implicit discretization scheme for (3.1)–(3.3) for the equidistant partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ with $\Delta t = \frac{T}{n}$ is given by

$$\frac{y(t_{i+1}) - y(t_i)}{\Delta t} - \Delta y(t_{i+1}) = f(t_{i+1}) \quad \text{in}\ \Omega\,, \tag{3.4}$$

$$y(t_{i+1}) = 0 \quad \text{on}\ \partial\Omega\,, \tag{3.5}$$

(the initial condition is automatically taken into account). For the sake of simplicity, we assume that $f \in C([0,T];L^2(\Omega))$ and $y_0 \in H_0^1(\Omega)$.

Apply the control variational method to (3.4), (3.5): in each discretization step $i = \overline{1,n}$:

$$\mathrm{Min}\left\{\Delta t \int_\Omega |h|_d^2\, dx + \int_\Omega |y|^2\, dx\right\}, \tag{3.6}$$

$$\nabla y = h + \ell \quad \text{in}\ \Omega\,, \tag{3.7}$$

$$y = 0 \quad \text{on}\ \partial\Omega\,, \tag{3.8}$$

where $|\cdot|_d$ is the Euclidean norm in $\mathbb{R}^d$, and where ℓ satisfies

$$-\mathrm{div}\,\ell = f(t_{i+1}) + \frac{1}{\Delta t} y(t_i)\,.$$

Notice that ℓ is not uniquely defined by the above relation, and we shall use in the following $\ell = \nabla p,$ where

$$-\Delta p \,=\, f(t_{i+1}) \,+\, \frac{1}{\Delta t}\, y(t_i) \quad \text{in}\, \Omega\,, \tag{3.9}$$

$$p \,=\, 0 \quad \text{on}\, \partial\Omega\,. \tag{3.10}$$

The problem (3.6)–(3.8) has the structure of an optimal control problem (cost functional and state equation, without constraints). However, it is easier to understand it as a minimization problem with the constraints (3.7), (3.8): for any $y \in H_0^1(\Omega),$ we can find $h \in L^2(\Omega)^d$ in a simple way. In this way, the set of admissible pairs $[y, h]$ is defined, and not every $h \in L^2(\Omega)^d$ is admissible. We also underline the iterative character of (3.6)–(3.8): indeed, in step $i+1,$ one needs the information from step $i,$ namely $y(t_i)$ (which is the optimal state in step i, also denoted by y_i^*).

The existence of a unique optimal pair for (3.6)–(3.8) is standard, due to the coercivity and the strict convexity.

Theorem 3.1 *The optimal state of* (3.6)–(3.8) *is the unique solution to* $(3.4), (3.5)$.

Proof. We take admissible variations of the form

$$[y_{i+1}^* + \lambda\, z\,,\; h_{i+1}^* + \lambda\, \nabla z]\,, \quad \forall\, z \in H_0^1(\Omega)\,, \quad \forall\, \lambda \in \mathbb{R}\,.$$

Here, $[y_{i+1}^*\,,\; h_{i+1}^*]$ denotes the optimal pair of (3.6)–(3.8). We obtain

$$\Delta t \int_\Omega |h_{i+1}^*|_d^2\, dx \,+\, \int_\Omega |y_{i+1}^*|^2\, dx \,\le\, \Delta t \int_\Omega |h_{i+1}^* + \lambda\, \nabla z|_d^2\, dx \,+\, \int_\Omega |y_{i+1}^* + \lambda\, z|^2\, dx\,.$$

Consequently, dividing by $\lambda \in \mathbb{R},$ and letting $\lambda \to 0,$ we get

$$\Delta t \int_\Omega h_{i+1}^* \cdot \nabla z\, dx \,+\, \int_\Omega y_{i+1}^*\, z\, dx \,=\, 0\,, \quad \forall\, z \in H_0^1(\Omega)\,, \tag{3.11}$$

that is, $y_{i+1}^* = \Delta t\, \mathrm{div}\; h_{i+1}^*$. Combining (3.11) and (3.7), we infer that

$$\Delta t\, \Delta y_{i+1}^* \,=\, \Delta t\, \mathrm{div}\; h_{i+1}^* \,+\, \Delta t\, \mathrm{div}\; \ell \,=\, y_{i+1}^* \,-\, y_i^* \,-\, \Delta t\, f(t_{i+1})\,. \tag{3.12}$$

