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Existence results for diffuse interface models describing phase separation and damage

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Abstract

In this paper, we analytically investigate multi-component Cahn-Hilliard and Allen-Cahn systems which are coupled with elasticity and uni-directional damage processes. The free energy of the system is of the form $\int_\Omega \frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e,c,z)\,\mathrm{d}x$ with a polynomial or logarithmic chemical energy density W^{ch}, an inhomogeneous elastic energy density W^{el} and a quadratic structure of the gradient of the damage variable z. For the corresponding elastic Cahn-Hilliard and Allen-Cahn systems coupled with uni-directional damage processes, we present an appropriate notion of weak solutions and prove existence results based on certain regularization methods and a higher integrability result for the strain e.

1 Introduction

Phase separation and damage are common phenomena in many fields, including material sciences, hydrodynamics, biology and chemical reactions. Such microstructural processes take place to reduce the total free energy, which may include bulk chemical energy, interfacial energy and elastic strain energy.

The knowledge of the mechanisms inducing phase separation and damage processes is very important for technological applications as for instance in the area of micro-electronics due to the ongoing miniaturization. The materials used in this area are typically alloys consisting of mixtures of several components (cf. [HCW91]).

Phase separation and damage processes are usually described by two separate models in the mathematical literature. To describe phase separation processes for alloys, phase-field models of Cahn-Hilliard and Allen-Cahn type coupled with elasticity are well adapted. On the other side, damage processes for standard materials are often modeled as unilateral processes within a gradient-theory. A phase-field approach which describes *both* phase separation and damage processes in an unifying model has been recently introduced in [HK11].

The main objective of this work is to prove under general assumptions existence results for multi-component systems where Cahn-Hilliard as well as Allen-Cahn equations are *coupled* with rate-dependent damage differential inclusions for elastic materials. We are interested in free energies of the system which may contain a chemical energy of *logarithmic* or *polynomial type*, an *inhomogeneous* elastic energy and a *quadratic term* of the gradient of the damage variable. To this end, we establish some regularization methods which enable us to show existence results for gradient terms $|\nabla z|^p$ of the damage variable z in the free energy even if the assumption $p > n$ (n space dimension) is dropped. In contrast to [MR06, HK11] now the physical meaningful term $|\nabla z|^2$ can be treated, cf. [Fre02]. In addition, we also provide a higher integrability result for the strain tensor. Therefore, chemical free energies may also have logarithmic structure such that we are not restricted to polynomial growth as in [HK11]. We focus on the modeling of rate-dependent damage processes but the results of this work can extended to rate-independent systems (i.e. the dissipation potential is homogeneous of degree one) by some modifications. In the following, we will introduce the model formally.

The elastic material we want to consider in this work is an N component alloy occupying a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. To account for phase separation, deformation and damage processes in *one* model, a state at a fixed time point is described by a triple (u, c, z) where $u : \Omega \to \mathbb{R}^n$ denotes the deformation, $c : \Omega \to \mathbb{R}^N$ the vector of the chemical concentrations and $z : \Omega \to \mathbb{R}$ the damage variable. The meaning of the variables and its governing evolutionary process is explained in the sequel more explicitly.

The mixture of the alloy is described by a phase field vector $c = (c_1, \ldots, c_N)$, where the elements c_k for $k = 1, \ldots, N$, denote the concentration of the component k. Therefore, we will restrict the state space for c to the physically meaningful condition $\sum_{j=1}^N c_j = 1$ in Ω. The constraint $c_k > 0$, $k = 1, \ldots, N$, in Ω is also used for logarithmic chemical potentials (see below).

If an alloy is cooled down below a critical temperature then usually spinodal decomposition and coarsening phenomena occur. Well established models for describing such effects are the Cahn-Hilliard and Allen-Cahn equations, which describe mass preserving and mass non-preserving phase separation in solids, cf. [Cah61, LC82, Gur96, AC79, CNC94] for modeling aspects. Analytical investigations of Cahn-Hilliard equations can be found in [BCD+02, Gar00, CMP00, Gar05a, Gar05b, BP05] and for Allen-Cahn equations in [BdS04, CP08, CGPGS10], respectively. The essential difference between these two equations is that that the Cahn-Hilliard equation is a fourth order parabolic evolutionary equation expressible as a H^{-1} gradient flow of the free energy with respect to c whereas the Allen-Cahn equation is a second order parabolic equation arising from an L^2 gradient flow. More precisely,

$$\begin{aligned} &\text{Allen-Cahn:} && \partial_t c = -\mathbb{M}\Big(- \operatorname{div}(\boldsymbol{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)\Big), \\ &\text{Cahn-Hilliard:} && \partial_t c = \operatorname{div}\Big(\mathbb{M}\nabla\Big(- \operatorname{div}(\boldsymbol{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)\Big)\Big). \end{aligned} \tag{1}$$

Here, W^{ch} denotes the chemical energy density, W^{el} the elastic energy density, $\mathbb{M}$ the mobility matrix satisfying $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1, \ldots, N$ and $\boldsymbol{\Gamma}$ the gradient energy tensor which is a fourth order symmetric and positive definite tensor, mapping matrices from $\mathbb{R}^{N\times n}$ into itself.

In this work, W^{ch} may be a chemical energy density of polynomial type, i.e. $W^{\mathrm{ch}}(c) = W^{\mathrm{ch,pol}}(c)$, or of logarithmic type, i.e. $W^{\mathrm{ch}}(c) = W^{\mathrm{ch,log}}(c)$ (see (A8)). Note that phase separation only arises if the matrix A in (A8) is non-positive definite since the first term in (A8) is convex.

Elastic behavior is modeled by a deformation variable u so that each material point $x \in \Omega$ from the reference configuration is located at $x + u(t, x)$. We use the assumption that the strain e is sufficiently small so that we can work with the linearized strain tensor given by $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$. In this work we will neglect inertia effects $\rho\ddot{u}$ and volume forces l so that the momentum balance equation from the continuum mechanics $\operatorname{div}(\sigma) + l = \rho\ddot{u}$ becomes a quasi-static force equation, i.e.

$$\operatorname{div}(\sigma) = 0. \tag{2}$$

The stress tensor σ is defined by $W^{\mathrm{el}}_{,e}$, i.e. as the derivative of the elastic energy with respect to the strain.

Analytical results for multi-component Cahn-Hilliard equations coupled with elastic deformations can be found in [Gar00] while Allen-Cahn systems with elasticity are studied in [BW05]. Numerical investigations are conducted in [GRW01] and finite element error estimates of Cahn-Hilliard and Allen-Cahn equations with logarithmic free energies are recently derived in [BM10]. It turns out that different elastic moduli of the phases in the mixture influence the rate of coarsening and the morphology of the phases decisively [DM00].

The damage process we want to consider in this paper is uni-directional, i.e. it can only increase in time and the material is not able to heal itself. The phase field variable z satisfying $0 \le z \le 1$ is interpreted as damage in a way that $z(x) = 1$ stands for a non-damaged and $z(x) = 0$ for a maximal damaged material point $x \in \Omega$. We assume that the damage in our model is *not complete* which means that a maximal damaged part has still elastic properties. These constraints lead to a differential inclusion formulation for the evolution of z which relates the energy dissipation of the system due to increasing damage with the derivative of the free energy with respect to z. More precisely, we consider the doubly nonlinear differential inclusion

$$0 \in \partial\rho(\partial_t z) - \Delta z + W^{\mathrm{el}}_{,z}(e(u), z) + \partial I_{[0,\infty)}(z). \tag{3}$$

The energy dissipation density due to damage progression is given by ρ where we use the ansatz

$$\rho(\dot{z}) = -\alpha\dot{z} + \frac{\beta}{2}|\dot{z}|^2 + I_{(-\infty,0]}(\dot{z})$$

with $\alpha, \beta > 0$. Because of the quadratic term $\frac{\beta}{2}|\dot{z}|^2$ the damage evolution is called rate-dependent whereas $\beta = 0$ would correspond to rate-independent systems. See [Mie05, EM06, MRZ10, MT10] for analytical results on rate-independent damage and numerical experiments (without phase separation). We also refer to [BSS05, FK09] for further analytical investigations of damage models. In comparison to [HK11] we use a gradient-of-damage theory with the Laplacian Δz in (3) instead of a p-Laplacian $\mathrm{div}(|\nabla z|^{p-2}\nabla z)$ with $p > n$.

In conclusion, the systems we would like to consider in this work are governed by (1), (2) and (3) and can be rewritten as

$$\begin{cases} \partial_t c = -\mathcal{S} w & \text{on } \Omega_T, \\ w = \mathbb{P}(-\mathrm{div}(\boldsymbol{\Gamma}\nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)) & \text{on } \Omega_T, \\ \mathrm{div}(\sigma) = 0 & \text{on } \Omega_T, \\ \partial\rho(\partial_t z) - \Delta z + W^{\mathrm{el}}_{,z}(e(u), z) + \partial I_{[0,\infty)}(z) \ni 0 & \text{on } \Omega_T, \end{cases} \qquad (S_0)$$

where w denotes the chemical potential. Here, the matrix $\mathbb{P}$ denotes the orthogonal projection of $\mathbb{R}^N$ onto the tangent space $T\Sigma = \{x \in \mathbb{R}^N \mid \sum_{k=1}^N x_k = 0\}$ of the affine plane $\Sigma := \{x \in \mathbb{R}^N \mid \sum_{l=1}^N x_k = 1\}$. The operator $\mathcal{S}$ determines whether we have an Allen-Cahn or a Cahn-Hilliard type diffusion of the system. More precisely,

$$\begin{aligned} &\text{Allen-Cahn:} && \mathcal{S} : L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N), && \mathcal{S}(f) := \mathbb{M}f, \\ &\text{Cahn-Hilliard:} && \mathcal{S} : H^1(\Omega;\mathbb{R}^N) \to \big(H^1(\Omega;\mathbb{R}^N)\big)^\star, && \mathcal{S}(f) := \langle \mathbb{M}\nabla f, \nabla\cdot\rangle_{L^2}, \end{aligned} \qquad (4)$$

In the Cahn-Hilliard case, the operator $\mathcal{S}$ is invertible when restricted to $\mathcal{S} : Y \to \mathcal{D}$ where the spaces Y and $\mathcal{D}$ are defined as

$$\begin{aligned} Y &:= \Big\{ c \in H^1(\Omega;\mathbb{R}^N) \mid \int_\Omega c = 0,\ \sum_{k=1}^N c_k = 0 \Big\}, \\ \mathcal{D} &:= \Big\{ c^\star \in \big(H^1(\Omega;\mathbb{R}^N)\big)^\star \mid \langle c^\star, c\rangle_{(H^1)^\star \times H^1} = 0 \text{ for all } c = d(x)(1,\dots,1), \\ &\qquad\qquad \text{where } d \in H^1(\Omega) \text{ and for all } c = e_k,\ k = 1,\dots N \Big\}. \end{aligned} \qquad (5)$$

We need to impose some restrictions to the mobility matrix $\mathbb{M}$. We assume that $\mathbb{M}$ is symmetric and positive definite on the tangent space $T\Sigma$. In addition, due to the constraint $\sum_{k=1}^N c_k = 1$, $\mathbb{M}$ has to satisfy the property $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1, \dots, N$. Note, that $\mathbb{M} = \mathbb{M}\,\mathbb{P}$.

We abbreviate $D_T := [0,T] \times D$ and $(\partial\Omega)_T := [0,T] \times \partial\Omega$, where $D \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$. The initial-boundary conditions of our systems are summarized as follows:

$$\begin{aligned} c(0) &= c^0 \text{ on } \Omega, & \sigma \cdot \overrightarrow{\nu} &= 0 \text{ on } (\partial\Omega)_T \setminus D_T, \\ z(0) &= z^0 \text{ on } \Omega, & \boldsymbol{\Gamma}\nabla c \cdot \overrightarrow{\nu} &= 0 \text{ on } (\partial\Omega)_T, \\ u &= b \text{ on } D_T, & \nabla z \cdot \overrightarrow{\nu} &= 0 \text{ on } (\partial\Omega)_T \end{aligned} \qquad \text{(IBC)}$$

and additionally for Cahn-Hilliard systems

$$\mathbb{M}\nabla w \cdot \overrightarrow{\nu} = 0 \quad \text{on } (\partial\Omega)_T, \qquad \text{(IBC)}$$

where $\overrightarrow{\nu}$ is the unit normal on $\partial\Omega$ pointing outward, b the boundary value function on the Dirichlet boundary D, $0 \le z_0 \le 1$ a.e. on Ω and $c^0 \in \Sigma \cap \mathbb{R}^N_{>0}$. In the following, we assume that b can be extended to $\overline{\Omega}_T$.

