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Global Lipschitz continuity for elliptic transmission problems with a boundary intersecting interface

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Abstract

We investigate the regularity of the weak solution to elliptic transmission problems that involve two layered anisotropic materials separated by a boundary intersecting interface. Under a compatibility condition for the angle of contact of the two surfaces and the boundary data, we prove the existence of square-integrable second derivatives, and the global Lipschitz continuity of the solution. We show that the second weak derivatives remain integrable to a certain power less than two if the compatibility condition is violated.

1 Introduction

The paper is concerned with the Lipschitz continuity of weak solutions to a class of elliptic equations with transmission conditions that occurs in manifold areas of mathematical physics. We consider a bounded domain $\Omega \subset \mathbb{R}^3$ partitioned by a $2-$dimensional interface S into two disjoint subdomains Ω_i ($i = 1,2$) that represent two materials, or two different phases or the same material. The materials are *layered*, in the sense that the interface S is a free surface, whose intersection with the outer boundary Γ of the domain Ω is a *closed* curve. We study the regularity of the function $u : \Omega \to \mathbb{R}$ that solves the problem

$$-\operatorname{div}(\kappa \nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$[u]_S = 0\,, \quad \left[-\kappa \frac{\partial u}{\partial n_S}\right]_S = 0 \quad \text{on } S\,, \tag{2}$$

in connection to one of the following boundary conditions on the surface $\Gamma := \partial\Omega$:

$$-\kappa \frac{\partial u}{\partial n_\Gamma} = Q \quad \text{on } \Gamma[= \text{ Problem } (P_N)], \tag{3}$$

$$u = u_e \quad \text{on } \Gamma[= \text{ Problem } (P_D)]\,. \tag{4}$$

In the equation (1), f is a function, and the coefficient κ is assumed to be material-dependent, that means

$$\kappa = \kappa_i(x) \quad \text{if } x \in \Omega_i\,, \tag{5}$$

with (at least) continuous mappings $\kappa_i : \overline{\Omega_i} \to \mathbb{R}^{3\times 3}$ ($i = 1,2$). The conditions (2) are the transmission conditions. The symbol $[\cdot]_S$ denotes the jump of a quantity across S, and n_S is the unit normal to the surface S that points into Ω_2.

In our regularity analysis, we want to include the boundary condition (3) where the flux Q is prescribed (Neumann problem (P_N)), as well as the Dirichlet boundary condition (4) ($= (P_D)$).

Conditions of Robin type such as $Q = \beta\,(u - u_e)$, or even $Q = \beta\,(u^4 - u_e^4)$ (in the context of heat-transfer), are also contained in the analysis via standard fixed-point arguments. We address the problem (1), (2), with either (3) or (4) as (P). The problem (P) is a classical elliptic transmission problem that for instance occurs in the context of phase transitions. For Stefan type problems, precisely for stationary heat transfer problems with a liquid-solid phase transition, it is essential to obtain bounded temperature gradients. But the relevance of (P) is obviously not restricted to the context of phase transitions, and there is a corresponding amount of worth mentioning mathematical studies.

For transmission conditions near surfaces of class $\mathcal{C}^2$, it has been known for a relatively long time that the solution is globally in $W^{2,2}$ up to the interface (the paper [Sav98] gives the reference [Sta56] as the pioneering work for this result; the transmission problem with interfaces in $\mathcal{C}^2$ and scalar, piecewise C^1 coefficients was also studied as an auxiliary problem in [LS60]; the full anisotropy was considered in Ch. 3, paragraph 16 of the book [LU68]). Lipschitz continuity, for the case that the jumps of the coefficients occurs at not intersecting surfaces of class $\mathcal{C}^2$, was proved in the book [LU68] Ch. 3, paragraph 16, and for surfaces of class $W^{2,q}$, $q > 3$ in the paper [LRU66]. These results were recently confirmed in [LV00] for interfaces of class $\mathcal{C}^{1,\alpha}$ ($\alpha > 0$). The $\mathcal{C}^{1,\alpha}$ assumption is crucial, since for anisotropic materials, the Lipschitz continuity seems to go lost if the interfaces are only of class $\mathcal{C}^1$. In the paper [ERS07], it was proved that $\nabla u \in W^{1,q}$ for all $q < \infty$ if the discontinuity of the coefficients occurs at not intersecting interfaces of class $\mathcal{C}^1$.

For transmission conditions at *piecewise smooth* surfaces, the higher integrability of ∇u to a power larger than two is known to become arbitrary little near interior edges, in dependence on the opening angle of the interface, and on a so-called 'quasi-monotonicity condition' for the coefficient matrix: see [EKRS07], [ERS07]. From this point of view, the Lipschitz continuity result in [LV00] in two space dimensions, section 8, where the interface is given by two circles that intersect in a point (cusp point), exhibits somewhat exceptional behaviour.

The junction $\Gamma \cap S$ of an interface and the outer boundary does not behave as bad as an interior edge. The research on regularity up to such junctions in three dimensions has recently attained very important successes.

In [Sav98], section 5, an analogon to the problem (P) is studied in general Lipschitz domains Ω with a scalar coefficient κ. Besov space methods are applied to prove that if Γ and S are Lipschitz surfaces, and if a local cone condition holds at every point of the curve $\Gamma \cap S$, then $u \in W^{s,2}(\Omega)$ globally, for $s \leq 3/2$ arbitrary. The paper [NS99] investigate a similar problem in a polyhedral domain. The methods of edge and corner asymptotics are applied to obtain the global $W^{3/2+\delta,2}(\Omega)$ regularity of the solution. In the model problem studied in the recent paper [HDKRS08], the outer boundary Γ is polyhedral, and the interface S is planar, but matrix-valued coefficients κ and mixed boundary conditions are also admitted (see also [EKRS07] for the Dirichlet problem). Via diffeomorphic transformations, very general constellations of curvilinear polyhedral domains with $\mathcal{C}^1$ interfaces are covered. In these papers the integrability of ∇u to a power $q_0 > 3$ is proved.

Beside the singularity of the outer boundary Γ, the problem that more than two materials are meeting each other near Γ (multimaterial edges), and even in the interior of Ω (intersecting surfaces), is another topic present in recent research on transmission problems. It seems that the integrability of ∇u to a power $q_0 > 3$ can be proved in very general settings ([EKRS07], [HDKRS08]), or the $W^{s,2}$ regularity ([Mer03]).

In the context of crystallization problems however, one has to rely on the *Lipschitz continuity* of u (see [LU70], [Ura73], [SS76]). On the one hand, regularity results near globally smooth surfaces can at most be used in the context of (P) to prove the regularity in domains $U \subset \Omega$ that have a positive distance to the curve $\Gamma \cap S$. This is also the case for the generalized transmission problem of [Sch60], even though the interface is smoothly embedded in the boundary of two touching $\mathcal{C}^\infty-$domains. On the other hand, the optimal results valid for Lipschitz domains, the $W^{s,2}$ regularity ($s < 3/2 + \delta$) or the W^{1,q_0} regularity with $q_0 > 3$, are still far away from the goal.

It therefore seems that *sufficient conditions* for the existence of square-integrable second derivatives, and for the boundedness of ∇u near to the intersection of two $\mathcal{C}^2$ surfaces is not yet investigated, and that to our best knowledge, the results of the present paper are new. Besides, they rely on the recent advances of [HDKRS08].

2 Notations and statement of the main result

2.1 Notations

Throughout the paper, Ω denote a bounded domain with boundary Γ of class $\mathcal{C}^2$. There are a free hypersurface $S \subset \overline{\Omega}$ of class $\mathcal{C}^2$ such that $S \cap \Gamma$ is a closed curve, and two disjoint open sets $\Omega_i \subset \Omega$ ($i = 1, 2$) such that the partition $\Omega \setminus S = \Omega_1 \cup \Omega_2$ is valid.

The outward unit normal to Γ is denoted by n_Γ, and n_S denotes the unit normal to S that points into Ω_2. We set $\Gamma_2 := \partial\Omega_2 \cap \Gamma$, $\Gamma_1 := \partial\Omega_1 \cap \Gamma$. The angle of contact $\alpha \in [0, \pi]$ of the surfaces Γ and S is defined on the curve $\Gamma \cap S$ via

$$\cos\alpha := n_S \cdot n_\Gamma, \quad \sin\alpha := \sqrt{1 - \cos^2\alpha} \quad \text{on } \Gamma \cap S\,. \tag{6}$$

Remark 2.1 (Data extension)**.** *Since S is of class $\mathcal{C}^2$, we loose no generality in assuming that S is also defined outside of Ω. Otherwise, we always will find an extension surface $S' \in \mathcal{C}^2$ such that S is compactly included in the interior of the surface S'. For $\rho > 0$, define $\Omega_\rho := \{x \in \Omega \,:\, \operatorname{dist}(x, S) < \rho\}$. Choosing $\rho \leq \rho_0(S)$ sufficiently small, there is for all $x \in \Omega_\rho$ a unique projection $y(x) \in S$ such that $|x - y| = \operatorname{dist}(x, S)$. Moreover, since S is defined in a neighbourhood of Ω, the point y belongs to the interior of S. In Ω_{ρ_0}, set $n_S := \nabla \operatorname{dist}(\cdot, S)$ so that $n_S \in [C^1(\overline{\Omega_{\rho_0}})]^3$ ([GT01], Lemma 14.16). From the neighbourhood Ω_{ρ_0}, it is then possible to extend n_S to the rest of Ω in order to obtain*

$$n_S \in [C^1(\overline{\Omega})]^3\,. \tag{7}$$

The normal n_Γ has by similar arguments a continuously differentiable extension into Ω (cp. for instance [GT01], Lemma 14.16). Due to (6)*, the functions $\cos\alpha$ and $\sin\alpha$ also possess natural extensions into Ω. In order to track the dependence on the surfaces Γ and S in the regularity estimate, we introduce*

$$g_0 := \|\nabla n_\Gamma\|_{[L^\infty(\Omega)]^3} + \|\nabla n_S\|_{[L^\infty(\Omega)]^3}\,. \tag{8}$$

Particular systems of tangential vectors arise naturally to derive estimates near the curve $\Gamma\cap S$. Those are

$$\tau^{(1)} := \frac{n_S\times n_\Gamma}{|n_S\times n_\Gamma|},\quad \tau^{(2)} := \frac{(n_S\times n_\Gamma)\times n_\Gamma}{|(n_S\times n_\Gamma)\times n_\Gamma|}\quad \text{on } \Gamma\,, \tag{9}$$

$$T^{(1)} := \frac{n_S\times n_\Gamma}{|n_S\times n_\Gamma|},\quad T^{(2)} := \frac{(n_S\times n_\Gamma)\times n_S}{|(n_S\times n_\Gamma)\times n_S|}\quad \text{on } S\,. \tag{10}$$

The Lemma C.3 in the appendix states the elementary relationships of these vectors. The orthogonal matrix that transforms the standard euclidian basis of $\mathbb{R}^3$ into the orthonormal system $\{T^{(1)},\, T^{(2)},\, n_S\}$ is denoted by O. Further relevant matrices are, at first, the matrix $A := O^T\,\kappa\,O$, the entries of which are given by,

$$A = \begin{pmatrix} \kappa T^{(1)}\cdot T^{(1)} & \kappa T^{(1)}\cdot T^{(2)} & \kappa T^{(1)}\cdot n_S \\ \kappa T^{(2)}\cdot T^{(1)} & \kappa T^{(2)}\cdot T^{(2)} & \kappa T^{(2)}\cdot n_S \\ \kappa n_S\cdot T^{(1)} & \kappa n_S\cdot T^{(2)} & \kappa n_S\cdot n_S \end{pmatrix}\,, \tag{11}$$

and, at second, the perturbation $\tilde\kappa$ of the matrix κ defined by

$$\tilde\kappa := O\begin{pmatrix} a^{1,1} & 2\,a^{1,2} & 2\,a^{1,3} \\ 0 & a^{2,2} & a^{2,3} \\ 0 & a^{3,2} & a^{3,3} \end{pmatrix} O^T\,. \tag{12}$$

For $B\in\mathbb{R}^{3\times3}$, the *minors* $m^{i,j}(B)$ ($i,j=1,2,3$) are the numbers

$$m^{i,j}(B) := \text{det}(B^{i,j})\quad B^{i,j} := \{b^{k,l}\}_{k\neq i,\, l\neq j}\quad \text{for } k,\ l,\ i,\ j=1,2,3\,, \tag{13}$$

Since almost exclusively the minors of the matrix A (cf. (11)) are needed in the paper, we define

$$m^{i,j} := m^{i,j}(A) \text{ for } i,\ j=1,2,3\,. \tag{14}$$

A function $\nu\in L^\infty(\Omega)$ is called *piecewise Lipschitz continuous* if there are $\nu_i\in W^{1,\infty}(\Omega_i)$ such that $\nu=\nu_i$ in Ω_i ($i=1,2$). For piecewise Lipschitz continuous ν, the jump across S is the quantity

$$[\nu]_S(x) = \nu_2(x)-\nu_1(x)\quad \text{for } x\in S\,. \tag{15}$$

Since the functions ν_i always have Lipschitz continuous extensions to $\overline{\Omega}$, the symbol $[\nu]_S$ still makes sense outside of S and

$$[\nu]_S\in C^{0,1}(\overline{\Omega})\,. \tag{16}$$

For symmetric and positive definite $B\in\mathbb{R}^{3\times3}$, and for $\theta\in]0,\ \pi[$ define

$$f_d(\theta,\ B) := \begin{cases} \cot\theta\,\dfrac{b^{3,3}}{m^{1,1}(B)} + \dfrac{b^{2,3}}{m^{1,1}(B)} & \text{for } (P_N)\,, \\ \cot\theta\, b^{3,3} + b^{3,2} & \text{for } (P_D)\,, \end{cases} \tag{17}$$

that plays the fundamental role with respect to compatibility conditions near $\Gamma \cap S$. We finally introduce some functional spaces. For $1 \leq q \leq \infty$, we denote by q' the number conjugated to $q \in]1, +\infty[$ in the sense that $1/q + 1/q' = 1$. The usual Lebesgue spaces $L^q(\Omega)$, the Sobolev spaces $W^{1,q}(\Omega)$, and their trace spaces $W^{1/q',q}(\partial\Omega)$, are needed. The definition and relevant properties of these spaces are to find in standard monographes (for instance [KJF77]). Maybe less well-known are the subspaces of $W^{1/q',q}(\Gamma)$ associated with the linear operators of extension by zero. Define

$$\gamma^-(u) := \begin{cases} u & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases}, \quad \gamma^+(u) := \begin{cases} 0 & \text{on } \Gamma_1 \\ u & \text{on } \Gamma_2 \end{cases}, \tag{18}$$

$$\begin{aligned} V^q(\Gamma) &:= \{u \in W^{1/q',q}(\Gamma) \,:\, \gamma^-(u) \in W^{1/q',q}(\Gamma)\}\,, \\ \|u\|_{V^q(\Gamma)} &:= \|u\|_{W^{1/q',q}(\Gamma)} + \|\gamma^-(u)\|_{W^{1/q',q}(\Gamma)}\,. \end{aligned} \tag{19}$$

Relevant properties of the V^q spaces are recalled in the appendix, Lemma B.2.

2.2 Statement of the main result

To investigate the regularity of the solution to (P_N) or (P_D), we at first formulate the required mathematical assumptions on the data. The surfaces Γ and S, and their angle of contact α, must satisfy

$$\Gamma,\, S \in \mathcal{C}^2, \quad \alpha \in W^{1,\infty}(\Gamma \cap S),\, \alpha \in]0,\, \pi[\text{ on } \Gamma \cap S\,. \tag{20}$$

Let moreover the matrix $\kappa(x)$ be symmetric for all $x \in \overline{\Omega}$, and satisfy

$$k_0\,|\eta|^2 \leq \kappa(x)\eta \cdot \eta \leq k_1\,|\eta|^2 \quad \text{for all } x \in \overline{\Omega},\, \eta \in \mathbb{R}^3\,, \tag{21}$$

with two constants $0 < k_0 \leq k_1 < \infty$. The matrix A (cf. (11)) is then also symmetric, and the matrices A and $\tilde{\kappa}$ (cf. (12)) uniformly satisfy the inequality (21) with the same constants $k_0,\, k_1$. For the matrix κ, we furthermore assume that

$$k_1' := \|\nabla\kappa_1\|_{L^\infty(\Omega_1)} + \|\nabla\kappa_2\|_{L^\infty(\Omega_2)} < \infty\,. \tag{22}$$

The right-hand side f of equation (1) is supposed to have the regularity

$$f \in L^q(\Omega)\,. \tag{23}$$

In (23), and also in the integrability conditions formulated hereafter, we focus on the cases $q = 2$ (for the $W^{2,p}$ analysis), and $q = q_0 > 3$ (for the $W^{1,\infty}$ analysis). We come now to the compatibility conditions that are essential for the higher regularity. The angle of contact α and the matrix κ (resp. the matrix A) must satisfy the compatibility condition

$$[f_d(\alpha,\, A)]_S := f_d(\alpha,\, A_2) - f_d(\alpha,\, A_1) \geq 0 \quad \text{on } \Gamma \cap S\,, \tag{24}$$

where $A_i := A_{|\Omega_i}$. For the problem (P_N), we additionally require on the surface Γ that

$$\exists Q_1 \in W^{1/q',q}(\Gamma),\, Q_2 \in V^q(\Gamma) \,:\, [\frac{a^{3,3}}{m^{1,1}}]_S \,\frac{Q}{\sin\alpha} = [f_d(\alpha,\, A)]_S\, Q_1 + Q_2\,, \tag{25}$$

$$\exists g_1 \in W^{1,\infty}(\Gamma) \,:\, [\frac{m^{2,1}}{m^{1,1}}]_S = g_1\, [f_d(\alpha,\, A)]_S\,. \tag{26}$$

For the problem (P_D), we require that

$$\begin{aligned} &\nabla u_e \in W^{1/q',q}(\Gamma) \text{ and } \exists\, U_1 \in W^{1/q',q}(\Gamma),\, U_2 \in V^q(\Gamma) \,:\\ &[a^{1,3}]_S\,(\tau^{(1)}\cdot\nabla u_e) - [a^{3,3}]_S\,\frac{\tau^{(2)}\cdot\nabla u_e}{\sin\alpha} = [f_d(\alpha,\, A)]_S\, U_1 + U_2\,. \end{aligned} \tag{27}$$

The main result of the paper is contained in the following theorem.

