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# The degenerate and non-degenerate Stefan problem with inhomogeneous and anisotropic Gibbs–Thomson law

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**Abstract**

The Stefan problem is coupled with a spatially inhomogeneous and anisotropic Gibbs–Thomson condition at the phase boundary. We show the long-time existence of weak solutions for the non-degenerate Stefan problem with a spatially inhomogeneous and anisotropic Gibbs–Thomson law and a conditional existence result for the corresponding degenerate Stefan problem. To this end, approximate solutions are constructed by means of variational problems for energy functionals with spatially inhomogeneous and anisotropic interfacial energy. By passing to the limit, we establish solutions of the Stefan problem with a spatially inhomogeneous and anisotropic Gibbs–Thomson law in a weak generalized $BV$-formulation.

# 1 Introduction

The Stefan problem models phase transitions in materials. To allow for superheating and undercooling, the Stefan problem is coupled with a geometrical condition at the phase boundary, the so-called Gibbs–Thomson law. This condition takes surface tension effects into account such that the temperature may differ from the melting temperature at the phase boundary. The Gibbs–Thomson law states that the system is in thermodynamic equilibrium.

The classical Gibbs–Thomson law accounts for isotropic surface tension effects. In this case, the temperature at the interface is proportional to the mean curvature. In many applications, however, such as the solidification of alloys, the surface energy density is spatially inhomogeneous and anisotropic, i.e. the density depends on the position in space and on the local orientation of the interface. This means that the Stefan problem with a generalized Gibbs–Thomson law has to be considered (see for instance [Gur88, Gur93] for a thermodynamic derivation). The temperature at the interface is then related to a spatially inhomogeneous and anisotropic mean curvature.

Heat conduction in materials often takes place on a much faster time scale than the evolution of the interface. Therefore, a quasi-static version of the Stefan problem, the so-called degenerate Stefan problem, is often used to describe melting and solidification processes.

To formulate the Stefan problem with Gibbs–Thomson law, let $(0,T)$ be a given time interval, $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $\Omega_T := (0,T) \times \Omega$. The phase field variables are the temperature

$$u : \Omega_T \to \mathbb{R}$$

and a phase function

$$\chi : \Omega_T \to \mathbb{R},$$

where the liquid phase is represented by the set $\{(t,x) \in \Omega_T : \chi(t,x) = 1\}$ and the solid phase by the set $\{(t,x) \in \Omega_T : \chi(t,x) = 0\}$.

The *(non-degenerate)* Stefan problem with isotropic Gibbs–Thomson law is formally described by

$$\partial_t\big(u + \chi\big) - \triangle u = f \qquad \text{in } \Omega_T, \tag{1.1}$$

$$u = H \qquad \text{on } \Gamma, \tag{1.2}$$

where $f : \Omega_T \to \mathbb{R}$ is a given heat source, $H : \Gamma \to \mathbb{R}$ is the mean curvature and $\Gamma$ denotes the phase boundary.

The *degenerate* Stefan problem models an infinite fast heat flow in the material, i.e. (1.1) is replaced by

$$\partial_t \chi - \triangle u = f \qquad \text{in } \Omega_T. \tag{1.3}$$

For a general theory of the Stefan problem, we refer to [Vis98, Mei92, Gup03]. Global existence results for the non-degenerate Stefan problem with isotropic Gibbs–Thomson law in a weak (generalized) $BV$-formulation are shown in [Luc90, Luc91, Rög04] and with anisotropic Gibbs–Thomson law in [GS11]. For the degenerate Stefan problem, existence of classical solutions locally in time has been proven by Chen, Hong and Yi [CHY96] and by Escher and Simonett [ES97a], [ES97b]. An existence result for global solutions of the degenerate problem can be found in [Che96], where the limit of a modified Cahn–Hilliard model is considered. However, the isotropic Gibbs–Thomson law is only fulfilled in a rather weak and complex formulation. Using the theory of varifolds, Röger [Rög05] established long-time existence of solutions of the degenerate Stefan problem with isotropic Gibbs–Thomson law in a weak generalized $BV$-formulation. In contrast to the classical Stefan problem, global weak solutions of the Stefan problem with Gibbs–Thomson law have sharp interfaces but are highly non-unique as discussed in [Luc90]. Uniqueness of classical solutions for the degenerate and non-degenerate Stefan problem with Gibbs–Thomson law is established in [EPS03], [CHY96] and [Kne07]. In addition, it is shown in [EPS03] that the free boundary is an analytic function in space and time.

The $BV$-formulation of the degenerate and non-degenerate Stefan problem with isotropic Gibbs–Thomson law was introduced by Luckhaus and considered for the non-degenerate problem in [Luc90, Luc91] and for the degenerate problem in [LS95] (see also [GS98] for a multiphase version): The temperature and the phase function

$$u \in u_D + L^2(0,T;H_0^1(\Omega)), \quad u_D \in H^1(0,T;H^1(\Omega)), \qquad \text{and} \qquad \chi \in L^\infty(0,T;BV(\Omega;\{0,1\}))$$

satisfy for the non-degenerate problem

$$\int_{\Omega_T} (u+\chi)\partial_t \xi + \int_\Omega \chi(0)\xi(0) = \int_{\Omega_T} \nabla u \nabla \xi - \int_{\Omega_T} f\xi \tag{1.4}$$
$$\text{for all } \xi \in C_c^\infty([0,T)\times\Omega),$$

for the degenerate problem

$$\int_{\Omega_T} \chi\partial_t \xi + \int_\Omega \chi(0)\xi(0) = \int_{\Omega_T} \nabla u \nabla \xi - \int_{\Omega_T} f\xi \tag{1.5}$$
$$\text{for all } \xi \in C_c^\infty([0,T)\times\Omega)$$

and for both problems

$$\int_0^T \int_\Omega \left( \nabla\cdot\xi - \frac{\nabla\chi}{|\nabla\chi|}\cdot\nabla\xi\frac{\nabla\chi}{|\nabla\chi|} + u\,\xi\cdot\frac{\nabla\chi}{|\nabla\chi|}\right)|\nabla\chi|\,dt = 0 \tag{1.6}$$
$$\text{for all } \xi \in C_c^\infty(\Omega_T;\mathbb{R}^n).$$

In this $BV$-setting, global solutions for the non-degenerate case are obtained in [Luc90, Luc91] by an implicit time discretization method. The time–discrete approximations $\chi^h$ and $u^h$ converge to weak solutions of (1.1) and (1.2). In particular, the exclusion of loss of surface area in the limit, i.e.

$$\lim_{h\to 0}\int_{\Omega_T}|\nabla\chi^h| = \int_{\Omega_T}|\nabla\chi|, \tag{1.7}$$

arises in a natural way from the discrete minimum problem.

For the degenerate system, i.e. (1.3) and (1.2), property (1.7) is in general not satisfied. However, assuming (1.7), existence of global solutions can be shown in the $BV$-setting (see [LS95]). Conditions of the form as in (1.7) are typical for such kind of geometric problems and have been applied to several other geometric problems (see [ATW93, LS95, GS98, BGS98, Ott98]).

In this paper we study the degenerate and non-degenerate Stefan problem with *spatially inhomogeneous* and anisotropic Gibbs–Thomson law. This generalized Gibbs–Thomson law results from an inhomogeneous and anisotropic surface energy, i.e.

$$\int_\Gamma \sigma(x,\nu)\,d\mathcal{H}^{n-1},$$

where $\nu$ is the outer unit normal of the liquid phase, $\mathcal{H}^{(n-1)}$ is the $(n-1)$–dimensional Hausdorff measure and $\sigma$ is an anisotropy function satisfying assumption A 2.1 (see Section 2.1). The corresponding generalized Gibbs–Thomson law at the phase boundary reads as

$$u = H_\sigma \qquad\qquad \text{on } \Gamma \tag{1.8}$$

with

$$H_\sigma = \nabla_\Gamma\cdot\sigma_{,p}(x,\nu) + \sigma_{,x}(x,\nu)\cdot\nu\,,$$

where $\nabla_\Gamma$ denotes the tangential gradient of $\Gamma$ and $\sigma_{,s}$ is the first partial derivative of $\sigma$ with respect to the variable $s$.

The aim of this work is to show existence of weak solutions for the Stefan problem with spatially inhomogeneous and anisotropic Gibbs–Thomson law and existence of weak solutions for the corresponding degenerate problem assuming a condition similar to (1.7). The results of [Luc90, Luc91, LS95, GS11] are generalized.

Our main results are under suitable assumptions as follows.

**Theorem 1.1**
*Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $\sigma$ be an anisotropy function satisfying assumption A 2.1 (see Section 2.1) and $f\in L^2(\Omega_T)$. Furthermore, let $u_D\in H^1(0,T;H^1(\Omega))$ and the initial data $u_0\in H^1(\Omega)\cap L^\infty(\Omega)$ and $\chi_0\in BV(\Omega;\{0,1\})$ be given. Then, there exist functions $\chi\in L^\infty(0,T;BV(\Omega;\{0,1\}))$ and $u\in\big(u_D+L^2(0,T;H^1_0(\Omega))\big)\cap L^\infty(0,T;L^2(\Omega))$ that are solutions of*

$$\int_{\Omega_T}(u+\chi)\partial_t\xi + \int_\Omega \chi(0)\xi(0) = \int_{\Omega_T}\nabla u\,\nabla\xi - \int_{\Omega_T} f\xi \tag{1.9}$$
$$\text{for all } \xi\in C^1_c([0,T)\times\Omega),$$

*and*

$$\int_0^T\int_\Omega\Big(\sigma\big(\cdot,\nu(t,\cdot)\big)\nabla\cdot\xi(t,\cdot) + \sigma_{,x}(\cdot,\nu(t,\cdot))\cdot\xi(t,\cdot) - \nu(t,\cdot)\cdot\nabla\xi(t,\cdot)\,\sigma_{,p}(\cdot,\nu(t,\cdot))$$
$$- u(t,\cdot)\,\xi(t,\cdot)\cdot\nu(t,\cdot)\Big)|\nabla\chi(t,\cdot)\,dt = 0 \tag{1.10}$$

*for all $\xi \in C^1_c(\Omega_T;\mathbb{R}^n)$ with $\nu = -\frac{\nabla\chi}{|\nabla\chi|}$.*

*If, in addition, $\Omega$ is a bounded domain with $C^1$–boundary then (1.10) even holds for all $\xi \in C^1(\overline{\Omega}_T;\mathbb{R}^n)$ with $\xi\cdot\nu_\Omega = 0$ on $\partial\Omega$, where $\nu_\Omega$ is the outer unit normal of $\partial\Omega$.*

The above existence result for the non-degenerate system is based on an implicit time discretization method. In this case, we obtain for the time discrete approximations $\chi^h$, $h>0$, the following generalized property of (1.7):

$$\lim_{h\to 0}\int_{\Omega_T}\sigma(x,\nu^h)|\nabla\chi^h| = \int_{\Omega_T}\sigma(x,\nu)|\nabla\chi|, \qquad \nu^h := -\frac{\nabla\chi^h}{|\nabla\chi^h|}. \tag{1.11}$$

Under this condition, we are also able to show existence of weak solutions for the degenerate problem.

**Theorem 1.2**
*Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $\sigma$ be an anisotropy function satisfying assumption A 2.1, see Section 2.1, and $f\in L^2(\Omega_T)$. Furthermore, let $u_D\in W^{1,1}(0,T;H^1(\Omega))$ and the initial datum $\chi_0\in BV(\Omega;\{0,1\})$ be given. If condition (1.11) (see Section 4 for the definition of $\chi^h$) is satisfied then there exist functions $\chi\in L^\infty(0,T;BV(\Omega;\{0,1\}))$ and $u\in u_D+L^2(0,T;H^1_0(\Omega))$ that are solutions of*

$$\int_{\Omega_T}\chi\partial_t\xi + \int_\Omega \chi(0)\xi(0) = \int_{\Omega_T}\nabla u\,\nabla\xi - \int_{\Omega_T} f\xi \tag{1.12}$$

*for all $\xi\in C^1_c([0,T)\times\Omega)$,*

*and*

$$\int_0^T\int_\Omega\Big(\sigma\big(\cdot,\nu(t,\cdot)\big)\nabla\cdot\xi(t,\cdot) + \sigma_{,x}(\cdot,\nu(t,\cdot))\cdot\xi(t,\cdot) - \nu(t,\cdot)\cdot\nabla\xi(t,\cdot)\,\sigma_{,p}(\cdot,\nu(t,\cdot))$$
$$- u(t,\cdot)\,\xi(t,\cdot)\cdot\nu(t,\cdot)\Big)|\nabla\chi(t,\cdot)|\,dt = 0 \tag{1.10}$$

*for all $\xi\in C^1_c(\Omega_T,\mathbb{R}^n)$ with $\nu = -\frac{\nabla\chi}{|\nabla\chi|}$.*

*If, in addition, $\Omega$ is a bounded domain with $C^1$–boundary then* (1.10) *even holds for all $\xi\in C^1(\overline{\Omega}_T,\mathbb{R}^n)$ with $\xi\cdot\nu_\Omega=0$ on $\partial\Omega$, where $\nu_\Omega$ is the outer unit normal of $\partial\Omega$.*

A major task of the proof of the existence results for both problems has been to assure convergence of the approximate terms, which arise from the spatially inhomogeneous character of the interfacial energy. To handle this convergence problem, we work with slicing and indicator measures and methods of geometric measure theory. We choose the notion of a generalized total variation for $BV$-functions. Our results are based on weak convergence theorems for homogeneous functions of measures, on geometric properties for anisotropic surface energies and on approaches of [GK09].

The paper is organized as follows: In Sections 2.1-2.2, we introduce some notation and the assumptions. Then, we state some properties for anisotropy functions and slicing and indicator measures, see Sections 2.3-2.4. In Section 3, we establish a suitable weak formulation of the Stefan problem with spatially inhomogeneous and anisotropic Gibbs–Thomson law in a generalized $BV$-setting. Section 4 is devoted to time-incremental minimization problems for energy functionals with spatially inhomogeneous and anisotropic interfacial energy. We construct time discretized solutions for (1.9), (1.10) and (1.12), (1.10) , respectively. Arguments similarly to [Luc90, Luc91, LS95, GS11] are only sketched. Finally, we pass to the limit in the time discretized problems, cf. Sections 5.1-5.3, and prove Theorems 1.1 and 1.2 in Section 5.4.