Relation (3.12) immediately gives (3.4), and the proof is finished. ∎

Remark. An unimportant modification of the cost functional (3.6) is the following (in step $i+1$):

$$\mathrm{Min} \left\{ \Delta t \int_\Omega |h|_d^2\, dx \,+\, \int_\Omega |y|^2\, dx \,-\, \int_\Omega |y_i^*|^2\, dx \right\}, \tag{3.13}$$

with the same state system (3.7), (3.8). If $[y_{i+1}\,,\ h_{i+1}]$ is any admissible pair, we may write the inequality

$$\Delta t \int\limits_\Omega |h^*_{i+1}|^2_d \, dx \,+\, \int\limits_\Omega |y^*_{i+1}|^2 \, dx \,-\, \int\limits_\Omega |y^*_i|^2 \, dx$$

$$\leq\ \Delta t \int\limits_\Omega |h_{i+1}|^2_d \, dx \,+\, \int\limits_\Omega |y_{i+1}|^2 \, dx \,-\, \int\limits_\Omega |y^*_i|^2 \, dx\,. \tag{3.14}$$

Summing up the inequalities (3.14) for $i = \overline{0, n-1}$, we obtain that

$$\Delta t \int\limits_\Omega \sum_{i=0}^{n-1} |h^*_{i+1}|^2_d \, dx \,+\, \int\limits_\Omega |y^*_n|^2 \, dx \,-\, \int\limits_\Omega |y_0|^2 \, dx$$

$$\leq\ \Delta t \int\limits_\Omega \sum_{i=0}^{n-1} |h_{i+1}|^2_d \, dx \,+\, \sum_{i=0}^{n-1} \int\limits_\Omega |y_{i+1}|^2 \, dx \,-\, \sum_{i=0}^{n-1} \int\limits_\Omega |y^*_i|^2 \, dx\,. \tag{3.15}$$

We define in $[0, T]$ the piecewise constant functions generated by $\{h_i\}$ and the piecewise linear functions generated by $\{y_i\}$, denoted by $\bar{h}_n\,,\ \bar{y}_n,$ and, similarly, for $\{y^*_i\}\,,\ \{h^*_i\},$ denoted by $\bar{h}^*_n\,,\ \bar{y}^*_n,$ respectively. By (3.15), we can infer that

$$\int\limits_0^T |\bar{h}^*_n|^2_{L^2(\Omega)^d} \, dt \,+\, \int\limits_\Omega |\bar{y}^*_n(T)|^2 \, dx \tag{3.16}$$

$$\leq\ \int\limits_0^T |\bar{h}_n|^2_{L^2(\Omega)^d} \, dt \,+\, \int\limits_\Omega |\bar{y}_n(T)|^2 \, dx \,+\, \sum_{i=1}^{n-1} \Big[\int\limits_\Omega |\bar{y}_n(t_i)|^2 \, dx \,-\, \int\limits_\Omega |\bar{y}^*_n(t_i)|^2 \, dx \Big]\,.$$

Taking into account **Theorem 3.1**, and multiplying by y^*_{i+1} in (3.4), we get the "generalized dissipativity" property

$$\frac{1}{2} \int\limits_\Omega \left[|y^*_{i+1}|^2 \,-\, |y^*_i|^2 \right] \, dx \,\leq\, \Delta t \int\limits_\Omega f(t_{i+1})\, y^*_{i+1} \, dx\,, \tag{3.17}$$

since

$$-\int\limits_\Omega \Delta y^*_{i+1}\, y^*_{i+1} \, dx \,\geq\, 0\,; \quad -\int\limits_\Omega y^*_i\, y^*_{i+1} \, dx \,\geq\, -\frac{1}{2} \int\limits_\Omega |y^*_i|^2 \, dx \,-\, \frac{1}{2} \int\limits_\Omega |y^*_{i+1}|^2 dx\ .$$