Theorem 2.2. *Let $u \in W^{1,2}(\Omega)$ denote the unique weak solution to (P_D) or to (P_N). Assume that the conditions* (20)*,* (21) *and* (22) *are satisfied, and that* (23) *is valid with $q = 2$. Assume that the condition* (24)*, and either* (25)*,* (26) *for the problem (P_N), or* (27) *for the problem (P_D), are valid with $q = 2$. Then $u \in W^{2,2}(\Omega_i)$ ($i = 1, 2$).*

Assume moreover that (23)*, and either* (25)*,* (26) *for the problem (P_N), or* (27) *for the problem (P_D), are satisfied for $q = q_0 > 3$. Then $u \in W^{1,\infty}(\Omega)$.*

For the case that the principal hypothesis (24) of Theorem 2.2 is violated, we can still prove that the weak solution to (P) has second derivatives at least integrable to the power $6/5$.

Theorem 2.3. *Except for* (24)*, same assumptions as in Theorem 2.2 with $q = 2$. Let $u \in W^{1,2}(\Omega)$ denote the unique weak solution to (P_D) or to (P_N). Then, there is $q_0 > 3$ such that $\nabla u \in L^{q_0}(\Omega)$. Define $s_0 := \min\{q_0, 6\}$. Then, for $1 \leq p < 2s_0/(s_0 + 2)$ arbitrary, $\nabla u \in W^{1,p}(\Omega_i)$ ($i = 1, 2$).*

2.3 Interpretation of the compatibility conditions

A few remarks can help better understand the conditions (24), (25) and (27).

In the case that κ is a scalar, one can verify that $f_d := \cot\alpha\,[\kappa]_S$, so that the condition (24) reduces to

$$\cot\alpha\,[\kappa]_S \leq 0 \text{ for } (P_N), \quad \cot\alpha\,[\kappa]_S \geq 0 \text{ for } (P_D) \quad \text{ on } \Gamma\cap S\,. \tag{28}$$

Elementary consequences of (28) for the isotropic problem are the following:

(1) For given data $(\kappa,\, \alpha)$, the result of Theorem 2.2 cannot be used to prove the regularity for the Neumann problem *and* for the Dirichlet problem, unless $\alpha \equiv \pi/2$ on $\Gamma\cap S$. Otherwise, the choice which quantity to prescribe on the outer boundary Γ has to follow the condition (28).

(2) If κ is moreover piecewise constant (that is, if $\kappa_1,\, \kappa_2 \in \mathbb{R}$), then the angle of contact α is not allowed to change the sign along $\Gamma\cap S$.

We now briefly comment on the conditions (25) and (27). In the scalar case, the condition (25) reduces to

$$[\kappa]_S\, Q = \cos\alpha\, [\kappa^{-1}]_S\, Q_1 + \sin\alpha\, Q_2 \quad \text{on } \Gamma\,, \tag{29}$$

and (27) reduces to

$$-[\kappa]_S\, \tau^{(2)} \cdot \nabla u_e = \cos\alpha\, [\kappa]_S\, U_1 + \sin\alpha\, U_2 \quad \text{on } \Gamma\,. \tag{30}$$

Thus, if $|\cos\alpha| \geq \delta_0 > 0$ on $\Gamma \cap S$, the condition (25) is trivially satisfied for every $Q \in W^{1/q',q}(\Gamma)$: set $Q_1 = -\kappa_2\, \kappa_1\, Q/\cos\alpha$ and $Q_2 = 0$. Similarly, set $U_1 := -\tau^{(2)} \cdot \nabla u_e/\cos\alpha$, $U_2 = 0$ to obtain (27). The compatibility conditions (25) and (27) are therefore only needed for the limiting case that the function $[f_d(\alpha, A)]_S$ tends to zero on some part of $\Gamma \cap S$.

The compatibility conditions shall be not easy to verify in practice. In particular, since the function $[f_d(\alpha, A)]_S$ is only given on $\Gamma \cap S$, the representations (25) and (27) depend on the choice of its extension to Γ. However, assuming additional regularity of the data Q, u_e, we show in the following Lemma that (25) and (27) more intrinsicly amount to require a certain decay along the curve $\Gamma \cap S$. To this aim denote

$$K := \Gamma \cap S\,, \quad K_0 := \{x \in K \,:\, |[f_d(\alpha, A)]_S| > 0\}\,,$$
$$d_{K_0}(x) := \operatorname{dist}(x,\, K \setminus K_0) \ \text{ for } x \in K\,.$$

For $s \in \mathbb{R}$, the properties of the spaces $W^{s,2}(U)$, $U \in \mathbb{R}^n$ have been studied in [LM68]. It is impossible to expose in a few lines the localization arguments that justify to extend these properties to the case that U is a $\mathcal{C}^2-$submanifold. We recall that our aim is here only to throw some light on the compatibility conditions, and that the next lemma does not affect in any respect the proof of the main result.

Define $W^{s,2}_K(\Gamma)$ as the space $W^{s,2}_0(\Gamma_1) \oplus W^{s,2}_0(\Gamma_2)$. If $s > 1/2$, every function $g \in W^{s,2}(\Gamma)$ has a trace tr$(g) \in W^{s-1/2,2}(K)$ (see [LM68], Th. 9.4).

Lemma 2.4. *Assume that there are $\beta \in]0,1]$ and constants $0 < c_1 \leq c_2$ such that $c_1\, d_{K_0}^{\beta} \leq [f_d(\alpha, A)]_S \leq c_2\, d_{K_0}^{\beta}$ on K_0. Assume that $g \in W^{s,2}(\Gamma)$, $s > 1/2$ is such that*

$$\text{tr}(g) \in \begin{cases} W^{s-1/2+\beta,2}_{00}(K_0) & \text{if } s - 1/2 + \beta = j + 1/2 \text{ for a } j \in \mathbb{N}\,,\\ W^{s-1/2+\beta,2}_{0}(K_0) & \text{otherwise.}\end{cases} \tag{31}$$

Then, for each extension of the function $[f_d(\alpha, A)]_S$ to Γ, there are $g_1 \in W^{s,2}(\Gamma)$ and $g_2 \in W^{s,2}_K(\Gamma)$ such that $g = [f_d(\alpha, A)]_S\, g_1 + g_2$.

Proof. Define $\tilde{g}_1 := g/[f_d(\alpha, A)]_S$ on K_0 and $\tilde{g}_1 := 0$ on $K \setminus K_0$. Then

$$\tilde{g}_1 \in W^{s-1/2}(K)\,. \tag{32}$$

To prove (32), we at first verify that the application $g \mapsto d_{K_0}^{-\beta}\, g$ is continuous from $W^{\beta,2}_0(K_0)$ into $L^2(K)$ and from $W^{1+\beta,2}_0(K_0)$ into $W^{1,2}(K)$ if $\beta \neq 1/2$. Otherwise, we verify that $g \mapsto d_{K_0}^{-\beta}\, g$

is continuous from $W_{00}^{\beta,2}(K_0)$ into $L^2(K)$ and from $W_{00}^{1+\beta,2}(K_0)$ into $W^{1,2}(K)$ ([LM68], Th. 11.7).

For $t \in [0,1]$, it follows from interpolation in Hilbert-spaces (cp. [LM68], Prop. 2.3) that

$$\|d_{K_0}^{-\beta}\, g\|_{W^{t,2}(K)} \leq \begin{cases} c\,\|g\|_{W_{00}^{t+\beta,2}(K_0)} & \text{if } t+\beta = j+1/2 \text{ for a } j \in \mathbb{N}\,, \\ c\,\|g\|_{W_{0}^{t+\beta,2}(K_0)} & \text{otherwise.} \end{cases}$$

Choose $t = s - 1/2$, then in view of the assumptions (31), the property (32) follows.

Due to (31) and to the trace theorem for $W^{s,2}$, there exists $g_1 \in W^{s,2}(\Gamma)$ such that $\mathrm{tr}(g_1) = \tilde{g}_1$ on K. Choosing an arbitrary extension of $[f_d(\alpha, A)]_S$ to Γ, we easily obtain that $g_2 := g - [f_d(\alpha, A)]_S\, g_1$ belongs to $W_K^{s,2}(\Gamma)$. ☐

3 Method of the proof

To prove Theorem 2.2, we investigate a regularization of the problems (P_N) and (P_D). For $\rho > 0$, $t \in \mathbb{R}$, define

$$I_\rho(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t}{\rho} & \text{for } t \in]0,\ \rho] \\ 1 & \text{for } t > \rho \end{cases} \tag{33}$$

For $\nu_1,\ \nu_2 \in L^\infty(\Omega)$, define $\nu := \nu_i$ in Ω_i. We introduce

$$L_\rho(\nu)(x) := \nu_1(x) + I_\rho(\mathrm{dist}(x,S))\,(\nu_2(x) - \nu_1(x)) \in L^\infty(\Omega)\,. \tag{34}$$

Note that

$$L_\rho(\nu) \longrightarrow \nu \text{ everywhere in } \Omega \setminus S\,, \tag{35}$$

and, also taking (7) into account, we obtain for piecewise Lipschitz continuous ν that

$$\begin{aligned} \nabla L_\rho(\nu) &= I_\rho'(\mathrm{dist}(x,S))\,\nabla\,\mathrm{dist}(\cdot,S) + L_\rho(\nabla\nu)(x) \\ &= \frac{[\nu]_S(x)}{\rho}\, b_\rho(x)\, n_S(x) + L_\rho(\nabla\nu)(x)\,. \end{aligned} \tag{36}$$

In (36), we have abbreviated $b_\rho := \chi_{\{0 \leq \mathrm{dist}(x,S) \leq \rho\}}(x)$, and L_ρ applies componentwise to vector fields. We now introduce a regularization of κ via the matrix A. For the problem (P_D), we apply the regularization (33), (34) to introduce the coefficients

$$a_\rho^{i,j} = L_\rho(a^{i,j}) \in C^{0,1}(\overline{\Omega}) \text{ for } i,j = 1,2,3\,, \tag{37}$$

where $a^{i,j}$ are taken from the matrix (11). For the problem (P_N), we introduce

$$a_\rho^{1,1} := L_\rho(a^{1,1}),\quad a_\rho^{3,1} := L_\rho(a^{3,1}),\quad m_\rho^{1,1} := L_\rho(m^{1,1})\,, \tag{38}$$

where the relevant $a^{i,j}$ are taken from the matrix (11), and $m^{1,1}$ is given by (14). The remaining entries are defined in the following way:

$$a_\rho^{3,3} := m_\rho^{1,1}\, L_\rho(\frac{a^{3,3}}{m^{1,1}})\,,\ a_\rho^{2,3} := m_\rho^{1,1}\, L_\rho(\frac{a^{2,3}}{m^{1,1}})\,,\ a_\rho^{2,1} := (a_\rho^{3,3})^{-1}\,(m_\rho^{1,1} + [a_\rho^{2,3}]^2)$$
$$a_\rho^{2,1} := (a_\rho^{3,3})^{-1}\,\left(m_\rho^{1,1}\, L_\rho(\frac{m^{2,1}}{m^{1,1}}) + a_\rho^{2,3}\, a_\rho^{3,1}\right)\,. \tag{39}$$

The construction (38), (39), has the properties

$$\frac{a_\rho^{3,3}}{m_\rho^{1,1}} = L_\rho(\frac{a^{3,3}}{m^{1,1}}),\quad \frac{a_\rho^{2,3}}{m_\rho^{1,1}} = L_\rho(\frac{a^{2,3}}{m^{1,1}}),\quad \frac{m_\rho^{2,1}}{m_\rho^{1,1}} = L_\rho(\frac{m^{2,1}}{m^{1,1}})\,. \tag{40}$$

In view of (36), the regularized coefficients have, for both (P_N) and (P_D), the important property

$$T^{(k)}\cdot\nabla a_\rho^{i,j} = L_\rho(T^{(k)}\cdot\nabla a^{i,j})\quad \text{for } i,j=1,2,3 \text{ and } k=1,2\,, \tag{41}$$

and therefore, due to (22),

$$|T^{(k)}\cdot\nabla a_\rho^{i,j}| \le k_1' \quad \text{for } i,j=1,2,3 \text{ and } k=1,2\,. \tag{42}$$

In view of (37), or of (38), (39), the matrix $A_\rho := \{a_\rho^{i,j}\}_{i,j=1,2,3}$ satisfies (cp. (35))

$$A_\rho \longrightarrow A \text{ everywhere in } \Omega\setminus S\,. \tag{43}$$

Define $\kappa_\rho := O\,A_\rho\,O^T$, and, similarly, $\tilde\kappa_\rho$ using (12). Then, κ_ρ, $\tilde\kappa_\rho$ belong to $C^{0,1}(\overline{\Omega};\,\mathbb{R}^{3\times3})$. Moreover $\kappa_\rho\to\kappa$ and $\tilde\kappa_\rho\to\tilde\kappa$ everywhere in $\Omega\setminus S$. We define $u_\rho\in W^{1,2}(\Omega)$ to be the unique weak solution to the following problem (P_ρ):

$$-\operatorname{div}(\kappa_\rho\,\nabla u_\rho) = f \text{ in } \Omega,\quad \left[-\kappa_\rho\,\frac{\partial u_\rho}{\partial n_S}\right]_S = 0 \text{ on } S, \tag{44}$$

together with one of the conditions

$$-\,\kappa_\rho\,\frac{\partial u_\rho}{\partial n_\Gamma} = Q \text{ on } \Gamma\ [\,=:(P_{N,\rho})],\quad u_\rho = u_e \text{ on } \Gamma\ [\,=:(P_{D,\rho})]\,. \tag{45}$$

Lemma 3.1. *Assume that κ satisfies* (21) *and* (22). *Let $f\in L^2(\Omega)$, $Q\in W^{1,2}(\Omega)$ and $u_e\in W^{2,2}(\Omega)$. Denote by $u\in W^{1,2}(\Omega)$ (resp. $u_\rho\in W^{1,2}(\Omega)$) the weak solution to (P) (resp. (P_ρ)). Then $u_\rho\in W^{2,2}(\Omega)$, and*

$$u_\rho \longrightarrow u \textit{ in } W^{1,2}(\Omega)\,. \tag{46}$$

Moreover, there is a constant c, depending only on Ω and on k_1/k_0, such that the function u_ρ satisfies the energy estimates

$$\|\nabla u_\rho\|_{L^2(\Omega)} \le c\,k_0^{-1}\,(\|f\|_{L^2(\Omega)} + \|Q\|_{L^2(\Gamma)}) \qquad \textit{in case of } (3),$$
$$\|\nabla u_\rho\|_{L^2(\Omega)} \le c\,(k_0^{-1}\,\|f\|_{L^2(\Omega)} + \|\nabla u_e\|_{L^2(\Omega)}) \qquad \textit{in case of } (4).$$

Proof. The matrix κ_ρ is symmetric and uniformely positive definite. Since $\kappa_\rho \in \mathcal{C}^{0,1}(\overline{\Omega};\, \mathbb{R}^{3\times 3})$, the standard regularity theory for second order elliptic equations in divergence form ([LU68], Ch. 3, Paragraph 10 or [Tro87], Ch, 2, section 2.5, a. o.) proves the $W^{2,2}$ regularity claim for the solution to (P_ρ). The strong convergence (46) for the entire sequence is obvious due to the uniqueness of the respective weak solutions to (P) and (P_ρ). ☐

Our method will consist in deriving uniform estimates for the main components of ∇u_ρ with respect to the system $\{T^{(1)},\, T^{(2)},\, n_S\}$, that means, the functions

$$\xi^{(1)}_\rho := T^{(1)} \cdot \nabla u_\rho, \quad \xi^{(2)}_\rho := T^{(2)} \cdot \nabla u_\rho, \quad \xi^{(3)}_\rho := \kappa_\rho \, n_S \cdot \nabla u_\rho \, . \tag{47}$$

In the section 4, we reformulate the problem of regularity in a more suitable coordinate system. The section 5 contains the core of the proof of the $W^{2,2}$ regularity, whereas the section 6 is dedicated to the boundedness of ∇u.

4 Auxiliary results

This section mainly contains the technical rearrangements needed to, so to say, restate the problem in a more convenient coordinates. Throughout the remaining sections, the matrices $A_\rho,\, \tilde{\kappa}_\rho$ are as defined in the section 3. In the following Lemma the basic relationships satisfied by the functions $\xi^{(i)}_\rho$ ($i = 1, 2, 3$) are derived. We recall the notation (8).