# 2 Preliminaries

If not otherwise mentioned, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz–boundary. The first and second partial derivatives of a function with respect to the variables $s$ and $p$ are abbreviated by $f_{,s}$ and $f_{,sp}$.
We begin with stating the hypotheses for the anisotropy function $\sigma$.

## 2.1 Anisotropy function

**Assumption A 2.1**
*The anisotropy function* $\sigma : \overline{\Omega} \times \mathbb{R}^n \to [0, +\infty)$ *satisfies the following properties:*

*(i)* $\sigma \in C(\overline{\Omega} \times \mathbb{R}^n)$ ,
$\sigma_{,x}, \sigma_{,p} \in C(\overline{\Omega} \times \mathbb{R}^n \backslash \{0\})$ ,
$\sigma_{,pp} \in C(\overline{\Omega} \times \mathbb{R}^n \backslash \{0\})$.

*(ii)* $\sigma$ *is 1-homogeneous in the second variable, i.e.* $\sigma(x, \lambda p) = \lambda \sigma(x, p)$ *for all* $p \in \mathbb{R}^n$ *and any* $\lambda > 0$.

*(iii) There exist constants* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that*

$$\lambda_1 |p| \leq \sigma(x, p) \leq \lambda_2 |p| \quad \text{for all } x \in \overline{\Omega} \text{ and all } p \in \mathbb{R}^n.$$

*(iv)* $\sigma$ *is convex as a 1-homogeneous function in the following sense: There exists a constant* $d_0 > 0$ *such that*

$$\sigma_{,pp}(x, p)\, q \cdot q \geq d_0 |q|^2$$

*for all* $x \in \Omega$ *and all* $p, q \in \mathbb{R}^n$ *with* $p \cdot q = 0$, $|p| = 1$.

Note that $\sigma$ is not differentiable at $0 \in \mathbb{R}^n$. However, if we set $\sigma\, \sigma_{,p} = 0$ and $g\, \sigma_{,p} = 0$ at $0 \in \mathbb{R}^n$ for $g \in C^1(\Omega)$ with $g = 0$ in some neighborhood of 0, then the expressions $\sigma\, \sigma_{,p}$ and $g\, \sigma_{,p}$ are well defined and continuous at 0.

## 2.2 Generalized total variation

To handle the spatially inhomogeneous and anisotropic Gibbs–Thomson law, we use the notion of the generalized total variation of $BV$-functions introduced in [AB94].

Let $\sigma : \Omega \times \mathbb{R}^n \to [0, +\infty)$ be a continuous anisotropy function fulfilling (ii) and (iii) of assumption A 2.1. Then the dual function $\sigma^* : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is given by

$$\sigma^*(x, q) = \sup \big\{ q \cdot p : p \in \mathbb{R}^n,\, \sigma(x, p) \leq 1 \big\} = \sup \left\{ \frac{q \cdot p}{\sigma(x, p)} : p \in \mathbb{R}^n \backslash \{0\} \right\}. \tag{2.1}$$

For any $f \in BV(\Omega)$ the *generalized total variation* of $f$ (with respect to $\sigma$) in $\Omega$ is defined by

$$\int_\Omega |\nabla f|_\sigma = \sup\Big\{ \int_\Omega f \,\mathrm{div}\eta \, dx : \eta \in K_\sigma(\Omega) \Big\},$$

where $K_\sigma(\Omega) = \big\{\eta \in C_c^1(\Omega, \mathbb{R}^n) : \sigma^*(x, \eta(x)) \le 1 \text{ for a.e. } x \in \Omega\big\}$. The generalized total variation can be represented by an integral formula in terms of the measure $|\nabla f|$, cf. [AB94, AB95]:

$$\int_\Omega |\nabla f|_\sigma = \int_\Omega \sigma(x, \nu_f)\, |\nabla f|, \tag{2.2}$$

where $\nu_f(x) = -\frac{\nabla f}{|\nabla f|}(x)$ for $|\nabla f|$–a.e. $x \in \Omega$.

We remark, $\int_\Omega |\nabla f|_\sigma$ is $L^1(\Omega)$–lower semicontinuous on $BV(\Omega)$.

## 2.3 Properties of anisotropy functions

In the sequel, we take advantage from the following properties for anisotropy functions, cf. [BP96], [Dzi99] and [Gig06]:

**Lemma 2.2**
*Let $\sigma$ be an anisotropy function satisfying assumption A 2.1. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for all $x \in \Omega$, $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$ and all $p, p_1, p_2 \in \mathbb{R}^n \backslash \{0\}$ the following properties are fulfilled:*

*(i)*

$$\sigma_{,p}(x,p) \cdot p = \sigma(x,p), \qquad \sigma^*_{,p}(x,p) \cdot p = \sigma^*(x,p), \tag{2.3}$$

*(ii)*

$$\sigma(x,\nu_1) - \sigma_{,p}(x,\nu_2) \cdot \nu_1 \ge C_1 |\nu_1 - \nu_2|^2, \tag{2.4}$$

*(iii)*

$$|\sigma_{,p}(x,\nu_1) - \sigma_{,p}(x,\nu_2)| \le C_2 |\nu_1 - \nu_2|, \tag{2.5}$$

*(iv)*

$$\sigma_{,p}(x, \lambda p) = \sigma_{,p}(x,p) \quad \textit{for } \lambda > 0, \tag{2.6}$$

*(v)*

$$\sigma\big(x, \sigma^*_{,p}(x,p_1)\big) = \sigma^*\big(x, \sigma_{,p}(x,p_2)\big) = 1. \tag{2.7}$$

*(vi)*

$$\sigma(x,p)\, \sigma^*_{,p}\big(x, s, \sigma_{,p}(x,p)\big) = p, \qquad \sigma^*(x,p)\, \sigma_{,p}\big(x, s, \sigma^*_{,p}(x,p)\big) = p. \tag{2.8}$$

Anisotropy can be visualized by the Wulff shape $W$ that varies in our situation with $x \in \Omega$:

$$W(x) = \{q \in \mathbb{R}^n : \sigma^*(x,q) \le 1\}.$$

The Wulff shape $W$ is convex and its boundary can be expressed as follows:

$$\partial W(x) = \big\{\sigma_{,p}(x, \tilde{\nu}) : \tilde{\nu} \in \mathbb{S}^{n-1}\big\}, \qquad x \in \Omega.$$

The outer unit normal at the point $\sigma_{,p}(x,\tilde{\nu})$ on $\partial W(x)$ is $\tilde{\nu}$. For more details on this topic, we refer to [Gur93] and [Gig06].

The following lemma is an essential tool for constructing suitable approximations of the Cahn-Hoffman vector $\sigma_{,p}$, cf. [GK09]. This auxiliary result is utilized to prove convergence of the time discretized solutions.

**Lemma 2.3 (cf. [GK09])**
*Let $\sigma$ be an anisotropy function satisfying assumption A 2.1. Then, there exists a constant $C > 0$ such that*

$$C\,|\sigma_{,\,p}(x,\nu) - p|^2 \leq \sigma(x,\nu) - p\cdot\nu$$

*for all $x \in \Omega$, $\nu \in \mathbb{S}^{n-1}$ and all $p \in \mathbb{R}^n\backslash\{0\}$ with $\sigma^*(x,p) \leq 1$.*

## 2.4 Slicing and indicator measures

We outline some properties on slicing and indicator measures, which are required in the limit process of the discrete spatially inhomogeneous and anisotropic Gibbs–Thomson law. For details we refer to [AFP00], [Eva90], [Fon91] and [Fon92].

Let $\Theta$ be a finite, nonnegative Radon measure on $\Omega \times \mathbb{R}^n$. The canonical projection onto $\Omega$ is denoted by $\pi$, i.e.

$$\pi(E) := \Theta(E \times \mathbb{R}^n)$$

for each Borel set $E \subset \Omega$.

**Proposition 2.4 (cf. [AFP00])**
*For $\pi$–a.e. point $x \in \Omega$, there exists a Radon probability measure $\lambda_x$ on $\mathbb{R}^n$ such that*

*(i) the mapping $x \to \int_{\mathbb{R}^n} f(x,y)\,d\lambda_x(y)$ is $\pi$ measurable,*

*(ii) $\int_{\Omega\times\mathbb{R}^n} f(x,y)\,d\Theta(x,y) = \int_\Omega \big(\int_{\mathbb{R}^n} f(x,y)d\lambda_x(y)\big)\,d\pi(x)$ (Fubini's decomposition)*

*for every continuous and bounded function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$.*

Let $\hat{\mu}$ be an $\mathbb{R}^n$–valued measure on $\Omega$ with polar decomposition $d\hat{\mu} = \alpha\, d\mu$. Then, the *indicator measure* of $\hat{\mu}$ is the finite, non-negative Radon measure $\Theta$ on $\Omega \times \mathbb{S}^{n-1}$ defined by

$$\langle\Theta, f\rangle = \int_\Omega f\big(x, \alpha(x)\big)d\mu(x)$$

for every continuous and bounded function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$. If $E \subset \Omega$ is a set with finite perimeter, i.e.

$$\mathrm{per}(E) = \int_\Omega |\nabla\chi_E| < \infty, \qquad \chi_E: \text{ characteristic function of } E,$$

then the indicator measure of $\nabla\chi_E$ has the form

$$\langle\Theta, f\rangle = \int_{\partial^* E} f\big(x, -\nu_E(x)\big)\,d\mathcal{H}^{n-1}(x)\,, \qquad \nu_E: \text{ unit outer normal of } E,$$

where $\partial^* E$ is the reduced boundary of $E$, cf. [Giu84, AFP00].

**Proposition 2.5 (cf. [AFP00], [Fon92])**
*Let $\{\hat{\mu}_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$–valued measures on $\Omega$ with polar decompositions $d\hat{\mu}_k = \alpha_k\, d\mu_k$ and suppose that $\hat{\mu}_k \to \hat{\mu}$ weakly* with $\hat{\mu} = \alpha\mu$. Then, there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ and a non-negative Radon measure $\Theta_\infty \equiv \pi_\infty \otimes \lambda_x^\infty$ on $\Omega \times \mathbb{S}^{n-1}$, $\lambda_x^\infty$ being probability measures, such that*

*(i)* $\Theta_{k_j} \equiv \mu_{k_j} \otimes \delta_{\alpha_{k_j}(x)} \to \Theta_\infty \equiv \pi_\infty \otimes \lambda_x^\infty$ *weakly**, $\qquad \delta_y$ *Dirac mass,*

*(ii)* $\mu_{k_j} \to \pi_\infty$ *weakly**,

*(iii)* $\pi_\infty \geq \mu$.

*Moreover, for every $f \in C_c(\Omega \times \mathbb{R}^n)$*

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega f\big(x, \alpha_{k_j}(x)\big) d\mu_{k_j} &= \int_{\Omega\times\mathbb{S}^{n-1}} f\big(x,y\big) d\Theta_\infty(x,y)\\ &= \int_\Omega \left( \int_{\mathbb{S}^{n-1}} f\big(x,y\big) d\lambda_x^\infty(y) \right) d\pi_\infty(x).\end{aligned}$$

# 3 Weak and strong formulations

In this section, we show that equation (1.10) is in fact a weak formulation of the spatially inhomogeneous and anisotropic Gibbs–Thomson law (see (1.8)). This weak generalized $BV$-formulation also includes a boundary condition for the interface with the outer boundary.

**Theorem 3.1**
*Let $\Omega$ be a bounded domain with $C^1$–boundary, $\Gamma$ be a $C^2$–hypersurface and let $\partial\Gamma$ consists of a finite number of $C^1$–$(n-2)$–dimensional surfaces. If $(\chi, u)$ is a solution of (1.9) and (1.10) or (1.12) and* (1.10) *then the following conditions are satisfied:*

*(i) Inhomogeneous and anisotropic Gibbs–Thomson law*

$$\sigma_{,x}(x,\nu(t))\cdot\nu(t) + \nabla_{\Gamma(t)}\cdot\sigma_{,p}(x,\nu(t)) = u(t) \quad \text{on } \Gamma(t)\ \mathcal{H}^{(n-1)}\text{–a.e. for a.e. } t\in(0,T),$$

*where $\nabla_\Gamma$ denotes the tangential gradient of $\Gamma$.*

*(ii) Force balance condition*

$$\sigma_{,p}(x,\nu(t))\cdot\nu_\Omega(t) = 0 \qquad \text{on } \partial\Gamma(t)\cap\partial\Omega\ \mathcal{H}^{(n-2)}\text{–a.e. for a.e. } t\in(0,T),$$

*where $\nu_\Omega$ is the outer unit normal of $\partial\Omega$.*

**Proof:**
We consider equation (1.10) and take test functions of the structure $\xi = \eta\nu$ on $\Gamma$, where $\eta$ is an arbitrary function of $C_c^1(\Omega_T;\mathbb{R})$. For the first and third summand of the area part of equation (1.10), we derive

$$\int_0^T \int_{\Gamma(t)} \nu(t)\cdot\nabla\xi(t)\,\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt = \int_0^T \int_{\Gamma(t)} \nabla\eta(t)\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt$$

and

$$\int_0^T \int_{\Gamma(t)} \sigma(x,\nu(t))\nabla\cdot\xi(t)\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\Gamma(t)} \nu(t)\cdot\big(\nabla\eta(t)\cdot\nu(t) + \eta(t)\,\nabla\cdot\nu(t)\,\big)\,\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\Gamma(t)} \big(\nabla\eta(t) - \nabla_{\Gamma(t)}\eta(t)\big)\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt$$

$$+ \int_0^T \int_{\Gamma(t)} \eta(t)\,\kappa(t)\,\big(\sigma_{,p}(x,\nu(t))\cdot\nu(t)\big)\,d\mathcal{H}^{n-1}(t)\,dt,$$

where $\kappa(t) = \nabla_{\Gamma(t)}\cdot\nu(t)$ is the mean curvature. Applying the divergence theorem on manifolds yields

$$\int_0^T \int_{\Gamma(t)} \nabla_{\Gamma(t)}\eta(t)\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt + \int_0^T \int_{\Gamma(t)} \eta(t)\;\nabla_{\Gamma(t)}\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\Gamma(t)} \nabla_{\Gamma(t)}\cdot\big(\eta(t)\,\sigma_{,p}(x,\nu(t))\big)\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\Gamma(t)} \kappa(t)\,\eta(t)\,\big(\sigma_{,p}(x,\nu(t))\cdot\nu(t)\big)\,d\mathcal{H}^{n-1}(t)\,dt.$$