The paper is organized as follows: In Chapter 2, we introduce an appropriate notion of weak solutions for the system (S_0). To handle the differential inclusion rigorously, we adapt the concept of energetic solutions originally introduced in the context of rate-independent systems (see for instance [Mie05]) to phase separation systems coupled with rate-dependent damage. This approach was firstly presented in [HK11]. The main result and their assumptions are stated at the end of Chapter 2.

In Chapter 3, we prove existence of weak solutions for a regularization of system (S_0) expressed in classical formulation as

$$\left.\begin{cases}\partial_t c = -\mathcal{S}w & \text{on } \Omega_T,\\ w = \mathbb{P}(-\mathrm{div}(\mathbf{\Gamma}\nabla c) + W^{\mathrm{ch,pol}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u),c,z) + \varepsilon\partial_t c) & \text{on } \Omega_T,\\ \mathrm{div}(\sigma) + \varepsilon\mathrm{div}(|\nabla u|^2\nabla u) = 0 & \text{on } \Omega_T,\\ \partial\rho(\partial_t z) - \Delta z - \varepsilon\mathrm{div}(|\nabla z|^{p-2}\nabla z) + W^{\mathrm{el}}_{,z}(e(u),z) + \partial I_{[0,\infty)}(z) \ni 0 & \text{on } \Omega_T,\end{cases}\right\} \tag{S_ε}$$

where $W^{\mathrm{ch,pol}}$ and W^{el} satisfy certain polynomial growth conditions and $p > n$. The initial-boundary conditions are

$$\text{(IBC) with } (\sigma + \varepsilon|\nabla u|^2\nabla u)\cdot\overrightarrow{\nu} = 0 \text{ instead of } \sigma\cdot\overrightarrow{\nu} = 0. \tag{IBC$_\varepsilon$}$$

It turns out that the weak solutions of the regularized system have better regularity:
$c \in H^1(0,T;L^2(\Omega;\mathbb{R}^N))$, $\nabla u \in L^4(\Omega_T;\mathbb{R}^{n\times n})$ and $\nabla z \in L^p(\Omega_T,\mathbb{R}^n)$ (with $p > n$ as above). They are constructed by adapting the approximation techniques developed in [HK11].

The limit problem $\varepsilon \searrow 0$ for (S_ε) corresponding to (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,pol}}$ is solved in Chapter 4. The displacement field u obtained in this process has $H^1(\Omega;\mathbb{R}^n)$-regularity in the first instance. To establish existence results for chemical free energies of logarithmic type, we prove a higher integrability result for ∇u in Chapter 5, which is based on some ideas of [EL91, Gar00, Gar05b].

Finally, Chapter 6 is devoted to logarithmic free energies for the concentration c. Following the approach in [Gar00, Gar05b], we use a suitable regularization $W^{\mathrm{ch},\delta}$ with polynomial growth of the logarithmic free energy density $W^{\mathrm{ch,log}}$ to obtain a solution for (S_0). Using this regularization, the chemical components c_k become strictly positive in the limit.

The notation, we will use throughout this paper, are collected in the following.

Spaces and sets.

$W^{1,r}(\Omega;\mathbb{R}^n)$	standard Sobolev space
$W^{1,r}_+(\Omega)$	functions of $W^{1,r}(\Omega)$ which are non-negative almost everywhere
$W^{1,r}_-(\Omega)$	functions of $W^{1,r}(\Omega)$ which are non-positive almost everywhere
$W^{1,r}_D(\Omega;\mathbb{R}^n)$	functions of $W^{1,r}(\Omega;\mathbb{R}^n)$ which vanishes on $D \subseteq \partial\Omega$ in the sense of traces
$B_R(A)$	open neighborhood of $A \subseteq \mathbb{R}^n$ with thickness R
$Q_R(x_0)$	open cube $\{x \in \mathbb{R}^n \,\vert\, \Vert x - x_0\Vert_\infty < R\}$
$\{f = 0\}$	zero set $\{x \in \overline{\Omega} \,\vert\, f(x) = 0\}$ of a function $f \in W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$
Ω_T	the set $[0,T]\times\Omega$

Functions, operations and measures.

$[f]^+$	non-negative part of f, i.e. $\max\{0,f\}$
I_M	indicator function of a subset $M \subseteq X$
χ_M	characteristic function of a subset $M \subseteq X$
$W_{,e}$	classical derivative of a function W with respect to the variable e
$\langle A, f\rangle$	dual pairing of $A \in (W^{1,r}(\Omega;\mathbb{R}^n))^\star$ and $f \in W^{1,r}(\Omega;\mathbb{R}^n)$
$\partial^{\mathrm{Cl}}E$	generalized Clarke's subdifferential of E
$\mathrm{d}E$	Gâteaux differential of E
$p^\star$	Sobolev critical exponent $\frac{np}{n-p}$ for $n > p$

$\mathrm{diam}(Q)$	diameter of a subset $Q \subseteq \mathbb{R}^n$
$\mathcal{H}^n$	Hausdorff measure of dimension n
$\mathcal{L}^n$	Lebesgue measure of dimension n

2 Existence theorem

2.1 Weak formulation

The weak notion, we will derive in this section for the doubly nonlinear differential inclusion occurring in (S_0), is inspired by the concept of energetic solutions for rate-independent systems (see for instance [Mie05]). In the rate-independent setting the differential inclusion is formulated by a global stability condition and an energy inequality. In [HK11] we introduced an approach which uses an energy inequality and a variational inequality to handle the rate-dependence coming from the viscosity term $\beta|\partial_t z|^2$ in the damage dissipation density function ρ.

The Gâteaux-differentiable free energy $\tilde{\mathcal{E}} : H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times (H^1(\Omega) \cap L^\infty(\Omega)) \to \mathbb{R}$ and dissipation functional $\tilde{\mathcal{R}} : L^2(\Omega) \to \mathbb{R}$ of the system is given by

$$\begin{aligned} \tilde{\mathcal{E}}(u,c,z) &:= \int_\Omega \frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e,c,z)\,\mathrm{d}x, \\ \tilde{\mathcal{R}}(\dot{z}) &:= \int_\Omega -\alpha \dot{z} + \frac{\beta}{2}|\dot{z}|^2\,\mathrm{d}x. \end{aligned}$$

with viscosity constants $\alpha, \beta > 0$. To account for constraints, we extend the functionals $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{R}}$ above for analytical reasons by indicator functions:

$$\mathcal{E}(u,c,z) := \tilde{\mathcal{E}}(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x, \qquad \mathcal{R}(\dot{z}) := \tilde{\mathcal{R}}(\dot{z}) + \int_\Omega I_{(-\infty,0]}(\dot{z})\,\mathrm{d}x.$$

If we equip the space $H^1(\Omega) \cap L^\infty(\Omega)$ with the norm $\|\cdot\|_{H^1 \cap L^\infty} := \|\cdot\|_{H^1} + \|\cdot\|_{L^\infty}$ the generalized subdifferential $\partial_z^{\mathrm{Cl}}\mathcal{E}$ at a point $(u,c,z) \in H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times (H^1(\Omega) \cap L^\infty(\Omega))$ is

$$\partial_z^{\mathrm{Cl}}\mathcal{E}(u,c,z) = \left\{ \mathrm{d}_z\tilde{\mathcal{E}}(u,c,z) + r \in (H^1(\Omega) \cap L^\infty(\Omega))^\star \,\middle|\, r \in \partial I_{H^1_+(\Omega)\cap L^\infty(\Omega)}(z) \right\}. \tag{6}$$

The inclusion $L^1(\Omega) \subset (H^1(\Omega) \cap L^\infty(\Omega))^\star$ will be later used for the construction of a specific r. Using property (6), the differential inclusion in (S_0) can be rewritten in a weaker form as

$$0 \in \partial_z^{\mathrm{Cl}}\mathcal{E}(u(t),c(t),z(t)) + \partial_{\dot{z}}\mathcal{R}(\dot{z}(t)).$$

The analytical basis for a formulation of a weak solution is the following proposition (a proof of a related proposition can be found in [HK11]):

Proposition 2.1 *Let $(u,c,w,z) \in \mathcal{C}^2(\Omega_T;\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$ be a smooth solution satisfying* (1) *and* (2) *with initial-boundary conditions* (IBC)*. Then the following two conditions are equivalent:*

(i) $0 \in \partial_z^{\mathrm{Cl}}\mathcal{E}(u(t),c(t),z(t)) + \partial_{\dot{z}}\mathcal{R}(\dot{z}(t))$ *for all* $t \in [0,T]$,

(ii) the energy inequality

$$\mathcal{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot{z}}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \mathrm{d}s$$

$$\leq \mathcal{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s$$

for all $0 \leq t \leq T$ *and the variational inequality*

$$0 \leq \left\langle \mathrm{d}_z\tilde{\mathcal{E}}(u(t),c(t),z(t)) + r(t) + \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z(t)), \zeta \right\rangle$$

for all $\zeta \in H^1_-(\Omega)\cap L^\infty(\Omega)$ *and* $r(t) \in N_{\mathrm{F}}(H^1_+(\Omega)\cap L^\infty(\Omega); z(t))$ *and for all* $0 \leq t \leq T$.

If one of the two conditions holds then the following energy balance equation is satisfied:

$$\mathcal{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t z), \partial_t z\rangle\,\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \mathrm{d}s$$
$$= \mathcal{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s$$

Remarks on the proof. In contrast to [HK11], the energy inequality compares the energy at the beginning $s = 0$ with the energy at an arbitrary time $s = t$ instead of $s = t_1$ with $s = t_2$ for $0 \leq t_1 < t_2 \leq T$.

Applying the chain rule in the right hand side of

$$\mathcal{E}(u(t),c(t),z(t)) - \mathcal{E}(u(0),c(0),z(0)) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}(u(s),c(s),z(s))\,\mathrm{d}s$$

and using (1), (2) as well as the variational inequality, the "$\geq$"-part of the energy balance can be shown.

We will see that in our approach the mathematical analysis of (S_0) requires several ε-regularization terms (see (S_ε)) to establish the energy and variational inequality for the differential inclusion and to handle the logarithmic free energy. A transition to $\varepsilon \searrow 0$ will finally give us solution of the limit problem (S_0).

Proposition 2.1 can also be formulated for the regularized system (S_ε) with the regularized energy

$$\tilde{\mathcal{E}}_\varepsilon(u,c,z) := \int_\Omega \frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch,pol}}(c) + W^{\mathrm{el}}(e,c,z) + \frac{\varepsilon}{4}|\nabla u|^4 + \frac{\varepsilon}{p}|\nabla z|^p\,\mathrm{d}x,$$
$$\mathcal{E}_\varepsilon(u,c,z) := \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x,$$

and the initial-boundary conditions (IBC_ε). Notice that $\mathbb{P}\partial_t c = \partial_t c$ because of $\partial_t c(t,x) \in \Sigma$.

We can now give a weak notion of (S_ε) and (S_0). (The energy densities $W^{\mathrm{ch,pol}}$ and W^{el} will satisfy some polynomial growth conditions which are specified in the next subsection.)

