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# A substitute for the maximum principle for singularly perturbed time-dependent semilinear reaction-diffusion problems II. Upper and lower solutions for problems with Neumann boundary conditions

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**Abstract**

The boundary function method for a singularly perturbed time dependent reaction-diffusion problem with Neumann boundary conditions is modified by means of a small parameter (p) featuring in the layer functions, which allows the indication of their exponential decay rates. These functions derived by first order perturbation theory are the ingredients of upper and lower solutions providing existence of solution in the absence of a maximum principle. This solution is also unique.

## Outline

The singular perturbation approach is an analytical tool for systems in which some states change faster than others, in that a full system can be divided into two time-scale systems - reduced (slow) and layer (fast) models, for instance, processes involving both electrical and mechanical phenomena, where electrical phenomena evolve much faster than the mechanical ones. In electroanalytical chemistry, singularly perturbed problems model diffusion processes complicated by chemical reactions. In physical chemistry or chemical engineering the singular perturbation parameters multiplying the highest derivatives describe the diffusion coefficients of substances, while reaction terms vary with temperature or chemical concentration in a reactor and their sign denote exothermic or endothermic reactions, respectively [5]. Discrepancies between magnitude orders of different components of a phenomenon are described by small parameters

within the mathematical model (singular perturbation parameters). They induce extremely narrow regions in the solution, namely layers, which are difficult to compute by standard methods. Based on perturbation theory, the boundary function method of Vasil'eva [10] prescribes problems for functions depending on rescaled variables which describe the layers.
We begin our analysis with the uniqueness of the solution. In Section 2 we define the lower and upper solutions. An asymptotic analysis is performed in Section 3 and perturbing the asymptotic expansion from [11], the upper and lower solutions are set. The main result encompassed by Theorem 3.6 is the existence of solution in a vicinity of the asymptotic expansion.

# 1 Uniqueness of solution and assumptions

For sufficiently smooth real functions $u = u(x,t)$, $\varphi(x)$ and $f = f(x,t,u)$, consider the non(semi)-linear initial-value problem with Neumann boundary conditions

$$\mathcal{F}u \equiv \varepsilon^2[u_t - u_{xx}] + f(x,t,u) = 0, \qquad (x,t) \in (0,1)\times(0,T], \tag{1}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad t \in (0,T]. \tag{3}$$

The small positive parameter $\varepsilon \ll 1$ induces initial and corner layers and a weak boundary layer at $x = 0$. For simplicity we neglect the boundary layer at $x = 1$.

**Proposition 1.1.** *Problem (1), (2), (3) has at most one solution.*

*Proof.* Assume that $\bar{u}$ and $\underline{u}$ are two solutions of (1), (2), (3). Set $d = \underline{u} - \bar{u}$, which is solution of the problem

$$\varepsilon^2[d_t - d_{xx}] + f(x,t,\underline{u}) - f(x,t,\bar{u}) = 0, \quad (x,t) \in (0,1)\times(0,T], \tag{4}$$

$$d(x,0) = 0, \qquad x \in [0,1], \tag{5}$$

$$\frac{\partial d}{\partial x}(0,t)0 = \frac{\partial d}{\partial x}(1,t), \qquad t \in (0,T]. \tag{6}$$

Further setting

$$p(x,t) := \int_0^1 f_u(x,t,\underline{u} + sd)ds,$$

by the mean value theorem, (4) yields

$$\varepsilon^2\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right]d + d\,p(x,t) = 0.$$

For some positive constants $K_1$ and $K_2$, which may depend on $\varepsilon$ and $T$, $|\underline{u}|$ and $|\bar{u}|$ are at least $K_1$, hence $f_u \geq -K_2$ in $[0,1]\times[0,T]\times[-K_1,K_1]$. By means of the transformation $y := d\, e^{-tK_2/\varepsilon^2}$, which satisfies

$$\begin{cases} \varepsilon^2 \left[ \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2} \right] y + (K_2 + p) y = 0 \\ y(x, 0) = 0 \\ \dfrac{\partial y}{\partial x}(0, t) = 0 = \dfrac{\partial y}{\partial x}(1, t), \end{cases}$$

the maximum principle [8] yields $y = 0$ for all $(x, t)$.