We infer

$$\int_0^T \int_{\Gamma(t)} \Big(\sigma\big(x,\nu(t)\big)\nabla\cdot\xi(t) - \nu(t)\cdot\nabla\,\xi(t)\sigma_{,p}\big(x,\nu(t)\big)\Big)\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\Gamma(t)} \eta(t)\;\nabla_{\Gamma(t)}\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt.$$

Since $\eta \in C_c^1(\Omega_T)$ was arbitrary, we end up with

$$\sigma_{,x}\big(x,\nu(t)\big)\cdot\nu(t) + \nabla_{\Gamma(t)}\cdot\sigma_{,p}(x,\nu(t)) = u(t)$$

on $\Gamma(t)$ $\mathcal{H}^{n-1}$-a.e. for a.e. $t \in (0,T)$.
To (ii): We choose arbitrary functions $\xi \in C^1(\overline{\Omega}_T;\mathbb{R}^n)$ with $\xi(t)\cdot\nu_\Omega(t) = 0$ on $\partial\Omega$ for a.e. $t \in (0,T)$ and an orthonormal basis $\tau_1(t) = \tau_\Gamma(t)$, $\tau_2(t)$, $\ldots$, $\tau_{n-1}(t)$ of the tangent space $T\,\Gamma(t)$, where $\tau_\Gamma(t)$ is the outer unit normal of $\partial\Gamma(t)$. Then, using the Einstein sum convention, we may express $\xi$ in the form $\xi = \eta_\nu\nu + \eta_{\tau_j}\tau_j$. Applying the divergence theorem on manifolds leads to

$$\int_0^T \int_{\Gamma(t)} \sigma(x,\nu(t))\nabla\cdot\big(\eta_{\tau_j}(t)\tau_j(t)\big)\,d\mathcal{H}^{n-1}(t)\,dt$$

$$= \int_0^T \int_{\partial\Gamma(t)} \sigma(x,\nu(t))\eta_{\tau_\Gamma}(t)\,d\mathcal{H}^{n-2}(t)$$

$$- \int_0^T \int_{\Gamma(t)} \nabla_{\Gamma(t)}\sigma(x,\nu(t))\cdot\eta_{\tau_j}(t)\tau_j(t)\,d\mathcal{H}^{n-1}(t)\,dt$$

$$+ \int_0^T \int_{\Gamma(t)} \sigma(x,\nu(t))\eta_{\tau_j}(t)\nu(t)\nabla\tau_j(t)\nu(t)\,d\mathcal{H}^{n-1}(t)dt.$$

Since $\big(\nabla(\eta_{\tau_j}\tau_j)\big)^T\nu = -(\nabla\nu)^T\big(\eta_{\tau_j}\tau_j\big)$, we have

$$\int_0^T\int_{\Gamma(t)}\nu(t)\cdot\nabla\big(\eta_{\tau_j}(t)\tau_j(t)\big)\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt$$
$$= -\int_0^T\int_{\Gamma(t)}\big(\eta_{\tau_j}(t)\tau_j(t)\big)\cdot\nabla\nu(t)\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)dt.$$

Thus, we get for (1.10) the following representation

$$\begin{aligned}
&\int_0^T\int_{\Gamma(t)}\Big(\sigma\big(x,\nu(t)\big)\nabla\cdot\xi(t)+\sigma_{,x}\big(x,\nu(t)\big)\cdot\xi(t)\\
&\qquad-\nu(t)\cdot\nabla\xi(t)\,\sigma_{,p}\big(x,\nu(t)\big)\Big)d\mathcal{H}^{n-1}(t)\,dt-\int_0^T\int_{\Gamma(t)}u(t)\,\xi(t)\cdot\nu(t)\,d\mathcal{H}^{n-1}(t)\,dt\\
&\quad=\int_0^T\int_{\partial\Gamma(t)}\Big(-\eta_\nu(t)\sigma_{,p}(x,\nu(t))\cdot\tau_\Gamma(t)+\sigma(x,\nu(t))\ \eta_{\tau_\Gamma}(t)\Big)\,d\mathcal{H}^{n-2}(t)\,dt\\
&\qquad+\int_0^T\int_{\Gamma(t)}\eta_\nu(t)\nabla_{\Gamma(t)}\cdot\sigma_{,p}(x,\nu(t))\,d\mathcal{H}^{n-1}(t)\,dt\\
&\qquad-\int_0^T\int_{\Gamma(t)}\big(\nabla_{\Gamma(t)}\sigma(x,\nu(t))-\nabla\nu(t)\sigma_{,p}(x,\nu(t))\big)\cdot\big(\eta_{\tau_j}(t)\tau_j(t)\big)\,d\mathcal{H}^{n-1}(t)\\
&\qquad+\int_0^T\int_{\Gamma(t)}\sigma(x,\nu(t))\eta_{\tau_j}(t)\nu(t)\nabla\tau_j(t)\nu(t)\,d\mathcal{H}^{n-1}(t)\,dt\\
&\qquad+\int_0^T\int_{\Gamma(t)}\sigma_{,x}\big(x,\nu(t)\big)\cdot\xi(t)\,d\mathcal{H}^{n-1}(t)\,dt-\int_0^T\int_{\Gamma(t)}u(t)\,\eta_\nu(t)d\mathcal{H}^{n-1}(t)\,dt=0.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^T\int_{\partial\Gamma(t)}\Big(\sigma(x,\nu(t))\ \eta_{\tau_\Gamma}(t)-\eta_\nu(t)\,\sigma_{,p}(x,\nu(t))\cdot\tau_\Gamma(t)\Big)\,d\mathcal{H}^{n-2}(t)dt\\
&\quad=\int_0^T\int_{\partial\Gamma(t)}\xi(t)\Big(\big(\sigma_{,p}(x,\nu(t))\cdot\nu(t)\big)\tau_\Gamma(t)-\nu(t)\big(\sigma_{,p}(x,\nu(t))\cdot\tau_\Gamma(t)\big)\Big)\,d\mathcal{H}^{n-2}(t)\,dt
\end{aligned}$$

we obtain by choosing suitable variations in the neighborhood of points of $\partial\Gamma$

$$(\sigma_{,p}(x,\nu(t))\cdot\nu(t))\tau_\Gamma(t)-(\sigma_{,p}(x,\nu(t))\cdot\tau_\Gamma(t))\,\nu(t)=l(t)\,\nu_\Omega(t)$$

with

$$l(t)=|(\sigma_{,p}(x,\nu(t))\cdot\nu(t))\tau_\Gamma(t)-(\sigma_{,p}(x,\nu(t))\cdot\tau_\Gamma(t))\,\nu(t)|$$

on $\Gamma(t)$ $\mathcal{H}^{n-1}$-a.e. for a.e. $t\in(0,T)$. It follows

$$l\,\nu_\Omega\cdot\tau_\Gamma=\sigma_{,p}(x,\nu)\cdot\nu,\qquad l\,\nu_\Omega\cdot\nu=-\sigma_{,p}(x,\nu)\cdot\tau_\Gamma,\qquad \nu_\Omega\cdot\tau_j=0\quad\text{ for } j\in\{2,\dots,n-1\}$$

on $\Gamma(t)$ $\mathcal{H}^{n-1}$-a.e. for a.e. $t\in(0,T)$. This shows

$$\begin{aligned}
\sigma_{,p}(x,\nu)\cdot\nu_\Omega&=(\sigma_{,p}(x,\nu)\cdot\nu)(\nu\cdot\nu_\Omega)+(\sigma_{,p}(x,\nu)\cdot\tau_j)\,(\tau_j\cdot\nu_\Omega)\\
&=\big(-(\sigma_{,p}(x,\nu)\cdot\nu)(\sigma_{,p}(x,\nu)\cdot\tau_\Gamma)+(\sigma_{,p}(x,\nu)\cdot\tau_\Gamma)\,(\sigma_{,p}(x,\nu)\cdot\nu)\big)/l\\
&=0
\end{aligned}$$

on $\Gamma(t)$ $\mathcal{H}^{n-1}$-a.e. for a.e. $t \in (0,T)$. ∎

We remark that the dependence of $\sigma$ on $x$ has no influence on the boundary condition at intersections of the interface with the outer boundary.

# 4 The discretization

The proofs of the existence theorems are based on minimization problems, cf. [LS95, Luc91, GS11]. For the degenerate problem, we choose an energy functional, which is similar to [LS95]. However, for the non-degenerate problem we introduce an energy functional, which differs from [Luc91, GS11].

Let $(0,T)$ be the time interval of interest with discretization fineness $h = \frac{T}{M}$, $M \in \mathbb{N}$. For $f$ and $u_D$ in Theorems 1.1 and 1.2, respectively, we choose discretizations $f^h$ and $u_D^h$ such that $f^h$ and $u_D^h$ are constant on the intervals $((k-1)h, kh], k = 1, \cdots, M$, and $f^h \to f$ in $L^2(\Omega_T)$ and $u_D^h \to u_D$ in $L^2(0,T;H^1(\Omega))$ as $h \to 0$. We also may assume that the boundary values of $u_D$ are extended in $\Omega$ such that $\triangle u_D(t) = 0$ for a.e. $t \in (0,T)$.

Now, we construct iteratively time discrete solutions $\chi^h$ and $u^h$ for time steps $h > 0$. To this end, we consider the following two minimization problems in each time step:

*Degenerate Stefan problem*

Minimize $\mathcal{F}_t^h : BV(\Omega;\{0,1\}) \to \mathbb{R}$,

$$\mathcal{F}_t^h(\chi) = \int_\Omega |\nabla\chi|_\sigma + \frac{h}{2}\int_\Omega \nabla v \nabla(v - u_D^h(t)) - \int_\Omega \chi u_D^h(t), \tag{4.1}$$

where $v \in H^1(\Omega)$ is the weak solution of

$$\chi - \chi^h(t-h) = h(\triangle v + f^h(t)), \qquad v = u_D^h(t)|_{\partial\Omega}. \tag{4.2}$$

Note that (4.2) is the implicit time discretization of (1.3) for $\chi = \chi^h(t)$ and $v = u^h(t)$.

*Non-degenerate Stefan problem*

Minimize $\mathcal{E}_t^h : BV(\Omega;\{0,1\}) \to \mathbb{R}$,

$$\mathcal{E}_t^h(\chi) = \int_\Omega |\nabla\chi|_\sigma + \frac{h}{2}\int_\Omega \nabla v \nabla(v - u_D^h(t)) + \frac{1}{2}\int_\Omega v^2 - \int_\Omega (v+\chi) u_D^h(t), \tag{4.3}$$

where $v \in H^1(\Omega)$ is the weak solution of

$$v + \chi - \chi^h(t-h) - u^h(t-h) = h(\triangle v + f^h(t)), \qquad v = u_D^h(t)|_{\partial\Omega}. \tag{4.4}$$

Note that (4.4) is the implicit time discretization of (1.1) for $\chi = \chi^h(t)$ and $v = u^h(t)$.

**Lemma 4.1**
*There exists a minimizer $\chi^h \in BV(\Omega;\{0,1\})$ of $\mathcal{F}_t^h$.*

**Proof:**
Let $\{\chi_k\}_{k\in\mathbb{N}}$, $\chi_k \in BV(\Omega;\{0,1\})$, be a minimizing sequence and $\{v_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of weak solutions of (4.2). In view of $\triangle u_D^h = 0$, we estimate

$$\mathcal{F}_t^h(\chi_k) \geq \int_\Omega |\nabla\chi_k|_\sigma + \frac{h}{2}\int_\Omega |\nabla(v_k - u_D^h(t))|^2 - \int_\Omega |u_D^h(t)|.$$

The uniform boundedness of $\{\chi_k\}_{\{k\in\mathbb{N}\}}$ in $L^2(\Omega;\{0,1\})$ and the $BV(\Omega)$–compactness imply that there exists a subsequence (still denoted by $\{\chi_k\}_{k\in\mathbb{N}}$) such that

$$\chi_k \to \hat\chi \quad \text{in} \;\; L^2(\Omega) \qquad \text{and} \qquad \hat\chi \in BV(\Omega;\{0,1\}).$$

In addition, by the uniform boundedness of $\{v_k\}_{k\in\mathbb{N}}$ in $H^1(\Omega)$ and by (4.2) we derive

$$v_k \to \hat v \qquad \text{in} \quad H^1(\Omega),$$

where $\hat v$ is the weak solution of (4.2) for $\chi = \hat\chi$. From this property and the lower semi-continuity of $\int_\Omega |\nabla\chi_k|_\sigma$, we conclude that $\hat\chi$ is a minimizer of $\mathcal{F}_t^h$. ■

**Lemma 4.2**
*There exists a minimizer* $\chi^h \in BV(\Omega;\{0,1\})$ *of* $\mathcal{E}_t^h$.

**Proof:**
Let $\{\chi_k\}_{k\in\mathbb{N}}$, $\chi_k \in BV(\Omega;\{0,1\})$, be a minimizing sequence and $\{v_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of weak solutions of (4.4). Due to $\triangle u_D^h = 0$, we have

$$\mathcal{F}_t^h(\chi_k) \geq \int_\Omega |\nabla\chi_k|_\sigma + \frac{h}{2}\int_\Omega |\nabla(v_k - u_D^h(t))|^2 + \frac{1}{2}\int_\Omega v_k^2 - \int_\Omega (|v_k|+1)|u_D^h(t)|.$$

Since $\{\chi_k\}_{\{k\in\mathbb{N}\}}$ is uniformly bounded in $L^2(\Omega;\{0,1\})$ and in $BV(\Omega)$, there exists a subsequence (still denoted by $\{\chi_k\}_{k\in\mathbb{N}}$) with

$$\chi_k \to \hat\chi \quad \text{in} \;\; L^2(\Omega) \qquad \text{and} \qquad \hat\chi \in BV(\Omega;\{0,1\}).$$

Moreover, the uniform boundedness of $\{v_k\}_{k\in\mathbb{N}}$ in $H^1(\Omega)$ implies that there exists a subsequence (still denoted by $\{\chi_k\}_{k\in\mathbb{N}}$) with

$$v_k \rightharpoonup \hat v \quad \text{in} \quad H^1(\Omega).$$

Since

$$\int_\Omega (\chi_k - \chi_l)(v_k - v_l) = -\int_\Omega (v_k - v_l)^2 - h\int_\Omega |\nabla(v_k - v_l)|^2 \to 0, \qquad \text{as} \quad k,l \to \infty,$$

we conclude

$$v_k \to \hat v \qquad \text{in} \quad H^1(\Omega),$$

where $\hat v$ is a weak solution of (4.4) for $\chi = \hat\chi$. This property and the lower semi-continuity of $\int_\Omega |\nabla\chi|_\sigma$ assures that $\hat\chi$ is a minimizer of $\mathcal{E}_t^h$. ■

From the minimization procedure, we obtain iteratively $\chi^h$ and $u^h$ ($u^h$ is the weak solution of

(4.2) and (4.4), respectively, for $\chi = \chi^h$) at the time steps $t = kh$, $k = 0, \ldots, M$. We extend $\chi^h$ and $u^h$ by $\chi^h(t) = \chi^h(kh)$ and $u^h(t) = u^h(kh)$ for $t \in ((k-1)h, kh]$, $k = 1, \ldots, M$, and abbreviate $\partial_t^{-h} g(t) := \frac{g(t)-g(t-h)}{h}$ for a function $g$.