Definition 2.2 (Weak solution for the regularized system (S_ε)) *We call a quadruple* $q = (u,c,w,z)$ *a weak solution of the regularized system* (S_ε) *with initial-boundary conditions* (IBC_ε) *if the following properties are satisfied:*

(i) the components of q *are in the following spaces:*

$$u \in L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n)),\ u|_{D_T} = b|_{D_T},$$
$$c \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^N)),\ c(0) = c^0,\ c \in \Sigma \text{ a.e. in } \Omega_T,$$
$$z \in L^\infty(0,T;W^{1,p}_+(\Omega)) \cap H^1(0,T;L^2(\Omega)),\ z(0) = z^0,\ \partial_t z \leq 0,$$

and

$$w \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \qquad \text{for C-H systems,}$$
$$w \in L^2(\Omega_T;\mathbb{R}^N) \qquad \text{for A-C systems}$$

(ii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ *and for a.e.* $t \in [0,T]$*:*

$$\int_\Omega \partial_t c(t)\cdot\zeta \,\mathrm{d}x = \begin{cases} \int_{\Omega_T} \mathbb{M}\nabla w(t) : \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t & \text{for C-H systems,} \\ \int_{\Omega_T} \mathbb{M}w(t)\cdot\zeta \,\mathrm{d}x\,\mathrm{d}t & \text{for A-H systems} \end{cases} \tag{7}$$

(iii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ *and for a.e.* $t \in [0,T]$*:*

$$\begin{aligned} \int_\Omega w(t)\cdot\zeta \,\mathrm{d}x = &\int_\Omega \mathbb{P}\mathbf{\Gamma}\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c(t))\cdot\zeta \,\mathrm{d}x \\ &+ \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)),c(t),z(t))\cdot\zeta + \varepsilon\partial_t c(t)\cdot\zeta \,\mathrm{d}x \end{aligned} \tag{8}$$

(iv) for all $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$ *and for a.e.* $t \in [0,T]$*:*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(\zeta) + \varepsilon|\nabla u(t)|^2\nabla u(t) : \nabla\zeta \,\mathrm{d}x = 0 \tag{9}$$

(v) for all $\zeta \in W^{1,p}_-(\Omega)$ *and for a.e.* $t \in [0,T]$*:*

$$\begin{aligned} &\int_\Omega (\varepsilon|\nabla z(t)|^{p-2}+1)\nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \\ &\quad\geq -\langle r(t),\zeta\rangle, \end{aligned} \tag{10}$$

where $r(t) \in (W^{1,p}(\Omega))^\star$ *satisfies* $\langle r(t), z(t)-\zeta\rangle \geq 0$ *for all* $\zeta \in W^{1,p}_+(\Omega)$

(vi) energy inequality for a.e. $t \in [0,T]$*:*

$$\begin{aligned} &\mathcal{E}_\varepsilon(u(t),c(t),z(t)) - \mathcal{E}_\varepsilon(u^0,c^0,z^0) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x \\ &+ \int_{\Omega_t} \beta|\partial_t z|^2 + \varepsilon|\partial_t c|^2 \,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \,\mathrm{d}s \\ &\qquad \leq \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s + \varepsilon\int_{\Omega_t} |\nabla u|^2\nabla u : \nabla\partial_t b \,\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{11}$$

where u^0 *the unique minimizer of* $\mathcal{E}_\varepsilon(\cdot,c^0,z^0)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$ *with trace* $b(0)|_D$*.*

With the help of the operator $\mathcal{S}$, the diffusion equation (7) can also be written as

$$\int_\Omega \partial_t c(t)\cdot\zeta \,\mathrm{d}x = -\langle \mathcal{S}w(t),\zeta\rangle,$$

which will be used in the following.

Definition 2.3 (Weak solution for the limit system (S_0)**)** *A quadruple* $q = (u,c,w,z)$ *is called a weak solution of the system* (S_0) *with initial-boundary conditions* (IBC) *if the following properties are satisfied:*

(i) the components of q *are in the following spaces:*

$$\begin{aligned} &u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)),\ u|_{D_T} = b|_{D_T}, \\ &c \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)),\ c \in \Sigma \text{ a.e. in } \Omega_T, \\ &z \in L^\infty(0,T;H^1_+(\Omega)) \cap H^1(0,T;L^2(\Omega)),\ z(0) = z^0,\ \partial_t z \leq 0 \end{aligned}$$

and

$$\begin{aligned} &w \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) && \text{for C-H systems,} \\ &w \in L^2(\Omega_T;\mathbb{R}^N) && \text{for A-C systems} \end{aligned}$$

(ii) for all $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ *with* $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ *and* $\zeta(T) = 0$*:*

$$\int_{\Omega_T} (c - c^0) \cdot \partial_t\zeta \,\mathrm{d}x = \int_0^T \langle \mathcal{S}w, \zeta\rangle \,\mathrm{d}t$$

(iii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ *and for a.e.* $t \in [0,T]$*:*

$$\begin{aligned}\int_\Omega w(t)\cdot\zeta \,\mathrm{d}x = \int_\Omega \mathbb{P}\boldsymbol{\Gamma}\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c(t))\cdot\zeta \,\mathrm{d}x \\ + \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)), c(t), z(t))\cdot\zeta \,\mathrm{d}x\end{aligned}$$

(iv) for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$ *and for a.e.* $t \in [0,T]$*:*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)), c(t), z(t)) : e(\zeta)\,\mathrm{d}x = 0$$

(v) for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ *and for a.e.* $t \in [0,T]$*:*

$$\int_\Omega \nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \geq -\langle r(t), \zeta\rangle,$$

where $r(t) \in (H^1(\Omega)\cap L^\infty(\Omega))^\star$ *satisfies* $\langle r(t), z(t) - \zeta\rangle \geq 0$ *for all* $\zeta \in H^1_+(\Omega)\cap L^\infty(\Omega)$

(vi) energy inequality for a.e. $t \in [0,T]$*:*

$$\begin{aligned}\mathcal{E}(u(t), c(t), z(t)) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x + \int_{\Omega_t} \beta|\partial_t z|^2 \,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathcal{S}w(s), w(s)\rangle \,\mathrm{d}s \\ \leq \mathcal{E}(u(0), c(0), z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

where u^0 *the unique minimizer of* $\mathcal{E}(\cdot, c^0, z^0)$ *in* $H^1(\Omega;\mathbb{R}^n)$ *with trace* $b(0)|_D$*.*

Note that both notions of weak solution imply chemical mass conservation, i.e.

$$\int_\Omega c(t)\,\mathrm{d}x \equiv const, \qquad c \in \Sigma \text{ a.e. in } \Omega_T.$$

2.2 Assumptions and main results

The general setting, the growth assumptions and the assumptions on the coefficient tensors which are mandatory for the existence theorems are summarized below.

(i) *Setting*

Space dimension	$n \in \mathbb{N}$,
Components in the alloy	$N \in \mathbb{N}$ with $N \geq 2$,
Regularization exponent	$p > n$,
Viscousity factors	$\alpha, \beta > 0$,
Domain	$\Omega \subseteq \mathbb{R}^n$ bounded Lipschitz domain,
Dirichlet boundary	$D \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$,
Time interval	$[0,T]$ with $T > 0$

(ii) *Energy densities*

Elastic energy density	$W^{\mathrm{el}} \in \mathcal{C}^1(\mathbb{R}^{n\times n} \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}_+)$ with
	$W^{\mathrm{el}}(e,c,z) = W^{\mathrm{el}}(e^t,c,z), \quad \text{(A1)}$
	$W^{\mathrm{el}}(e,c,z) \le C(\lvert e\rvert^2 + \lvert c\rvert^2 + 1), \quad \text{(A2)}$
	$\eta\lvert e_1 - e_2\rvert^2 \le (W^{\mathrm{el}}_{,e}(e_1,c,z) - W^{\mathrm{el}}_{,e}(e_2,c,z)) : (e_1 - e_2), \quad \text{(A3)}$
	$\lvert W^{\mathrm{el}}_{,e}(e_1+e_2,c,z)\rvert \le C(W^{\mathrm{el}}(e_1,c,z) + \lvert e_2\rvert + 1), \quad \text{(A4)}$
	$\lvert W^{\mathrm{el}}_{,c}(e,c,z)\rvert \le C(\lvert e\rvert^2 + \lvert c\rvert^2 + 1), \quad \text{(A5)}$
	$\lvert W^{\mathrm{el}}_{,z}(e,c,z)\rvert \le C(\lvert e\rvert^2 + \lvert c\rvert^2 + 1), \quad \text{(A6)}$
Chemical energy densities	$W^{\mathrm{ch,pol}}, W^{\mathrm{ch,log}} \in \mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$ with $W^{\mathrm{ch,pol}} \ge -C$,
	$\lvert W^{\mathrm{ch,pol}}_{,c}(c)\rvert \le C(\lvert c\rvert^{2^\star/2} + 1), \quad \text{(A7)}$
	$W^{\mathrm{ch,log}}(c) = \theta \sum_{k=1}^N c_k \log c_k + \frac{1}{2} c\cdot Ac,\ \theta > 0,\ A \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \quad \text{(A8)}$

(iii) *Tensors*

Mobility tensor	$\mathbb{M} \in \mathbb{R}^{N\times N}$ symmetric and positive definite on $T\Sigma$ and
	$\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1,\dots,N$,
Energy gradient tensor	$\mathbf{\Gamma} \in \mathcal{L}(\mathbb{R}^{N\times n};\mathbb{R}^{N\times n})$ symmetric and positive definite fourth order tensor

Remark 2.4 *Due to the effect of damage on the elastic response of the material,* W^{el} *is often modeled by the following ansatz:*

$$W^{\mathrm{el}} = (\Phi(z) + \tilde{\eta})\,\hat{W}^{\mathrm{el}},$$

where $\Phi : [0,1] \to \mathbb{R}_+$ *is a continuously differentiable and monotonically increasing function with* $\Phi(0) = 0$ *and* $\tilde{\eta} > 0$ *is a small value.*

A typically form of the elastically stored energy density $\hat{W}^{\mathrm{el}}$ *is as follows:*

$$\hat{W}^{\mathrm{el}}(c,e) = \frac{1}{2}\big(e - e^*(c)\big) : \mathbb{C}(c)\big(e - e^*(c)\big).$$

Here, $e^*(c)$ *denotes the eigenstrain, which is usually linear in* c*, and* $\mathbb{C}(c) \in \mathcal{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ *is a fourth order stiffness tensor, which is symmetric and positive definite. The elastic energy density is called homogeneous if the stiffness tensor does not depend on the concentration, i. e.* $\mathbb{C}(c) = \mathbb{C}$*.*

Note that due to the previous growth assumptions (A1)-(A6) *this type of energy fits into our setting. In particular, we are not confined to homogeneous elasticity as in [HK11]. There, the more restrictive growth condition* $\lvert W^{\mathrm{el}}_{,c}(e,c,z)\rvert \le C(\lvert e\rvert + \lvert c\rvert^2 + 1)$ *is used instead of* (A5)*.*

The main results of this work are summarized in the following theorems:

Theorem 2.5 (Existence theorem - polynomial case) *Let the above assumptions be satisfied. Then for every*

$$b \in W^{1,1}(0,T; W^{1,\infty}(\Omega;\mathbb{R}^n)),$$
$$c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ a.e. in } \Omega,$$

$$z^0 \in H^1(\Omega) \text{ with } 0 \le z^0 \le 1,$$

there exists a weak solution q of the system (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,pol}}$ and initial-boundary condition (IBC) *in the sense of Definition 2.3.*

Theorem 2.6 (Existence theorem - logarithmic case) *Let the above assumptions be satisfied and, additionally, let $D = \partial\Omega$ and $\mathbf{\Gamma} = \gamma\,\mathrm{Id}$ with a constant $\gamma > 0$. Then for every*

$$\begin{aligned}
&b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n)),\\
&c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ and } c_k^0 > 0 \text{ a.e. in } \Omega \text{ for } k = 1,\ldots,N,\\
&z^0 \in H^1(\Omega) \text{ with } 0 \le z^0 \le 1,
\end{aligned}$$

there exists a weak solution q of the system (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,log}}$ and initial-boundary condition (IBC) *in the sense of Definition 2.3. Additionally, $c_k > 0$ a.e. in Ω_T for $k = 1,\ldots,N$.*

3 Existence of weak solutions of (S_ε)

The proof is based on [HK11]. Arguments similar to [HK11] are only sketched.

Since $\varepsilon > 0$ is fixed in this section, we omit the ε-dependence in the notation, e.g. $\mathcal{E}$ always means here $\mathcal{E}_\varepsilon$ and so on. Furthermore, z^0 is assumed to be in $W^{1,p}(\Omega)$ in this section.

1. Step: constructing time-discrete solutions.

   Set u^0 to be a minimizer of $u \mapsto \mathcal{E}(u, c^0, z^0)$ defined on the space $W^{1,4}(\Omega)$ with the constraint $u|_D = b(0)|_D$ in the sense of traces.