Lemma 4.1. *Let $u_\rho \in W^{2,2}(\Omega)$ denote the weak solution to (P_ρ). Then, there are $G^{(i)}_\rho \in [L^2(\Omega)]^3$ ($i = 1, 2, 3$) and $M^{(3)}_\rho \in [L^2(\Omega)]^9$ such that*

$$|G^{(i)}_\rho| + k_0^2 \, |G^{(3)}_\rho| \leq c \, (|f| + g_0 \, \kappa_1 \, |\nabla u_\rho|), \quad |M^{(3)}_\rho| \leq c \, k_0^{-1} \, |\nabla u_\rho| \, , \tag{48}$$

with $c = c(\Omega, k_1/k_0, k_1'/k_0)$, and such that the following identities are valid almost everywhere in Ω:

$$\begin{aligned} \kappa_\rho \nabla \xi^{(1)}_\rho &= G^{(1)}_\rho + (T^{(2)} - \frac{a^{2,3}_\rho}{a^{3,3}_\rho} \, n_S) \times \nabla \xi^{(3)}_\rho - \sum_{i=1}^{2} (a^{2,i}_\rho - \frac{a^{i,3}_\rho \, a^{3,2}_\rho}{a^{3,3}_\rho}) \, (n_S \times \nabla \xi^{(i)}_\rho) \\ \tilde{\kappa}_\rho \nabla \xi^{(2)}_\rho &= G^{(2)}_\rho - T^{(1)} \times \nabla \xi^{(3)}_\rho + (a^{1,1}_\rho \, n_S - a^{1,3}_\rho \, T^{(1)}) \times \nabla \xi^{(1)}_\rho \, , \\ \tilde{\kappa}_\rho \nabla \xi^{(3)}_\rho &= m^{1,1}_\rho \, (G^{(3)}_\rho + M^{(3)}_\rho \nabla \xi^{(1)}_\rho + T^{(1)} \times \nabla \xi^{(2)}) \, . \end{aligned} \tag{49}$$

Proof. **First step.** In the proof, $g_\rho,\, \bar{g}_\rho$ denote generic functions, and $G_\rho,\, \bar{G}_\rho$ generic vector fields, that may change from line to line, but that satisfy the estimates

$$|g_\rho| + |G_\rho| \leq c \, g_0 \, |\nabla u_\rho|, \quad \bar{g}_\rho + |\bar{G}_\rho| \leq c \, (|f| + \, g_0 \, \kappa_1 \, |\nabla u_\rho|) \, , \tag{50}$$

with a constant c only dependent on k_1/k_0 and k_1'/k_0. An important device in the proof is the orthonormality of the system $\{T^{(1)},\, T^{(2)},\, n_S\}$ everywhere in Ω. Every vector field V defined in Ω has a decomposition

$$V = \sum_{j=1}^{2} (T^{(j)} \cdot V) \, T^{(j)} + (n_S \cdot V) \, n_S \, . \tag{51}$$

We furtheron introduce the differential operators $\partial^{(i)} := T^{(i)} \cdot \nabla$ for $i = 1, 2$ and $\partial^{(3)} := n_S \cdot \nabla$. If $V_1,\, V_2$ are two vector fields among $\{T^{(1)},\, T^{(2)},\, n_S\}$, the permutation formula

$$V_1 \cdot \nabla(V_2 \cdot \nabla u_\rho) = V_2 \cdot \nabla(V_1 \cdot \nabla u_\rho) + [(V_1 \cdot \nabla) V_2 - (V_2 \cdot \nabla) V_1] \cdot \nabla u_\rho \,,$$

is valid, so that, in view of the convention (50),

$$\partial^{(i)}\, \partial^{(j)}\, u_\rho = \partial^{(j)}\, \partial^{(i)}\, u_\rho + g_\rho \quad \text{for } i, j = 1, 2, 3 \,. \tag{52}$$

Second step. Due to (52) and the definition (47),

$$\partial^{(i)}\, \xi^{(j)}_\rho - \partial^{(j)}\, \xi^{(i)}_\rho = g_\rho \quad \text{for } i, j = 1, 2 \,. \tag{53}$$

The definition of the function $\xi^{(3)}_\rho$, and the property (51) imply that

$$\begin{aligned}
\xi^{(3)}_\rho &= \sum_{i=1}^{2} (\kappa_\rho n_S \cdot T^{(i)})\, \xi^{(i)}_\rho + (\kappa_\rho n_S \cdot n_S)\, \partial^{(3)} u_\rho \\
&= \sum_{i=1}^{2} a^{3,i}_\rho\, \xi^{(i)}_\rho + a^{3,3}_\rho\, \partial^{(3)} u_\rho \,.
\end{aligned}$$

Thus

$$\partial^{(3)}\, u_\rho = \frac{1}{a^{3,3}_\rho}\, (\xi^{(3)}_\rho - \sum_{i=1}^{2} a^{3,i}_\rho\, \xi^{(i)}_\rho) \,, \tag{54}$$

and it follows for $i = 1, 2$ from (54) and (52) that

$$\partial^{(3)}\, \xi^{(i)}_\rho = \partial^{(i)}\, \partial^{(3)}\, u_\rho + g_\rho = \frac{1}{a^{3,3}_\rho}\, \partial^{(i)}\, \xi^{(3)}_\rho - \sum_{j=1}^{2} \frac{a^{3,j}_\rho}{a^{3,3}_\rho}\, \partial^{(i)}\, \xi^{(j)}_\rho + g_\rho \,, \tag{55}$$

$$g_\rho := \partial^{(i)}\, \frac{1}{a^{3,3}_\rho}\, \xi^{(3)}_\rho + \sum_{j=1}^{2} \partial^{(i)}\, \frac{a^{3,j}_\rho}{a^{3,3}_\rho}\, \xi^{(j)}_\rho \,.$$

The properties (41), (42) yield for $i = 1, 2$

$$|\partial^{(i)}\, \frac{1}{a^{3,3}_\rho}| + \sum_{j=1}^{2} |\partial^{(i)}\, \frac{a^{3,j}_\rho}{a^{3,3}_\rho}| \leq 3\, \frac{k_1\, k_1'}{k_0^2} \,, \tag{56}$$

which can be used to prove in (55) that g_ρ still satisfies (50). Thanks to the property (41) and the definition of $\xi^{(3)}_\rho$, similar arguments show, for $i = 1, 2$, the permutation formula

$$\begin{aligned}
\partial^{(i)}\, \xi^{(3)}_\rho &= \partial^{(i)}\, (a^{3,3}_\rho\, \partial^{(3)}\, u_\rho + \sum_{j=1}^{2} a^{3,j}_\rho\, \partial^{(j)}\, u_\rho) \\
&= a^{3,3}_\rho\, \partial^{(i)}\, \partial^{(3)}\, u_\rho + \sum_{j=1}^{2} a^{3,j}_\rho\, \partial^{(i)}\, \partial^{(j)}\, u_\rho + \bar{g}_{\rho,i} \qquad (57) \\
&= a^{3,3}_\rho\, \partial^{(3)}\, \xi^{(i)}_\rho + \sum_{j=1}^{2} a^{3,j}_\rho\, \partial^{(j)}\, \xi^{(i)}_\rho + \bar{g}_{\rho,i} \,,
\end{aligned}$$

Third step. In (53), (55) and (57), we have obtained useful information about the derivatives $\partial^{(i)}\xi_\rho^{(j)}$ for $i \neq j$. The idea of the proof is now to make use of the information contained in the equation (44) about the symmetric derivatives $\partial^{(i)}\,\xi_\rho^{(i)}$.

Decomposition of ∇u_ρ according to (51), joined to the relation (54), yields

$$\kappa_\rho \nabla u_\rho = \sum_{i=1}^{2} \xi_\rho^{(i)}\,(\kappa_\rho T^{(i)} - \frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,\kappa_\rho n_S) + \frac{\xi_\rho^{(3)}}{a_\rho^{3,3}}\,\kappa_\rho n_S\,.$$

Again decomposing the vectors $\kappa_\rho T^{(i)}$ ($i = 1, 2$), and $\kappa_\rho n_S$ it follows that

$$\kappa_\rho \nabla u_\rho = \sum_{i,j=1}^{2} (a_\rho^{i,j} - \frac{a_\rho^{i,3}\,a_\rho^{j,3}}{a_\rho^{3,3}})\,\xi_\rho^{(i)}\,T^{(j)} + \sum_{j=1}^{2} \frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,T^{(j)}\,\xi_\rho^{(3)} + n_S\,\xi_\rho^{(3)}\,. \tag{58}$$

According to Lemma 3.1, $u_\rho \in W^{2,2}(\Omega)$, and $-\operatorname{div}(\kappa_\rho\,\nabla u_\rho) = f$ almost everywhere in Ω. Therefore, (58) implies that

$$\begin{aligned}
&\sum_{i,j=1}^{2} (a_\rho^{i,j} - \frac{a_\rho^{i,3}\,a_\rho^{j,3}}{a_\rho^{3,3}})\,\partial^{(j)}\,\xi_\rho^{(i)} + \sum_{j=1}^{2} \frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,\partial^{(j)}\,\xi_\rho^{(3)} + \partial^{(3)}\,\xi_\rho^{(3)} = \bar{g}_\rho \\
&:= -f - \sum_{i,j=1}^{2} \operatorname{div}\big((a_\rho^{i,j} - \frac{a_\rho^{i,3}\,a_\rho^{j,3}}{a_\rho^{3,3}})\,T^{(j)}\big)\,\xi_\rho^{(i)} - \operatorname{div}(\sum_{j=1}^{2} \frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,T^{(j)} + n_S)\,\xi_\rho^{(3)}\,.
\end{aligned} \tag{59}$$

Due to (41), $\bar{g}_\rho$ satisfies the estimate (50) again (cp. the computation (56)). Fix an indices $i \in \{1, 2\}$, and define i' by requesting that $\{i\} \cup \{i'\} = \{1, 2\}$. From (59), it follows for $i = 1, 2$ that

$$\begin{aligned}
(a_\rho^{i,i} - \frac{[a_\rho^{i,3}]^2}{a_\rho^{3,3}})\,\partial^{(i)}\,\xi_\rho^{(i)} &= \bar{g}_\rho - \nabla\xi_\rho^{(3)} \cdot (n_S + \sum_{j=1}^{2} \frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,T^{(j)}) \\
&- (a_\rho^{i,i'} - \frac{a_\rho^{i,3}\,a_\rho^{i',3}}{a_\rho^{3,3}})\,\partial^{(i')}\,\xi_\rho^{(i)} - \sum_{j=1}^{2} (a_\rho^{i',j} - \frac{a_\rho^{i',3}\,a_\rho^{j,3}}{a_\rho^{3,3}})\,\partial^{(j)}\,\xi_\rho^{(i')}\,.
\end{aligned} \tag{60}$$

In (60), permutation of $\partial^{(i')}$ and $\partial^{(i)}$ with the formula (52) yields

$$\begin{aligned}
(a_\rho^{i,i} - \frac{[a_\rho^{i,3}]^2}{a_\rho^{3,3}})\,\partial^{(i)}\,\xi_\rho^{(i)} &= \bar{g}_\rho - \nabla\xi_\rho^{(3)} \cdot (n_S + \sum_{j=1}^{2} \frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,T^{(j)}) \\
&- (a_\rho^{i',i'} - \frac{[a_\rho^{i',3}]^2}{a_\rho^{3,3}})\,\partial^{(i')}\,\xi_\rho^{(i')} - 2\,(a_\rho^{i,i'} - \frac{a_\rho^{i,3}\,a_\rho^{i',3}}{a_\rho^{3,3}})\,\partial^{(i)}\,\xi_\rho^{(i')}\,.
\end{aligned} \tag{61}$$

Using the formula (57) we can also reexpress the term $\partial^{(i')}\,\xi_\rho^{(3)}$ in the formula (61) to obtain, for $i = 1, 2$, the decomposition

$$\begin{aligned}
(a_\rho^{i,i} - \frac{[a_\rho^{i,3}]^2}{a_\rho^{3,3}})\,\partial^{(i)}\,\xi_\rho^{(i)} &= \bar{g}_\rho - \nabla\xi_\rho^{(3)} \cdot (n_S + \frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,T^{(i)}) \\
&- \nabla\xi_\rho^{(i')} \cdot (a_\rho^{i',i'}\,T^{(i')} + [2\,a_\rho^{i',i} - \frac{a_\rho^{3,i}\,a_\rho^{3,i'}}{a_\rho^{3,3}}]\,T^{(i)} + a_\rho^{3,i'}\,n_S)\,.
\end{aligned} \tag{62}$$

In the case $i=3$, we conclude from (59) and (57) that

$$\begin{aligned}\partial^{(3)}\,\xi_\rho^{(3)} &= -\sum_{i,j=1}^{2}(a_\rho^{i,j}-\frac{a_\rho^{i,3}\,a_\rho^{j,3}}{a_\rho^{3,3}})\,\partial^{(j)}\,\xi_\rho^{(i)}-\sum_{i=1}^{2}\frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,\partial^{(i)}\,\xi_\rho^{(3)}+\bar{g}_\rho\\ &= -\sum_{i,j=1}^{2}a_\rho^{i,j}\,\partial^{(j)}\,\xi_\rho^{(i)}-\sum_{i=1}^{2}a_\rho^{3,i}\,\partial^{(3)}\,\xi_\rho^{(i)}+\bar{g}_\rho = -\sum_{i=1}^{2}\kappa_\rho T^{(i)}\cdot\nabla\xi_\rho^{(i)}+\bar{g}_\rho\,.\end{aligned}\tag{63}$$

Fourth step. For $i=1,2$, the relation

$$(a_\rho^{i,i}-\frac{[a_\rho^{3,i}]^2}{a_\rho^{3,3}})\,\partial^{(i)}\,\xi_\rho^{(i)} = -\nabla\xi_\rho^{(3)}\cdot V_\rho-\nabla\xi_\rho^{(i')}\cdot W_\rho+\bar{g}_\rho\,,\tag{64}$$

follows from (61) for the choice

$$\begin{aligned}V_\rho &:= n_S+\sum_{j=1}^{2}\frac{a_\rho^{3,j}}{a_\rho^{3,3}}\,T^{(j)}\,,\\ W_\rho &:= (a_\rho^{i',i'}-\frac{[a_\rho^{3,i'}]^2}{a_\rho^{3,3}})\,T^{(i')}+2\,(a_\rho^{i,i'}-\frac{a_\rho^{3,i'}\,a_\rho^{3,i}}{a_\rho^{3,3}})\,T^{(i)}\,,\end{aligned}\tag{65}$$

whereas (64) is a consequence of (62) for the choice

$$\begin{aligned}V_\rho &:= n_S+\frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,T^{(i)}\,,\\ W_\rho &:= a_\rho^{i',i'}\,T^{(i')}+[2\,a_\rho^{i',i}-\frac{a_\rho^{3,i}\,a_\rho^{3,i'}}{a_\rho^{3,3}}]\,T^{(i)}+a_\rho^{3,i'}\,n_S\,.\end{aligned}\tag{66}$$

We decompose the vector $\nabla\xi_\rho^{(i)}$ in the way of (51) and we use the representation (55) to show for $i=1,2$ that

$$\begin{aligned}\nabla\xi_\rho^{(i)} =&\partial^{(i)}\,\xi_\rho^{(i)}\,(T^{(i)}-\frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,n_S)+\partial^{(i)}\,\xi_\rho^{(i')}\,(T^{(i')}-\frac{a_\rho^{3,i'}}{a_\rho^{3,3}}\,n_S)\\ &+\partial^{(i)}\,\xi_\rho^{(3)}\,\frac{n_S}{a_\rho^{3,3}}+g_\rho\,n_S\,.\end{aligned}\tag{67}$$

The representation (64) and the formula (67), imply for $i=1,2$ that

$$\begin{aligned}\nabla\xi_\rho^{(i)} =&(a_\rho^{i,i}-\frac{[a_\rho^{3,i}]^2}{a_\rho^{3,3}})^{-1}\,(-\nabla\xi_\rho^{(3)}\cdot V_\rho-\nabla\xi_\rho^{(i')}\cdot W_\rho+\bar{g}_\rho)\,(T^{(i)}-\frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,n_S)\\ &+\partial^{(i)}\,\xi_\rho^{(i')}\,(T^{(i')}-\frac{a_\rho^{3,i'}}{a_\rho^{3,3}}\,n_S)+\partial^{(i)}\,\xi_\rho^{(3)}\,\frac{n_S}{a_\rho^{3,3}}+g_\rho\,n_S\,.\end{aligned}\tag{68}$$

Let $B_\rho^{(i)}$ be the matrix that satisfies

$$\begin{aligned}&B_\rho^{(i)}(T^{(i)}-\frac{a_\rho^{3,i}}{a_\rho^{3,3}}\,n_S) = (a_\rho^{i,i}-\frac{[a_\rho^{3,i}]^2}{a_\rho^{3,3}})\,T^{(i)}\\ &B_\rho^{(i)}(T^{(i')}-\frac{a_\rho^{3,i'}}{a_\rho^{3,3}}\,n_S) = W_\rho\,,\qquad B_\rho^{(i)}n_S = a_\rho^{3,3}\,V_\rho\,.\end{aligned}\tag{69}$$

Multiply the relation (68) with $B^{(i)}$ to see that

$$\begin{aligned} B^{(i)}_\rho \nabla \xi^{(i)}_\rho &= (-\nabla \xi^{(3)}_\rho \cdot V_\rho - \nabla \xi^{(i')}_\rho \cdot W_\rho)\, T^{(i)} + \partial^{(i)}\, \xi^{(i')}_\rho\, W_\rho + \partial^{(i)}\, \xi^{(3)}_\rho\, V_\rho + \bar{G}_\rho \\ &= (T^{(i)} \times V_\rho) \times \nabla \xi^{(3)}_\rho + (T^{(i)} \times W_\rho) \times \nabla \xi^{(i')}_\rho + \bar{G}_\rho\,. \end{aligned} \tag{70}$$

Fifth step. In **the case $\mathbf{i = 1}$**, the formula (65) yields

$$T^{(1)} \times V_\rho = T^{(2)} - \frac{a^{2,3}_\rho}{a^{3,3}_\rho}\, n_S\,, \quad T^{(1)} \times W_\rho = -(a^{2,2}_\rho - \frac{[a^{3,2}_\rho]^2}{a^{3,3}_\rho})\, n_S\,. \tag{71}$$

Moreover, the conditions (69) imply the identity

$$O^T\, B^{(1)}_\rho\, O = A_\rho + \begin{pmatrix} 0 & -b^{(1)}_\rho & 0 \\ b^{(1)}_\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad b^{(1)}_\rho := a^{2,1}_\rho - \frac{a^{3,1}_\rho\, a^{3,2}_\rho}{a^{3,3}_\rho} = \frac{m^{1,1}_\rho}{a^{3,3}_\rho}\,.$$

Elementary calculations with the skew-symmetric matrix part show that

$$B^{(1)}_\rho \nabla \xi^{(1)}_\rho = \kappa_\rho \nabla \xi^{(1)}_\rho + b^{(1)}_\rho\, (T^{(2)} \times T^{(1)}) \times \nabla \xi^{(1)}_\rho\,. \tag{72}$$

Observe that $T^{(2)} \times T^{(1)} = -n_S$. Putting (72) and (71) into (70), the claim (49) follows for $\xi^{(1)}_\rho$.