Next, we establish weak formulations of the Euler–Lagrange equations for $\mathcal{F}_t^h$ and $\mathcal{E}_t^h$, which are connected to (1.8) and (1.10), respectively. To determine the first variation of the spatially inhomogeneous and anisotropic interfacial energy, we fall back on the following variational property, cf. [GK09]:

**Lemma 4.3**
*Let $\Phi : [-\tau_0, \tau_0] \times G \to G$ be a family of diffeomorphisms of $G$ onto itself with $G = \Omega$ or $G = \overline{\Omega}$. If $g \in BV(\Omega; \{0,1\})$ then*

$$
\begin{aligned}
&\frac{d}{d\tau} \int_\Omega \left|\nabla g\big(\Phi^{-1}(\tau, \cdot)\big)\right|_\sigma \Big|_{\tau=0} \\
&\quad = \int_\Omega \Bigg( \sigma\big(\Phi(\tau,x), \Psi(\tau,x)\nu_g(x)\big)\, tr\left(\frac{\partial \Phi_{,\tau}(\tau,x)}{\partial x}\right) + \sigma_{,x}\big(\Phi(\tau,x), \Psi(\tau,x)\nu_g(x)\big) \cdot \frac{d}{d\tau}\Phi(\tau,x) \\
&\qquad\qquad + \sigma_{,p}\big(\Phi(\tau,x), \Psi(\tau,x)\nu_g(x)\big) \cdot \frac{d}{d\tau}\big(\Phi_{,x}(\tau,x)\big)^{-T}\nu_g(x) \Bigg)\Bigg|_{\tau=0} \big|\nabla g(x)\big|,
\end{aligned}
$$

*where $tr$ denotes the trace, $\Psi(\tau,x) = |\det \Phi_{,x}(\tau,x)|\big(\Phi_{,x}(\tau,x)\big)^{-T}$ and $\nu_g = -\frac{\nabla g}{|\nabla g|}$ for $|\nabla g|$–a.e. $x \in \Omega$.*

Note that if $M$ is an $n \times n$–matrix then $Id + \eta M$, $\eta \in \mathbb{R}$, is invertible for $|\eta|$ sufficiently small. In addition,

$$
\det(Id + \eta M) = 1 + \eta \operatorname{tr}(M) + \frac{1}{2}\eta^2\Big((\operatorname{tr} M)^2 - \operatorname{tr}(M^2)\Big) + O(\eta^3),
$$

and

$$
(Id + \eta M)^{-1} = Id - \eta M + \eta^2 M^2 + O(\eta^3).
$$

**Theorem 4.4**
*Let $\Omega$ be a domain with Lipschitz–boundary. Further, let assumption A 2.1 be satisfied. If $\chi^h(t) \in BV(\Omega; \{0,1\})$ is a minimizer of $\mathcal{F}_t^h$ or $\mathcal{E}_t^h$ and $v = u^h(t)$ is the corresponding weak solution of (4.2) and* (4.4), *respectively, then*

$$
\begin{aligned}
&\int_\Omega \Big(\sigma\big(\cdot, \nu^h(t,\cdot)\big)\nabla \cdot \xi(\cdot) + \sigma_{,x}\big(\cdot, \nu^h(t,\cdot)\big) \cdot \xi(\cdot) - \nu^h(t,\cdot) \cdot \nabla\xi(\cdot)\, \sigma_{,p}\big(\cdot, \nu^h(t,\cdot)\big)\Big)|\nabla \chi^h(t,\cdot)| \\
&\qquad\qquad - \int_\Omega u^h(t,\cdot)\, \xi(\cdot) \cdot \nu^h(t,\cdot)|\nabla \chi^h(t,\cdot)| = 0 \qquad (4.5)
\end{aligned}
$$

*for all $\xi \in C_c^1(\Omega; \mathbb{R}^n)$, where $\nu^h(t) = -\frac{\nabla \chi^h(t)}{|\nabla \chi^h(t)|}$.*
*If, in addition, $\Omega$ is a bounded domain with $C^1$–boundary then (4.5) even holds for all $\xi \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$, where $\nu_\Omega$ is the outer unit normal of $\partial\Omega$.*

**Proof:**
Let $\xi \in C_c^1(\Omega;\mathbb{R}^n)$ and consider

$$\Phi(x\,;\tau) = x + \tau\,\xi(x) \tag{4.6}$$

for $x \in \Omega$ and $\tau \in \mathbb{R}$. Then, $\Phi(\cdot\,;\tau)$ is a diffeomorphism of $\Omega$ onto itself if $|\tau|$ is sufficiently small. Via the above diffeomorphism, we define

$$\chi_\tau^h(t,x) = \chi^h\big(t, \Phi^{-1}(x\,;\tau)\big).$$

Furthermore,

$$\nu_\tau^h(t,x) = -\frac{\nabla\chi_\tau^h(t,x)}{\big|\nabla\chi_\tau^h(t,x)\big|}.$$

We denote the weak solution of (4.2) and (4.4) for $\chi = \chi_\tau^h(t)$ by $u_\tau^h(t)$. Since $\chi^h(t) = \chi_\tau^h(t)|_{\tau=0}$ is a minimizer of $\mathcal{F}_t^h$ and $\mathcal{E}_t^h$, respectively, we obtain

$$0 = \frac{d}{d\tau}\mathcal{F}_t^h(\chi_\tau^h(t))\Big|_{\tau=0} \qquad \text{and} \qquad 0 = \frac{d}{d\tau}\mathcal{E}_t^h(\chi_\tau^h(t))\Big|_{\tau=0}, \quad \text{respectively.}$$

Next, we compute the above derivatives. Here, we take advantage from the following properties of $\Phi$:

(i) $\big|\det \Phi_{,x}(x\,;0)\big| = 1$,

(ii) $\Phi_{,x}^{-1}(\Phi(x\,;\tau)\,;\tau) = \Big(\Phi_{,x}(x\,;\tau)\Big)^{-1}$,

(iii) $\frac{d}{d\tau}\left(\Phi_{,x}(x\,;\tau)\right)^{-1}\Big|_{\tau=0} = -\nabla\xi(x)$.

Lemma 4.3 gives

$$\begin{aligned}
&\frac{d}{d\tau}\int_\Omega \sigma\Bigg(z, -\frac{\nabla_z\chi^h\big(t,\Phi^{-1}(z\,;\tau)\big)}{\big|\nabla_z\chi^h\big(t,\Phi^{-1}(z\,;\tau)\big)\big|}\Bigg)\big|\nabla_z\chi^h\big(t,\Phi^{-1}(z\,;\tau)\big)\big|\,\Bigg|_{\tau=0}\\
&\qquad = \int_\Omega \Big(\sigma\big(x,\nu^h(t)\big)\nabla\cdot\xi + \sigma_{,x}\big(x,\nu^h(t)\big)\cdot\xi - \nu^h(t)\cdot\nabla\xi\,\sigma_{,p}\big(x,\nu^h(t)\big)\Big)|\nabla\chi^h(t)|.
\end{aligned}$$

We abbreviate $w_\tau^h(t) = u_\tau^h(t) - u_D^h(t)$, $w^h(t) = u^h(t) - u_D^h(t)$ and utilize $\triangle u_D^h(t) = 0$.
Hence, the remaining parts of $\mathcal{F}_t^h$ at $\chi = \chi_\tau^h$ can be rewritten as

$$\begin{aligned}
&\frac{h}{2}\int_\Omega \nabla u_\tau^h(t)\nabla(u_\tau^h(t) - u_D^h(t)) - \int_\Omega \chi_\tau^h(t)u_D^h(t)\\
&= \frac{h}{2}\int_\Omega |\nabla w_\tau^h(t)|^2 - \int_\Omega \chi_\tau^h(t)u_D^h(t)\\
&= \frac{h}{2}\int_\Omega |\nabla\big(w_\tau^h(t) - w^h(t)\big)|^2 + h\int_\Omega \nabla\big(w_\tau^h(t) - w^h(t)\big)\nabla w^h(t) + \frac{h}{2}\int_\Omega |\nabla w^h(t)|^2 - \int_\Omega \chi_\tau^h(t)u_D^h(t)\\
&= \frac{h}{2}\int_\Omega |\nabla\big(w_\tau^h(t) - w^h(t)\big)|^2 - \int_\Omega (\chi_\tau^h(t) - \chi^h(t))w^h(t) + \frac{h}{2}\int_\Omega |\nabla w^h(t)|^2 - \int_\Omega \chi_\tau^h(t)u_D^h(t)\\
&= \frac{h}{2}\int_\Omega |\nabla\big(w_\tau^h(t) - w^h(t)\big)|^2 - \int_\Omega \chi_\tau^h(t)u^h(t) + \int_\Omega \chi^h(t)w^h(t) + \frac{h}{2}\int_\Omega |\nabla w^h(t)|^2.
\end{aligned} \tag{4.7}$$

Next, we compute the $\tau$-derivative of the first term in (4.7).
In the following, we denote by $C > 0$ some constant, which may differ from estimate to estimate. Note that

$$\begin{aligned}
&\frac{h}{\tau}\int_\Omega \Big|\nabla\big(w^h_\tau(t,z) - w^h(t,z)\big)\Big|^2 dz \\
&\qquad = -\int_\Omega \Big(\frac{\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)}{\sqrt{\tau}}\Big)\Big(\frac{w^h_\tau(t,z) - w^h(t,z)}{\sqrt{\tau}}\Big) dz \\
&\qquad \le C_\delta \int_\Omega \Big(\frac{\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)}{\sqrt{\tau}}\Big)^2 + \delta \int_\Omega \Big(\frac{w^h_\tau(t,z) - w^h(t,z)}{\sqrt{\tau}}\Big)^2 dz
\end{aligned}$$

for any $\delta > 0$ and some $C_\delta > 0$. In consequence, by Poincaré's inequality

$$\frac{1}{\tau}\int_\Omega \Big|\nabla\big(w^h_\tau(t,z) - w^h(t,z)\big)\Big|^2 dz \le C\int_\Omega \Big(\frac{\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)}{\sqrt{\tau}}\Big)^2 dz \tag{4.8}$$

for some constant $C > 0$.
Now, we show that the term on the right-hand side of (4.8) is uniformly bounded as $\tau \to 0$. Denoting $\Omega_0(t) = \{x \in \Omega : \chi^h(t,x) = 0\}$ and $\Omega_1(t) = \{x \in \Omega : \chi^h(t,x) = 1\}$ we estimate

$$\begin{aligned}
&\int_\Omega \big(\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)\big)^2 dz \\
&\qquad = \int_{\Omega_0(t)} \big|\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)\big| dz + \int_{\Omega_1(t)} \big|\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)\big| dz \\
&\qquad \le \Big|\Phi^{-1}(\Omega_0(t);\tau))\backslash\Omega_0(t)\Big| + \Big|\Phi^{-1}(\Omega_1(t);\tau)\backslash\Omega_1(t)\Big| \\
&\qquad \le 2\int_\Omega |\nabla\chi^h(t,z)| \max_{z\in\overline{\Omega}} \Big|\Phi^{-1}(z;\tau) - \Phi^{-1}(z;0)\Big| \\
&\qquad \le 2\int_\Omega |\nabla\chi^h(t,z)| \max_{z\in\overline{\Omega}} \Big|x - \Phi(z;\tau)\Big| \\
&\qquad \le 2\int_\Omega |\nabla\chi^h(t,z)| \tau \max_{z\in\overline{\Omega}} \big|\xi(z)\big| \\
&\qquad \le C\tau
\end{aligned}$$

for some constant $C > 0$ (independent of $t$). Hence,

$$\frac{1}{\tau}\int_\Omega \Big|\nabla\big(w^h_\tau(t,z) - w^h(t,z)\big)\Big|^2 dz \le C.$$

Furthermore, for any $q \in (2, 2^*]$ with $2^* = \frac{2n}{n-2}$ if $n \ge 3$ or any $q \in (2,\infty)$ if $n = 2$, we obtain

$$\begin{aligned}
&\frac{h}{\tau}\int_\Omega \big|\nabla\big(w^h_\tau(t,z) - w^h(t,z)\big)\big|^2 dz \\
&\qquad \le \int_\Omega \Big|\frac{\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)}{\sqrt{\tau}}\Big|\Big|\frac{w^h_\tau(t,z) - w^h(t,z)}{\sqrt{\tau}}\Big| dz \\
&\qquad \le \Big\|\frac{\chi^h(t,\Phi^{-1}(z;\tau)) - \chi^h(t,z)}{\sqrt{\tau}}\Big\|_{L^{\frac{q}{q-1}}(\Omega)} \Big\|\frac{w^h_\tau(t,z) - w^h(t,z)}{\sqrt{\tau}}\Big\|_{L^q(\Omega)} \\
&\qquad \le C\frac{1}{\sqrt{\tau}}|\tau|^{\frac{q-1}{q}} \Big\|\nabla\Big(\frac{w^h_\tau(t,z) - w^h(t,z)}{\sqrt{\tau}}\Big)\Big\|_{L^2(\Omega)} \\
&\qquad \to 0 \qquad \text{for } \tau \to 0.
\end{aligned}$$

In consequence,

$$\frac{d}{d\tau}h\int_\Omega\left|\nabla\Big(w^h_\tau(t,z)-w^h(t,z)\Big)\right|^2dz\bigg|_{\tau=0}=\lim_{\tau\to0}\frac{1}{\tau}h\int_\Omega\left|\nabla\Big(w^h_\tau(t,z)-w^h(t,z)\Big)\right|^2dz=0$$

In addition,

$$\begin{aligned}\frac{d}{d\tau}\int_\Omega\chi^h_\tau(t)u^h(t)dz\bigg|_{\tau=0}&=\int_\Omega\chi^h(t,x)u^h(t,x)\nabla\cdot\xi(x)dx+\int_\Omega\chi^h(t,x)\nabla u^h(t,x)\cdot\xi(x)\,dx\\&=\int_\Omega u^h(t)\,\xi(x)\cdot\nu^h(t)|\nabla\chi^h(t)|\,.\end{aligned}\tag{4.9}$$

This shows the claim for $\mathcal{F}^h_t$ since the remaining terms of (4.7) do not depend on $\tau$.