□

We consider the problem (1), (2), (3) under the following assumptions:
A1. The reduced equation, obtained by setting $\varepsilon = 0$ in (1)

$$f(x, t, u(x, t)) = 0, \qquad (x, t) \in (0, 1) \times (0, T) \tag{7}$$

has a sufficiently smooth solution $u_0(x, t)$ that is stable, i.e., for some positive $\gamma$

$$f_u(x, t, u_0(x, t)) > \gamma^2 > 0 \quad \forall\, (x, t) \in [0, 1] \times [0, T]; \tag{8}$$

A2. The initial condition belongs to the domain of attraction of the reduced solution $u_0$ :

$$\frac{f(x, 0, u_0(x, 0) + s)}{s} > 0 \quad \forall\, s \in (0, \varphi(x) - u_0(x, 0)], \;\; x \in [0, 1], \tag{9}$$

with $\varphi(x) - u_0(x, 0) \geq 0$, otherwise (9) is satisfied for $s \in [\varphi(x) - u_0(x, 0), 0)$. We further assume the compatibility conditions

$$\varphi_x(0) = 0, \qquad \varphi_x(1) = 0. \tag{10}$$

Differentiating (1) with respect to $x$, we obtain

$$\varepsilon^2 (u_{tx}(x, t) - u_{xxx}(x, t)) + f_x(x, t, u(x, t)) + f_u(x, t, u(x, t)) u_x(x, t) = 0,$$

where we set $(x, t) = (0, 0)$ and use (3) and the first condition in (10). Thus, skipping the first term of order $O(\varepsilon^2)$, we also impose the compatibility condition

$$f_x(0, 0, \varphi(0)) = 0. \tag{11}$$

## 2 Sub-solutions and super-solutions

**Definition 2.1.** Let $\alpha(x, t)$ and $\beta(x, t)$ be functions continuously mapping $[0, 1] \times [0, T]$ into $\mathbb{R}$. Function $\alpha(x, t)$ is sub-solution of the problem (1), (2), (3) and $\beta(x, t)$ is super-solution of the problem if:

$$\alpha(x, t) \leq \beta(x, t) \qquad \forall\, (x, t) \in [0, 1] \times [0, T]; \tag{12}$$

$$\mathcal{F}\alpha \leq 0, \quad \mathcal{F}\beta \geq 0 \qquad \forall\, (x, t) \in [0, 1] \times [0, T]; \tag{13}$$

$$-\frac{\partial \alpha}{\partial x}(0, t) \leq 0 \leq -\frac{\partial \beta}{\partial x}(0, t), \;\; \frac{\partial \alpha}{\partial x}(1, t) \leq 0 \leq \frac{\partial \beta}{\partial x}(1, t) \quad \forall\, t \in [0, T]; \tag{14}$$

$$\alpha(x, 0) \leq \varphi(x) \leq \beta(x, 0) \qquad \forall\, x \in [0, 1]. \tag{15}$$

Adapting Pao's theorem from [7], we prove that existence of sub-solutions and super-solutions provides existence of a unique solution of problem (1), (2), (3) located between them.

**Theorem 2.1.** *Existence of sub-solutions $\alpha$ and super-solutions $\beta$ provides existence of a solution $u(x,t)$ of the problem (1), (2), (3), with*

$$\alpha(x,t) \leq u(x,t) \leq \beta(x,t). \tag{16}$$

*Proof.* While $u$ is in the sector $[\alpha, \beta]$, the function $f_u(x,t,u)$ is bounded by

$$K(x,t) = \max_{u \in [\alpha,\beta]} |f_u(x,t,u)|$$

because it is continuous. Then for all $(x,t) \in (0,1) \times (0,T)$, and , $\alpha \leq u_2 \leq u_1 \leq \beta$,

$$f(x,t,u_1) - f(x,t,u_2) \geq -(u_1 - u_2)K(x,t). \tag{17}$$

Function $K(x,t)$ being continuous,

$$B(x,t,u) \equiv K(x,t)u - f(x,t,u) \tag{18}$$

is also continuous in $[0,1] \times [0,T] \times [\alpha,\beta]$ and monotone nondecreasing in $u \in [\alpha,\beta]$ :

$$B(x,t,u_1) - B(x,t,u_2) \geq 0, \quad \beta \geq u_1 \geq u_2 \geq \alpha. \tag{19}$$