In **the case $\mathbf{i = 2}$**, the formula (66) implies that

$$T^{(2)} \times V_\rho = T^{(2)} \times n_S = -T^{(1)}\,, \quad T^{(2)} \times W_\rho = a^{1,1}_\rho\, n_S - a^{3,1}_\rho\, T^{(1)}\,. \tag{73}$$

Moreover, it can easily be shown that the matrix $B^{(2)}_\rho$, that is uniquely defined by the conditions (69), is nothing else but the matrix $\tilde{\kappa}_\rho$ introduced in section 3. The claim (49) for $\xi^{(2)}_\rho$ follows from (70).

In **the case $\mathbf{i = 3}$**, orthonormal decomposition and the formula (57) imply that

$$\begin{aligned} \nabla \xi^{(3)}_\rho &= \sum_{i=1}^{2} (a^{3,3}_\rho\, \partial^{(3)}\, \xi^{(i)}_\rho + \sum_{j=1}^{2} a^{3,j}_\rho\, \partial^{(j)}\, \xi^{(i)}_\rho)\, T^{(i)} + \partial^{(3)}\, \xi^{(3)}_\rho\, n_S + \bar{G}_\rho \\ &= \sum_{i=1}^{2} \kappa_\rho n_S \cdot \nabla \xi^{(i)}\, T^{(i)} + \partial^{(3)}\, \xi^{(3)}_\rho\, n_S + \bar{G}_\rho\,. \end{aligned} \tag{74}$$

Insert (63) into (74) to obtain the equivalent representation

$$\nabla \xi^{(3)}_\rho = \sum_{i=1}^{2} (\kappa_\rho n_S \cdot \nabla \xi^{(i)}_\rho\, T^{(i)} - \kappa_\rho T_i \cdot \nabla \xi^{(i)}_\rho\, n_S) + \bar{G}_\rho\,. \tag{75}$$

The permutation formula (52) implies that

$$\begin{aligned} &\kappa_\rho n_S \cdot \nabla \xi^{(2)}_\rho\, T^{(2)} - \kappa_\rho T_2 \cdot \nabla \xi^{(2)}_\rho\, n_S = \\ &\quad (\kappa_\rho n_S - a^{3,1}_\rho\, T^{(1)}) \cdot \nabla \xi^{(2)}_\rho\, T^{(2)} - (\kappa_\rho T^{(2)} - a^{2,1}_\rho\, T^{(1)}) \cdot \nabla \xi^{(2)}_\rho\, n_S \\ &\qquad + a^{3,1}_\rho\, T^{(2)} \cdot \nabla \xi^{(1)}_\rho\, T^{(2)} - a^{2,1}_\rho\, T^{(2)} \cdot \nabla \xi^{(1)}_\rho\, n_S + \bar{G}_\rho\,. \end{aligned} \tag{76}$$

From (75) and (76), it follows that

$$
\begin{aligned}
\nabla\xi_\rho^{(3)} =&(\kappa_\rho n_S - a_\rho^{3,1}\, T^{(1)}) \cdot \nabla\xi_\rho^{(2)}\, T^{(2)} - (\kappa_\rho T^{(2)} - a_\rho^{2,1}\, T^{(1)}) \cdot \nabla\xi_\rho^{(2)}\, n_S\\
&+ \bar{G}_\rho + \tilde{M}_\rho^{(3)} \nabla\xi_\rho^{(1)}\,, \qquad (77)\\
(\tilde{M}_\rho^{(3)})^{i,j} :=& T_i^{(1)}\, (\kappa_\rho n_S)_j - n_{S,i}\, (\kappa_\rho T^{(1)} + a_\rho^{2,1}\, T^{(2)})_j + a_\rho^{3,1}\, T_i^{(2)}\, T_j^{(2)}\,.
\end{aligned}
$$

Let $B_\rho^{(3)}$ be a matrix that satisfies

$$
B_\rho^{(3)} T^{(2)} = \frac{\kappa_\rho T^{(2)} - a_\rho^{2,1}\, T^{(1)}}{m_\rho^{1,1}}, \quad B_\rho^{(3)} n_S = \frac{\kappa_\rho n_S - a_\rho^{3,1}\, T^{(1)}}{m_\rho^{1,1}}\,. \qquad (78)
$$

Apply $B_\rho^{(3)}$ to (77), and define $M_\rho^{(3)} := B_\rho^{(3)}\, \tilde{M}_\rho^{(3)}$, then

$$
B_\rho^{(3)} \nabla\xi_\rho^{(3)} = B_\rho^{(3)} \bar{G}_\rho + M_\rho^{(3)} \nabla\xi_\rho^{(1)} + (m_\rho^{1,1})^{-1}\, (B_\rho^{(3)} n_S \times B_\rho^{(3)} T^{(2)}) \times \nabla\xi_\rho^{(2)}\,.
$$

Observe that

$$
\begin{aligned}
B_\rho^{(3)} n_S \times B_\rho^{(3)} T^{(2)} &= [a_\rho^{2,3}]^2\, (T^{(2)} \times n_S) + a_\rho^{2,2}\, a_\rho^{3,3}\, (n_S \times T^{(2)})\\
&= (a_\rho^{2,2}\, a_\rho^{3,3} - [a_\rho^{2,3}]^2)\, T^{(1)} = m_\rho^{1,1}\, T^{(1)}\,.
\end{aligned}
$$

We at last notice using (12) that the choice $B_\rho^{(3)} = (m_\rho^{1,1})^{-1}\, \tilde{\kappa}_\rho$ satisfies (78). The claim (49) for $\xi_\rho^{(3)}$ follows easily. ☐

In the following Lemmas, we use the result of Lemma 4.1 to derive integral relations satisfied by the functions $\xi_\rho^{(i)}$ ($i = 1, 2, 3$).

Lemma 4.2. *Same assumptions as in Lemma 4.1. Then, there is*

$$
\bar{G}_\rho^{(1)} \in [L^2(\Omega)]^3\,, \quad |\bar{G}_\rho^{(1)}| \leq c\, (|f| + g_0\, k_1\, |\nabla u_\rho|) \text{ a. e. in } \Omega\,, \qquad (79)
$$

such that for all $v \in W^{2,2}(\Omega)$

$$
\int_\Omega \kappa_\rho\, \nabla\xi_\rho^{(1)} \cdot \nabla v = \int_\Omega \bar{G}_\rho^{(1)} \cdot \nabla v - \int_\Gamma (\kappa_\rho\, n_\Gamma \cdot \nabla u_\rho)\, (\tau^{(1)} \cdot \nabla v)\,. \qquad (80)
$$

Proof. Choose $v \in W^{2,2}(\Omega)$ arbitrary, and multiply the relation (49) for $\xi_\rho^{(1)}$ with ∇v. Due to integration by parts, and to the vector identity $\operatorname{div}(a \times b) = \operatorname{curl} a \cdot b + \operatorname{curl} b \cdot a$,

$$
\begin{aligned}
&\int_\Omega (T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S) \times \nabla\xi_\rho^{(3)} \cdot \nabla v = -\int_\Omega (T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S) \times \nabla v \cdot \nabla\xi_\rho^{(3)}\\
&\quad = \int_\Omega \operatorname{curl}(T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S) \cdot \nabla v\, \xi_\rho^{(3)} - \int_\Gamma (T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S) \times \nabla v \cdot n_\Gamma\, \xi_\rho^{(3)}\,.
\end{aligned}
$$

With similar arguments, and abbreviating $p_{i,\rho} := a_\rho^{2,i} - a_\rho^{3,i}\, a_\rho^{3,2}/a_\rho^{3,3}$, it follows for $i = 1,2$ that

$$
\begin{aligned}
-\int_\Omega p_{i,\rho}\,(n_S \times \nabla \xi_\rho^{(i)}) \cdot \nabla v = &\int_\Gamma p_{i,\rho}\,(n_S \times \nabla v) \cdot n_\Gamma\, \xi_\rho^{(i)} \\
&- \int_\Omega \operatorname{curl}(p_{i,\rho} n_S) \cdot \nabla v\, \xi_\rho^{(i)}\,.
\end{aligned}
$$

Choosing $G_\rho^{(1)}$ as in Lemma 4.1, we define

$$
\bar G_\rho^{(1)} := G_\rho^{(1)} - \sum_{i=1}^{2} \operatorname{curl}((a_\rho^{2,i} - \frac{a_\rho^{3,i}\, a_\rho^{3,2}}{a_\rho^{3,3}})\, n_S)\, \xi_\rho^{(i)} + \operatorname{curl}(T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S)\, \xi_\rho^{(3)}\,.
$$

For $g \in C^{0,1}(\Omega)$, observe that $\operatorname{curl}(g\, n_S) = g\, n_S + \nabla g \times n_S$. Thus, only tangential derivatives of the regularized coefficients occur in the definition of $\bar G_\rho^{(1)}$, and (42) can be used to prove the estimate (79). In order to reformulate the integrals over Γ, observe that

$$
\begin{aligned}
(n_S \times \nabla v) \cdot n_\Gamma &= -(n_S \times n_\Gamma) \cdot \nabla v = -|n_S \times n_\Gamma|\, \tau^{(1)} \cdot \nabla v \\
(T^{(2)} \times \nabla v) \cdot n_\Gamma &= -(T^{(2)} \times n_\Gamma) \cdot \nabla v = (T^{(2)} \cdot \tau^{(2)})\, \tau^{(1)} \cdot \nabla v\,.
\end{aligned}
$$

Lemma C.3 in the appendix implies that

$$
\begin{aligned}
&\sum_{i=1}^{2} (a_\rho^{2,i} - \frac{a_\rho^{3,i}\, a_\rho^{3,2}}{a_\rho^{3,3}})\,(n_S \times \nabla v) \cdot n_\Gamma\, \xi_\rho^{(i)} - (T^{(2)} - \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, n_S) \times \nabla v \cdot n_\Gamma\, \xi_\rho^{(3)} \\
&= (-\sin\alpha\, \left[\sum_{i=1}^{2} (a_\rho^{2,i} - \frac{a_\rho^{3,i}\, a_\rho^{3,2}}{a_\rho^{3,3}})\, \xi_\rho^{(i)} + \frac{a_\rho^{2,3}}{a_\rho^{3,3}}\, \xi_\rho^{(3)}\right] - \cos\alpha\, \xi_\rho^{(3)})\,(\tau^{(1)} \cdot \nabla v)\,.
\end{aligned}
$$

Using orthonormal decomposition for the vector $-\kappa_\rho\, n_\Gamma \cdot \nabla u_\rho$, the relation (80) is obvious. $\square$

Lemma 4.3. *Same assumptions as in Lemma 4.1. Then, there are $\bar G_\rho^{(2)},\ \bar G_\rho^{(3)} \in [L^2(\Omega)]^3$ such that*

$$
|\bar G_\rho^{(2)}| + k_0\, |\bar G_\rho^{(3)}| \leq c\,(|f| + g_0\, k_1\, |\nabla u_\rho|) \text{ a. e. in } \Omega\,, \tag{81}
$$

and such that for all $v \in W^{2,2}(\Omega)$

$$
\begin{aligned}
\int_\Omega \tilde\kappa_\rho\, \nabla \xi_\rho^{(2)} \cdot \nabla v = &\int_\Omega \{\bar G_\rho^{(2)} + (a_\rho^{1,1}\, n_S - a_\rho^{1,3}\, T^{(1)}) \times \nabla \xi_\rho^{(1)}\} \cdot \nabla v \\
&- \int_\Gamma \xi_\rho^{(3)}\,(\tau^{(2)} \cdot \nabla v)
\end{aligned} \tag{82}
$$

$$
\int_\Omega [m_\rho^{1,1}]^{-1}\, \tilde\kappa_\rho \nabla \xi_\rho^{(3)} \cdot \nabla v = \int_\Omega \{\bar G_\rho^{(3)} + M_\rho^{(3)} \nabla \xi_\rho^{(1)}\} \cdot \nabla v + \int_\Gamma \xi_\rho^{(2)}\,(\tau^{(2)} \cdot \nabla v)\,. \tag{83}
$$

Proof. We multiply the relation (49) for $\xi_\rho^{(2)}$ with ∇v, $v \in W^{2,2}(\Omega)$ arbitrary. Integration by parts, and the fact that $T^{(1)} \times n_\Gamma = \tau^{(2)}$, yield

$$
\int_\Omega (T^{(1)} \times \nabla \xi_\rho^{(3)}) \cdot \nabla v = \int_\Omega \operatorname{curl} T^{(1)} \cdot \nabla v\, \xi_\rho^{(3)} + \int_\Gamma \xi_\rho^{(3)}\, \tau^{(2)} \cdot \nabla v\,, \tag{84}
$$

Choosing $G^{(2)}_\rho$ as in Lemma 4.1, we define $\bar{G}^{(2)}_\rho := G^{(2)}_\rho - \operatorname{curl} T^{(1)}\, \xi^{(3)}_\rho$. The estimate (81) is readily checked. The relation (82) is obvious.

In order to prove (83), multiply the relation (49) for $\xi^{(3)}_\rho$ with ∇v, $v \in W^{2,2}(\Omega)$ arbitrary. As in (84),

$$\int_\Omega (T^{(1)} \times \nabla \xi^{(2)}_\rho) \cdot \nabla v = \int_\Omega \operatorname{curl} T^{(1)} \cdot \nabla v\, \xi^{(2)}_\rho + \int_\Gamma \xi^{(2)}_\rho\, \tau^{(2)} \cdot \nabla v\,.$$

Define $\bar{G}^{(3)}_\rho := G^{(3)}_\rho + \operatorname{curl} T^{(1)}\, \xi^{(2)}_\rho$. The estimate (81) is readily checked, finishing the proof. □

We now prove two Lemmas concerning the boundary data u_e and Q. The compatibility conditions (25), (26), (27) come here into the play.

Lemma 4.4. *In addition to the hypotheses of Lemma 4.1, assume that the conditions* (25), (26) *are satisfied for the problem* (P_N)*, or that* (27) *is valid for the problem* (P_D)*. Then, there are* $\tilde{Q}_{1,\rho},\, \tilde{Q}_{2,\rho} \in W^{1/2,2}(\Gamma)$ *and* $\tilde{U}_{2,\rho} \in W^{1/2,2}(\Gamma)$ *such that*

$$\frac{m^{2,1}_\rho}{m^{1,1}_\rho}\, \xi^{(1)}_\rho + \frac{a^{3,3}_\rho}{m^{1,1}_\rho\, \sin\alpha}\, Q = f_d(\alpha,\, A_\rho)\, \tilde{Q}_{1,\rho} + \tilde{Q}_{2,\rho} \tag{85}$$

$$a^{3,1}_\rho\, (\tau^{(1)} \cdot \nabla u_e) - \frac{a^{3,3}_\rho}{\sin\alpha}\, (\tau^{(2)} \cdot \nabla u_e) = f_d(\alpha,\, A_\rho)\, U_1 + \tilde{U}_{2,\rho}\,. \tag{86}$$

Moreover, there is $c = c(\Omega,\, k_1/k_0)$ *such that*

$$\begin{aligned}
\|\tilde{U}_{2,\rho}\|_{W^{1/2,2}(\Gamma)} \leq& \|U_2\|_{V^2(\Gamma)} + c\, k_1\, g_0\, \|\nabla u_e\|_{W^{1/2,2}(\Gamma)} + C_{1,\rho}\,, &(87)\\
\|\tilde{Q}_{1,\rho}\|_{W^{1/2,2}(\Gamma)} \leq& \|Q_1\|_{W^{1/2,2}(\Gamma)} + c\, g_1\, \|\xi^{(1)}_\rho\|_{W^{1/2,2}(\Gamma)}\,, &(88)\\
\|\tilde{Q}_{2,\rho}\|_{W^{1/2,2}(\Gamma)} \leq& c\, k_0^{-1}\, (1+g_0)(\|Q\|_{W^{1/2,2}(\Gamma)} + \|Q_1\|_{W^{1/2,2}(\Gamma)} \\
&+ (1+g_1)\, \|\xi^{(1)}_\rho\|_{W^{1/2,2}(\Gamma)}) + \|Q_2\|_{W^{1/2,2}(\Gamma)} + C_{2,\rho}\,, &(89)
\end{aligned}$$

where $C_{1,\rho},\, C_{2,\rho} \to 0$.