   Let the closed subspace $\mathcal{Q}_M^m$ of $H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times W^{1,p}(\Omega)$ be defined by:

$$\mathcal{Q}_M^m = \left\{ \begin{array}{l|l} u \in H^1(\Omega;\mathbb{R}^n), & u|_D = b(m\tau)|_D, \\ c \in H^1(\Omega;\mathbb{R}^N), & \int_\Omega c - c^0 \,\mathrm{d}x = 0 \text{ for C-H systems}, \\ z \in W^{1,p}(\Omega) & 0 \le z \le z_M^{m-1}. \end{array} \right\}$$

   Based on the initial triple (u^0, c^0, z^0), we construct (u_M^m, c_M^m, z_M^m) for $m = 1,\ldots,M$ recursively by minimizing the following functional $\mathbb{E}_M^m : \mathcal{Q}_M^m \to \mathbb{R}$:

$$\mathbb{E}_M^m(u,c,z) := \tilde{\mathcal{E}}(u,c,z) + \tau\tilde{\mathcal{R}}\left(\frac{z - z_M^{m-1}}{\tau}\right) + \frac{\tau}{2}\left\|\frac{c - c_M^{m-1}}{\tau}\right\|_X^2 + \frac{\varepsilon\tau}{2}\left\|\frac{c - c_M^{m-1}}{\tau}\right\|_{L^2}^2, \tag{12}$$

   where X denotes the space $\mathcal{D}$ (see (5)) with the scalar-product

$$(c_1 \,|\, c_2)_X := \int_\Omega \mathbb{M}\nabla\mathcal{S}^{-1}c_1 \cdot \nabla\mathcal{S}^{-1}c_2 \,\mathrm{d}x$$

   for Cahn-Hilliard systems and $X = L^2(\Omega;\mathbb{R}^N)$ with the scalar-product

$$(c_1 \,|\, c_2)_X := \int_\Omega \mathbb{M}c_1 \cdot c_2 \,\mathrm{d}x$$

   for Allen-Cahn systems.

   Note that the last regularization term in (12) is not necessary for Allen-Cahn equations due to the term with the X-norm. To use a uniform approach, we consider this term in both systems. By direct methods in the calculus of variations the triple

$$(u_M^m, c_M^m, z_M^m) := \underset{(u,c,z)\in\mathcal{Q}_M^m}{\arg\min}\ \mathbb{E}_M^m(u,c,z)$$

exists, cf. [HK11]. Furthermore, we set

$$w_M^m := \begin{cases} -\mathcal{S}^{-1}\left(\frac{c_M^m - c_M^{m-1}}{\tau}\right) + \lambda_M^m & \text{for C-H systems,} \\ -\mathcal{S}^{-1}\left(\frac{c_M^m - c_M^{m-1}}{\tau}\right) & \text{for A-C systems,} \end{cases}$$

with the Lagrange multiplier λ_M^m (associated with the mass constraint for C-H systems) given by

$$\lambda_M^m := \fint_\Omega W_{,c}^{\mathrm{ch,pol}}(c_M^m) + W_{,c}^{\mathrm{el}}(e(u_M^m), c_M^m, z_M^m)\,\mathrm{d}x.$$

We define the time incremental solutions as

$$q_M^m := (u_M^m, c_M^m, w_M^m, z_M^m)$$

and introduce the piecewise constant interpolations q_M, q_M^-, t_M, t_M^- and the linear interpolation $\hat{q}_M$ as

$$\begin{aligned} t_M &:= \min\{m\tau \mid m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}, \\ t_M^- &:= \min\{(m-1)\tau \mid m \in \mathbb{N}_0 \text{ and } m\tau \geq t\}, \\ q_M(t) &:= q_M^m \text{ for } t \in \big((m-1)\tau, m\tau\big], \\ q_M^-(t) &:= q_M^m \text{ for } t \in \big[m\tau, (m+1)\tau\big), \\ \hat{q}_M(t) &:= \beta q_M^m + (1-\beta) q_M^{m-1} \text{ for } t \in \big[(m-1)\tau, m\tau\big) \text{ and } \beta = \frac{t}{\tau} - (m-1). \end{aligned}$$

Due to the minimization properties of (u_M^m, c_M^m, z_M^m), we establish the following variational formulas and energy estimate (cf. [HK11, Lemma 6.2]):

Lemma 3.1 (Euler-Lagrange equation, energy estimate) *The functions q_M, q_M^- and $\hat{q}_M$ satisfy the following properties for all $t \in (0,T)$:*

(i) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:

$$\int_\Omega (\partial_t \hat{c}_M(t)) \cdot \zeta \,\mathrm{d}x = -\langle \mathcal{S} w_M(t), \zeta\rangle \tag{13}$$

(ii) for all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:

$$\begin{aligned} \int_\Omega w_M(t)\cdot\zeta\,\mathrm{d}x = &\int_\Omega \mathbb{P}\mathbf{\Gamma}\nabla c_M(t) : \nabla\zeta + \mathbb{P}W_{,c}^{\mathrm{ch,pol}}(c_M(t))\cdot\zeta\,\mathrm{d}x \\ &+ \int_\Omega \mathbb{P}W_{,c}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\cdot\zeta + \varepsilon\partial_t\hat{c}_M(t)\cdot\zeta\,\mathrm{d}x \end{aligned} \tag{14}$$

(iii) for all $\zeta \in W_D^{1,4}(\Omega;\mathbb{R}^n)$:

$$\int_\Omega W_{,e}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t)) : e(\zeta) + \varepsilon|\nabla u_M(t)|^2\nabla u_M(t) : \nabla\zeta\,\mathrm{d}x = 0 \tag{15}$$

(iv) for all $\zeta \in W^{1,p}(\Omega)$ with $0 \leq \zeta + z_M(t) \leq z_M^-(t)$:

$$\begin{aligned} &\int_\Omega (\varepsilon|\nabla z_M(t)|^{p-2} + 1)\nabla z_M(t)\cdot\nabla\zeta + W_{,z}^{\mathrm{el}}(e(u_M(t)), c_M(t), z_M(t))\zeta\,\mathrm{d}x \\ &\quad + \int_\Omega (-\alpha + \beta(\partial_t\hat{z}_M(t)))\zeta\,\mathrm{d}x \geq 0 \end{aligned} \tag{16}$$

(v) *energy estimate:*

$$
\begin{aligned}
&\mathcal{E}(u_M(t),c_M(t),z_M(t))+\int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M+\frac{\beta}{2}|\partial_t\hat{z}_M|^2+\frac{\varepsilon}{2}|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s\\
&+\int_0^{t_M}\frac{1}{2}\langle\mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s-\mathcal{E}(u^0,c^0,z^0)\\
&\qquad\le\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^-+b-b_M^-),c_M^-,z_M):e(\partial_t b)\,\mathrm{d}x\mathrm{d}s\\
&\qquad\quad+\varepsilon\int_0^{t_M}\int_\Omega|\nabla u_M^-+\nabla b-\nabla b_M^-|^2\nabla(u_M^-+b-b_M^-):\nabla\partial_t b\,\mathrm{d}x\mathrm{d}s
\end{aligned}
\tag{17}
$$

2. Step: identifying convergent subsequences.

The energy estimate (v) in Lemma 3.1, growth condition (A4) and a Gronwall estimation argument lead to a-priori estimates for the energy $\mathcal{E}(u_M(t),c_M(t),z_M(t))$ and for $\|\partial_t\hat{z}_M\|_{L^2(\Omega_T)}$, $\|\partial_t\hat{c}_M\|_{L^2(\Omega_T)}$ and $\int_0^T\langle\mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s$. By standard compactness arguments and a compactness theorem from Aubin and Lions [Sim86], we deduce the following weak convergence properties, cf. [HK11]:

Lemma 3.2 *There exists a subsequence $\{M_k\}$ and an element $(u,c,w,z)=q$ satisfying (i) from Definition 2.2 such that for a.e. $t\in[0,T]$*

(i) $u_{M_k}\overset{\star}{\rightharpoonup}u$ *in* $L^\infty(0,T;W^{1,4}(\Omega))$,

(ii) $c_{M_k},c_{M_k}^-\overset{\star}{\rightharpoonup}c$ *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$,
$c_{M_k}(t),c_{M_k}^-(t)\rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$,
$c_{M_k},c_{M_k}^-\to c$ *a.e. in* Ω_T,
$\hat{c}_{M_k}\rightharpoonup c$ *in* $H^1(0,T;L^2(\Omega;\mathbb{R}^N))$,

(iii) $z_{M_k},z_{M_k}^-\overset{\star}{\rightharpoonup}z$ *in* $L^\infty(0,T;W^{1,p}(\Omega))$,
$z_{M_k}(t),z_{M_k}^-(t)\rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$,
$z_{M_k},z_{M_k}^-\to z$ *a.e. in* Ω_T,
$\hat{z}_{M_k}\rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

and

(iv) $w_{M_k}\rightharpoonup w$ *in* $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ *for C-H systems,*
$w_{M_k}\rightharpoonup w$ *in* $L^2(\Omega_T;\mathbb{R}^N)$ *for A-C systems.*

as $k\to\infty$.

Exploiting the Euler-Lagrange equations, we can even prove stronger convergence properties. To proceed, we recall an approximation lemma from [HK11].

Lemma 3.3 ([HK11, Lemma 5.2]) *Let $q\ge1$, $p>n$ and $f,\zeta\in L^q(0,T;W^{1,p}_+(\Omega))$ with $\{\zeta=0\}\supseteq\{f=0\}$. Furthermore, let $\{f_M\}_{M\in\mathbb{N}}\subseteq L^q(0,T;W^{1,p}_+(\Omega))$ be a sequence with $f_M(t)\rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M\to\infty$ for a.e. $t\in[0,T]$. Then there exists a sequence $\{\zeta_M\}_{M\in\mathbb{N}}\subseteq L^q(0,T;W^{1,p}_+(\Omega))$ and constants $\nu_{M,t}>0$ such that*

(i) $\zeta_M\to\zeta$ *in* $L^q(0,T;W^{1,p}(\Omega))$ *as* $M\to\infty$,

(ii) $\zeta_M\le\zeta$ *a.e. in* Ω_T *for all* $M\in\mathbb{N}$,

(iii) $\nu_{M,t}\zeta_M(t)\le f_M(t)$ *a.e. in* Ω *for a.e.* $t\in[0,T]$ *and for all* $M\in\mathbb{N}$.

If, in addition, $\zeta\le f$ a.e. in Ω_T then condition (iii) can be refined to

(iii)' $\zeta_M\le f_M$ *a.e. in* Ω_T *for all* $M\in\mathbb{N}$.

We are now able to prove strong convergence results by using uniform convexity estimates.

Lemma 3.4 (Strong convergence of the time incremental solutions) *There exists a subsequence $\{M_k\}$ such that for a.e. $t\in[0,T]$*

(i) $u_{M_k}, u^-_{M_k} \to u$ *in* $L^4(0,T;W^{1,4}(\Omega;\mathbb{R}^n))$,
$u_{M_k}(t), u^-_{M_k}(t) \to u(t)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$,
$u_{M_k}, u^-_{M_k} \to u$ *a.e. in* Ω_T,

(ii) $c_{M_k}, c^-_{M_k} \to c$ *in* $L^{2^\star}(0,T;H^1(\Omega;\mathbb{R}^N))$,
$c_{M_k}(t), c^-_{M_k}(t) \to c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$,
$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T,
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1(0,T;L^2(\Omega;\mathbb{R}^N))$,

(iii) $z_{M_k}, z^-_{M_k} \to z$ *in* $L^p(0,T;W^{1,p}(\Omega))$,
$z_{M_k}(t), z^-_{M_k}(t) \to z(t)$ *in* $W^{1,p}(\Omega)$,
$z_{M_k}, z^-_{M_k} \to z$ *a.e. in* Ω_T,
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

as $k \to \infty$.

Proof. We omit the index k in the proof.

(i) We refer to [HK11, Lemma 6.6].