To verify the claim for $\mathcal{E}^h_t$, we observe

$$\begin{aligned}&\frac{h}{2}\int_\Omega\nabla u^h_\tau(t)\nabla(u^h_\tau(t)-u^h_D(t))+\frac12\int_\Omega\big(u^h_\tau(t)\big)^2-\int_\Omega(u^h_\tau(t)+\chi^h_\tau(t))u^h_D(t)\\&=\frac{h}{2}\int_\Omega|\nabla w^h_\tau(t)|^2+\frac12\int_\Omega(w^h_\tau(t))^2-\frac12\int_\Omega(u^h_D(t))^2-\int_\Omega\chi^h_\tau(t)u^h_D(t)\\&=\frac{h}{2}\int_\Omega|\nabla\big(w^h_\tau(t)-w^h(t)\big)|^2+h\int_\Omega\nabla\big(w^h_\tau(t)-w^h(t)\big)\nabla w^h(t)+\frac{h}{2}\int_\Omega|\nabla w^h(t)|^2+\frac12\int_\Omega(w^h_\tau(t))^2\\&\qquad-\frac12\int_\Omega(u^h_D(t))^2-\int_\Omega\chi^h_\tau(t)u^h_D(t)\\&=\frac{h}{2}\int_\Omega|\nabla\big(w^h_\tau(t)-w^h(t)\big)|^2-\int_\Omega\big(w^h_\tau(t)-w^h(t)\big)w^h(t)-\int_\Omega(\chi^h_\tau(t)-\chi^h(t))w^h(t)\\&\qquad+\frac12\int_\Omega(w^h_\tau(t))^2-\frac12\int_\Omega(u^h_D(t))^2+\frac{h}{2}\int_\Omega|\nabla w^h(t)|^2-\int_\Omega\chi^h_\tau(t)u^h_D(t)\\&=\frac{h}{2}\int_\Omega|\nabla\big(w^h_\tau(t)-w^h(t)\big)|^2+\frac12\int_\Omega(w^h_\tau(t)-w^h(t))^2+\frac12\int_\Omega(w^h(t))^2-\int_\Omega\chi^h_\tau(t)u^h(t)\\&\qquad+\int_\Omega\chi^h(t)w^h(t)-\frac12\int_\Omega(u^h_D(t))^2+\frac{h}{2}\int_\Omega|\nabla w^h(t)|^2.\end{aligned}\tag{4.10}$$

Since

$$\begin{aligned}&h\int_\Omega\big|\nabla(w^h_\tau(t,z)-w^h(t,z))\big|^2dz+\int_\Omega(w^h_\tau(t,z)-w^h(t,z))^2dz\\&\qquad=-\int_\Omega\big(\chi^h(t,\Phi^{-1}(z;\tau))-\chi^h(t,z)\big)\big(w^h_\tau(t,z)-w^h(t,z)\big)dz\end{aligned}$$

we may use the same argumentation as before to derive

$$\frac{d}{d\tau}\left(h\int_\Omega\left|\nabla\Big(w^h_\tau(t,z)-w^h(t,z)\Big)\right|^2dz+\int_\Omega\left|\Big(w^h_\tau(t,z)-w^h(t,z)\Big)\right|^2dz\right)\Bigg|_{\tau=0}=0$$

Due to (4.9), the assertion also follows for $\mathcal{E}^h_t$ since the remaining terms of (4.10) do not depend on $\tau$.

If $\Omega$ is a bounded domain with $C^1$–boundary, we may choose a family of diffeomorphisms $\Phi(\tau, \cdot)$, $\tau \in [-\tau_0, \tau_0]$, of $\overline{\Omega}$ onto itself given by the initial value problem

$$\Phi(0, x) = x \qquad \text{and} \qquad \Phi_{,\tau}(\tau, x) = \xi(\Phi(\tau, x)), \qquad x \in \overline{\Omega},$$

with $\xi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$. Then, $\Phi$ also fulfills the above properties (i)–(iii) and $|\Phi(x;\tau) - \Phi(x;0)| \leq \tau \max_{x\in\overline{\Omega}} |\xi(x)|$. Thus,

$$\int_\Omega \Big(\sigma\big(\cdot, \nu^h(t,\cdot)\big)\nabla\cdot\xi(\cdot) + \sigma_{,x}\big(\cdot, \nu^h(t,\cdot)\big)\cdot\xi(\cdot) - \nu^h(t,\cdot)\cdot\nabla\xi(\cdot)\,\sigma_{,p}\big(\cdot, \nu^h(t,\cdot)\big)\Big)|\nabla\chi^h(t,\cdot)|$$
$$- \int_\Omega u^h(t,\cdot)\,\xi(\cdot)\cdot\nu^h(t,\cdot)|\nabla\chi^h(t,\cdot)| = 0$$

for all $\xi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ with $\xi \cdot \nu_\Omega = 0$ on $\partial\Omega$, as required. ■

# 5 Convergence to solutions

## 5.1 The degenerate case

We are going to establish compactness of the discrete solutions $\chi^h$, $h > 0$, in $L^1(\Omega_T)$ similarly to [LS95].

**Lemma 5.1 (Uniform bound)**
*There exists a constant $C > 0$ (depending only on $\int_\Omega |\nabla\chi(0)|_\sigma$, $||u_D||_{W^{1,1}(0,T;H^1(\Omega))}$, $||f||_{L^2(\Omega_T)}$) such that*

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_\Omega |\nabla\chi^h(t)|_\sigma \;+\; \int_{\Omega_T} |\nabla u^h(t)|^2 \leq C. \tag{5.1}$$

**Proof:**
We first like to mention that for weak solutions $\tilde{u}^h(t)$, $h > 0$, of $-\triangle v = f^h(t)$ with $v = u^h_D(t)|_{\partial\Omega}$ it holds

$$\int_0^T ||\tilde{u}^h(t)||^2_{H^1(\Omega)}\,dt \leq D_1,$$

where $D_1 > 0$ is some constant depending on $||u_D||_{W^{1,1}(0,T;H^1(\Omega))}$ and $||f||_{L^2(\Omega_T)}$. In view of $\mathcal{F}^h_t(\chi^h(t)) \leq \mathcal{F}^h_t(\chi^h(t-h))$, we obtain

$$\int_\Omega |\nabla\chi^h(t)|_\sigma + \frac{h}{2}\int_\Omega \nabla u^h(t)\nabla\big(u^h(t) - u^h_D(t)\big)$$
$$\leq \int_\Omega |\nabla\chi^h(t-h)|_\sigma + \frac{h}{2}\int_\Omega f^h(t)\big(\tilde{u}^h(t) - u^h_D(t)\big) + \int_\Omega \big(\chi^h(t) - \chi^h(t-h)\big)u^h_D(t).$$

By Young's and Poincaré's inequality, we estimate

$$\int_\Omega |\nabla\chi^h(t)|_\sigma + hD_2\int_\Omega |\nabla u^h(t)|^2 \leq \int_\Omega |\nabla\chi^h(t-h)|_\sigma + hD_3||f^h(t)||^2_{L^2(\Omega)} + hD_3\,||u^h_D(t)||^2_{H^1(\Omega)}$$
$$+ h||\tilde{u}^h(t)||^2_{L^2(\Omega)} + \int_\Omega \big(\chi^h(t) - \chi^h(t-h)\big)u^h_D(t) \tag{5.2}$$

with some constants $D_2, D_3 > 0$. Since

$$\int_0^{jh} \int_\Omega |\partial_t^{-h} u_D^h(t)| \le \int_0^{jh} \int_\Omega |\partial_t u_D(t)|,$$

we obtain for $k = 1, 2, \dots, j$, $j \le M$,

$$\begin{aligned}
&\sum_{k=1}^{j} \int_\Omega \Big(\chi^h(kh) - \chi^h((k-1)h)\Big) u_D^h(kh) \\
&\qquad = - \int_h^{jh} \int_\Omega \partial_t^{-h} u_D^h(t) \chi^h(t-h) + \int_\Omega \chi^h(jh)\, u_D^h(jh) - \int_\Omega \chi^h(0)\, u_D^h(h) \\
&\qquad \le \int_h^{jh} \int_\Omega |\partial_t^{-h} u_D^h(t)| \; + 2\, ||u_D||_{L^\infty(0,T;L^1(\Omega))} \\
&\qquad \le D_4 ||u_D||_{W^{1,1}(0,T;L^1(\Omega))},
\end{aligned}$$

where $D_4 > 0$ is some constant.

Now we take inequality (5.2) iteratively for $t = kh$, $k \in \mathbb{N}$, and sum over $k = 1, 2, \dots, j$, $j \le M$, which leads to

$$\begin{aligned}
\int_\Omega |\nabla \chi^h(jh)|_\sigma + D_2 \int_{\Omega_{jh}} |\nabla u^h(t)|^2 \le \int_\Omega |\nabla \chi(0)|_\sigma + D_3 ||f||^2_{L^2(\Omega_T)} \\
+ D_5\, ||u_D||_{W^{1,1}(0,T;H^1(\Omega))} + D_6
\end{aligned}$$

for some constants $D_5 > 0$ and $D_6 > 0$. Hence, the assertion is obvious. ■

The following lemma is used to control time differences of $\chi^h$, see [LS95].

**Lemma 5.2 ([LS95])**
*Let $\varphi \in BV(\Omega)$ with $||\varphi||_{L^\infty(\Omega)} \le M$ for some constant $M > 0$. Then, there exist constants $C > 0$ and $\rho_0 > 0$ (depending only on $\Omega$ and $M$) such that for all $\rho \le \rho_0$*

$$\int_\Omega |\varphi| \le \rho \Big( \int_\Omega |\nabla \varphi| + C\, \mathcal{H}^{n-1}(\partial\Omega) \Big) + \frac{C}{\rho} ||\varphi||_{H^{-1}(\Omega)}.$$

**Lemma 5.3** (Compactness in $L^1(\Omega_T)$)

*(i) (Compactness in space)*
*The discrete solutions $\chi^h$, $h > 0$, are bounded in $L^1(0,T; BV(\Omega))$.*

*(ii) (Compactness in time, cf. [LS95])*
*The discrete solutions $\chi^h$, $h > 0$, fulfill*

$$\int_0^{T-\tau} \int_\Omega |\chi^h(\cdot + \tau) - \chi^h(\cdot)| \le C \tau^{1/4}$$

*for some $C > 0$.*

*In consequence,*

$$\chi^h \to \chi \qquad \text{in } L^1(\Omega_T) \tag{5.3}$$

*for a subsequence as $h \to 0$.*

**Proof:**
To (i): This property immediately follows from Lemma 5.1.
To (ii): Without loss of generality, we may assume $\tau = kh$ and $t = lh$. From (4.2) and Lemma 5.1, we infer

$$\begin{aligned}
||\chi^h(t+\tau) - \chi^h(t)||_{H^{-1}(\Omega)} &= \sup_{||g||_{H_0^1(\Omega)}=1} \left| \int_\Omega \big(\chi^h(t+\tau) - \chi^h(t)\big) g \right| \\
&= \sup_{||g||_{H_0^1(\Omega)}=1} \left| \int_t^{t+\tau} \int_\Omega \frac{\chi^h(s) - \chi^h(s-h)}{h} g \, ds \right| \\
&\leq \int_t^{t+\tau} \left|\left| \frac{\chi^h(s) - \chi^h(s-h)}{h} \right|\right|_{H^{-1}(\Omega)} ds \\
&\leq \tau^{\frac{1}{2}} \left( \int_t^{t+\tau} \Big( ||u^h(s)||^2_{H^1(\Omega)} + ||f^h(s)||^2_{L^2(\Omega)} \Big) \right)^{\frac{1}{2}} \leq C\tau^{\frac{1}{2}}.
\end{aligned} \tag{5.4}$$

Choosing $\rho = \tau^{1/4}$ in Lemma 5.2 shows (ii).

We infer from (i) and (ii) that $\{\chi^h\}$ is relatively compact in $L^1(\Omega_T)$ (cf. [Sim78, Sim87]), i.e. there exists a subsequence $\{\chi^{h_k}\}_{\{k\in\mathbb{N}\}}$ such that

$$\chi^{h_k} \to \chi \qquad \text{in} \quad L^1(\Omega_T).$$

■

## 5.2 The non-degenerate case

To pass to the continuous problem, we first establish a priori estimates for $u^h$ and $\chi^h$.