Proof. The condition (27) is by assumption valid on Γ. Recalling the definition (33), we multiply (27) with the function $I_\rho(\operatorname{dist}(\cdot, S))$, and we then add on both sides of the new relation the term $a^{3,1}_1\, \xi^{(1)}_e - a^{3,3}_1\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha$. We obtain that

$$\begin{aligned}
&(a^{3,1}_1 + I_\rho\, [a^{3,1}]_S)\, \xi^{(1)}_e - a^{3,3}_1 + I_\rho\, [a^{3,3}]_S\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha \\
&\quad = (\cot\alpha\, (a^{3,3}_1 + I_\rho\, [a^{3,3}]_S) + (a^{2,3}_1 + I_\rho\, [a^{2,3}]_S))\, U_1 + I_\rho\, U_2 \\
&\qquad + a^{3,1}_1\, \xi^{(1)}_e - a^{3,3}_1\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - (\cot\alpha\, a^{3,3}_1 + a^{2,3}_1)\, U_1
\end{aligned} \tag{90}$$

Due to (37), $a^{3,1}_1 + I_\rho\, [a^{3,1}]_S = a^{3,1}_\rho = L_\rho(a^{3,1})$ (etc.), so that

$$\begin{aligned}
a^{3,1}_\rho\, \xi^{(1)}_e - a^{3,3}_\rho\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha &= (\cot\alpha\, a^{3,3}_\rho + a^{2,3}_\rho)\, U_1 + \tilde{U}_{2,\rho} \\
\tilde{U}_{2,\rho} &:= I_\rho\, U_2 + a^{3,1}_1\, \xi^{(1)}_e - a^{3,3}_1\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - (\cot\alpha\, a^{3,3}_1 + a^{2,3}_1)\, U_1\,,
\end{aligned}$$

which proves (86) on Γ. Thanks to Lemma B.5 in the appendix, we verify that

$$\|I_\rho\, U_2\|_{W^{1/2,2}(\Gamma)} \leq \|U_2\|_{V^2(\Gamma)} + C_{1,\rho}\,, \qquad C_{1,\rho} \to 0\,.$$

Using also Lemma B.1, the norm estimate (87) follows. In order to prove (85), use the assumption (26) to define

$$\tilde{Q}_{1,\rho} := Q_1 + [\frac{m^{2,1}}{m^{1,1}}]_S\, [f_d(\alpha,\, A)]_S^{-1}\, \xi_\rho^{(1)} = Q_1 + g_1\, \xi_\rho^{(1)}\,.$$

Due to (B.1), we readily verify the estimate (88). It then follows from (25) that

$$\begin{aligned}[f_d(\alpha,\, A)]_S\, \tilde{Q}_{1,\rho} &= [f_d(\alpha,\, A)]_S\, Q_1 + [\frac{m^{2,1}}{m^{1,1}}]_S\, \xi_\rho^{(1)} \\ &= [\frac{a^{3,3}}{m^{1,1}}]_S\, \frac{Q}{\sin\alpha} - Q_2 + [\frac{m^{2,1}}{m^{1,1}}]_S\, \xi_\rho^{(1)}\,. \end{aligned} \tag{91}$$

Multiply (91) with $I_\rho(\mathrm{dist}(\cdot, S)$, then with the help of (40), argue as previously (cf. (90)) to obtain that

$$\begin{aligned} f_d(\alpha,\, A_\rho)\, \tilde{Q}_{1,\rho} &= \frac{a_\rho^{3,3}}{m_\rho^{1,1}}\, \frac{Q}{\sin\alpha} + \frac{m_\rho^{2,1}}{m_\rho^{1,1}}\, \xi_\rho^{(1)} - \tilde{Q}_{2,\rho}\,, \\ Q_{2,\rho} &:= I_\rho\, Q_2 + \frac{a_1^{3,3}\, Q}{m_1^{1,1}\, \sin\alpha} - (\cot\alpha\, \frac{a_1^{3,3}}{m_1^{1,1}} + \frac{a_1^{2,3}}{m_1^{1,1}})\, \tilde{Q}_{1,\rho} + \frac{m_1^{2,1}}{m_1^{1,1}}\, \xi_\rho^{(1)}\,. \end{aligned} \tag{92}$$

The construction of the regularization (38), (39) plays here the essential role. The inequality (89) is derived in the same fashion as (87), using Lemma B.5, , Lemma B.1 and (88). ☐

Lemma 4.5. *Same assumption as in Lemma 4.4. Let $u_\rho \in W^{2,2}(\Omega)$ be the weak solution to (P_ρ). If u_ρ satisfies the condition* (3)*, then*

$$-\xi_\rho^{(2)} = (\cot\alpha\, \frac{a_\rho^{3,3}}{m_\rho^{1,1}} + \frac{a_\rho^{3,2}}{m_\rho^{1,1}})\, (\xi_\rho^{(3)} + \tilde{Q}_{1,\rho}) + \tilde{Q}_{2,\rho} \quad \textit{a. e. on } \Gamma\,. \tag{93}$$

If u_ρ satisfies the condition (4)*, then*

$$\xi_\rho^{(3)} = (\cot\alpha\, a_\rho^{3,3} + a_\rho^{3,2})\, (\xi_\rho^{(2)} + U_1) + \tilde{U}_{2,\rho} \quad \textit{a. e. on } \Gamma\,. \tag{94}$$

Proof. We recall the notations (9) and (10). If (3) is satisfied in the sense of traces, then

$$Q = -\kappa_\rho n_\Gamma \cdot \nabla u_\rho = -(n_\Gamma \cdot n_S)\, \kappa_\rho n_S \cdot \nabla u_\rho - (n_\Gamma \cdot T^{(2)})\, \kappa_\rho T^{(2)} \cdot \nabla u_\rho\,, \tag{95}$$

thanks to orthonormal decomposition on Γ. For the same reason, the equivalence (54) yields

$$\kappa_\rho T^{(2)} \cdot \nabla u_\rho = \sum_{i=1}^{2} (a_\rho^{2,i} - \frac{a_\rho^{3,i}\, a_\rho^{3,2}}{a_\rho^{3,3}})\, \xi_\rho^{(i)} + \frac{a_\rho^{3,2}}{a_\rho^{3,3}}\, \xi_\rho^{(3)}\,.$$

Using Lemma C.3 and the definition (47) of $\xi_\rho^{(3)}$, we easily deduce from (95) that

$$-\xi_\rho^{(2)} = (\cot\alpha\, \frac{a_\rho^{3,3}}{m_\rho^{1,1}} + \frac{a_\rho^{3,2}}{m_\rho^{1,1}})\, \xi_\rho^{(3)} + \frac{m_\rho^{2,1}}{m_\rho^{1,1}}\, \xi_\rho^{(1)} + \frac{a_\rho^{3,3}}{m_\rho^{1,1}\, \sin\alpha}\, Q\,,$$

and (93) follows from Lemma 4.4, (85). With the help of orthonormal decomposition, (54), and Lemma C.3

$$\begin{aligned}\tau^{(2)} \cdot \nabla u_\rho &= (\tau^{(2)} \cdot T^{(2)})\, \xi_\rho^{(2)} + (\tau^{(2)} \cdot n_S)\,(n_S \cdot \nabla u_\rho)\\ &= \cos\alpha\, \xi^{(2)} - \frac{\sin\alpha}{a_\rho^{3,3}}\,(\xi_\rho^{(3)} - \sum_{i=1}^{2} a_\rho^{3,i}\, \xi_\rho^{(i)})\,.\end{aligned}$$

If (4) is satisfied in the sense of traces, then

$$\xi_\rho^{(3)} = (\cot\alpha\, a_\rho^{3,3} + a_\rho^{3,2})\, \xi_\rho^{(2)} + a_\rho^{3,1}\, \xi_e^{(1)} - \frac{a_\rho^{3,3}}{\sin\alpha}\, \tau^{(2)} \cdot \nabla u_e\,,$$

and (94) follows from Lemma 4.4, (86). □

5 $W^{2,2}$ regularity

In this section, we prove the convergence of the approximation method (P_ρ) in the space $W^{2,2}$. In order to abbreviate in our estimates, we introduce for the problem (P_N) the quantities

$$\begin{aligned}N_q &:= k_0^{-1}\,(\|f\|_{L^q(\Omega)} + \|Q\|_{W^{1/q',q}(\Gamma)})\\ \tilde N_q &:= k_0^{-1}\,(\|f\|_{L^q(\Omega)} + \|Q_1\|_{W^{1/q',q}(\Gamma)} + \|Q_2\|_{V^q(\Gamma)})\,,\end{aligned}$$

and for the problem (P_D) the quantities

$$\begin{aligned}N_q &:= k_0^{-1}\,\|f\|_{L^q(\Omega)} + \|\nabla u_e\|_{W^{1,q}(\Omega)}\\ \tilde N_q &:= N_q + \|U_1\|_{W^{1/q',q}(\Gamma)} + \|U_2\|_{V^q(\Gamma)}\,.\end{aligned}$$

Here, the functions Q_i and U_i are taken from (25), (26) and (27). The main result of the section is the following:

Theorem 5.1. *Assume that $S \in \mathcal{C}^2$ and that $f \in L^2(\Omega)$. Let u be the weak solution to (P). Assume that the condition* (24) *is valid, and that one of the following assumptions is satisfied:*

(1) u satisfies (3) *on Γ, and the conditions* (25), (26) *hold with $q = 2$.*

(2) u satisfies (4) *on Γ, and the condition* (27) *holds with $q = 2$.*

Then, u belongs to $W^{2,2}(\Omega_i)$ for $i = 1, 2$ and moreover satisfies the continuous estimate

$$\|D^2 u\|_{L^2(\Omega)} \le c\,(1 + g_0)\,\tilde N_2\,. \tag{96}$$

with a constant c that depends on Ω, k_1/k_0, k_1'/k_0, and additionally on g_1 for the problem (P_N).

The proof of the theorem is carried out in the following four propositions.

Proposition 5.2. *Assume that $S \in \mathcal{C}^2$ and that $f \in L^2(\Omega)$. Let u_ρ be the weak solution to (P_ρ). Assume that u_ρ either satisfies* (3) *with $Q \in W^{1/2,2}(\Gamma)$ or* (4) *with $u_e \in W^{3/2,2}(\Gamma)$. Then, there is a constant c, depending only on Ω and on κ_1/κ_0, such that*

$$\|\nabla \xi_\rho^{(1)}\|_{L^2(\Omega)} \leq c\,(1+g_0)\,N_2\,. \tag{97}$$

Proof. Let u_ρ denote the solution to the problem (P_ρ). We at first consider the boundary condition (3). For $v \in W^{2,2}(\Omega)$, introduce the linear functional

$$F_Q^{(1)}(v) := \int_\Gamma Q\,(\tau^{(1)} \cdot \nabla v)\,. \tag{98}$$

The continuity estimate

$$|F_Q^{(1)}(v)| \leq c\,g_0\,\|Q\|_{W^{1/2,2}(\Gamma)}\,\|\nabla v\|_{L^2(\Omega)}\,, \tag{99}$$

follows from Lemma C.1, and implies (via standard density results for Sobolev spaces) that the functional $F_Q^{(1)}$ extends by density on $W^{1,2}(\Omega)$. Due to (80),

$$\int_\Omega \kappa_\rho \nabla \xi_\rho^{(1)} \cdot \nabla v = \int_\Omega \bar{G}_\rho^{(1)} \cdot \nabla v + F_Q^{(1)}(v) \quad \forall v \in W^{1,2}(\Omega)\,. \tag{100}$$

In (100), we are allowed to choose $v := \xi_\rho^{(1)}$. To derive (97) from the estimates (99) and (79) and Lemma 3.1 is a straightforward exercise on Young's inequality.

For the boundary condition (4), we introduce the extension $u_e \in W^{2,2}(\Omega)$ of the boundary data, and $\xi_e^{(1)} := \tau^{(1)} \cdot \nabla u_e \in W^{1,2}(\Omega)$. Due to (80),

$$\int_\Omega \kappa_\rho\,\nabla(\xi_\rho^{(1)} - \xi_e^{(1)}) \cdot \nabla v = \int_\Omega (\bar{G}_\rho^{(1)} - \kappa_\rho\,\nabla \xi_e^{(1)}) \cdot \nabla v \quad \forall v \in W_0^{1,2}(\Omega)\,, \tag{101}$$

and (97) follows. ☐

Ahead of the statement of the following Lemma, we recall the definition (17) of the function f_d.

Lemma 5.3. *Let the hypotheses of Proposition 5.2 be valid. Assume in addition that the condition* (24) *is valid. For $u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$, define*

$$(B_\rho(u),\,v) := -\int_\Gamma f_d(\alpha,\,A_\rho)\,u\,(\tau^{(2)} \cdot \nabla v)\,. \tag{102}$$

Then, the mapping B_ρ extends to an element of $\mathcal{L}(W^{1,2}(\Omega),\,[W^{1,2}(\Omega)]^*)$. Moreover, there is $\rho_0 = \rho_0(S,\,\kappa_2,\,\kappa_1,\,\alpha)$ such that for all $\rho \leq \rho_0$ the inequalities

$$(B_\rho(u),\,(u-m)^+) \leq \tilde{c}\,(1+g_0)\,\int_\Gamma u\,(u-m)^+\,, \tag{103}$$

$$(B_\rho(u),\,(u+m)^-) \leq \tilde{c}\,(1+g_0)\,\int_\Gamma u\,(u+m)^-\,, \tag{104}$$

are valid for all $u \in W^{1,2}(\Omega)$ and all $m \in \mathbb{N}$, with $\tilde{c} := c\,k_0^{-1}$ for (P_N), and $\tilde{c} := c\,k_1$ for (P_D).

Proof. Due to Lemma C.1 and Lemma B.1,

$$\begin{aligned}|(B_\rho(u),\, v)| &\leq c\, g_0\, \|f_d(\alpha,\, A_\rho)\, u\|_{W^{1/2,2}(\Gamma)}\, \|\nabla v\|_{L^2(\Omega)}\\ &\leq c_\rho\, \|u\|_{W^{1/2,2}(\Gamma)}\, \|\nabla v\|_{L^2(\Omega)}\,,\end{aligned} \tag{105}$$

for all $u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$. Therefore, the mapping B_ρ extends by density to an element of $\mathcal{L}(W^{1,2}(\Omega),\, [W^{1,2}(\Omega)]^*)$.