Inequalities (3.16), (3.17) suggest the consideration of the following continuous optimal control problem (with respect to both the time and space variables):

$$\mathrm{Min} \Big\{ \int\limits_0^T |h(x,t)|^2_{L^2(\Omega)^d} \, dt \,+\, \int\limits_\Omega |y(x,T)|^2 \, dx \Big\}\,, \tag{3.18}$$

$$\nabla_x y(x,t) \;=\; h(x,t) \;+\; \ell(x,t) \quad \text{in } \Omega\times]0,T[\,, \tag{3.19}$$

$$-\mathrm{div}_x \ell(x,t) \;=\; f(x,t) \quad \text{in } \Omega\times]0,T[\,, \tag{3.20}$$

for $h \in L^2(0,T;L^2(\Omega)^d)$.

The function ℓ in (3.20) is not uniquely determined, but we may fix it by putting $\ell(x,t) := \nabla_x p(x,t)$, where p satisfies $-\Delta p(x,t) = f(x,t)$ with zero boundary conditions, and t is interpreted as a parameter. The problem (3.18)–(3.20) has a nonstandard character (since t appears just as a parameter), and it should be understood as a constrained minimization problem over the set of admissible pairs $[y,h] \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;L^2(\Omega)^d)$ satisfying (3.19), (3.20). We assume that $f \in L^2(0,T;L^2(\Omega))$, and we impose no initial condition in (3.18)–(3.20).

Theorem 3.2 *If $[y^*,h^*]$ is a smooth optimal pair for (3.18)–(3.20), then*

$$\int_0^T y_t^*(x,t)\,dt \;-\; \int_0^T \Delta y^*(x,t)\,dt \;=\; \int_0^T f(x,t)\,dt \;-\; y^*(x,0) \quad \textit{in } \Omega\,.$$

Proof. We consider admissible variations of the form $y^* + \lambda z$, $h^* + \lambda\nabla_x z$, $\lambda \in \mathbb{R}$, $z \in H_0^1(\Omega)$ (time independent). We obtain the inequality

$$\int_0^T |h^*|^2_{L^2(\Omega)^d}\,dt + \int_\Omega |y^*(x,T)|^2\,dx \;\le\; \int_0^T |h^* + \lambda\,\nabla_x z|^2_{L^2(\Omega)^d}\,dt + \int_\Omega |y^*(x,T) + \lambda\,z|^2\,dx\,.$$

Dividing by $\lambda > 0$, $\lambda < 0$, and letting $\lambda \to 0$, we get

$$0 \;=\; \int_0^T\int_\Omega h^*(x,t)\,\cdot\,\nabla_x z(x)\,dx\,dt \;+\; \int_\Omega y^*(x,T)\,z(x)\,dx\,.$$

Integration by parts with respect to x shows that

$$y^*(x,T) \;=\; \int_0^T \mathrm{div}_x h^*(x,t)\,dt \quad \text{in } \Omega\,. \tag{3.21}$$

From (3.19)–(3.21) we infer that

$$\begin{aligned}
-\int_0^T \Delta y^*(x,t)\,dt \;&=\; -\int_0^T \mathrm{div}_x h^*(x,t)\,dt \;-\; \int_0^T \mathrm{div}_x \ell(x,t)\,dt \\
&=\; -y^*(x,T) \;+\; \int_0^T f(x,t)\,dx \\
&=\; -\int_0^T y_t^*(x,t)\,dt \;+\; \int_0^T f(x,t)\,dt \;-\; y^*(x,0)\,.
\end{aligned}$$

This ends the proof. ∎

Remark. In case that an initial condition $y(x,0) = y_0(x)$ is added to (3.18)–(3.20), we may assume that $y_0 = 0$ by shifting y and f. We may say that the variational problem (3.18)–(3.20) defines a generalized solution to (3.1)–(3.3) in the sense defined in **Theorem 3.2**. This provides a partial answer to the question of finding a variational formulation to the parabolic equation (3.1) via the control variational method.

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