Following [7], we define the operator

$$\mathbb{L}[u] \equiv \varepsilon^2[u_t - u_{xx}] + K(x,t)u \tag{20}$$

and consider the differential equation

$$\mathbb{L}[u] = B(x,t,u) \tag{21}$$

with $B$ given by (18), which is equivalent to (1), and the same initial and boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad u(0,x) = \varphi(x). \tag{22}$$

The sub-solutions and super-solutions of the problem (21), (22) must satisfy the same conditions as in Definition 2.1 for (1), (2), (3), with the only difference that in (13) the operator $\mathcal{F}$ is replaced by $\mathbb{L}$. We construct the sequences $\{\alpha^{(k)}\}$ by

$$\mathbb{L}[\alpha^{(k)}] = B(x,t,\alpha^{(k-1)}), \quad \alpha^{(0)} = \alpha \tag{23}$$

and $\{\beta^{(k)}\}$ by

$$\mathbb{L}[\beta^{(k)}] = B(x,t,\beta^{(k-1)}), \quad \beta^{(0)} = \beta, \tag{24}$$

where $k = 1, 2, ...$, with the boundary and initial conditions

$$\frac{\partial \alpha^{(k)}}{\partial x}(0,t) = \frac{\partial \alpha^{(k)}}{\partial x}(1,t) = 0,$$

$$\alpha^{(k)}(0,x) = \varphi(x), \tag{25}$$

$$\frac{\partial \beta^{(k)}}{\partial x}(0,t) = \frac{\partial \beta^{(k)}}{\partial x}(1,t) = 0,$$
$$\beta^{(k)}(0,x) = \varphi(x), \tag{26}$$

and refer to them as lower and upper sequences.
We prove that
(i) Each $\alpha^{(k)}$ is a sub-solution and each $\beta^{(k)}$ is a super-solution; the lower and upper sequences possess the monotone property

$$\alpha \le \alpha^{(k)} \le \alpha^{(k+1)} \le \beta^{(k+1)} \le \beta^{(k)} \le \beta \quad \text{ in } [0,1] \times [0,T]. \tag{27}$$

Let $w = \beta^{(0)} - \beta^{(1)} = \beta - \beta^{(1)}$. In (13) of Definition (2.1) we replaced the operator $\mathcal{F}$ by $\mathbb{L}$, yielding

$$\mathbb{L}[w] = \mathbb{L}[\beta^{(0)}] - B(x,t,\beta^{(0)}) \ge 0.$$

From (24) and (26)

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial \beta}{\partial x}(0,t) - \frac{\partial \beta^{(1)}}{\partial x}(0,t) \le 0, \quad \frac{\partial w}{\partial x}(1,t) = \frac{\partial \beta}{\partial x}(1,t) - \frac{\partial \beta^{(1)}}{\partial x}(1,t) \ge 0,$$
$$w(x,0) = \beta(x,0) - \varphi(x) \ge 0.$$

By the maximum principle [8, ch. 3], $w(x,t) \ge 0$, so $\beta^{(1)} \le \beta^{(0)}$. Similarly, $\alpha^{(1)} \ge \alpha^{(0)}$. Let $w^{(1)} = \beta^{(1)} - \alpha^{(1)}$. From (23), (24), (25), (26) and the monotone property of $B$ in (19) we have

$$\mathbb{L}[w^{(1)}] = B(x,t,\beta^{(0)}) - B(x,t,\alpha^{(0)}) \ge 0,$$
$$\frac{\partial w^{(1)}}{\partial x}(0,t) = 0, \quad \frac{\partial w^{(1)}}{\partial x}(1,t) = 0,$$
$$w^{(1)}(x,0) = \varphi(x) - \varphi(x) = 0$$

and from the maximum principle it follows that $w^{(1)} \ge 0$ in $[0,1] \times [0,T]$. Hence

$$\alpha^{(0)} \le \alpha^{(1)} \le \beta^{(1)} \le \beta^{(0)} \quad \text{ in } [0,1] \times [0,T].$$

Assume now by induction that

$$\alpha^{(k-1)}(x,t) \le \alpha^{(k)}(x,t) \le \beta^{(k)}(x,t) \le \beta^{(k-1)}(x,t) \quad \text{in } [0,1] \times [0,T].$$