For $u \in W^{2,2}(\Omega)$, $m \in \mathbb{N}$,

$$(B_\rho(u),\, (u-m)^+) = \frac{-1}{2} \int_\Gamma f_d(\alpha,\, A_\rho)\, \tau^{(2)} \cdot \nabla((u+m)\,(u-m)^+)\,. \tag{106}$$

For the $(P_N)-$case of (17), integration by parts yields

$$\begin{aligned}(B_\rho(u),\, (u-m)^+) &= \int_\Gamma (\cot\alpha\, \tau^{(2)} \cdot \nabla \frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \tau^{(2)} \cdot \nabla \frac{a^{2,3}_\rho}{m^{1,1}_\rho})\, \frac{(u+m)}{2}\, (u-m)^+\\ &+ \int_\Gamma (\mathrm{div}_\Gamma(\cot\alpha\, \tau^{(2)})\, \frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \mathrm{div}_\Gamma(\tau^{(2)})\, \frac{a^{2,3}_\rho}{m^{1,1}_\rho})\, \frac{(u+m)}{2}\, (u-m)^+\,.\end{aligned} \tag{107}$$

Using (36), the fact that $\tau^{(2)} \cdot n_S = -\sin\alpha$ on Γ, and (40), we compute

$$\begin{aligned}\cot\alpha\, \tau^{(2)} \cdot \nabla \frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \tau^{(2)} \cdot \nabla \frac{a^{2,3}_\rho}{m^{1,1}_\rho} &= -\sin\alpha\, b_\rho\, \rho^{-1}\, [f_d(\alpha,\, A)]_S\\ &+ \cot\alpha\, L_\rho(\tau^{(2)} \cdot \nabla \frac{a^{3,3}}{m^{1,1}}) + L_\rho(\tau^{(2)} \cdot \nabla \frac{a^{2,3}}{m^{1,1}})\,.\end{aligned} \tag{108}$$

Due to the uniform continuity of the data $A_1,\, A_2,\, \alpha$, there is a neighbourhood D of the curve $\Gamma \cap S$ such that (24) is valid in the domain $\overline{D \cap \Omega}$. Therefore, if $\rho \leq \rho_0(A_1,\, A_2,\, \alpha)$,

$$-\sin\alpha\, b_\rho\, \rho^{-1}\, [f_d(\alpha,\, A)]_S\, (u+m)\, (u-m)^+ \leq 0\,. \tag{109}$$

The estimate

$$\cot\alpha\, L_\rho(\tau^{(2)} \cdot \nabla \frac{a^{3,3}}{m^{1,1}}) + L_\rho(\tau^{(2)} \cdot \nabla \frac{a^{2,3}}{m^{1,1}}) \leq \frac{k_1^2\, k_1'}{k_0^4}\,,$$

together with (108) and (109), yields

$$(\cot\alpha\, \tau^{(2)} \cdot \nabla \frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \tau^{(2)} \cdot \nabla \frac{a^{2,3}_\rho}{m^{1,1}_\rho})\, (u+m)\, (u-m)^+ \leq \frac{2\, k_1^2\, k_1'}{k_0^4}\, u\, (u-m)^+\,. \tag{110}$$

The estimate (103) follows from (107) and (110). For the problem (P_D), we can reformulate

$$\begin{aligned}(B_\rho(u),\, (u-m)^+) &= \frac{1}{2} \int_\Gamma (\cot\alpha\, \tau^{(2)} \cdot \nabla a^{3,3}_\rho + \tau^{(2)} \cdot \nabla a^{2,3}_\rho)\, (u+m)\, (u-m)^+\\ &+ \frac{1}{2} \int_\Gamma (\mathrm{div}_\Gamma(\cot\alpha\, \tau^{(2)})\, a^{3,3}_\rho + \mathrm{div}_\Gamma(\tau^{(2)})\, a^{2,3}_\rho)\, (u+m)\, (u-m)^+\,.\end{aligned} \tag{111}$$

Under the assumption (24), we verify for $\rho \le \rho_0$ (cf. (109)) that

$$(\cot \alpha\, \tau^{(2)} \cdot \nabla a^{3,3}_{\rho} + \tau^{(2)} \cdot \nabla a^{3,2}_{\rho}) \le c\, k_1' \,. \tag{112}$$

Here again, the estimate (103) follows from (111) thanks to standard inequalities. Due to the formula

$$(B_{\rho}(u),\, (u+m)^{-}) = \frac{-1}{2} \int_{\Gamma} f_d(\alpha,\, A_{\rho})\, \tau^{(2)} \cdot \nabla((u-m)\,(u+m)^{-})\,,$$

we similarly verify (104). Finally, in view of the continuity property (105), the inequalities (103) and (104) hold true for all $u \in W^{1,2}(\Omega)$. □

Proposition 5.4. *Assume that $S \in C^2(\overline{G})$, and that $f \in L^2(\Omega)$. Let u_{ρ} be the weak solution to (P_{ρ}). Assume that u_{ρ} satisfies* (3)*, and that the condition* (24), (25), (26) *are valid with $q = 2$. Then there is a constant $c = c(\Omega,\, k_1/k_0,\, k_1'/k_0)$, and a sequence of numbers $\{C_{\rho}\}$ that tends to zero, such that*

$$\|\nabla \xi^{(3)}_{\rho}\|_{L^2(\Omega)} \le c\,(1+g_0)\,\tilde{N}_2 + C_{\rho}\,. \tag{113}$$

Proof. Thanks to the relation (93), the operator B_{ρ} of Lemma 5.3, and to the functional

$$F^{(2)}_{\tilde{Q}_{2,\rho}} := -\int_{\Gamma} \tilde{Q}_{2,\rho}\,(\tau^{(2)} \cdot \nabla v)\,, \tag{114}$$

(cf. (98), and (99) for a norm estimate on $F^{(2)}$), (83) is equivalent to

$$\begin{aligned} \int_{\Omega} (m^{1,1}_{\rho})^{-1}\, \tilde{\kappa}_{\rho} \nabla \xi^{(3)}_{\rho} \cdot \nabla v = \int_{\Omega} \{\bar{G}^{(3)}_{\rho} + M^{(3)}_{\rho} \nabla \xi^{(1)}_{\rho}\} \cdot \nabla v + (B_{\rho}(\xi^{(3)}_{\rho} + \tilde{Q}_{1,\rho}),\, v) \\ + F^{(2)}_{\tilde{Q}_{2,\rho}}(v)\,, \quad \forall\, v \in W^{1,2}(\Omega)\,, \end{aligned} \tag{115}$$

or, for the variable $w_{\rho} := \xi^{(3)}_{\rho} + \tilde{Q}_{1,\rho}$, to

$$\begin{aligned} \int_{\Omega} (m^{1,1}_{\rho})^{-1}\, \tilde{\kappa}_{\rho} \nabla w_{\rho} \cdot \nabla v = \int_{\Omega} \{\bar{G}^{(3)}_{\rho} + M^{(3)}_{\rho} \nabla \xi^{(1)}_{\rho} + [m^{1,1}_{\rho}]^{-1}\, \tilde{\kappa}_{\rho} \nabla \tilde{Q}_{1,\rho}\} \cdot \nabla v \\ + (B_{\rho}(w_{\rho}), v) + F^{(2)}_{\tilde{Q}_{2,\rho}}(v)\,, \quad \forall\, v \in W^{1,2}(\Omega)\,. \end{aligned} \tag{116}$$

In the relation (116), it is possible to choose $v := w_{\rho}$. In view of (103) and (104) with $m = 0$, and of the interpolation inequality (C.2),

$$\begin{aligned} (B_{\rho}(w_{\rho}),\, w_{\rho}) &\le c\, \kappa_0^{-1}\,(1+g_0)\, \|w_{\rho}\|^2_{L^2(\Gamma)} \\ &\le c\, c_0^2\, k_0^{-1}\,(1+g_0)\, \|w_{\rho}\|_{L^2(\Omega)}\, \|\nabla w_{\rho}\|_{L^2(\Omega)}\,. \end{aligned} \tag{117}$$

Employing from now on Young's inequality as in the proof of Proposition 5.2, Lemma 4.4 and Proposition 5.2 to bound the quantities $\tilde{Q}_{1,\rho}$, $\tilde{Q}_{2,\rho}$ and $\xi^{(1)}_{\rho}$, the estimate (113) immediately follows. □

Proposition 5.5. *Assume that $S \in \mathcal{C}^2$, and that $f \in L^2(\Omega)$. Let u_ρ be the weak solution to (P_ρ). Assume that u_ρ satisfies* (24)*, and the condition* (27) *with $q = 2$. Then there is a constant $c = c(\Omega,\, k_1/k_0,\, k_1'/k_0)$, and a sequence C_ρ that converges to zero, such that*

$$\|\nabla \xi_\rho^{(2)}\|_{L^2(\Omega)} \leq c\,(1+g_0)\,\tilde{N}_2 + C_\rho\,. \tag{118}$$

Proof. The proof is very similar to the proof of Proposition 5.4.

Using the relation (94), the operator B_ρ of Lemma 5.3 and a functional $F^{(2)}_{\tilde{U}_{2,\rho}}$ (cf. (114)), the relation (83) is equivalent to

$$\begin{aligned}\int_\Omega \tilde{\kappa}_\rho \nabla \xi_\rho^{(2)} \cdot \nabla v = \int_\Omega \{\bar{G}_\rho^{(2)} + (a_\rho^{1,1}\, n_S - a_\rho^{3,1}\, T^{(1)}) \times \nabla \xi_\rho^{(1)}\} \cdot \nabla v \\ + (B_\rho(\xi_\rho^{(2)} + U_1), v) + F^{(2)}_{\tilde{U}_{2,\rho}}(v)\,, \quad \forall\, v \in W^{1,2}(\Omega)\,.\end{aligned} \tag{119}$$

For the variable $w_\rho := \xi_\rho^{(2)} + U_1$, it follows that

$$\begin{aligned}\int_\Omega \tilde{\kappa}_\rho\, \nabla w_\rho \cdot \nabla v = \int_\Omega \{\bar{G}_\rho^{(2)} + (a_\rho^{1,1}\, n_S - a_\rho^{3,1}\, T^{(1)}) \times \nabla \xi_\rho^{(1)} + \tilde{\kappa}_\rho \nabla U_1\} \cdot \nabla v \\ + (B_\rho(w_\rho), v) + F^{(2)}_{\tilde{U}_{2,\rho}}(v)\,, \quad \forall\, v \in W^{1,2}(\Omega)\,,\end{aligned} \tag{120}$$

where it is possible to choose $v := w_\rho$. The estimate (118) follows with arguments similar to the proof of Proposition 5.4. ☐

Proposition 5.6. *Let the assumptions of Theorem 5.1 be satisfied. If u denotes the weak solution to (P), then $u \in W^{2,2}(\Omega_i)$ for $i = 1, 2$. Moreover*

$$\xi_\rho^{(i)} \rightharpoonup T^{(i)} \cdot \nabla u \text{ in } W^{1,2}(\Omega) (\text{for } i = 1, 2)\,, \;\; \xi_\rho^{(3)} \rightharpoonup \kappa n_S \cdot \nabla u \text{ in } W^{1,2}(\Omega)\,.$$

Proof. Proposition 5.2, and either Proposition 5.4 in the case of $(P_{N,\rho})$, or Proposition 5.5 in the case of $(P_{D,\rho})$, provide uniform bounds for the sequences $\{\xi_\rho^{(1)}\}$, and either $\{\xi_\rho^{(2)}\}$ or $\{\xi_\rho^{(3)}\}$, in the space $W^{1,2}(\Omega)$. Due to the gradient representations of Lemma 4.1, it then follows for both problems that there is $C > 0$ independent of ρ such that $\|\nabla \xi_\rho^{(i)}\|_{L^2(\Omega)} \leq C$ for $i = 1, 2, 3$. Thanks to the reflexivity of $W^{1,2}(\Omega)$, we find $\xi^{(i)} \in W^{1,2}(\Omega)$ such that

$$\xi_\rho^{(i)} \rightharpoonup \xi^{(i)} \text{ in } W^{1,2}(\Omega) \text{ for } i = 1, 2, 3\,.$$

On the other hand $\xi_\rho^{(i)} \to T^{(i)} \cdot \nabla u$ for $i = 1, 2$ and $\xi_\rho^{(3)} \to \kappa n_S \cdot \nabla u$ almost everywhere in Ω (cp. Lemma 3.1). Thus

$$T^{(i)} \cdot \nabla u = \xi^{(i)} \in W^{1,2}(\Omega)\ (i = 1, 2), \quad \kappa n_S \cdot \nabla u \in W^{1,2}(\Omega)\,.$$

☐

6 $W^{1,\infty}$ regularity

Theorem 6.1. *Same assumptions as in the Theorem 5.1. Assume that there is $q_0 > 3$ such that $f \in L^{q_0}(\Omega)$. For the problem* (3)*, let Q satisfy* (25) *with $q = q_0$; for the problem* (4)*, let u_e satisfy* (27) *with $q = q_0$. Then $\nabla u \in L^{\infty}(\Omega)$, and the estimate*

$$\|\nabla u\|_{L^{\infty}(\Omega)} \leq c\, \tilde{N}_{q_0}\,,$$

is valid, with a constant c that depends continuously on Ω, on g_0, on κ_1/κ_0, on κ_1'/κ_0, and also on g_1 for the problem (P_N).

For the proof, we will show that the functions $\xi^{(i)}$, $i = 1, 2, 3$, belong to $L^{\infty}(\Omega)$. However, we cannot prove that the approximation method (P_ρ) converges in the space $W^{1,\infty}(\Omega)$. Fortunately, once the result of Theorem 5.1 is ensured, we can derive in the limit new regularity properties that turn out to be sufficient for the result.

Proposition 6.2. *Assume that $S \in \mathcal{C}^2$. Let u denote the weak solution to (P). Assume that there is $q_0 > 3$ such that $f \in L^{q_0}(\Omega)$, and such that u either satisfies* (3) *with $Q \in W^{1/q_0',q_0}(\Gamma)$ or* (4) *with $u_e \in W^{2,q_0}(\Omega)$. Then $\xi^{(1)}$ belongs to $W^{1,q_0}(\Omega)$ and satisfies the estimate*

$$\|\xi^{(1)}\|_{W^{1,q_0}(\Omega)} \leq c\, \tilde{N}_{q_0}\,. \tag{121}$$

Here, the constant c depends continuously on Ω, on g_0, on k_1/k_0, and on k_1'/k_0.

Proof. We let $\rho \to 0$ in (80) to see in the case of the boundary condition (3) that $\xi^{(1)} \in W^{1,2}(\Omega)$ satisfies

$$\int_{\Omega} \kappa \nabla \xi^{(1)} \cdot \nabla v = \int_{\Omega} \{\bar{G}^{(1)} + \operatorname{curl}(Q\,(\tau^{(1)} \times n_{\Gamma}))\} \cdot \nabla v \quad \forall\, v \in W^{2,2}(\Omega)\,, \tag{122}$$

where Lemma C.1 was used to rewrite the functional $F_Q^{(1)}$. The estimate (79) ensures that

$$\|\bar{G}^{(1)}\|_{L^{q_0}(\Omega)} \leq \|f\|_{L^{q_0}(\Omega)} + c\, g_0\, \kappa_1\, \|\nabla u\|_{L^{q_0}(\Omega)}\,.$$

Since we can obtain a bound on $\|\nabla u\|_{L^{q_0}(\Omega)}$ with the arguments of Lemma A.1, the right-hand of (122) belongs to $[W^{1,q_0'}(\Omega)]^*$, with a corresponding norm estimate. The result now follows in principle from the Theorem 1. 2 in [HDKRS08]. We give the idea of the proof in the appendix, Lemma A.1. In the case of the boundary condition (4), introduce $\xi_e^{(1)} := \tau^{(1)} \cdot \nabla u_e$ to see that the function $\xi^{(1)} - \xi_e^{(1)}$ satisfies

$$\int_{\Omega} \kappa \nabla(\xi^{(1)} - \xi_e^{(1)}) \cdot \nabla v = \int_{\Omega} \{\bar{G}^{(1)} - \kappa \nabla \xi_e^{(1)}\} \cdot \nabla v\,, \quad \forall\, v \in W_0^{1,2}(\Omega)\,. \tag{123}$$

Here again, the right-hand of (123) extends by continuity to an element of the space $[W_0^{1,q_0'}(\Omega)]^*$, and the regularity follows from the same fundamental result in [HDKRS08]. □

For the regularity of $\xi^{(2)}$ and $\xi^{(3)}$, we need to state a further properties valid on the surface Γ. We introduce a weighted space (cp. (19))

$$\begin{aligned} V^q_\alpha(\Gamma) &:= \{u \in W^{1/q',q}(\Gamma) \,:\, f_d(\alpha,\, A)\, u \in W^{1/q',q}(\Gamma)\}\,, \\ \|u\|_{V^q_\alpha(\Gamma)} &:= \|u\|_{W^{1/q',q}(\Gamma)} + \|f_d(\alpha,\, A)\, u\|_{W^{1/q',q}(\Gamma)}\,. \end{aligned} \tag{124}$$

Lemma 6.3. *Let $u \in W^{1,2}(\Omega)$ be the weak solution to (P). Assume that the hypotheses of Theorem 6.1 are valid. If u is associated to (P_N), then there are $\tilde{Q}_1,\, \tilde{Q}_2 \in W^{1/q_0',q_0}(\Gamma)$ such that*

$$\xi^{(2)} = -f_d(\alpha, A)\, (\xi^{(3)} + \tilde{Q}_1) - \tilde{Q}_2 \quad \textit{a. e. on } \Gamma \tag{125}$$

$$f_d(\alpha,\, A)\, (\xi^{(3)} + \tilde{Q}_1) \in W^{1/2,2}(\Gamma)\,. \tag{126}$$

If u is associated with (P_D), then there is $\tilde{U}_2 \in W^{1/q_0',q_0}(\Gamma)$ such that

$$\xi^{(3)} = f_d(\alpha, A)\, (\xi^{(2)} + U_1) + \tilde{U}_2 \quad \textit{a. e. on } \Gamma\,, \tag{127}$$

$$f_d(\alpha,\, A)\, (\xi^{(2)} + U_1) \in W^{1/2,2}(\Gamma)\,. \tag{128}$$

Proof. The relations (125) and (127) are easy consequences of Lemma 4.5 and of the convergence in Proposition 5.6. We recall the notations (18). Defining $\tilde{Q}_1,\, \tilde{Q}_2,\, \tilde{U}_2$ as accumulation points of the sequences $\tilde{Q}_{1,\rho},\, \tilde{Q}_{2,\rho},\, \tilde{U}_{2,\rho}$, the representations derived in Lemma 4.4 yield

$$\begin{aligned} &\tilde{Q}_1 = Q_1 + g_1\, \xi^{(1)} \\ &\tilde{Q}_2 = \gamma^+(Q_2) + \frac{a_1^{3,3}}{m_1^{1,1}}\, Q/\sin\alpha - f_d(\alpha,\, A_1)\, \tilde{Q}_1 + \frac{m_1^{2,1}}{m_1^{1,1}}\, \xi^{(1)} \\ &\tilde{U}_2 = \gamma^+(U_2) + a_1^{3,1}\, \xi_e^{(1)} - a_1^{3,3}\, (\tau^{(2)} \cdot \nabla u_e)/\sin\alpha - f_d(\alpha,\, A_1)\, U_1\,. \end{aligned}$$

Thus, the assumptions on $Q_1,\, Q_2, U_1,\, U_2$, the result of Proposition 6.2, and the property (B.1), are sufficient to verify the $W^{1/q_0',q_0}$ regularity of $\tilde{Q}_1,\, \tilde{Q}_2,\, \tilde{U}_2$.