**Lemma 5.4 (Uniform bound)**
*There exists a constant $C > 0$ (depending only on $\int_\Omega |u(0)|^2$, $\int_\Omega |\nabla\chi(0)|_\sigma$, $||u_D||_{H^1(0,T;H^1(\Omega))}$, $||f||_{L^2(\Omega_T)}$) such that*

$$\operatorname{ess\,sup}_{t\in(0,T)} \left( \int_\Omega \big( (u^h(t))^2 + |\nabla\chi^h(t)| \big) \right) + \int_{\Omega_T} |\nabla u^h(t)|^2 \leq C \tag{5.5}$$

*and*

$$\int_0^T ||\partial_t^{-h}\big(u^h(t) + \chi^h(t)\big)||^2_{H^{-1}(\Omega)} \leq C. \tag{5.6}$$

**Proof:**
Equation (4.4) yields

$$\begin{aligned}
\frac{h}{2} \int_\Omega \big|\nabla\big(v - u_D^h(t)\big)\big|^2 = -\frac{1}{2} \int_\Omega \big(v + \chi - u^h(t-h) - \chi^h(t-h)\big)\big(v - u_D^h(t)\big) \\
+ \frac{h}{2} \int_\Omega f^h(t)\big(v - u_D^h(t)\big).
\end{aligned} \tag{5.7}$$

Utilizing (5.7), $\mathcal{E}_t^h$ can be rewritten in the following form:

$$\begin{aligned}\mathcal{E}_t^h(\chi) &= \int_\Omega |\nabla\chi|_\sigma + \frac{1}{2}\int_\Omega \big(u^h(t-h) + \chi^h(t-h) + hf^h(t)\big)\big(v - u_D^h(t)\big) \\ &\qquad - \frac{1}{2}\int_\Omega \big(v+\chi\big)\big(v - u_D^h(t)\big) + \frac{1}{2}\int_\Omega v^2 - \int_\Omega (v+\chi)\, u_D^h(t) \\ &= \int_\Omega |\nabla\chi|_\sigma + \frac{1}{2}\int_\Omega \big(u^h(t-h) + \chi^h(t-h) + hf^h(t)\big)\big(v - u_D^h(t)\big) \\ &\qquad - \frac{1}{2}\int_\Omega v u_D^h(t) - \frac{1}{2}\int_\Omega \chi\big(v + u_D^h(t)\big)\end{aligned}$$

Note that

$$\begin{aligned}\mathcal{E}_t^h(\chi^h(t-h)) &= \int_\Omega |\nabla\chi^h(t-h)|_\sigma - \frac{1}{2}\big(\hat{u}^h(t) - u^h(t-h)\big)\big(\hat{u}^h(t) - u_D^h(t)\big) \\ &\qquad + \frac{h}{2}\int_\Omega f^h(t)\big(\hat{u}^h(t) - u_D^h(t)\big) + \frac{1}{2}\int_\Omega \big(\hat{u}^h(t)\big)^2 - \int_\Omega \big(\hat{u}^h(t) + \chi^h(t-h)\big)u_D^h(t) \\ &= \int_\Omega |\nabla\chi^h(t-h)|_\sigma + \frac{1}{2}\int_\Omega \big(u^h(t-h) + hf^h(t)\big)\big(\hat{u}^h(t) - u_D^h(t)\big) \\ &\qquad - \frac{1}{2}\int_\Omega \big(\hat{u}^h(t) + \chi^h(t-h)\big)u_D^h(t) - \frac{1}{2}\int_\Omega \chi^h(t-h)u_D^h(t),\end{aligned} \tag{5.8}$$

where $\hat{u}^h(t)$ is the weak solution of

$$v - u^h(t-h) = h\,\big(\triangle v + f^h(t)\big), \qquad v = u_D^h(t)|_{\partial\Omega}. \tag{5.9}$$

Due to $\mathcal{E}_t^h(\chi^h(t)) \le \mathcal{E}_t^h(\chi^h(t-h))$ we conclude

$$\begin{aligned}&\frac{2}{h}\big(\mathcal{E}_t^h(\chi^h(t)) - \mathcal{E}_t^h(\chi^h(t-h))\big) = \\ &\qquad \frac{2}{h}\int_\Omega \big(|\nabla\chi^h(t)|_\sigma - |\nabla\chi^h(t-h)|_\sigma\big) - \int_\Omega \frac{\chi^h(t) - \chi^h(t-h)}{h}u^h(t) \\ &\qquad + \int_\Omega \big(u^h(t-h) + hf^h(t)\big)\frac{u^h(t) - \hat{u}^h(t)}{h} \\ &\qquad - \int_\Omega \left(\frac{u^h(t) - \hat{u}^h(t)}{h} + \frac{\chi^h(t) - \chi^h(t-h)}{h}\right)u_D^h(t) \le 0.\end{aligned} \tag{5.10}$$

Multiplying (4.4) by $(u^h(t) - u_D^h(t))$ gives

$$\begin{aligned}&\frac{u^h(t) - u^h(t-h)}{h}u^h(t) - \frac{u^h(t) - u^h(t-h)}{h}u_D^h(t) + \frac{\chi^h(t) - \chi^h(t-h)}{h}\big(u^h(t) - u_D^h(t)\big) \\ &\qquad = -\int_\Omega |\nabla\big(u^h(t) - u_D^h(t)\big)|^2 + \int_\Omega f^h(t)\big(u^h(t) - u_D^h(t)\big).\end{aligned} \tag{5.11}$$

In addition, testing (5.9) with $\big(\hat{u}^h(t) - u_D^h(t)\big)$ yields

$$\begin{aligned}&\frac{\hat{u}^h(t) - u^h(t-h)}{h}\hat{u}^h(t) - \frac{\hat{u}^h(t) - u^h(t-h)}{h}u_D^h(t) \\ &\qquad = -\int_\Omega |\nabla\big(\hat{u}^h(t) - u_D^h(t)\big)|^2 + \int_\Omega f^h(t)\big(\hat{u}^h(t) - u_D^h(t)\big).\end{aligned} \tag{5.12}$$

Adding (5.11) and (5.12) shows

$$
\begin{aligned}
-\int_\Omega |\nabla\big(u^h(t)-u^h_D(t)\big)|^2 &- \int_\Omega |\nabla\big(\hat u^h(t)-u^h_D(t)\big)|^2 + \int_\Omega f^h(t)\big(u^h(t)-2u^h_D(t)+\hat u^h(t)\big)\\
&= \frac{1}{h}\Big(\big(u^h(t)\big)^2 - u^h(t-h)u^h(t) + \big(\hat u^h(t)\big)^2 - u^h(t-h)\hat u^h(t)\\
&\qquad - \big(u^h(t)-2u^h(t-h)+\hat u^h(t)\big)u^h_D(t) + h\,\partial_t^{-h}\chi\big(u^h(t)-u^h_D(t)\big)\Big) \qquad (5.13)\\
&\ge \frac{1}{h}\Big(\big(u^h(t)\big)^2 - \big(u^h(t-h)\big)^2 - u^h(t-h)\big(u^h(t)-\hat u^h(t)\big)\\
&\qquad - \big(u^h(t)-2u^h(t-h)+\hat u^h(t)\big)u^h_D(t) + h\,\partial_t^{-h}\chi^h(t)\big(u^h(t)-u^h_D(t)\big)\Big).
\end{aligned}
$$

Moreover, adding (5.10) and (5.13) leads to

$$
\begin{aligned}
&\frac{2}{h}\int_\Omega \big(|\nabla\chi^h(t)|_\sigma - |\nabla\chi^h(t-h)|_\sigma\big) - 2\int_\Omega \partial_t^{-h}\big(u^h_D(t)(u^h(t)+\chi^h(t))\big)\\
&\qquad + 2\int_\Omega \partial_t^{-h}u^h_D(t)\,\big(u^h(t-h)+\chi^h(t-h)\big) + \int_\Omega \frac{(u^h(t))^2-(u^h(t-h))^2}{h}\\
&\quad\le -\int_\Omega \big(|\nabla(u^h(t)-u^h_D(t))|^2 + |\nabla(\hat u^h(t)-u^h_D(t))|^2\big) + 2\int_\Omega f^h(t)\big(\hat u^h(t)-u^h_D(t)\big).
\end{aligned}
$$

From (4.4), we deduce

$$
\begin{aligned}
||\hat u^h(t)-u^h(t)||^2_{L^2(\Omega)} \le ||\chi^h(t)-\chi^h(t-h)||_{L^2(\Omega)}\,||\hat u^h(t)-u^h(t)||_{L^2(\Omega)}\\
- h||\nabla\big(\hat u^h(t)-u^h(t)\big)||^2_{L^2(\Omega)}
\end{aligned}
$$

and therefore,

$$
||\hat u^h(t)-u^h(t)||_{L^2(\Omega)} \le ||\chi^h(t)-\chi^h(t-h)||_{L^2(\Omega)}.
$$

Hence, we obtain

$$
\begin{aligned}
\int_\Omega |f^h(t)\big(\hat u^h(t)-u^h_D(t)\big)| \le\ & ||f^h(t)||_{L^2(\Omega)}||\chi^h(t)-\chi^h(t-h)||_{L^2(\Omega)}\\
&+ C_\delta||f^h(t)||^2_{L^2(\Omega)} + \delta||u^h(t)-u^h_D(t)||^2_{L^2(\Omega)}
\end{aligned}
$$

for any $\delta>0$ and some $C_\delta>0$. Note that

$$
\int_0^t\int_\Omega |\partial_t^{-h}u^h_D(s)|^2 \le ||\partial_t u_D||^2_{L^2(\Omega_t)}.
$$

By means of Poincaré's and Young's inequality, we finally establish

$$
\begin{aligned}
&\operatorname{ess\,sup}_{t\in(0,T)}\Big(\int_\Omega \big((u^h(t))^2 + |\nabla\chi^h(t)|\big)\Big) + \int_0^T\int_\Omega |\nabla u^h(t)|^2 dx\,dt\\
&\qquad\le C_1\bigg(\int_\Omega |\nabla\chi(0)|_\sigma + \int_\Omega |u(0)|^2 + ||u_D||^2_{H^1(0,T;H^1(\Omega))} + ||f||^2_{L^2(\Omega_T)}\bigg) + C_2,
\end{aligned}
$$

where $C_1, C_2>0$ are some constants and (5.5) is established.
Due to (4.4), we obtain for $\eta\in H^1_0(\Omega)$ with $||\eta||_{H^1_0(\Omega)}\le 1$

$$
\int_\Omega \partial_t^{-h}\big(u^h(t)+\chi^h(t)\big)\eta \le \Big(\int_\Omega \big(|\nabla u^h(t)|^2 + |f^h(t)|^2\big)\Big)^{1/2}.
$$

From (5.5), we infer

$$\int_0^T ||\partial_t^{-h}\big(u^h(t)+\chi^h(t)\big)||^2_{H^{-1}(\Omega)} \le C_3$$

for some constant $C_3 > 0$. ∎

Next we take advantage from an $L^1$-bound for fractional time derivatives of $\chi^h$ and $u^h$ (see [Luc90, Luc91] ), which ensures compactness of $\chi^h$ and $u^h$ in $L^1(\Omega_T)$.

**Lemma 5.5 (Compactness in time, cf. [Luc90, Luc91])**
*Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-boundary. Furthermore, let*

$$\begin{aligned}&u_D \in H^1(\Omega_T), \quad u \in L^\infty(0,T;L^2(\Omega)), \quad u-u_D \in L^2(0,T,H^1_0(\Omega)),\\ &\chi \in L^\infty(0,T;BV(\Omega;\{0,1\}))\end{aligned}$$

*and*

$$\partial_t(u+\chi) \in L^2(0,T;H^{-1}(\Omega)).$$

*Then, there exists a constant $C>0$ (depending on the above norms) such that*

$$\int_0^{T-\tau}\int_\Omega |\chi(\cdot+\tau)-\chi(\cdot)| + |u(\cdot+\tau)-u(\cdot)| \le C\tau^{\delta_n}$$

*with $1/\delta_n = 13 - \frac{8}{n}$.*

Due to the a priori estimates and Lemma 5.5 we can select (weakly) convergent subsequences as following.

**Corollary 5.6**
*There exist*

$$u \in \big(u_D + L^2(0,T;H^1_0(\Omega))\big) \cap L^\infty(0,T;L^2(\Omega)), \quad u_D \in H^1(\Omega_T),$$

*and*

$$\chi \in L^\infty\big(0,T;BV(\Omega;\{0,1\})\big)$$

*such that*

*(i)* $u^h \rightharpoonup u$ *in* $L^2\big(0,T;H^1(\Omega)\big)$,
*(ii)* $u^h \to u$ *in* $L^1\big(0,T;L^1(\Omega)\big)$,
*(iii)* $\chi^h \to \chi$ *in* $L^2\big(0,T;L^2(\Omega)\big)$,
*(iv)* $u^h(t) \to u(t)$ *in* $L^1(\Omega)$ *for a.e.* $t \in (0,T)$,
*(v)* $\chi^h(t) \to \chi(t)$ *in* $L^2(\Omega)$ *for a.e.* $t \in (0,T)$,

*for some subsequence as $h \to \infty$.*

In the following lemma, we show that for the non-degenerate problem loss of surface area is excluded in the limit.

**Lemma 5.7**
*The functions $\chi^h(t)$, $h>0$, fulfill for a.e. $t \in (0,T)$:*

$$\int_\Omega |\nabla\chi^h(t)|_\sigma \to \int_\Omega |\nabla\chi(t)|_\sigma \qquad \text{as } h \to 0\,.$$

**Proof:**
Since $\chi^h(t) \to \chi(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$, we immediately get

$$\int_\Omega |\nabla \chi(t)|_\sigma \le \liminf_{h\to 0} \int_\Omega |\nabla \chi^h(t)|_\sigma \quad \text{for a.e. } t \in (0,T)$$

by the lower semicontinuity property of $\int_\Omega |\nabla \chi^h(t)|_\sigma$.
Now, we prove the opposite inequality. Since

$$\mathcal{E}_t^h\big(\chi^h(t)\big) \le \mathcal{E}_t^h\big(\chi(t)\big)$$

we derive

$$\int_\Omega \Big( |\nabla \chi^h(t)|_\sigma + \frac{1}{2}(u^h(t))^2 + \frac{h}{2}\big|\nabla(u^h(t) - u_D^h(t))\big|^2 - \big(u^h(t) + \chi^h(t)\big)u_D^h(t)\Big) \le \int_\Omega \Big( |\nabla \chi(t)|_\sigma + \frac{1}{2}(\hat{v}^h(t))^2 + \frac{h}{2}\big|\nabla(\hat{v}^h(t) - u_D^h(t))\big|^2 - \big(\hat{v}^h(t) + \chi(t)\big)u_D^h(t)\Big), \quad (5.14)$$

where $\hat{v}^h(t)$ is the weak solution of

$$\frac{v - u^h(t-h)}{h} + \frac{\chi(t) - \chi^h(t-h)}{h} = \triangle v + f^h(t), \qquad v = u_D^h(t)|_{\partial\Omega}.$$

Note that from (4.4), we conclude

$$\int_\Omega \big(u^h(t) - \hat{v}^h(t)\big)^2 = -\int_\Omega \big(\chi^h(t) - \chi(t)\big)\big(u^h(t) - \hat{v}^h(t)\big) - h\int_\Omega |\nabla\big(u^h(t) - \hat{v}^h(t)\big)|^2.$$