(ii) By Lebesgue's convergence theorem and the weak convergence properties, we have $c_M \to c$ in $L^{2^\star/2+1}(\Omega_T;\mathbb{R}^N)$. Testing (14) with $\zeta = c_M(t)$ and with $\zeta = c(t)$ gives after integration from $t = 0$ to $t = T$:

$$\begin{aligned}\int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T} w_M \cdot c_M - \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot c_M \,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M)\cdot c_M + \varepsilon\partial_t \hat{c}_M \cdot c_M \,\mathrm{d}x\mathrm{d}t \\ \int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c \,\mathrm{d}x\mathrm{d}t &= \int_{\Omega_T} w_M \cdot c - \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot c \,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M)\cdot c + \varepsilon\partial_t \hat{c}_M \cdot c \,\mathrm{d}x\mathrm{d}t\end{aligned}$$

Passing to $M \to \infty$ and comparing the right sides of the equations shows

$$\int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} \mathbb{P}\mathbf{\Gamma}\nabla c : \nabla c \,\mathrm{d}x\mathrm{d}t.$$

By using the properties $\mathbb{P}\nabla c_M = \nabla c_M$ and $\mathbb{P}\nabla c = \nabla c$, we eventually obtain

$$\int_{\Omega} \mathbf{\Gamma}\nabla c_M : \nabla c_M \,\mathrm{d}x \to \int_{\Omega} \mathbf{\Gamma}\nabla c : \nabla c \,\mathrm{d}x.$$

We end up with

$$\int_{\Omega} \mathbf{\Gamma}(\nabla c_M - \nabla c) : (\nabla c_M - \nabla c) \,\mathrm{d}x \to 0.$$

Therefore $\nabla c_M \to \nabla c$ in $L^2(\Omega_T;\mathbb{R}^N)$ since $\mathbf{\Gamma}$ is positive definite.

(iii) Applying Lemma 3.3 with $f = z$ and $f_M = z^-_M$ and $\zeta = z$ gives an approximation sequence $\{\zeta_M\} \subseteq L^p(0,T;W^{1,p}_+(\Omega))$ with the properties:

$$\zeta_M \to z \text{ in } L^p(0,T;W^{1,p}(\Omega)), \tag{18a}$$

$$0 \le \zeta_M \le z^-_M \text{ for all } M \in \mathbb{N}. \tag{18b}$$

The estimate

$$C_{\mathrm{uc}}|\nabla z_M - \nabla z|^p \le (|\nabla z_M|^{p-2}\nabla z_M - |\nabla z|^{p-2}\nabla z)\cdot\nabla(z_M - z)$$

where $C_{\mathrm{uc}} > 0$ is a constant and equation (16) tested with $\zeta = \zeta_M(t) - z_M(t)$ (possible due to (18b)) yield:

$$C_{\mathrm{uc}}\int_{\Omega_T} \varepsilon|\nabla z_M - \nabla z|^p \,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} |\nabla z_M - \nabla z|^2 \,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&\le \int_{\Omega_T} \big((\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\big)\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\\
&\le \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(z_M-\zeta_M)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(\zeta_M - z) - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M-z)\,\mathrm{d}x\mathrm{d}t\\
&\le \int_{\Omega_T} (W^{\mathrm{el}}_{,\mathrm{z}}(e(u_M),c_M,z_M)-\alpha+\beta\partial_t\hat{z}_M)(\zeta_M - z_M)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{\Omega_T} (\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(\zeta_M - z) - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M-z)\,\mathrm{d}x\mathrm{d}t\\
&\le \underbrace{\|W^{\mathrm{el}}_{,\mathrm{z}}(e(u_M),c_M,z_M)-\alpha+\beta\partial_t\hat{z}_M\|_{L^2(\Omega_T)}}_{\text{bounded}}\|\zeta_M - z_M\|_{L^2(\Omega_T)}\\
&\quad + \underbrace{(\varepsilon\|\nabla z_M\|^{p-1}_{L^p(\Omega_T)} + \|\nabla z_M\|_{L^{p/(p-1)}(\Omega_T)})}_{\text{bounded}}\|\nabla\zeta_M - \nabla z\|_{L^p(\Omega_T)}\\
&\quad - \int_{\Omega_T} (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Due to (18a) and $z_M \overset{\star}{\rightharpoonup} z$ in $L^\infty(0,T;W^{1,p}(\Omega))$ as well as $z_M \to z$ in $L^2(\Omega_T)$, each term on the right hand side converges to 0 as $M \to \infty$. ■

3. STEP: ESTABLISHING A PRECISE ENERGY INEQUALITY.

In this step we establish an asymptotic energy inequality, which is sharper than the energy inequality in (17). Note, that compared to (17) the factor $1/2$ in front of $\langle \mathcal{S}w_M(s), w_M(s)\rangle$ is missing. To simplify notation, we omit the index k in the following.

Lemma 3.5 *For every $t \in [0,T]$:*

$$
\begin{aligned}
&\mathcal{E}(u_M(t),c_M(t),z_M(t)) + \int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\mathrm{d}s\\
&+ \int_0^{t_M}\langle\mathcal{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s - \mathcal{E}(u^0,c^0,z^0)\\
&\qquad\le \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-),c_M^-,z_M):e(\partial_t b)\,\mathrm{d}x\mathrm{d}s\\
&\qquad\quad + \varepsilon\int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-):\nabla\partial_t b\,\mathrm{d}x\mathrm{d}s + \kappa_M
\end{aligned}
$$

with $\kappa_M \to 0$ as $M \to \infty$.

Proof. Applying the estimate $\mathbb{E}_M^m(q_M^m) \le \mathbb{E}_M^m(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^m)$ for $m = 1$ to $\frac{t_M}{\tau}$ yields (cf. [HK11, Lemma 6.10]):

$$
\begin{aligned}
&\mathcal{E}(u_M(t),c_M(t),z_M(t)) - \mathcal{E}(u^0,c^0,z^0)\\
&\quad\le \varepsilon\int_0^{t_M}\int_\Omega |\nabla(u_M^- + b(s) - b_M^-)|^2\nabla(u_M^- + b(s) - b_M^-):\nabla\partial_t b(s)\,\mathrm{d}x\mathrm{d}s\\
&\quad\quad + \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-),c_M^-,z_M^-):e(\partial_t b)\,\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$+\underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M^- + b_M - b_M^-), \hat c_M, z_M^-)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{(\star)_1}$$

$$+\underbrace{\int_0^{t_M}\int_\Omega \mathbf{\Gamma}\nabla\hat c_M : \nabla\partial_t\hat c_M + W^{\mathrm{ch,pol}}_{,c}(\hat c_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{(\star)_2}$$

$$+\underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,z}(e(u_M^- + b_M - b_M^-), c_M, \hat z_M)\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{(\star\star)_1}$$

$$+\underbrace{\int_0^{t_M}\int_\Omega \varepsilon|\nabla\hat z_M|^{p-2}\nabla\hat z_M\cdot\nabla\partial_t\hat z_M + \nabla\hat z_M\cdot\nabla\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{(\star\star)_2} \tag{19}$$

The elementary inequalities

$$(|\nabla\hat z_M|^{p-2}\nabla\hat z_M - |\nabla z_M|^{p-2}\nabla z_M)\cdot\nabla\partial_t\hat z_M \le 0 \quad\text{and}\quad (\nabla\hat z_M - \nabla z_M)\cdot\nabla\partial_t\hat z_M\le 0$$

and (16) tested with $\zeta := -\partial_t\hat z_M(t)\tau$ leads to the estimate:

$$\begin{aligned}&(\star\star)_1 + (\star\star)_2\\ &\quad\le -\int_0^{t_M}\int_\Omega -\alpha\partial_t\hat z_M + \beta|\partial_t\hat z_M|^2\,\mathrm{d}x\mathrm{d}s\\ &\qquad+\underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{el}}_{,z}(e(u_M^- + b_M - b_M^-), c_M, \hat z_M) - W^{\mathrm{el}}_{,z}(e(u_M), c_M, z_M))\partial_t\hat z_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^3}\end{aligned}$$

Furthermore,

$$\begin{aligned}(\star)_1 \le{}& \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s\\ &+\underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{el}}_{,c}(e(u_M^- + b_M - b_M^-), \hat c_M, z_M^-) - W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M))\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^1}\end{aligned}$$

Using the elementary estimate $\mathbf{\Gamma}(\nabla\hat c_M - \nabla c_M) : \nabla\partial_t\hat c_M \le 0$ gives

$$\begin{aligned}(\star)_2 \le{}& \int_0^{t_M}\int_\Omega \mathbf{\Gamma}\nabla c_M : \nabla\partial_t\hat c_M + W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s\\ &+\underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{ch,pol}}_{,c}(\hat c_M) - W^{\mathrm{ch,pol}}_{,c}(c_M))\cdot\partial_t\hat c_M\,\mathrm{d}x\mathrm{d}s}_{=:\kappa_M^2}.\end{aligned}$$

Hence, applying equations (14) with $\zeta = \partial_t\hat c_M(t)$ and (13) with $\zeta = w_M(t)$ by noticing $\mathbb{P}\partial_t\hat c_M(t) = \partial_t\hat c_M(t)$ shows

$$(\star)_1 + (\star)_2 \le -\int_0^{t_M}\langle\mathcal{S}w_M(s), w_M(s)\rangle\,\mathrm{d}s - \int_0^{t_M}\int_\Omega \varepsilon|\partial_t\hat c_M|^2\,\mathrm{d}x\mathrm{d}s + \kappa_M^1 + \kappa_M^2.$$

Lebesgue's generalized convergence theorem, growth conditions (A5)-(A7) and Lemma 3.4 shows $\kappa_M := \kappa_M^1 + \kappa_M^2 + \kappa_M^3 \to 0$ as $M\to\infty$. We would like to emphasize that we need the boundedness of ∇u_M in $L^4(\Omega_T;\mathbb{R}^{n\times n})$ and the boundedness of $\partial_t\hat c_M$ and $\partial_t\hat z_M$ in $L^2(\Omega_T)$ with respect to M.■

4. Step: passing to $M \to \infty$. Using Lemma 3.2, Lemma 3.4 and (13), (14) and (15) we establish (ii), (iii) and (iv) of Definition 2.2. Moreover, Lemma 3.5 implies

$$\begin{aligned}
&\mathcal{E}(u_M(t), c_M(t), z_M(t)) + \int_{\Omega_t} -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \varepsilon|\partial_t \hat{c}_M|^2 \,\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \langle \mathcal{S}w_M(s), w_M(s)\rangle \,\mathrm{d}s - \mathcal{E}(u^0, c^0, z^0) \\
&\qquad \le \int_0^{t_M} \int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b) \,\mathrm{d}x\mathrm{d}s \\
&\qquad\quad + \varepsilon \int_0^{t_M} \int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla\partial_t b \,\mathrm{d}x\mathrm{d}s + \kappa_M.
\end{aligned}$$

The energy estimate (vi) from Definition 2.2 follows from above by using the known convergence properties and weakly semi-continuity arguments.

It remains to show (v) of Definition 2.2. To proceed, we cite the following lemma from [HK11] which provides a tool to drop a restriction in the space of test-function for a variational inequality of a specific form.

Lemma 3.6 ([HK11, Lemma 5.3]) *Let $f \in L^p(\Omega;\mathbb{R}^n)$, $g \in L^p(\Omega)$ and $z \in W^{1,p}(\Omega)$ with $f \cdot \nabla z \ge 0$ and $\{f = 0\} \supseteq \{z = 0\}$ a.e.. Furthermore, we assume that*

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \ge 0 \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega) \textit{ with } \{\zeta = 0\} \supseteq \{z = 0\}.$$

Then

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \ge \int_{\{z=0\}} [g]^+\zeta \,\mathrm{d}x \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega).$$

We are now able to prove the remaining property.

Lemma 3.7 *It holds*

$$\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \\
&\qquad \ge -\langle r(t), \zeta\rangle, \qquad (20)
\end{aligned}$$

for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in [0,T]$, where $r(t) \in L^1(\Omega) \subseteq (W^{1,p}(\Omega))^\star$ is given by

$$r(t) := -\chi_{\{z(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+. \qquad (21)$$

Proof. First of all, we take any test-function $\zeta \in L^p(0,T;W^{1,p}_-(\Omega))$ with $\{\zeta = 0\} \supseteq \{z = 0\}$. Lemma 3.3 gives a sequence $\{\zeta_M\} \subseteq L^p(0,T;W^{1,p}_-(\Omega))$ with $\zeta_M \to \zeta$ in $L^p(0,T;W^{1,p}(\Omega))$ and $0 \ge \nu\zeta_M(t) \ge -z_M(t)$ where ν depends on M and t. Therefore (16) holds for $\zeta = \zeta_M(t)$. Integration from 0 to T and passing to $M \to \infty$ gives

$$\int_{\Omega_T} (\varepsilon|\nabla z|^{p-2} + 1)\nabla z \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u), c, z) - \alpha + \beta(\partial_t z))\zeta \,\mathrm{d}x\mathrm{d}t \ge 0.$$

In other words,

$$\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))\zeta \,\mathrm{d}x \\
&\qquad + \int_\Omega (-\alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \ge 0
\end{aligned}$$

holds every $\zeta \in W^{1,p}_-(\Omega)$ with $\{\zeta = 0\} \supseteq \{z(t) = 0\}$ and a.e. $t \in [0,T]$. To finish the proof, we need to extend the variational inequality to the whole space $W^{1,p}_-(\Omega)$.