Since $\xi^{(2)} + \tilde{Q}_2 \in W^{1/2,2}(\Gamma)$, (125) directly proves (126). The proof of (128) is completely similar. ☐

Lemma 6.4. *For $u \in V^2_\alpha(\Gamma)$ and $v \in W^{2,2}(\Omega)$, define the bilinear form*

$$(B(u),\, v) = -\int_\Gamma f_d(\alpha,\, A)\, u\, (\tau^{(2)} \cdot \nabla v)\,. \tag{129}$$

Then, B extends by density to an element of $\mathcal{L}(V^2_\alpha(\Gamma),\, [W^{1,2}(\Omega)]^*)$, and for $2 \le q_0 \le 6$, the inequalities

$$(B(u),\, (u-m)^+) \le \tilde{c}\, (1+g_0)\, \|u\|_{L^{2\,q_0/3}(\Gamma)}\, \|\nabla(u-m)^+\|_{L^{q_0'}(\Omega)} \tag{130}$$

$$(B(u),\, (u+m)^-) \le \tilde{c}\, (1+g_0)\, \|u\|_{L^{2\,q_0/3}(\Gamma)}\, \|\nabla(u+m)^-\|_{L^{q_0'}(\Omega)} \tag{131}$$

are valid for all $u \in W^{1,2}(\Omega)$ such that $u \in V^2_\alpha(\Gamma)$, and for all $m \in \mathbb{N}$. Here $\tilde{c} := c\, k_0^{-1}$ for (P_N), and $\tilde{c} := c\, k_1$ for (P_D).

Proof. For $u \in V^2_\alpha(\Gamma)$ and $v \in W^{2,2}(\Omega)$, Lemma C.1 implies the inequality

$$|(B(u),\, v)| \leq c\, g_0\, \|f_d(\alpha,\, A)\, u\|_{W^{1/2,2}(\Gamma)}\, \|\nabla v\|_{L^2(\Omega)}\,, \tag{132}$$

so that B extends by density to an element of $\mathcal{L}(V^2_\alpha(\Gamma),\, [W^{1,2}(\Omega)]^*)$.

For $u \in W^{2,2}(\Gamma)$ such that $u \in V^2_\alpha(\Gamma)$, and for $m \in \mathbb{N}$ (cp. (102))

$$(B(u),\, (u-m)^+) = \lim_{\rho\to 0}(B_\rho(u),\, (u-m)^+),\, . \tag{133}$$

The inequalities (130) and (131) therefore immediately follows from (103) and (104) and Hölder's inequality. Due to the density Lemma B.4 these inequalities remain valid for all $u \in W^{1,2}(\Omega)$ such that $u \in V^2_\alpha(\Gamma)$. ☐

Proposition 6.5. *Same assumptions as in Theorem 6.1 for the problem (P_N). Then $\xi^{(3)}$ belongs to $L^\infty(\Omega)$ with estimate*

$$\sup_\Omega |\xi^{(3)}| \leq c\, \tilde{N}_{q_0}\,. \tag{134}$$

Proof. Denote $w := \xi^{(3)} + \tilde{Q}_1$. Passing to the limit $\rho \longrightarrow 0$ in the relation (116) for test functions $v \in W^{2,2}(\Omega)$, it follows that

$$\begin{aligned}\int_\Omega [m^{1,1}]^{-1}\, \tilde{\kappa}\nabla w \cdot \nabla v = & \int_\Omega \{\bar{G}^{(3)} + M^{(3)}\nabla\xi^{(1)} + [m^{1,1}]^{-1}\, \tilde{\kappa}\nabla\tilde{Q}_1\} \cdot \nabla v \\ & - \int_\Gamma (\cot\alpha\, \frac{a^{3,3}}{m^{1,1}} + \frac{a^{3,2}}{m^{1,1}})\, w\, (\tau^{(2)} \cdot \nabla v) - \int_\Gamma \tilde{Q}_2\, (\tau^{(2)} \cdot \nabla v)\,.\end{aligned} \tag{135}$$

In view of Lemma 6.4, (135) is equivalent to

$$\begin{aligned}\int_\Omega (m^{1,1})^{-1}\, \tilde{\kappa}\nabla w \cdot \nabla v = & \int_\Omega \{\bar{G}^{(3)} + M^{(3)}\nabla\xi^{(1)} + [m^{1,1}]^{-1}\, \tilde{\kappa}\nabla\tilde{Q}_1\} \cdot \nabla v \\ & + (B(w),\, v) + F^{(2)}_{\tilde{Q}_2}(v)\,,\end{aligned}$$

where the choices $v := (w-m)^+$ and $v := (w+m)^-$ are possible for all $m \in \mathbb{N}$. The claim follows using Lemma C.4, in connection with the estimates (130), (131), (99), as well as (81) and the Proposition 6.2. ☐

Proposition 6.6. *Let the hypotheses of Theorem 6.1 for the problem (P_D) be valid. Then $\xi^{(2)}$ belongs to $L^\infty(\Omega)$ and satisfies the estimate*

$$\sup_\Omega |\xi^{(2)}| \leq c\, \tilde{N}_{q_0}\,. \tag{136}$$

Proof. Define $w := \xi^{(2)} + U_1$. Passage to the limit in the relation (120) for test functions $v \in W^{2,2}_{\Gamma_2}(\Omega)$, and Lemma 6.4 yield

$$\begin{aligned}\int_\Omega \tilde{\kappa}\nabla w \cdot \nabla v = & \int_\Omega \{\bar{G}^{(2)} + (a^{1,3}\, T^{(1)} - a^{1,1}\, n_S) \times \nabla\xi^{(1)} + \tilde{\kappa}\nabla U_1\} \cdot \nabla v \\ & + (B(w),\, v) + F^{(2)}_{\tilde{U}_2}(v)\,.\end{aligned} \tag{137}$$

We finish the proof as in Proposition 6.5. ☐

We are now able to finish the proof of Theorem 6.1.

Proof of Theorem 6.1. We first consider the case of the boundary condition (3). Due to the Propositions 6.2 and 6.5, $\xi^{(1)}$, $\xi^{(3)}$ are globally bounded in the domain Ω. The relation (125) and the triangle inequality yield

$$\sup_{\Gamma} |\xi^{(2)}| \leq \frac{k_1}{k_0^2} \sup_{\Omega} (|\xi^{(3)}| + |\tilde{Q}_1|) + \|\tilde{Q}_2\|_{L^\infty(\Gamma)} \,. \tag{138}$$

On the other hand, we can pass to the limit in the relation (82) to see that $\xi^{(2)} \in W^{1,2}(\Omega)$ satisfies, for all $v \in W^{1,2}_0(\Omega)$

$$\int_\Omega \tilde{\kappa} \nabla \xi^{(2)} \cdot \nabla v = \int_\Omega \{\bar{G}^{(2)} + (a^{3,1}\, T^{(1)} - a^{1,1}\, n_S) \times \nabla \xi^{(1)}\} \cdot \nabla v \,.$$

Lemma C.4 implies that

$$\|\xi^{(2)}\|_{L^\infty(\Omega)} \leq \sup_{\Gamma} |\xi^{(2)}| + c\,(\|\bar{G}^{(2)}\|_{L^{q_0}(\Omega)} + \|\nabla \xi^{(1)}\|_{L^{q_0}(\Omega)})\,,$$

and the claim follows from the estimate (138) and Proposition 6.2.

In the case of the boundary condition (4), the Propositions 6.2 and 6.6 yield the global boundedness of the components $\xi^{(1)}$, $\xi^{(2)}$. Using the relation (127) and the triangle inequality,

$$\sup_{\Gamma} |\xi^{(3)}| \leq k_1 \sup_{\Omega} (|\xi^{(2)}| + |U_1|) + \|\tilde{U}_2\|_{L^\infty(\Gamma)})\,, \tag{139}$$

and the claim follows from (83) and Proposition 6.2. ☐

7 $W^{2,p}-$ regularity

This section is essentially devoted to the proof of the Theorem 2.3. In the case that the compatibility condition (24) is violated, it is still possible to prove the existence of second weak derivatives for the weak solution to (P). This is based on the following observation.

Lemma 7.1. *Let $g \in C^1(\mathbb{R})$ be nonnegative and nondecreasing, and assume moreover that $M_g := \int_{-\infty}^{+\infty} |t|\, g'(t)\, dt < \infty$. Then the mapping B_ρ of Lemma 5.3 satisfies for all $u \in W^{1,2}(\Omega)$ the inequality*

$$(B_\rho(u),\, g(u)) \leq c\, M_g \,. \tag{140}$$

Proof. For $t \in \mathbb{R}$, define $G(t) := \int_0^t s\, g'(s)\, ds$. The function G is by assumption bounded by the number M_g, and for $u \in W^{2,2}(\Omega)$ arbitrary, the identity

$$(B_\rho(u),\, g(u)) = \int_\Gamma f_d(\alpha,\, A_\rho)\, \tau^{(2)} \cdot \nabla G(u)\,, \tag{141}$$

is valid. For the $(P_N)-$case of (17), integration by parts yields (cp. (107))

$$(B_\rho(u),\, g(u)) = \int_\Gamma (\cot\alpha\,\tau^{(2)}\cdot\nabla\frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \tau^{(2)}\cdot\nabla\frac{a^{2,3}_\rho}{m^{1,1}_\rho})\, G(u)$$
$$+ \int_\Gamma (\mathrm{div}_\Gamma(\cot\alpha\,\tau^{(2)})\,\frac{a^{3,3}_\rho}{m^{1,1}_\rho} + \mathrm{div}_\Gamma(\tau^{(2)})\,\frac{a^{2,3}_\rho}{m^{1,1}_\rho})\, G(u)\,. \tag{142}$$

Observe that under the assumptions of the present Lemma

$$\rho^{-1}\int_{\{x\in\Gamma\,:\,\mathrm{dist}(x,\Gamma\cap S)\leq\rho\}} G(u) \leq M_g\,\rho^{-1}\,\mathrm{meas}(\{x\in\Gamma\,:\,\mathrm{dist}(x,\Gamma\cap S)\leq\rho\})$$
$$\to M_g\,\mathrm{meas}(\Gamma\cap S)\,. \tag{143}$$

Arguing as in (108), (110), the inequality (140) follows. The arguments for (P_D) are completely similar. In Lemma 5.3, we have already proved that the mapping B_ρ extends by density to an element of $\mathcal{L}(W^{1,2}(\Omega),\,[W^{1,2}(\Omega)]^*)$. In view of the continuity property (105), the inequality (140) is valid for all $u\in W^{1,2}(\Omega)$. □

Proof of Theorem 2.3. For $\delta\in]0,1[$, consider the function

$$g_\delta(t) := \mathrm{sign}(t)\,(1-\frac{1}{(1+|t|)^{1+\delta}})\,. \tag{144}$$

Then, $g_\delta'(t) = (1+\delta)\,(1+|t|)^{-2-\delta}$, and it follows that $M_{g_\delta}<\infty$. We consider the relation (116) in the case of (P_N). In the case of (P_D), we start from (120) and the arguments are completely similar. In (116), choose $v := g_\delta(w_\rho)$ as the test function. Using in particular Lemma 7.1, we can prove that there is C independent of ρ such that

$$\int_\Omega (m^{1,1}_\rho)^{-1}\,g_\delta'(w_\rho)\,\tilde{\kappa}_\rho\nabla w_\rho\cdot\nabla w_\rho \leq C\,. \tag{145}$$

It is to note here that the uniform bounds on $\bar{G}^{(3)}_\rho$ (Lemma 4.3), on $\tilde{Q}_{2,\rho}$ (Lemma 4.4) and on $\nabla\xi^{(1)}_\rho$ (Prop. 5.2) are still valid since obtained independently of the condition (24). Denote $h_\delta(t) := \int_0^t\sqrt{g_\delta'(s)}\,ds$. The function h_δ is globally bounded, and the inequality (145) shows that there is $\tilde{C}$ independent of ρ such that $\|\nabla h_\delta(w_\rho)\|_{L^2(\Omega)}\leq\tilde{C}$. Therefore, $h_\delta(w_\rho)\to\chi\in W^{1,2}(\Omega)$. Moreover, using Lemma 3.1 and Lemma 4.4, we can show that $\chi = h_\delta(w)$, where $w = \xi^{(3)}+\tilde{Q}_1$. Using the lower semicontinuity of the norm, the latest and (145) yield

$$\int_\Omega g_\delta'(w)\,|\nabla w|^2 \leq \tilde{C}\,. \tag{146}$$

Let $p<2$. Then, Hölder's inequality and (146) imply that

$$\int_\Omega |\nabla w|^p \leq \left(\int_\Omega g_\delta'(w)\,|\nabla w|^2\right)^{p/2}\,\left(\int_\Omega |g_\delta'(w)|^{-p/(2-p)}\right)^{(2-p)/2}$$
$$\leq \tilde{C}^{2/p}\,\left(\int_\Omega |1+|w||^{p\,(2+\delta)/(2-p)}\right)^{(2-p)/2}\,. \tag{147}$$

The main Theorem of [HDKRS08] implies, via arguments similar to Lemma A.1, that there is $q_0 > 3$ such that the weak solution to (P) satisfies $u \in W^{1,q_0}(\Omega)$. This yields $\xi^{(3)} \in L^{q_0}(\Omega)$. Thanks to Lemma 4.4, $\tilde{Q}_1 \in L^6(\Omega)$. Therefore, $w \in L^s(\Omega)$, $s = \min\{q_0, 6\}$. If $p < 2\,s/(s+2)$, then there is $\delta > 0$ such that the right-hand of (147) is finite, which implies that $\nabla w \in L^p(\Omega)$. We obtain that $\xi^{(3)} \in L^s(\Omega) \cap W^{1,p}(\Omega)$. Due to Lemma 4.1, also $\xi^{(2)} \in L^s(\Omega) \cap W^{1,p}(\Omega)$. Therefore, $\nabla u \in W^{1,p}(\Omega_i)$ for $i = 1, 2$. □

A An auxiliary regularity result

Lemma A.1. *Let $F \in [L^{q_0}(\Omega)]^3$ with $3 < q_0 \leq 3 + \delta$ (δ = a positive constant defined in the paper [HDKRS08]). Assume that $u \in W^{1,2}(\Omega)$ satisfies*

$$\int_\Omega \kappa \nabla u \cdot \nabla v = \int_\Omega F \cdot \nabla v \,, \quad \forall\, v \in W^{1,2}(\Omega)\,. \tag{148}$$

Then u belongs to $W^{1,q_0}(\Omega)$, and it satisfies the estimate

$$\|u\|_{W^{1,q_0}(\Omega)} \leq c\left(\|F\|_{L^{q_0}(\Omega)} + c_S\left\{\|F\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right\}\right). \tag{149}$$

The constant c depends on Ω, κ_0 and κ_1. The constant c_S depends on the surface S only upon its $\mathcal{C}^1-$norm, and on the matrices κ_i ($i = 1, 2$) upon their $C-$norm.

Proof. For simplicity, we only prove the regularity in a neighbourhood D ($D \subset \mathbb{R}^3$ open) of the curve $\Gamma \cap S$, which is clearly the challenging point. For $x_0 \in \Gamma \cap S$, there are, due to the definition of $\mathcal{C}^2$ surfaces, a neighbourhood U of x_0 and a C^2-Diffeomorphism ϕ maps U onto the unit cube Q_1, and such that $\phi(x_0) = 0$, $\phi(\Gamma \cap U) =]-1, 1[\times\{0\}\times]-1, 1[$ and $\phi(S \cap U) =]-1, 1[\times]-1, 1[\times\{0\}$. Define $\psi := \phi^{-1}$.

To attain the model configuration of the paper [HDKRS08], consider for $0 < r < 1$ a prism $P_r := \triangle_r\times]-r, r[\subset Q_1$, where $\triangle_r$ is an equilateral triangle with sidelength $= r$, and with its base located in the line $]-1, 1[\times\{0\}\times\{0\}$. Denote, $\Gamma_r := \partial P_r \cap]-1, 1[\times\{0\}\times]-1, 1[$, and $\Sigma_r := \partial P_r \setminus \Gamma_r$. Due to the choice of P_r, there is $r_0 = r_0(S, \Gamma)$ such that $\psi(P_r) \subset U$ for all $r \leq r_0$.

Transforming the formula (148), we obtain that

$$\int_{P_r} \mu \nabla \tilde{u} \cdot \nabla \tilde{v} = \int_{P_r} \tilde{F} \cdot \nabla \tilde{v}\,, \quad \forall \tilde{v} \in W^{1,2}_{\Sigma_r}(P_r)\,. \tag{150}$$

where $\tilde{u} = u \circ \psi$, and μ is the piecewise Lipschitz continuous, symmetric, and uniformely positive definite matrix $|\det \psi'|\,(\psi')^{-1}\,\kappa\,(\psi')^{-T}$, and $\tilde{F}$ is the vector field $|\det \psi'|\,(\psi')^{-T}\,F$.

Introduce the (in P_r) piecewise constant matrix μ^0 such that $\mu^0_i := \mu_i(0)$ for $i = 1, 2$. If $w \in W^{1,2}_{\Sigma_r}(P_r)$, satisfies

$$\int_{P_r} \mu^0 \nabla w \cdot \nabla \tilde{v} = \int_{P_r} \tilde{F} \cdot \nabla \tilde{v}, \quad \forall \tilde{v} \in W^{1,2}_{\Sigma_r}(P_r)\,, \tag{151}$$

the Theorem 1.2 in [HDKRS08] implies that there is a constant $c_0 = c_0(\mu^0)$ such that

$$\|w\|_{W^{1,q_0}(P_r)} \leq c_0 \, \|\tilde{F}\|_{L^{q_0}(P_r)} \, . \tag{152}$$

(The independence of c_0 on r is easy to check: use the transformation $\Psi_r(x) := r\,x$ from the unit prism P_1 onto P_r, and apply on P_1 the Theorem 1.2 of [HDKRS08]).