In consequence,

$$||u^h(t) - \hat{v}^h(t)||_{L^2(\Omega)} \le ||\chi^h(t) - \chi(t)||_{L^2(\Omega)} \to 0 \qquad \text{as } h \to 0$$

for a.e. $t \in (0,T)$. We estimate

$$\Big| \int_\Omega \Big(\frac{1}{2}u^h(t) - u_D^h(t)\Big)u^h(t) - \int_\Omega \Big(\frac{1}{2}\hat{v}^h(t) - u_D^h(t)\Big)\,\hat{v}^h(t)\Big| \le ||u_D^h(t)||_{L^2(\Omega)}||u^h(t) - \hat{v}^h(t)||_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \big(|u^h(t)| + |\hat{v}^h(t)|\big)\,|u^h(t) - \hat{v}^h(t)| \to 0 \qquad \text{as } h \to 0,$$

and

$$\Big| \int_\Omega \big(\chi^h(t) - \chi(t)\big)u_D^h(t)\Big| \le ||\chi^h(t) - \chi(t)||_{L^2(\Omega)}\,||u_D^h(t)||_{L^2(\Omega)} \to 0 \qquad \text{as } h \to 0$$

for a.e. $t \in (0,T)$ since $u^h(t) - \hat{v}^h(t) \to 0$, $\chi^h(t) \to \chi(t)$ and $u_D^h(t) \to u_D(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$. In addition,

$$\Big| h\int_\Omega \big|\nabla(\hat{v}^h(t) - u_D^h(t))\big|^2 - h\int_\Omega \big|\nabla(u^h(t) - u_D^h(t))\big|^2\Big| = \Big| \int_\Omega \Big( (\hat{v}^h(t))^2 - (u^h(t))^2 - \big(u^h(t-h) + u_D^h(t)\big)\big(\hat{v}^h(t) - u^h(t)\big) - hf^h(t)\big(\hat{v}^h(t) - u^h(t)\big) - \big(\chi(t) - \chi^h(t)\big)u_D^h(t) + \chi(t)\,\hat{v}^h(t) - \chi^h(t)\,u^h(t) - \chi^h(t-h)\big(\hat{v}^h(t) - u^h(t)\big)\Big)\Big| \to 0 \quad \text{as } h \to 0$$

for a.e. $t \in (0,T)$. From (5.14), we conclude

$$\int_\Omega |\nabla\chi(t)|_\sigma \geq \limsup_{h\to 0} \int_\Omega |\nabla\chi^h(t)|_\sigma$$

for a.e. $t \in (0,T)$. ■

### 5.3 The spatially inhomogeneous and anisotropic Gibbs–Thomson law

Before we pass to the limit in the weak formulation of the discrete spatially inhomogeneous and anisotropic Gibbs–Thomson law, we verify some approximation properties, cf. [GK09].

**Lemma 5.8**
*Suppose*

$$\int_\Omega \sigma(\cdot,\nu^h(t,\cdot))\big|\nabla\chi^h(t,\cdot)\big| \to \int_\Omega \sigma(\cdot,\nu(t,\cdot))\big|\nabla\chi(t,\cdot)\big|, \qquad h\to 0, \tag{5.15}$$

*for a.e. $t \in (0,T)$, where $\nu^h = -\nabla\chi^h/|\nabla\chi^h|$ and $\nu = -\nabla\chi/|\nabla\chi|$.*
*Then, using the same notation as in Proposition 2.5:*

*(i)* $\int_{\Omega\times\mathbb{S}^{n-1}} \sigma(\cdot,\cdot) d\Theta_\infty(t,\cdot,\cdot) \leq \int_\Omega \sigma(\cdot,\nu(t,\cdot))|\nabla\chi(t,\cdot)|$ *for a.e. $t \in (0,T)$.*

*(ii) There exists a sequence $\{g_t^l\}_{l\in\mathbb{N}}$ of functions $g_t^l \in C_c^1(\Omega)$, $t\in(0,T)$, such that*

$$g_t^l \to \sigma_{,p}(\cdot,\nu(t,\cdot)) \qquad \text{in } L^1(|\nabla\chi(t,\cdot)|)$$

*for a.e. $t \in (0,T)$.*

*(iii)* $\lambda_x^\infty(t) = \delta_{y=\nu(t,x)}$ *for $|\nabla\chi(t)|$-a.e. $x\in\Omega$ and a.e. $t\in(0,T)$.*

**Proof:**
To (i): Due to Proposition 2.5, we infer

$$\begin{aligned}\int_{\Omega\times\mathbb{S}^{n-1}} \sigma(\cdot,\cdot)\, d\Theta_\infty(t,\cdot,\cdot) &\leq \liminf_{j\to\infty} \int_{\Omega\times\mathbb{S}^{n-1}} \sigma(\cdot,\cdot)\; d\Theta_{h_j}(t,\cdot,\cdot) \\ &= \liminf_{j\to\infty} \int_\Omega \sigma(\cdot,\nu^{h_j}(t,\cdot))\, \big|\nabla\chi^{h_j}(t,\cdot)\big| \\ &= \int_\Omega \sigma(\cdot,\nu(t,\cdot))\, |\nabla\chi(t,\cdot)|\end{aligned}$$

for a.e. $t\in(0,T)$.
To (ii): Smooth approximations $g_t^l$ for the Cahn–Hoffman vector $\sigma_{,p}$ can be constructed as follows: Due to (2.2), there exists for every $\delta>0$ and a.e. $t\in(0,T)$ approximative functions $g_t^\delta \in K_\sigma$ such that

$$\int_\Omega \big(\sigma(\cdot,\nu(t,\cdot)) - g_t^\delta(\cdot)\cdot\nu(t,\cdot)\big)\, |\nabla\chi(t,\cdot)| \leq \delta^2.$$

Thus, by Lemma 2.3,

$$\int_\Omega |\sigma_{,p}(\cdot,\nu(t,\cdot)) - g_t^\delta(\cdot)|\, |\nabla\chi(t,\cdot)| \leq C_1\,\delta$$

for some constant $C_1 > 0$ and a.e. $t \in (0,T)$. This implies the existence of a sequence $\{g_t^l\}_{l\in\mathbb{N}}$, $g_t^l \in C_c^1(\Omega;\mathbb{R}^n)$, with $g_t^l \to \sigma_{,p}(\cdot,\nu(t,\cdot))$ in $L^1(|\nabla\chi(t,\cdot)|)$ for a.e. $t \in (0,T)$ since $\delta > 0$ may be chosen arbitrarily small.
To (iii): Since $\chi^h(t) \to \chi(t)$ in $L^1(\Omega)$ for a.e. $t \in (0,T)$ and $\limsup_{h\to 0}\int_\Omega |\nabla\chi^h(t)|$ is bounded for a.e. $t \in (0,T)$, we obtain

$$\nabla\chi^h(t) \to \nabla\chi(t) \qquad \text{weakly}^*$$

for a.e. $t \in (0,T)$. Hence, we can choose a set $S \subset (0,T)$ of Lebesgue measure zero such that $\chi^h(t) \to \chi(t)$ in $L^1(\Omega)$ and $\nabla\chi^h(t) \to \nabla\chi(t)$ weakly$^*$ for $t \in (0,T)\backslash S$.
From Proposition 2.5, we conclude that there exist a sequence $\{h_j\}_{j\in\mathbb{N}}$ and a non-negative Radon measure $\Theta_\infty(t) \equiv \pi_\infty(t)\otimes\lambda_x^\infty(t)$ on $\Omega\times\mathbb{S}^{n-1}$, $t \in (0,T)\backslash S$, such that

(a) $\Theta_{h_j}(t) \equiv |\nabla\chi^{h_j}(t)| \otimes \delta_{\nu^{h_j}(t)} \to \Theta_\infty(t) \equiv \pi_\infty(t)\otimes\lambda_x^\infty(t)$ weakly$^*$, $\delta_y$ Dirac mass,

(b) $|\nabla\chi^{h_j}(t)| \to \pi_\infty(t)$ weakly$^*$,

(c) $\pi_\infty(t) \geq |\nabla\chi(t)|$,

(d)

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega F\big(x,\nu^{h_j}(t,x)\big)|\nabla\chi^{h_j}(t,x)| &= \int_{\Omega\times\mathbb{S}^{n-1}} F\big(x,y\big)d\Theta_\infty(t,x,y)\\ &= \int_\Omega\bigg(\int_{\mathbb{S}^{n-1}} F\big(x,y\big)d\lambda_x^\infty(t,y)\bigg)d\pi_\infty(t,x)\end{aligned}$$

for any $F \in C_c(\Omega\times\mathbb{R}^n)$ and all $t \in (0,T)\backslash S$.

For any $\hat{x} \in \Omega$, we take $r > 0$ such that $B(\hat{x},r) = \{x \in \mathbb{R}^n : ||x-\hat{x}|| < r\} \Subset \Omega$ and set

$$F_g(x,y;t) = \Phi_1(x)\Phi_2(y)|\sigma_{,p}(x,y) - g_t(x)|^2,$$

where $\Phi_1 \in C_c(\Omega)$ with $0 \leq \Phi_1 \leq 1$ in $\Omega$ and $\Phi_1 \equiv 1$ in $B(\hat{x},r)$ and $\Phi_2 \in C_c(\mathbb{R}^n)$ with $\Phi_2(y) = 0$ in $\{y \in \mathbb{R}^n : ||y|| < h\}$ for some $h > 0$, $\Phi_2(y) = 1$ on $\mathbb{S}^{n-1}$ and $g_t \in K_\sigma(\Omega)$. Consequently, $F_g(\cdot,\cdot;t) \in C_c(\Omega\times\mathbb{R}^n)$. Proposition 2.5 assures (modulo a subsequence)

$$\begin{aligned}&\int_\Omega \Phi_1(x)\bigg(\int_{\mathbb{S}^{n-1}}\Phi_2(y)|\sigma_{,p}(x,y)-g_t(x)|^2 d\lambda_x^\infty(t,y)\bigg)\big|\nabla\chi(t,x)\big|\\ &\qquad\leq \int_\Omega \Phi_1(x)\bigg(\int_{\mathbb{S}^{n-1}}\Phi_2(y)|\sigma_{,p}(x,y)-g_t(x)|^2 d\lambda_x^\infty(t,y)\bigg)d\pi_\infty(t,x)\\ &\qquad= \lim_{j\to\infty}\int_\Omega \Phi_1(x)\Phi_2(\nu^{h_j}(t,x))|\sigma_{,p}(x,\nu^{h_j}(t,x)) - g_t(x)|^2|\nabla\chi^{h_j}(t,x)|\\ &\qquad\leq \lim_{j\to\infty}\int_\Omega |\sigma_{,p}(x,\nu^{h_j}(t,x)) - g_t(x)|^2|\nabla\chi^{h_j}(t,x)|\end{aligned} \tag{5.16}$$

for every $t \in (0,T)\backslash S$. Taking advantage from Lemma 2.3, we estimate

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega C\,|\sigma_{,p}(x,\nu^{h_j}(t,x))-g_t(x)|^2|\nabla\chi^{h_j}(t,x)|\\ &\le \lim_{j\to\infty}\int_\Omega \big(\sigma(x,\nu^{h_j}(t,x))-g_t(x)\cdot\nu^{h_j}(t,x)\big)|\nabla\chi^{h_j}(t,x)|\\ &=\int_\Omega\Big(\sigma(x,\nu(t,x))-g_t(x)\cdot\nu(t,x)\Big)|\nabla\chi(t,x)|\\ &\le\int_\Omega\big|\sigma_{,p}(x,\nu(t,x))-g_t(x)\big||\nabla\chi(t,x)|\end{aligned}\tag{5.17}$$

for every $t \in (0,T)\backslash S$, where $C > 0$ is some constant. Hence, (ii) combined with (5.16) and (5.17) shows

$$\int_\Omega \Phi_1(x)\bigg(\int_{\mathbb{S}^{n-1}}|\sigma_{,p}(x,y)-\sigma_{,p}(x,\nu(t,x))|^2 d\lambda_x^\infty(t,y)\bigg)\,|\nabla\chi(t,x)|=0$$

for $t \in (0,T)\backslash S$. In particular

$$\int_\Omega \Phi_1(x)\bigg(\int_{\mathbb{S}^{n-1}}|\sigma_{,p}(x,y)\cdot y-\sigma_{,p}(x,\nu(t,x))\cdot y|^2 d\lambda_x^\infty(t,y)\bigg)\,|\nabla\chi(t,x)|=0$$

for $t \in (0,T)\backslash S$. This implies, according to Lemma 2.2 (ii),

$$\int_{\mathbb{S}^{n-1}}\big|\nu(t,x)-y\big|^4 d\lambda_x^\infty(t,y)=0 \quad \text{for } |\nabla\chi(t)|\text{–a.e. } x\in B(\hat{x},r) \ \text{ and } \ t\in(0,T)\backslash S.$$

Hence we obtain that $\lambda_x^\infty$ is a Dirac mass, i.e. $\lambda_x^\infty(t)=\delta_{y=\nu(t,x)}$, for $|\nabla\chi(t)|$–a.e. $x \in B(\hat{x},r)$ and $t \in (0,T)\backslash S$ and the claim follows as $\hat{x} \in \Omega$ was arbitrary. ■

**Lemma 5.9**
*Let $\Omega$ be a bounded domain with Lipschitz–boundary and suppose assumption A 2.1 is satisfied. If $\chi^h(t) \in BV(\Omega;\{0,1\})$ is a minimizer of $\mathcal{F}_t^h$ and condition (5.15) is satisfied, or if $\chi^h(t) \in BV(\Omega;\{0,1\})$ is a minimizer of $\mathcal{E}_t^h$, then*

$$\begin{aligned}\lim_{h\to0}\int_{\Omega_T}\Big(\sigma\big(\cdot,\nu^h(t,\cdot)\big)\nabla\cdot\xi(t,\cdot)+\sigma_{,x}\big(\cdot,\nu^h(t,\cdot)\big)\cdot\xi(t,\cdot)-\nu^h(t,\cdot)\cdot\nabla\xi(t,\cdot)\,\sigma_{,p}\big(\cdot,\nu^h(t,\cdot)\big)\Big)|\nabla\chi^h(t,\cdot)|\\ =\int_{\Omega_T}\Big(\sigma\big(\cdot,\nu(t,\cdot)\big)\nabla\cdot\xi(t,\cdot)+\sigma_{,x}\big(\cdot,\nu(t,\cdot)\big)\cdot\xi(t,\cdot)-\nu(t,\cdot)\cdot\nabla\xi(t,\cdot)\,\sigma_{,p}\big(\cdot,\nu(t,\cdot)\big)\Big)|\nabla\chi(t,\cdot)|\end{aligned}\tag{5.18}$$