Setting $f = (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t)$ and $g = W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))$, Lemma 3.6 shows for every $\zeta \in W^{1,p}_-(\Omega)$

$$\begin{aligned}\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \\ \geq \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))]^+\zeta\,\mathrm{d}x \\ \geq \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+\zeta\,\mathrm{d}x.\end{aligned}$$

Now, variational inequality (20) follows by setting

$$r(t) := -\chi_{\{z(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+.$$

■

Remark 3.8 *Lemma 3.7 gives more information than (v) from Definition 2.2. It provides a special choice for $r(t)$ given by* (21).

4 Existence of weak solutions of (S_0) - polynomial case

In this chapter, we show that an appropriate subsequence of the regularized solutions q_ε for $\varepsilon \in (0,1]$ of Definition 2.2 converges in "some sense" to q which satisfies the limit equations given in Definition 2.3. Beside that, the initial damage profile z^0 in this chapter is in $H^1(\Omega)$. We approximate $z^0 \in H^1(\Omega)$ by a sequence $\{z^0_\varepsilon\}$ in $W^{1,p}(\Omega)$ such that $z^0_\varepsilon \to z^0$ in $H^1(\Omega)$ as $\varepsilon \searrow 0$.

Using the energy inequality and Gronwall's inequality, we establish again the following energy estimate.

Lemma 4.1 *It holds*

$$\begin{aligned}\mathcal{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) + \int_0^t\int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle \mathcal{S}w_\varepsilon(s), w_\varepsilon(s)\rangle\,\mathrm{d}s \\ \leq C(\mathcal{E}_\varepsilon(u^0_\varepsilon, c^0, z^0_\varepsilon) + 1)\end{aligned}$$

for a.e. $t \in [0,T]$ and every $\varepsilon \in (0,1]$.

Since $\mathcal{E}_\varepsilon(u^0_\varepsilon, c^0, z^0_\varepsilon) \leq \mathcal{E}_\varepsilon(u^0_1, c^0, z^0_\varepsilon) \leq \mathcal{E}_1(u^0_1, c^0, z^0_\varepsilon)$, the left hand side is also uniformly bounded with respect to a.e. $t \in [0,T]$ and every $\varepsilon \in (0,1]$. By using standard compactness theorems and uniform convexity properties of W^{el} (see (A3)), this results in the following convergence properties (cf. [HK11]).

Lemma 4.2 (Convergence properties of q_ε) *There exists a subsequence $\{\varepsilon_k\}$ with $\varepsilon_k \searrow 0$ as $k \to \infty$ and an element $(u,c,w,z) = q$ satisfying (i) of Definition 2.3 such that for a.e. $t \in [0,T]$*

(i) $u_{\varepsilon_k} \to u$ *in* $L^2(0,T;H^1(\Omega;\mathbb{R}^n))$,
$\sqrt[3]{\varepsilon_k}\nabla u_{\varepsilon_k} \to 0$ *in* $L^\infty(0,T;L^4(\Omega;\mathbb{R}^n))$,
$u_{\varepsilon_k}(t) \to u(t)$ *in* $H^1(\Omega;\mathbb{R}^n)$,
$u_{\varepsilon_k} \to u$ *a.e. in* Ω_T,
$u^0_{\varepsilon_k} \to u^0$ *in* $H^1(\Omega;\mathbb{R}^n)$,
$\sqrt[3]{\varepsilon_k}\nabla u^0_{\varepsilon_k} \to 0$ *in* $L^4(\Omega;\mathbb{R}^n)$,

(ii) $c_{\varepsilon_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$,
$\varepsilon_k\partial_t c_{\varepsilon_k} \to 0$ *in* $L^2(\Omega_T;\mathbb{R}^N)$,
$c_{\varepsilon_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$,
$c_{\varepsilon_k} \to c$ *a.e. in* Ω_T,

(iii) $z_{\varepsilon_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0,T;H^1(\Omega))$,
$\sqrt[p-1]{\varepsilon_k}\nabla z_{\varepsilon_k} \to 0$ *in* $L^\infty(0,T;L^p(\Omega))$,
$z_{\varepsilon_k}(t) \rightharpoonup z(t)$ *in* $H^1(\Omega)$,
$z_{\varepsilon_k} \to z$ *a.e. in* Ω_T,
$z_{\varepsilon_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

as $k \to \infty$*. We additionally obtain for Cahn-Hilliard systems*

$$w_{\varepsilon_k} \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega;\mathbb{R}^N))$$

and for Allen-Cahn systems

$$\begin{aligned} &w_{\varepsilon_k} \rightharpoonup w \text{ in } L^2(\Omega_T;\mathbb{R}^N),\\ &c_{\varepsilon_k} \rightharpoonup c \text{ in } H^1(0,T;L^2(\Omega;\mathbb{R}^N)). \end{aligned}$$

as $k \to \infty$*.*

As usual, we omit the index k in the subscripts.

Remark 4.3 *We would like to mention that the arguments in [HK11, Lemma 6.14] cannot be adapted to prove strong convergence properties of* ∇c_ε *and* ∇z_ε *due to the more generous growth condition* (A5) *as well as the usage of Lemma 3.3 where the compact embedding* $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ *for* $p > n$ *is exploited.*

We are now able to establish existence of weak solutions of (S_0) in the polynomial case.

Proof of Theorem 2.5. Whenever we refer in the following to (7)-(11) the functions u, c, w, z and r are substituted by $u_\varepsilon, c_\varepsilon, w_\varepsilon, z_\varepsilon$ and r_ε. Moreover, Lemma 4.2 is used without mentioning in the following.

(i) Let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ with $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$. Integration from $t = 0$ to $t = T$ of (7) and integration by parts yields

$$\int_{\Omega_T} (c_\varepsilon - c^0)\cdot\partial_t\zeta \,\mathrm{d}x\mathrm{d}s = \int_0^T \langle \mathcal{S}w_\varepsilon, \zeta\rangle \,\mathrm{d}s.$$

Passing to $\varepsilon \searrow 0$ shows (ii) of Definition 2.3.

(ii) Let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \cap L^\infty(\Omega_T;\mathbb{R}^N)$. Integration from $t = 0$ to $t = T$ of (8) and passing to $\varepsilon \searrow 0$ yields

$$\int_{\Omega_T} w\cdot\zeta \,\mathrm{d}x\mathrm{d}s = \int_{\Omega_T} \mathbb{P}\boldsymbol{\Gamma}\nabla c : \nabla\zeta + (\mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u),c,z))\cdot\zeta \,\mathrm{d}x\mathrm{d}s.$$

Note that

$$\left|\int_{\Omega_T} \varepsilon\partial_t c_\varepsilon\cdot\zeta \,\mathrm{d}x\mathrm{d}s\right| \le \varepsilon\|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)}\|\zeta\|_{L^2(\Omega_T;\mathbb{R}^N)} \to 0$$

as $\varepsilon \searrow 0$. This shows (iii) of Definition 2.3 with $W^{\mathrm{ch}}_{,c} = W^{\mathrm{ch,pol}}_{,c}$.

(iii) Let $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$ be arbitrary. Passing to $\varepsilon \searrow 0$ in (9) yields for a.e. $t \in [0,T]$

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)), c(t), z(t)) : e(\zeta)\,\mathrm{d}x = 0, \tag{22}$$

by noticing

$$\left|\int_\Omega \varepsilon |\nabla u_\varepsilon(t)|^2 \nabla u_\varepsilon(t) : \nabla\zeta\,\mathrm{d}x\right| \le \varepsilon \|\nabla u_\varepsilon(t)\|^3_{L^4(\Omega)} \|\zeta\|_{L^4(\Omega)} \to 0.$$

A density argument shows that (22) also holds for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. Therefore, (iv) of Definition 2.3 is shown.

(iv) The characteristic functions $\chi_{\{z_\varepsilon = 0\}}$ are bounded in $L^\infty(\Omega_T)$ with respect to $\varepsilon \in (0,1]$. We select a subsequence such that $\chi_{\{z_{\varepsilon_k}=0\}} \overset{\star}{\rightharpoonup} \chi$ in $L^\infty(\Omega_T)$ as $k \to \infty$. We omit the index k in the notation. Integrating (10) from $t=0$ to $t=T$ and passing to $\varepsilon \searrow 0$ shows

$$\int_{\Omega_T} \nabla z \cdot \nabla \zeta + (W^{\mathrm{el}}_{,z}(e(u), c, z) - \alpha + \beta(\partial_t z))\zeta\,\mathrm{d}x \ge \int_{\Omega_T} \chi [W^{\mathrm{el}}_{,z}(e(u), c, z)]^+ \zeta\,\mathrm{d}x\mathrm{d}s \tag{23}$$

for all $\zeta \in L^p(0,T;W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$. We also use the fact that

$$\left|\int_{\Omega_T} \varepsilon |\nabla z_\varepsilon|^{p-2} \nabla z_\varepsilon \cdot \nabla\zeta\,\mathrm{d}x\mathrm{d}s\right| \le \varepsilon \|\nabla z_\varepsilon\|^{p-1}_{L^p(\Omega_T)} \|\nabla\zeta\|_{L^p(\Omega_T)} \to 0.$$

It follows

$$\begin{aligned}\int_\Omega \nabla z(t) \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x \\ \ge \int_\Omega \chi(t)[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+ \zeta\,\mathrm{d}x\end{aligned}$$

for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and a.e. $t \in [0,T]$. Set $r := -\chi[W^{\mathrm{el}}_{,z}(e(u), c, z)]^+$. For every $\xi \in L^\infty([0,T])$ with $\xi \ge 0$ a.e. on $[0,T]$ and every $\zeta \in H^1_+(\Omega) \cap L^\infty(\Omega)$ we also have

$$\begin{aligned}0 \ge \int_0^T \left(\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t = \int_{\Omega_T} r_\varepsilon(\zeta - z_\varepsilon)\xi\,\mathrm{d}x\mathrm{d}t \\ \to \int_{\Omega_T} r(\zeta - z)\xi\,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x\right)\xi(t)\,\mathrm{d}t.\end{aligned}$$

This shows $\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x \le 0$ for a.e. $t \in [0,T]$. We obtain the inequalities (v) of Definition 2.3.

(v) Weakly semi-continuity arguments lead to

$$\begin{aligned}\liminf_{\varepsilon \searrow 0} \Big(\mathcal{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) + \int_{\Omega_t} \alpha|\partial_t z_\varepsilon| + \beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathcal{S}w_\varepsilon, w_\varepsilon\rangle\,\mathrm{d}s\Big) \\ \ge \mathcal{E}(u(t), c(t), z(t)) + \int_{\Omega_t} \alpha|\partial_t z_\varepsilon| + \beta|\partial_t z|^2 + \int_0^t \langle \mathcal{S}w, w\rangle\,\mathrm{d}s.\end{aligned}$$

Testing (9) with $\zeta = u^0_\varepsilon - b(0)$ and (iv) of Definition 2.3 with $\zeta = u^0 - b(0)$ yield

$$\begin{aligned}\varepsilon \int_\Omega |\nabla u^0_\varepsilon|^4\,\mathrm{d}x = \varepsilon \int_\Omega |\nabla u^0_\varepsilon|^2 \nabla u^0_\varepsilon : \nabla b(0)\,\mathrm{d}x \\ - \int_\Omega W^{\mathrm{el}}_{,e}(e(u^0_\varepsilon), c^0, z^0_\varepsilon) : e(u^0_\varepsilon - b(0))\,\mathrm{d}x\end{aligned}$$

$$\to -\int_\Omega W^{\mathrm{el}}_{,e}(e(u^0),c^0,z^0):e(u^0-b(0))\,\mathrm{d}x=0$$

as $\varepsilon\searrow 0$.