It has been shown for instance in [ERS07] that the Banach perturbation arguments implies the existence of a positive $r_0 = r_0(\mu)$, such that for all $r \leq r_0$, and for $\tilde{u}$ satisfying (150)

$$\|\nabla\tilde{u}\|_{W^{1,q_0}(P_r)} \leq \frac{c_0}{1 - c_0\, f(r)} \, (\|\bar{F}\|_{L^{q_0}(P_r)} + \frac{1}{r}\, \{\|\tilde{F}\|_{L^2(P_r)} + \|\nabla\tilde{u}\|_{L^2(P_r)}\}) \, ,$$

where $f(r) := \|\mu - \mu_0\|_{L^\infty(P_r)}$.

The maximal allowed size of r depends only on the surfaces $S, \, \Gamma$, and on the uniform continuity of the matrices κ_i, so that finite covering of a neighbourhood of the curve $\Gamma \cap S$ is possible. ☐

B Auxiliary propositions concerning trace spaces

We at first note a useful elementary property of the spaces $W^{1/q',q}(\Gamma)$.

Lemma B.1. *Let $1 \leq q \leq \infty$ arbitrary. If $u \in W^{1/q',q}(\Gamma)$ and $g \in C^{0,1}(\Gamma)$, then $g\,u$ belongs to $W^{1/q',q}(\Gamma)$, and there is a constant $c = c(q, \Gamma)$ such that*

$$\|g\,u\|_{W^{1/q',q}(\Gamma)} \leq c_q \, \|g\|_{C^{0,1}(\Gamma)} \, \|u\|_{W^{1/q',q}(\Gamma)}$$

Proof. For $q = \infty$ the claim is obvious. Otherwise, the triangle inequality implies that

$$\|g\,u\|^q_{W^{1/q',q}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|u(x)\,g(x) - u(y)\,g(y)|^q}{|x-y|^{2+q/q'}} \, dy\,dx$$
$$\leq \int_\Gamma |u(x)|^q \, (\int_\Gamma \frac{|g(x) - g(y)|^q}{|x-y|^{2+q/q'}} dy) dx + \int_\Gamma |g(x)|^q \, (\int_\Gamma \frac{|u(x) - u(y)|^q}{|x-y|^{2+q/q'}} dy) dx \, .$$

Define $\tilde{c}_q := \sup_{x\in\Gamma}(\int_\Gamma |x-y|^{-1}\, dy)^{1/q}$. Due to Lipschitz continuity of g, it follows that

$$\|g\,u\|_{W^{1/q',q}(\Gamma)} \leq \tilde{c}_q \, \|\nabla g\|_{L^\infty(\Gamma)} \, \|u\|_{L^q(\Gamma)} + \|g\|_{L^\infty(\Gamma)} \, \|u\|_{W^{1/q',q}(\Gamma)} \, ,$$

and (B.1) follows easily. ☐

The following Lemma states basic properties of the spaces $V^q(\Gamma)$, and of the operators γ^+ and γ^- (cp. (18)).

Lemma B.2. *Let $\mu \in L^\infty(\Omega)$ be piecewise Lipschitz continuous, that is, $\mu := \mu_i \in C^{0,1}(\Omega_i)$ for $i = 1, 2$. Then,*

(1) The mapping $u \mapsto \mu\, u$ is continuous from $V^q(\Gamma)$ into $W^{1/q',q}(\Gamma)$ for all $1 \leq q \leq \infty$.

(2) Define $d(x) := \mathrm{dist}(x,\, \Gamma\cap S)$ for $x \in \Gamma$. For $1 \le q < \infty$, a function $u \in W^{1/q',q}(\Gamma)$ belongs to $V^q(\Gamma)$ if, and only if, $\|\frac{u}{d^{1/q'}}\|_{L^q(\Gamma_1)} < \infty$.

Proof. (1): On Γ, one has $\mu\, u = \mu_1\, \gamma^-(u) + \mu_2\, \gamma^+(u)$. Due to Lemma (B.1) and the triangle inequality, it follows that

$$\begin{aligned}\|\mu\, u\|_{W^{1/q',q}(\Gamma)} &\le \|\mu_1\, \gamma^-(u)\|_{W^{1/q',q}(\Gamma)} + \|\mu_2\, (u-\gamma^-(u))\|_{W^{1/q',q}(\Gamma)}\\ &\le c\, (\|\mu_1\|_{W^{1,\infty}(\Gamma)} + \|\mu_2\|_{W^{1,\infty}(\Gamma)})\, \|u\|_{V^q(\Gamma)}\,. \end{aligned} \tag{153}$$

(2): The definition of γ^- implies that

$$\begin{aligned}\|\gamma^-(u)\|^q_{W^{1/q',q}(\Gamma)} &= \int_{\Gamma_1}\int_{\Gamma_1} \frac{|u(x)-u(y)|^q}{|x-y|^{2+q/q'}} dxdy + 2\int_{\Gamma_1} |u(x)|^q\, \bar{d}_{\Gamma_1}(x)^{q/q'}\, dx\,,\\ \bar{d}_{\Gamma_1}(x) &:= (\int_{\Gamma_2} |x-y|^{-(2+q/q')}\, dy)^{q'/q}, \quad x\in\Gamma_1\,.\end{aligned}$$

There are constants $c_1,\, c_2$ such that $c_1\, d(x) \le \bar{d}_{\Gamma_1}(x) \le c_2\, d(x)$ on Γ_1, proving the claim. ☐

Remark B.3. *The operators of extension by zero in spaces $W^{s,q}$ ($s\in\mathbb{R}$, $q\in[1,\infty]$) have been extensively studied in. The elements of the space $V^q(\Gamma)$ satisfy a critical decay property $u/d^{1/q'} \in L^q(\Gamma_1)$ (cp. [LM61], Cor. 5.1). In the case $q = 2$, it is possible to relate the space $V^2(\Gamma)$ to the space $W^{1/2,2}_{00}$.*

In the following Lemma, we note a density property of the space $V^q_\alpha(\Gamma)$ (cp. (124)).

Lemma B.4. *Assume that $u \in V^2_\alpha(\Gamma)$. Then, there is a sequence $\{v_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})\cap V^2_\alpha(\Gamma)$ such that $v_k \to u$ in $V^2_\alpha(\Gamma)$.*

Proof. We first show some preliminaries. With the abbreviation $\mu := f_d(\alpha,\, A)$, the definition of V^2_α implies that

$$\|u\|_{V^2_\alpha(\Gamma)} = \|u\|_{W^{1/2,2}(\Gamma)} + \|\mu\, u\|_{W^{1/2,2}(\Gamma)}\,,$$

and since $\mu\, u = \mu_1\, \gamma^-(u) + \mu_2\, \gamma^+(u)$, it follows that

$$\|u\|_{V^2_\alpha(\Gamma)} \le \|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1\, \gamma^-(u)\|_{W^{1/2,2}(\Gamma)} + \|\mu_2\, \gamma^+(u)\|_{W^{1/2,2}(\Gamma)}\,. \tag{154}$$

Lemma B.2, (2) and LemmaB.1 yield

$$\begin{aligned}\|\mu_1\, \gamma^-(u)\|_{W^{1/2,2}(\Gamma)} &\le \|\mu_1\, u\|_{W^{1/2,2}(\Gamma_1)} + \|\mu_1\, u/d^{1/2}\|_{L^2(\Gamma_1)}\\ &\le c\, \|\mu_1\|_{C^{0,1}(\Gamma)}\, \|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1\, u/d^{1/2}\|_{L^2(\Gamma_1)}\,.\end{aligned}$$

With similar arguments, it follows from (154) that

$$\|u\|_{V^2_\alpha(\Gamma)} \le c_1\, (\|u\|_{W^{1/2,2}(\Gamma)} + \|\mu_1\, u/d^{1/2}\|_{L^2(\Gamma_1)} + \|\mu_2\, u/d^{1/2}\|_{L^2(\Gamma_2)})\,. \tag{155}$$

To start the proof of the approximation property, consider at first the truncature $T_k(u) := \mathrm{sign}(u)\,\min\{|u|,\,k\}$, at level $k \in \mathbb{N}$. Due to dominated convergence, note that

$$\|\mu_1\,(T_k(u)-u)/d^{1/2}\|^2_{L^2(\Gamma_1)} = \int_{\{x\in\Gamma:|u(x)|>k\}} \frac{\mu_1^2\,u^2}{d} \to 0\,.$$

Since it is well-known that $T_k(u) \to u$ in $W^{1,2}(\Omega)$, respectively in $W^{1/2,2}(\Gamma)$, the inequality (155) shows that $T_k(u) \to u$ in $V^2_\alpha(\Gamma)$ as $k \to \infty$. Therefore, there is no loss of generality in assuming $u \in V^2_\alpha(\Gamma) \cap L^\infty(\Gamma)$.

Since $\mu \in L^\infty(\Gamma)$, (155) implies immediately that

$$\|u\|_{V^2_\alpha(\Gamma)} \le c_2\,\|u\|_{V^2(\Gamma)}, \quad V^2(\Gamma) \subseteq V^2_\alpha(\Gamma)\,. \tag{156}$$

In the second step, we prove that $V^2(\Gamma)$ is dense in $V^2_\alpha(\Gamma)$.

For $k \in \mathbb{N}$, we choose a Lipschitz continuous function $\psi_k \in C^{0,1}(\overline{\Omega})$ such that

$$\psi_k(x) \begin{cases} = 1 & \text{if } \mathrm{dist}(x,\,\Gamma\cap S) > 1/k\,,\\ \in [0,\,1] & \text{if } 1/2k \le \mathrm{dist}(x,\,\Gamma\cap S) \le 1/k\,,\\ = 0 & \text{if } \mathrm{dist}(x,\,\Gamma\cap S) < 1/2k\,,\end{cases}$$
$$|\nabla\psi_k| \le k, \quad \text{supp}(\nabla\psi_k) \subseteq \{x\in\Omega\,:\,\mathrm{dist}(x,\,\Gamma\cap S) \le 1/k\}\,. \tag{157}$$

Then the sequence $\{\psi_k\,u\}$ is uniformly bounded in $W^{1,2}(\Omega)$ and in $W^{1/2,2}(\Gamma)$, since

$$\|u\,\nabla\psi_k\|_{L^2(\Omega)} \le \|u\|_{L^\infty(\Omega)}\,k\ \mathrm{meas}(\text{supp}(\nabla\psi_k))^{1/2} \le C\,,$$

Since also $\psi_k\,|u| \le |u|$ on Γ, the inequality (155) shows that the sequence $\{\psi_k\,u\}$ is uniformly bounded in $V^2_\alpha(\Gamma)$ as well. Due to the Hilbert space structure of $V^2_\alpha(\Gamma)$, $\psi_k\,u \rightharpoonup u$ in $V^2_\alpha(\Gamma)$ for a subsequence.

Weak and strong closure are identical for convex sets (an argument sometimes called Mazur's Lemma), and we can extract a sequence of convex combinations of $\psi_k\,u$ that strongly converges to u in V^2_α.

In the third step, we show that $C^\infty(\overline{\Omega}) \cap V^2(\Gamma)$ is dense in $V^2(\Gamma)$. If $\tilde{u} \in V^2(\Gamma)$, then the extension by zero on S (same denotation) satisfies $\tilde{u} \in W^{1/2,2}(\partial\Omega_i)$ for $i = 1, 2$. Therefore, via extension into Ω with the trace theorem, there is a sequence $\{\zeta_k\} \subset C^\infty_c(\Omega\setminus S)$ such that $\zeta_k \to \tilde{u}$ in $W^{1,2}(\Omega_i)$. Thus, with the argument of Lemma B.2, (2)

$$\|\gamma^-(\zeta_k - \tilde{u})\|_{W^{1/2,2}(\Gamma)} = \|\zeta_k - \tilde{u}\|_{W^{1/2,2}(\partial\Omega_1)} \to 0\,,$$

establishing the density in V^2.

For $\epsilon > 0$, there is thanks to the first and second steps of this proof a $\tilde{u}_\epsilon \in V^2(\Gamma)$, such that $\|u - \tilde{u}_\epsilon\|_{V^2_\alpha(\Gamma)} \le \epsilon$. Due to the third step, there is $\zeta_\epsilon \in C^\infty(\overline{\Omega}) \cap V^2(\Gamma)$ such that $\|\zeta_\epsilon - \tilde{u}_\epsilon\|_{V^2(\Gamma)} \le \epsilon$. It follows from (156) that

$$\|u - \zeta_\epsilon\|_{V^2_\alpha(\Gamma)} \le \|u - \tilde{u}_\epsilon\|_{V^2_\alpha(\Gamma)} + \|\zeta_\epsilon - \tilde{u}_\epsilon\|_{V^2_\alpha(\Gamma)} \le (1+c_2)\,\epsilon\,,$$

proving the approximation property. ☐

Lemma B.5. *Let $\mu_\rho := L_\rho(\mu)$ denote the approximation* (34) *of the coefficient μ. Assume that $u \in V^2(\Gamma)$. Then, $\lim_{\rho\to 0} \|\mu_\rho\, u\|_{W^{1/2,2}(\Gamma)} = \|\mu\, u\|_{W^{1/2,2}(\Gamma)}$.*

Proof. In the step three of the proof of Lemma B.4, we have proved that $C_c^\infty(\Omega \setminus S)$ is dense in $V^2(\Gamma)$. Let $u_k \in C_c^\infty(\Omega \setminus S)$, $u_k \to u$ in $V^2(\Gamma)$. For k fixed and ρ sufficiently small, we have

$$\mu_\rho\, u_k = \mu\, u \quad \text{on } \Gamma, \qquad \|\mu_\rho\, u_k\|_{V^2(\Gamma)} = \|\mu\, u\|_{V^2(\Gamma)}\,.$$

Since $u_k \to u$ in $V^2(\Gamma)$, then $\mu\, u_k \to \mu\, u$ in $W^{1/2,2}(\Gamma)$ due to Lemma B.2, (1), and the claim follows. ☐

C Some useful properties

Lemma C.1. *Let $1 \le q \le \infty$ arbitrary, let $g \in W^{1,q}(\Omega)$, and let $\tau \in \{\tau^{(1)},\, \tau^{(2)}\}$ where $\tau^{(i)}$ is defined by* (9). *For $v \in W^{2,q'}(\Omega)$,*

$$\int_\Gamma g\,(\tau\cdot\nabla v) = \int_\Omega \mathrm{curl}(g\,(\tau\times n_\Gamma))\cdot\nabla v\,, \tag{158}$$

$$\left|\int_\Gamma g\,(\tau\cdot\nabla v)\right| \le (g_0\,\|g\|_{L^q(\Omega)} + \|\nabla g\|_{L^q(\Omega)})\,\|\nabla v\|_{L^{q'}(\Omega)}\,. \tag{159}$$

Proof. The representation (158) follows from integration by parts. The estimate (159) is obvious due to Hölder's inequality. ☐

Lemma C.2. *For all $u \in W^{1,2}(\Omega)$, we have the estimate*

$$\|u\|^2_{L^2(\Gamma)} \le c_0\,\|\nabla u\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}\,. \tag{160}$$

Proof. The inequality (160) is proved in [LU68], Ch. 2, Paragraph 2. ☐

The proof of the following Lemma follows from elementary vector identities.

Lemma C.3. *Let $T^{(1)},\, T^{(2)}$ be given by* (10)*, and let $\tau^{(1)},\, \tau^{(2)}$ be given by* (9)*. Then, we have on Γ*

$$T^{(1)}\cdot\tau^{(1)} = 1, \quad T^{(1)}\cdot\tau^{(2)} = 0, \quad T^{(1)}\cdot n_\Gamma = 0\,, \tag{161}$$

$$T^{(2)}\cdot\tau^{(1)} = 0, \quad T^{(2)}\cdot\tau^{(2)} = \cos\alpha, \quad T^{(2)}\cdot n_\Gamma = \sin\alpha \tag{162}$$

$$n_S\cdot\tau^{(1)} = 0, \quad n_S\cdot\tau^{(2)} = -\sin\alpha, \quad n_S\cdot n_\Gamma = \cos\alpha\,. \tag{163}$$

Lemma C.4. *Let $q_0 > 3$ be an arbitrary real number, and let $m_0 \in \mathbb{N}$. For all $m\mathbb{N}$ such that $m \ge m_0$, let $u \in W^{1,2}(\Omega)$ satisfy*

$$\int_\Omega |\nabla(u-m)^+|^2 \le K\,\|\nabla(u-m)^+\|_{L^{q_0'}(\Omega)}\,, \tag{164}$$

Then, there is a consant c depending on Ω such that $\sup_\Omega u \le m_0 + c\,K$. Under the same conditions, let u satisfy

$$\int_\Omega |\nabla(u+m)^-|^2 \le K \, \|\nabla(u+m)^-\|_{L^{q_0'}(\Omega)}\,, \tag{165}$$

Then, $\inf_\Omega u \ge -m_0 - c\,K$ with a constant c depending on Ω.

Proof. Lemma C.4 follows from a (nowadays classical) Lemma by G. Stampacchia [Sta65]. Complements to the original proof are to find, for instance, in [Tro87], Ch. 2, Section 2.3. Similar results were obtained in [LU68], Ch. 3, Paragraph 13. ☐

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