*for all $\xi \in C_c^1(\Omega_T;\mathbb{R}^n)$, where $\nu^h = -\frac{\nabla\chi^h}{|\nabla\chi^h|}$ and $\nu = -\frac{\nabla\chi}{|\nabla\chi|}$.*
*If, in addition, $\Omega$ is a bounded domain with $C^1$–boundary then (5.18) is satisfied for all $\xi \in C^1(\overline{\Omega}_T;\mathbb{R}^n)$ with $\xi\cdot\nu_\Omega = 0$ on $\partial\Omega$, where $\nu_\Omega$ is the outer unit normal of $\partial\Omega$.*

**Proof:**
In view of Lemma 5.8 (i), we have

$$\int_{\Omega\times\mathbb{S}^{n-1}} \sigma(x,y)\,d\Theta_\infty(t,x,y) \le \int_\Omega \sigma(x,\nu(t,x))|\nabla\chi(t,x)|$$

for a.e. $t\in(0,T)$. Since, by Lemma 5.8, $\lambda_x^\infty(t)=\delta_{y=\nu(t,x)}$ for $\big|\nabla\chi(t)\big|$–a.e. $x\in\Omega$ and a.e. $t\in(0,T)$, we infer from Lemma 2.5

$$\begin{aligned}\int_\Omega \sigma(x,\nu(t,x))|\nabla\chi(t,x)| &= \int_\Omega\left(\int_{\mathbb{S}^{n-1}}\sigma(x,y)\,d\lambda_x^\infty(t,y)\right)\big|\nabla\chi(t,x)\big|\\ &= \int_\Omega\left(\int_{\mathbb{S}^{n-1}}\sigma(x,y)\,d\lambda_x^\infty(t,y)\right)g(t,x)\,d\pi_\infty(t,x)\\ &\le \int_{\Omega\times\mathbb{S}^{n-1}}\sigma(x,y)\,d\Theta_\infty(t,x,y),\end{aligned}$$

where $g$ is the density of $|\nabla\chi|$ with respect to $\pi_\infty$ and $0\le g(t,x)\le 1$ for $\pi_\infty$–a.e. $x\in\Omega$ and a.e. $t\in(0,T)$. Consequently, as $\int_{\mathbb{S}^{n-1}}\sigma(x,y)\,d\lambda_x^\infty(t,y)>0$ for $\pi_\infty$–a.e. $x\in\Omega$ and a.e. $t\in(0,T)$, we deduce

$$g\equiv 1 \quad\text{and}\quad |\nabla\chi|=\pi_\infty \quad\text{for }\ \pi_\infty\text{–a.e. }\ x\in\Omega \text{ and a.e. }\ t\in(0,T).$$

Moreover, $\Theta_{h_j}(t,\Omega\times\mathbb{S}^{n-1})=\big|\nabla\chi^{h_j}(t)\big|(\Omega)$ converges to $|\nabla\chi(t)|(\Omega)=\Theta_\infty(t,\Omega\times\mathbb{S}^{n-1})$ for a.e. $t\in(0,T)$.
Next we utilize the property that $\lim_{j\to\infty}\Theta_{h_j}(t,\Omega\times\mathbb{S}^{n-1})=\Theta_\infty(t,\Omega\times\mathbb{S}^{n-1})$ and $\Theta_{h_j}(t)\to\Theta_\infty(t)$ weakly$^*$, $t\in(0,T)$, implies

$$\lim_{j\to\infty}\int_{\Omega\times\mathbb{S}^{n-1}}u(x,y)\,d\Theta_{h_j}(t,x,y)=\int_{\Omega\times\mathbb{S}^{n-1}}u(x,y)\,\Theta_\infty(t,x,y)$$

for every continuous and bounded function $u:\Omega\times\mathbb{S}^{n-1}\to\mathbb{R}$. We conclude

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega f(x,\nu^{h_j}(t,x))\big|\nabla\chi^{h_j}(t,x)\big| &= \lim_{j\to\infty}\int_{\Omega\times\mathbb{S}^{n-1}}f(x,y)\,d\Theta_{h_j}(t,x,y)\\ =\int_{\Omega\times\mathbb{S}^{n-1}}f(x,y)\,\Theta_\infty(t,x,y) &= \int_\Omega f(x,\nu(t,x))|\nabla\chi(t,x)|\end{aligned}$$

for every continuous and bounded function $f:\Omega\times\mathbb{S}^{n-1}\to\mathbb{R}$ and a.e. $t\in(0,T)$. Thus, we infer

$$\begin{aligned}\lim_{h\to0}\int_\Omega \sigma\big(x,\nu^h(t,x)\big)\nabla\cdot\xi(t,x)|\nabla\chi^h(t,x)| &= \int_\Omega\sigma\big(x,\nu(t,x)\big)\nabla\cdot\xi(t,x)|\nabla\chi(t,x)|\\ \lim_{h\to0}\int_\Omega \sigma_{,x}\big(x,\nu^h(t,x)\big)\cdot\xi(t,x)|\nabla\chi^h(t,x)| &= \int_\Omega\sigma_{,x}\big(x,\nu(t,x)\big)\cdot\xi(t,x)|\nabla\chi(t,x)|\\ \lim_{h\to0}\int_\Omega \nu^h(t)\cdot\nabla\xi(t,x)\,\sigma_{,p}\big(x,\nu^h(t)\big)|\nabla\chi^h(t,x)| &= \int_\Omega\nu(t,x)\cdot\nabla\xi(t,x)\,\sigma_{,p}\big(x,\nu(t,x)\big)|\nabla\chi(t,x)|\end{aligned}$$

for $h\to0$ and the claim is established by Lebesgue's convergence theorem. ∎

### 5.4 Proofs of Theorems 1.1 and 1.2

Now, we are well prepared to prove Theorems 1.1 and 1.2.

**Proof of Theorems 1.1 and 1.2:** From Lemma 5.1 and Lemma 5.4, respectively, we conclude

$$u^h \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \chi^h \to \chi \quad \text{in } L^2(0,T;L^2(\Omega)).$$

The weak compactness of $L^2(0,T;H_0^1(\Omega))$, in turn, implies

$$u \in u_D + L^2(0,T;H_0^1(\Omega)).$$

To establish (1.12) and (1.9), respectively, we consider the time discretization of the diffusion equations, see (4.2) and (4.4), for $\chi = \chi^h(t)$ and $v = u^h(t)$. Discrete integration of the terms $\int_{\Omega_T} \partial_t^{-h}(\chi^h)\xi$ and $\int_{\Omega_T} \partial_t^{-h}\big(u^h + \chi^h\big)\xi$ by parts and passing to the limit $h \to 0$ in (4.2) and (4.4) shows (1.12) and (1.9), respectively.
Now, we show equation (1.10). From equation (5.18) of Lemma 5.9, we derive the convergence of the discrete curvature term to the corresponding expression in (1.10). In addition,

$$\begin{aligned}\lim_{h\to 0}\int_{\Omega_T} u^h(t,\cdot)\,\xi(t,\cdot)\cdot\nu^h(t,\cdot)\big|\nabla\chi^h(t,\cdot)\big| &= \lim_{h\to 0}\int_{\Omega_T} \operatorname{div}\big(u^h(t,\cdot)\,\xi(t,\cdot)\big)\,\chi^h(t,\cdot)\\ &= \int_{\Omega_T} \operatorname{div}\big(u(t,\cdot)\,\xi(t,\cdot)\big)\,\chi(t,\cdot) = \int_{\Omega_T} u(t,\cdot)\,\xi(t,\cdot)\cdot\nu(t,\cdot)\big|\nabla\chi(t,\cdot)\big|.\end{aligned}$$

Hence the assertion follows. ■

### 5.5 Conclusion

The Stefan problem with Gibbs–Thomson law has many applications in material sciences, i.e. describing melting and solidification processes in materials. It has been addressed mathematically by several authors. For a realistic modeling, such as solidification of alloys, it is quite important to take surface tension effects into account, which are spatially inhomogeneous and anisotropic. In this work, we have presented existence results for Stefan problems with spatially inhomogeneous and anisotropic Gibbs–Thomson law. Previous results to this topic (cf. [Luc90, Luc91, LS95, GS11]) have been generalized. We like to mention that in contrast to the isotropic case we cannot apply the Reshetnyak convergence theorem [AFP00] since we do not directly obtain the property $\int_\Omega |\nabla\chi^h(t)| \to \int_\Omega |\nabla\chi(t)|$ as $h \to 0$. To tackle both inhomogeneity and anisotropy, we have used slicing and indicator measures and methods of geometric measure theory.

## 6 References


[AB94] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré, Analyse Non-Linéaire*, 11:91–133, 1994.

[AB95] M. Amar and G. Bellettini. Approximation by Γ–convergence of a total variation with discontinuous coefficients. *Asymptotic Anal.*, 10(3):225–243, 1995.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. Oxford: Clarendon Press, 434 p., 2000.

[ATW93] F. Almgren, J. E. Taylor, and L. Wang. Curvature-driven flows: A variational approach. *SIAM J. Control Optimization*, 31(2):387–438, 1993.

[BGS98] L. Bronsard, H. Garcke, and B. Stoth. A multi-phase Mullins–Sekerka system: Matched asymptotic expansions and an implicit time discretisation for the geometric evolution problem. *Proc. R. Soc. Edinb., Sect. A, Math.*, 128(3):481–506, 1998.

[BP96] G. Bellettini and P. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.*, 25(3):537–566, 1996.

[Che96] X. Chen. Global asymptotic limit of solutions of the Cahn–Hilliard equation. *J. Differ. Geom.*, 44(2):262–311, 1996.

[CHY96] X. Chen, J. Hong, and F. Yi. Existence, uniqueness, and regularity of classical solutions of the Mullins–Sekerka problem. *Commun. Partial Differ. Equations*, 21(11-12):1705–1727, 1996.

[Dzi99] G. Dziuk. Discrete anisotropic curve shortening flow. *SIAM Numer. Anal.*, 36(6):199–227, 1999.

[EPS03] J. Escher, J. Prüss, and G. Simonett. Analytic solutions for a Stefan problem with Gibbs–Thomson correction. *J. Reine Angew. Math*, 563(5):1–52, 2003.

[ES97a] J. Escher and G. Simonett. Classical solutions for the Hele–Shaw models with surface tension. *Adv. Differ. Equ.*, 2(4):619–647, 1997.

[ES97b] J. Escher and G. Simonett. Classical solutions of multidimensional Hele–Shaw models. *SIAM J. Math. Anal.*, 28(5):1028–1047, 1997.

[Eva90] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations.* Conference Board of the Mathematical Sciences. Regional Conference Series in Mathematics, AMS, 80 p., 1990.

[Fon91] I. Fonseca. The Wulff theorem revisited. *Proc. R. Soc. Lond., Ser. A*, 432(1884):125–145, 1991.

[Fon92] I. Fonseca. Lower semicontinuity of surface energies. *Proc. R. Soc. Edinb., Sect. A*, 120(1-2):99–115, 1992.

[Gig06] Y. Giga. *Surface Evolution Equations, A Level Set Approach*, volume 99 of *Monographs in Mathematics*. Birkhäuser, Basel, 264 p., 2006.

[Giu84] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser, Boston, 240 p., 1984.

[GK09] H. Garcke and C. Kraus. An anisotropic, inhomogeneous, elastically modified Gibbs–Thomson law as singular limit of a diffuse interface model. *WIAS-Preprint 1467*, 2009.

[GS98] H. Garcke and T. Sturzenhecker. The degenerate multi-phase Stefan problem with Gibbs–Thomson law. *Adv. Math. Sci. Appl.*, 8(2):929–941, 1998.

[GS11] H. Garcke and S. Schaubeck. Existence of weak solutions for the Stefan problem with anisotropic Gibbs–Thomson law. *Preprint, Regensburg*, 2011.

[Gup03] S. C. Gupta. *The classical Stefan problem. Basic concepts, modelling and analysis.* North-Holland Series in Applied Mathematics and Mechanics 45. Amsterdam: Elsevier. xvii, 385 p., 2003.

[Gur88] M. E. Gurtin. Multiphase thermomechanics with interfacial structure. I: Heat conduction and the capillary balance law. *Arch. Ration. Mech. Anal.*, 104(3):195–221, 1988.

[Gur93] M. E. Gurtin. *Thermomechanics of Evolving Phase Boundaries in the Plane.* Oxford Mathematical Monographs. Oxford, 148 p., 1993.

[Kne07] Carlheinz Kneisel. Über das Stefan-Problem mit Oberflächenspannung und thermischer Unterkühkung. *PhD-thesis, Kassel*, 2007.

[LS95] S. Luckhaus and T. Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differ. Equ.*, 3(2):253–271, 1995.

[Luc90] S. Luckhaus. Solutions for the two-phase Stefan problem with the Gibbs–Thomson law for the melting temperature. *Eur. J. Appl. Math.*, 1(2):101–111, 1990.

[Luc91] S. Luckhaus. The Stefan problem with Gibbs–Thomson law. *Sezione di Analisi Matematica e Probabilitita*, Universita die Pisa, 2.75 (591), 1991.

[Mei92] A. M. Meirmanov. *The Stefan problem. (Translated from the Russian).* De Gruyter Expositions in Mathematics. 3. Berlin etc.: Walter de Gruyter. ix, 245 p., 1992.

[Ott98] F. Otto. Dynamics of labyrinthine pattern formation in magnetic fluids: A mean-field theory. *Arch. Ration. Mech. Anal.*, 141(1):63–103, 1998.

[Rög04] M. Röger. Solutions for the Stefan problem with Gibbs–Thomson law by a local minimisation. *Interfaces Free Bound.*, 6(1):105–133, 2004.

[Rög05] M. Röger. Existence of weak solutions for the Mullins–Sekerka flow. *SIAM J. Math. Anal.*, 37(1):291–301, 2005.

[Sim78] J. Simon. Ecoulement d'un fluide non homogene avec une densite initiale s'annulant. *C. R. Acad. Sci. Paris*, 287:1009–1012, 1978.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl., IV. Ser.*, 146:65–96, 1987.

[Vis98] A. Visintin. Models of phase transitions. *Progress in Nonlin. Diff. Equ. and their Appl.*, 28:322, Birkhäuser, Boston, 1998.