Therefore, we can pass $\varepsilon\searrow 0$ in (11) and obtain (vi) from Definition 2.3. ■

5 Higher integrability of the strain tensor

To prove existence results for chemical free energies of logarithmic type, a higher integrability result for the strain tensor based on [Gar00, Gar05b] will be established. We adapt the higher integrability result for solutions of the elliptic equation of the form

$$\begin{Bmatrix}\mathrm{div}(W^{\mathrm{el}}_{,e}(e(u),c)) & \text{on } \Omega_T,\\ W^{\mathrm{el}}_{,e}(e(u),c)\cdot\overrightarrow{\nu}=\sigma^\star\cdot\overrightarrow{\nu} & \text{on } (\partial\Omega)_T\end{Bmatrix}$$

to our setting with non-constant Dirichlet boundary data b and the additional damage variable z in (S_0). In the sequel, we will use the assumption $D=\partial\Omega$.

The proof of the higher integrability result is based on the following special cases of the Sobolev-Poincaré inequalities and on a reverse Hölder inequality.

Theorem 5.1 (Sobolev-Poincaré type inequalities) *Let $1\le p<n$. There exists a constant $C>0$ such that*

(i) for all rectangles $Q\subseteq\mathbb{R}^n$ and all $u\in W^{1,p}(Q)$:

$$\left(\fint_Q\Big|u-\fint_Q u\Big|^{p^\star}\right)^{\frac{1}{p^\star}}\le C\left(\fint_Q|\nabla u|^p\right)^{\frac1p}(\mathrm{diam}Q)$$

(ii) for all rectangles $Q=\prod_{i=1}^n(a_i,b_i)\subseteq\mathbb{R}^n$ and all $u\in W^{1,p}(Q)$ with $u=0$ on $\big\{(x_1,\ldots,x_{n-1},a_n)\,|\,a_i\le x_i\le b_i,\ i=1,\ldots,n-1\big\}\subseteq\partial Q$ (in the sense of traces):

$$\left(\fint_Q|u|^{p^\star}\right)^{\frac{1}{p^\star}}\le C\left(\fint_Q|\nabla u|^p\right)^{\frac1p}(\mathrm{diam}Q)$$

Theorem 5.1 can be obtained by considering the corresponding inequalities on the unit cube $(0,1)^n$ (for instance the case $1<p<n$ was proven by Sobolev [Sob38] while Nirenberg [Nir59] gave a proof to $p=1$) and then using a scaling argument.

Theorem 5.2 (Reverse Hölder inequality, see [Gia83]) *Let $Q\subseteq\mathbb{R}^n$ be a cube, $g\in L^q_{\mathrm{loc}}(Q)$ for a $q>1$ and $g\ge 0$. Suppose that there exist a constant $b>0$ and a function $f\in L^r_{\mathrm{loc}}(Q)$ with $r>q$ and $f\ge 0$ such that*

$$\fint_{Q_R(x_0)}g^q\,\mathrm{d}x\le b\left(\fint_{Q_{2R}(x_0)}g\,\mathrm{d}x\right)^q+\fint_{Q_{2R}(x_0)}f^q\,\mathrm{d}x$$

for each $x_0\in Q$ and all $R>0$ with $2R<\mathrm{dist}(x_0,\partial Q)$. Then $g\in L^s_{\mathrm{loc}}(Q)$ for $s\in[q,q+\varepsilon)$ for some $\varepsilon>0$ and

$$\left(\fint_{Q_R(x_0)}g^s\,\mathrm{d}x\right)^{\frac1s}\le c\left(\left(\fint_{Q_{2R}(x_0)}g^q\,\mathrm{d}x\right)^{\frac1q}+\left(\fint_{Q_{2R}(x_0)}f^s\,\mathrm{d}x\right)^{\frac1s}\right)$$

for all $x_0\in Q$ and $R>0$ such that $Q_{2R}(x_0)\subseteq Q$. The positive constants $c,\varepsilon>0$ depend on b, q, n and r.

Theorem 5.3 (Higher integrability) *Let $b \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, $z \in L^\infty(\Omega)$ with $0 \le z \le 1$ a.e. in Ω and $c \in L^\mu(\Omega;\mathbb{R}^N)$ for a $\mu > 4$. Then there exists some $p \in (2,\mu/2]$ such that for all $u \in H^1(\Omega;\mathbb{R}^n)$ which satisfies $u|_D = b|_D$ and*

$$\int_\Omega W^{\rm el}_{,e}(e(u),c,z) : e(\zeta)\,\mathrm{d}x = 0 \text{ for all } \zeta \in H^1_D(\Omega;\mathbb{R}^n), \tag{24}$$

we obtain $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ and

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \le C(\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})} + \|c\|^2_{L^{2p}(\Omega;\mathbb{R}^N)} + 1). \tag{25}$$

The positive constants p and C are independent of u, c, z.

Proof. The proof is based on [Gar00, Lemma 4.4 and Theorem 4.3] and uses a covering argument. Due to the non-constant boundary condition, we need to apply a more general Sobolev-Poincaré inequality (see Theorem 5.1 (ii)) than in [Gar00].

(i) Higher integrability at the boundary.

Let $x_0 \in \partial\Omega$. Then there exists an $R_0 > 0$ and a bi-Lipschitz function $\tau : Q \to \mathbb{R}^n$ with the open cube $Q := Q_{R_0}(0)$ such that $x_0 \in \tau(Q)$ and

$$\tau(Q^+) \subseteq \Omega,$$
$$\tau(Q^-) \subseteq \mathbb{R}^n \setminus \overline{\Omega}.$$

where $Q^+ := \{x \in Q \,|\, x_n > 0\}$ and $Q^- := \{x \in Q \,|\, x_n < 0\}$. Define the transformated functions $\tilde{u}, \tilde{b} \in H^1(Q^+;\mathbb{R}^n)$, $\tilde{c} \in H^1(Q^+)$ and $\tilde{z} \in L^\infty(Q^+)$ as

$$(\tilde{u},\tilde{b},\tilde{c},\tilde{z})(x) := (u,b,c,z)(\tau(x)).$$

To proceed, let $y_0 \in Q$ and $R < \frac{1}{2}\mathrm{dist}(y_0,\partial Q)$ and define for each $R' > 0$ the sets

$$Q^\pm_{R'}(y_0) := \{x \in Q_{R'}(y_0) \,|\, x_n \gtrless 0\}.$$

We distinguish three cases:

Case 1. We first consider the case $Q^+_R(y_0) \neq \emptyset$ and $Q^-_{\frac{3}{2}R}(y_0) \neq \emptyset$.

The bi-Lipschitz continuity of τ ensures

$$\mathrm{dist}(\tau(\partial Q^+_{2R}(y_0)) \cap \Omega, \tau(\partial Q^+_R(y_0)) \cap \Omega) > RC_1,$$

where $C_1 > 0$ is independent of R and y_0. Let $\xi \in \mathcal{C}^\infty_0(\Omega)$ be a cutoff function with the properties:

(a) $\xi = 0$ in $\Omega \setminus \tau(Q_{2R}(y_0))$,

(b) $0 \le \xi \le 1$ in Ω,

(c) $\xi \equiv 1$ in $\tau(Q_R(y_0)) \cap \Omega$,

(d) $|\nabla\xi| \le \frac{2}{C_1}R^{-1}$.

Testing (24) with $\zeta = \xi^2(u-b)$, using the computation

$$e(\zeta) = \xi^2 e(u) - \xi^2 e(b) + \xi((u-b)(\nabla\xi)^t + \nabla\xi(u-b)^t),$$

and (A1), we obtain

$$\int_\Omega \xi^2 W^{\rm el}_{,e}(e(u),c,z) : e(u)\,\mathrm{d}x$$
$$= \int_\Omega \xi^2 W^{\rm el}_{,e}(e(u),c,z) : e(b)\,\mathrm{d}x - 2\int_\Omega \xi W^{\rm el}_{,e}(e(u),c,z) : ((u-b)(\nabla\xi)^t)\,\mathrm{d}x \tag{26}$$

By (A3), (A4) and (A2) we also have the estimates

$$\eta|e(u)|^2 \le W^{\rm el}_{,e}(e(u),c,z):e(u) + C(|c|^2+1)|e(u)|,$$
$$|W^{\rm el}_{,e}(e(u),c,z):((u-b)(\nabla\xi)^t| \le \frac{C}{R}(|e(u)|+|c|^2+1)|u-b|,$$
$$|W^{\rm el}_{,e}(e(u),c,z):e(b)| \le (|e(u)|+|c|^2+1)|e(b)|.$$

Therefore, (26) can be estimated to

$$\eta\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x \le C\int_\Omega \xi^2(|c|^2+1)|e(u)|\,\mathrm{d}x + \frac{C}{R}\int_\Omega \xi(|e(u)|+|c|^2+1)|u-b|\,\mathrm{d}x$$
$$+ C\int_\Omega \xi^2(|e(u)|+|c|^2+1)|e(b)|\,\mathrm{d}x.$$

Young's inequality yields

$$c_1\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x \le C\int_\Omega \xi^2(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_\Omega |u-b|^2\,\mathrm{d}x. \tag{27}$$

We choose $\mu = ⨍_{Q^+_{2R}(y_0)} \tilde u\,\mathrm{d}x$. The calculation $e(\xi(u-\mu)) = \xi e(u) + \frac{1}{2}((u-\mu)(\nabla\xi)^t + \nabla\xi(u-\mu)^t)$ leads to

$$\int_\Omega |e(\xi(u-\mu))|^2\,\mathrm{d}x \le 2\left(\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x + \int_\Omega |u-\mu|^2|\nabla\xi|^2\,\mathrm{d}x\right). \tag{28}$$

Combining (27) and (28), applying Korn's inequality for H^1-functions with zero boundary values and using (a) and (b) gives

$$\int_\Omega |\nabla(\xi(u-\mu))|^2\,\mathrm{d}x \le C\int_{\tau(Q^+_{2R}(y_0))}(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-b|^2\,\mathrm{d}x$$
$$+ \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-\mu|^2\,\mathrm{d}x.$$

Because of $\nabla(\xi(u-\mu)) = \xi\nabla u + (u-\mu)(\nabla\xi)^t$ we derive by (a) and (c) the following type of Caccioppoli-inequality:

$$\int_{\tau(Q^+_R(y_0))}|\nabla u|^2\,\mathrm{d}x \le C\int_{\tau(Q^+_{2R}(y_0))}(|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-b|^2\,\mathrm{d}x$$
$$+ \frac{C}{R^2}\int_{\tau(Q^+_{2R}(y_0))}|u-\mu|^2\,\mathrm{d}x.$$

Integral transformation by τ implies

$$\int_{Q^+_R(y_0)}|\nabla\tilde u|^2\,\mathrm{d}x \le C\int_{Q^+_{2R}(y_0)}(|\tilde c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{Q^+_{2R}(y_0)}|\tilde u-\tilde b|^2\,\mathrm{d}x$$
$$+ \frac{C}{R^2}\int_{Q^+_{2R}(y_0)}|\tilde u-\mu|^2\,\mathrm{d}x.$$

The condition $Q^-_{\frac{3}{2}R}(y_0) \neq \emptyset$ and $D = \partial\Omega$ imply that $\tilde u - \tilde b$ vanishes on $\partial\big(Q^+_{2R}(y_0)\big) \cap \mathbb{R}^{n-1}\times\{0\}$. Therefore, we obtain by applying both variants of the Poincaré-Sobolev inequality in Theorem 5.1 for $p = 2n/(n+2)$

$$\int_{Q^+_R(y_0)}|\nabla\tilde u|^2\,\mathrm{d}x \le C\int_{Q^+_{2R}(y_0)}(|\tilde c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\mathcal{L}^n(Q^+_{2R}(y_0))^{-\frac{2}{n}}\operatorname{diam}(Q^+_{2R}(y_0))^2$$

$$\cdot \left[\left(\int_{Q_{2R}^+(y_0)} |\nabla \tilde{u} - \nabla \tilde{b}|^{\frac{2n}{n+2}} \, \mathrm{d}x \right)^{\frac{n+2}{n}} + \left(\int_{Q_{2R}^+(y_0)} |\nabla \tilde{u}|^{\frac{2n}{n+2}} \, \mathrm{d}x \right)^{\frac{n+2}{n}} \right]. \tag{29}$$

Note that if $n = 1$ we cannot apply Theorem 5.1 because of $p = 2n/(n+2) < 1$. In this case, we can work with the inequalities in Theorem 5.1 where p is substituted by 1 and $p^\star$ is substituted by 2. However, we will only treat the more delicate case $n \geq 2$ in the following.

The estimates $\mathrm{diam}(Q_{2R}^+(y_0)) \leq CR$ and $\mathcal{L}^n(Q_{2R}^+(y_0)) \geq R^n$ (because of $Q_R^+(y_0) \neq \emptyset$) show

$$\mathcal{L}^n(Q_{2R}^+(y_0))^{-\frac{2}{n}} \mathrm{diam}(Q_{2R}^+(y_0))^2 \leq C. \tag{30}$$

Now, dividing (29) by $\mathcal{L}^n(Q_R(y_0))$ and using (30) and

$$\frac{1}{R^2} \frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \leq C \left(\frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \right)^{\frac{n+2}{n}}$$

gives

$$\begin{aligned} \frac{1}{\mathcal{L}^n(Q_R(y_0))} \int_{Q_R^+(y_0)} |\nabla \tilde{u}|^2 \, \mathrm{d}x \leq{}& \frac{C}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} (|\tilde{c}|^4 + 1) \, \mathrm{d}x \\ &+ C \left(\frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla \tilde{u}|^{\frac{2n}{n+2}} \, \mathrm{d}x \right)^{\frac{n+2}{n}} \\ &+ C \left(\frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla \tilde{b}|^{\frac{2n}{n+2}} \, \mathrm{d}x \right)^{\frac{n+2}{n}}. \end{aligned}$$

Observe that

$$\left(\frac{1}{\mathcal{L}^n(Q_{2R}(y_0))} \int_{Q_{2R}^+(y_0)} |\nabla \tilde{b}|^{\frac{2n}{n+2}} \, \mathrm{d}x \right)^{\frac{n+2}{n}} \leq \|\nabla b\|_{L^\infty(\Omega)}^2.$$

Define the following functions on Q:

$$g(x) := \begin{cases} |\nabla \tilde{u}(x)|^{\frac{2n}{n+2}} & \text{for } x \in Q^+, \\ 0 & \text{for } x \in Q \setminus Q^+ \end{cases}$$

and

$$f(x) := \begin{cases} C(|\tilde{c}|^4 + \|\nabla b\|_{L^\infty(\Omega)}^2 + 1)^{\frac{n}{n+2}} & \text{for } x \in Q^+, \\ 0 & \text{for } x \in Q \setminus Q^+. \end{cases}$$

We eventually get

$$⨍_{Q_R(y_0)} g^{\frac{n+2}{n}} \, \mathrm{d}x \leq ⨍_{Q_{2R}(y_0)} f^{\frac{n+2}{n}} \, \mathrm{d}x + C \left(⨍_{Q_{2R}(y_0)} g \, \mathrm{d}x \right)^{\frac{n+2}{n}}. \tag{31}$$

Case 2. Assume $Q_R^+(y_0) \neq \emptyset$ and $Q_{\frac{3}{2}R}^-(y_0) = \emptyset$.

The bi-Lipschitz continuity of τ implies

$$\mathrm{dist}(\tau(\partial Q_{\frac{3}{2}R}(y_0)), \tau(\partial Q_R(y_0))) > RC_1,$$

where $C_1 > 0$ is independent of R and y_0. Therefore, we can choose a cutoff function $\xi \in \mathcal{C}_0^\infty(\Omega)$ which satisfies

(a) $\xi = 0$ in $\Omega \setminus \tau(Q_{\frac{3}{2}R}(x_0))$,

(b) $0 \le \xi \le 1$ in Ω,

(c) $\xi \equiv 1$ in $\tau(Q_R(x_0))$,

(d) $|\nabla \xi| \le \frac{2}{C_1} R^{-1}$.

Testing (24) with $\xi = \zeta^2(u-\mu)$ and $\mu := \fint_{Q_{\frac{3}{2}R}(x_0)} \tilde{u}\,\mathrm{d}x$ yields as in the previous case

$$\int_{\tau(Q_R(x_0))} |\nabla u|^2\,\mathrm{d}x \le C \int_{\tau(Q_{\frac{3}{2}R}(x_0))} (|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2} \int_{\tau(Q_{\frac{3}{2}R}(x_0))} |u-\mu|^2\,\mathrm{d}x.$$

Consequently,

$$\fint_{Q_R(x_0)} |\nabla \tilde{u}|^2\,\mathrm{d}x \le C \fint_{Q_{\frac{3}{2}R}(x_0)} (|\tilde{c}|^4+1)\,\mathrm{d}x + C\left(\fint_{Q_{\frac{3}{2}R}(x_0)} |\nabla \tilde{u}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}.$$

Therefore, the inequality (31) is also satisfied in this case.

Case 3. Assume $Q_R^+(y_0) = \emptyset$.

In this case, inequality (31) trivially holds.

In all three cases, the reverse Hölder inequality (see Theorem 5.2) shows $g \in L^s_{\mathrm{loc}}(Q)$ for all $s \in \left[\frac{n+2}{n}, \frac{n+2}{n} + \varepsilon\right)$ and some $\varepsilon > 0$ depending on R_0 and n.

(ii) Higher integrability in the interior.

This case follows with much less effort and is only sketched here.

Let $x_0 \in \Omega$ arbitrary and $R > 0$ such that $Q_{2R}(x_0) \subseteq \Omega$. We take a cutoff function $\xi \in \mathcal{C}_0^\infty(\Omega)$ with

(a) $\xi = 0$ in $\Omega \setminus Q_{2R}(x_0)$,

(b) $0 \le \xi \le 1$ in Ω,

(c) $\xi \equiv 1$ in $Q_R(x_0)$,

(d) $|\nabla \xi| \le \frac{2}{R}$.

Testing (24) with $\xi = \zeta^2(u-\mu)$ and $\mu = \fint_{Q_{2R}(x_0)} u\,\mathrm{d}x$ yields with the same computation as in the case (i):

$$\int_{Q_R(x_0)} |\nabla u|^2\,\mathrm{d}x \le C \int_{Q_{2R}(x_0)} (|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2} \int_{Q_{2R}(x_0)} |u-\mu|^2\,\mathrm{d}x$$

Poincaré-Sobolev inequality implies

$$\fint_{Q_R(x_0)} |\nabla u|^2\,\mathrm{d}x \le C \fint_{Q_{2R}(x_0)} (|c|^4+1)\,\mathrm{d}x + C\left(\fint_{Q_{2R}(x_0)} |\nabla u|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}.$$

Applying Theorem 5.2 with $g = |\nabla u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$ and $f = C(|c|^4+1)^{\frac{n}{n+2}}$ finishes the proof. ■

6 Existence of weak solutions of (S_0) - logarithmic case

The challenge here is to establish the integral equation (iii) in Definition 2.3 because the derivative of the logarithmic free chemical energy (A8) becomes singular if one of the c_k's approaches 0. We only sketch the proof in this section since all essential ideas can be found in [Gar00, Gar05b]. We use a regularization method suggested in [EL91] and also used in [Gar00, Gar05b].

The energy gradient tensor is assumed to be of the form $\mathbf{\Gamma} = \gamma\,\mathrm{Id}$ with a constant $\gamma > 0$. Define a $\mathcal{C}^2(\mathbb{R}^N)$ regularization with the regularization parameter $\delta > 0$ as

$$W^{\mathrm{ch},\delta}(c) := \theta \sum_{k=1}^{N} \phi^\delta(c^k) + \frac{1}{2} c \cdot Ac,$$

with

$$\phi^\delta(x) := \begin{cases} x\log(x) & \text{for } d \geq \delta, \\ x\log(\delta) - \frac{\delta}{2} + \frac{x^2}{2\delta} & \text{for } x < \delta. \end{cases}$$

Elliott and Luckhaus showed that the regularization $W^{\mathrm{ch},\delta}$ is uniformly bounded from below.

Lemma 6.1 (cf. [EL91]) *There exists constants $\delta_0 > 0$ and $C > 0$ such that*

$$W^{\mathrm{ch},\delta}(c) \geq -C \qquad \text{for all } c \in \Sigma,\ \delta \in (0, \delta_0).$$

Let q_δ denote a weak solution in the sense of Definition 2.3 with the free chemical energy $W^{\mathrm{ch}} = W^{\mathrm{ch},\delta}$. By applying Lemma 6.1 and using Gronwall's inequality in the energy inequality (vi) of Definition 2.3, we can show a-priori estimates analogous as in Section 4 except the a-priori estimate of w_δ.

In the Allen-Cahn case, we have $\partial_t c_\delta = -\mathbb{M} w_\delta$ and, consequently, the boundedness of c_δ in $L^2(\Omega;\mathbb{R}^N)$ and $w_\delta \in T\Sigma$ pointwise lead to boundedness of w_δ in $L^2(\Omega;\mathbb{R}^N)$.

However, in the case of Cahn-Hilliard systems, we can use the following lemma.

Lemma 6.2 ([Gar00, Lemma 4.3]) *There exists a constant $C > 0$ such that for all $\delta \in (0, \delta_0)$*

$$\int_0^T \left(\fint_\Omega \mathbb{P} W^{\mathrm{ch},\delta}_{,c}(c_\delta(t))\,\mathrm{d}x \right)^2 \mathrm{d}t < C.$$

The proof of this lemma is similar to [Gar00, Lemma 4.3], since all arguments can be adapted to our case. Therefore we will omit the proof.

This lemma and the integral equation

$$\int_\Omega w_\delta(t)\,\mathrm{d}x = \int_\Omega \mathbb{P} W^{\mathrm{ch},\delta}_{,c}(c_\delta(t)) + \mathbb{P} W^{\mathrm{el}}_{,c}(e(u_\delta(t)), c_\delta(t), z_\delta(t))\,\mathrm{d}x$$

together with the already known boundedness properties shows

$$\int_0^T \left(\fint_\Omega w_\delta(t)\,\mathrm{d}x \right)^2 \mathrm{d}t < C$$

for a constant $C > 0$. Therefore w_δ is bounded in $L^2(0,T;H^1(\Omega))$ by Poincaré's inequality. In conclusion, we can extract a subsequence $\{q_{\delta_k}\}$ such that we have the same convergence properties as in Lemma 4.2. We omit the subscript k.

Proof of Theorem 2.6. The remaining crucial step is to show that the limit c satisfies $c_k > 0$ a.e. on Ω_T for all $k = 1, \ldots, N$ and $W^{\mathrm{ch},\delta}_{,c}(c_\delta) \to W^{\mathrm{ch},\log}_{,c}(c)$ in $L^1(\Omega_T)$ as $\varepsilon \searrow 0$. Then the rest follows as in Section 4.

To this end, we need an additional boundedness property.

Lemma 6.3 *There exists constants $q > 1$ and $C > 0$ such that for all $\delta \in (0, \delta_0)$ and all $k = 1, \ldots, N$*

$$\|(\phi^\delta)'(c^k_\delta)\|_{L^q(\Omega_T)} < C.$$

We omit the proof of this lemma, since by utilizing Theorem 5.3 the arguments are analogous to [Gar00, Lemma 4.5].

Note that

$$\lim_{\delta\searrow 0}(\phi^\delta)'(c_\delta^k) = \begin{cases} \log(c^k)+1 & \text{if } \lim_{\delta\searrow 0} c_\delta^k = c^k > 0, \\ \infty & \text{else.} \end{cases}$$

holds pointwise a.e. on Ω_T and for all $k = 1, \ldots, N$. Together with Lemma 6.3 we obtain

$$c^k > 0 \text{ a.e. on } \Omega_T$$

and

$$(\phi^\delta)'(c_\delta^k) \to \log(c^k) + 1 \text{ a.e. on } \Omega_T.$$

This and Lemma 6.3 further shows

$$(\phi^\delta)'(c_\delta^k) \to \log(c^k) + 1 \text{ in } L^1(\Omega_T)$$

by Vitali's convergence theorem. Finally, we can pass to $\delta \searrow 0$ in the equation

$$\int_{\Omega_T} w_\delta \cdot \zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \gamma\nabla c_\delta : \nabla\zeta + \mathbb{P}W_{,c}^{\mathrm{ch},\delta}(c_\delta)\cdot\zeta + \mathbb{P}W_{,c}^{\mathrm{el}}(e(u_\delta), c_\delta, z_\delta)\cdot\zeta \,\mathrm{d}x\mathrm{d}t$$

and obtain (iii) from Definition 2.3. The remaining properties can be easily established as in Section 4.

Theorem 2.6 is proven. ■

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