Then by (23), (24), (25), (26) and from the monotone property of $B$ in (19), $w^{(k)} = \beta^{(k)} - \beta^{(k+1)}$ satisfies

$$\mathbb{L}w^{(k)} = B(x,t,\beta^{(k-1)}) - B(x,t,\beta^{(k)}) \ge 0$$

and the boundary and initial conditions as for $w^{(1)}$. Hence $w^{(k)} \ge 0$ and

$$\beta^{(k+1)} \le \beta^{(k)}.$$

Similarly,

$$\alpha^{(k+1)} \ge \alpha^{(k)},$$
$$\beta^{(k+1)} \ge \alpha^{(k+1)}.$$

By the principle of induction, assertion $(i)$ is established for all $k$.
To prove that each element of the lower sequence is sub-solution, from (23) and (25) we have

$$\varepsilon^2[\alpha_t^{(k)} - \alpha_{xx}^{(k)}] = K(x,t)(\alpha^{(k-1)} - \alpha^{(k)}) - f(x,t,\alpha^{(k-1)})$$

$$= K(x,t)(\alpha^{(k-1)} - \alpha^{(k)}) + [f(x,t,\alpha^{(k)}) - f(x,t,\alpha^{(k-1)})] - f(x,t,\alpha^{(k)}),$$

so by (17) and (27)

$$\mathcal{F}\alpha^{(k)} = \varepsilon^2[\alpha_t^{(k)} - \alpha_{xx}^{(k)}] + f(x,t,\alpha^{(k)}) \le 0.$$

Hence (13) is satisfied. Due to the boundary and initial conditions (25), (14) and (15) are satisfied by $\alpha^{(k)}$.
From (24) and (26) we have

$$\varepsilon^2[\beta_t^{(k)} - \beta_{xx}^{(k)}] = K(x,t)(\beta^{(k-1)} - \beta^{(k)}) - f(x,t,\beta^{(k-1)})$$

$$= K(\beta^{(k-1)} - \beta^{(k)}) + [f(x,t,\beta^{(k)}) - f(x,t,\beta^{(k-1)})] - f(x,t,\beta^{(k)}),$$

so by (17) and (27)

$$\mathcal{F}\beta^{(k)} = \varepsilon^2[\beta_t^{(k)} - \beta_{xx}^{(k)}] + f(x,t,\beta^{(k)}) \ge 0.$$

From the boundary and initial conditions (26) it follows that (14) and (15) are satisfied by $\beta^{(k)}$ and from (27), that $\alpha^{(k)} \le \beta^{(k)}$.
Therefore, by Definition (2.1), $\alpha^{(k)}$ is sub-solution and $\beta^{(k)}$ is super-solution of (1), (2), (3).
(ii) The pointwise limits

$$\lim_{k\to\infty} \beta^{(k)}(x,t) \qquad \text{and} \qquad \lim_{k\to\infty} \alpha^{(k)}(x,t) \tag{28}$$

exist and satisfy in $[0,1] \times [0,T]$

$$\alpha(x,t) \le \alpha^{(k)}(x,t) \le \alpha^{(k+1)}(x,t) \le \lim_{k\to\infty} \alpha^{(k)}(x,t) \le$$

$$\le \lim_{k\to\infty} \beta^{(k)}(x,t) \le \beta^{(k+1)}(x,t) \le \beta^{(k)}(x,t) \le \beta(x,t). \tag{29}$$

Indeed, since by (i) the sequence $\{\beta^{(k)}\}$ is monotone nonincreasing and is bounded from below and the sequence $\{\alpha^{(k)}\}$ is monotone nondecreasing and is bounded from above, the pointwise limits of these sequences exist and satisfy (29).
(iii) If the limits (28) are solutions of (1), (2), (3), then

$$\lim_{k\to\infty} \beta^{(k)}(x,t) = \lim_{k\to\infty} \alpha^{(k)}(x,t)$$

and is the solution in the $[\alpha, \beta]$.
Indeed, let

$$d = \lim_{k\to\infty} \alpha^{(k)}(x,t) - \lim_{k\to\infty} \beta^{(k)}(x,t) \le 0.$$

Then $d$ satisfies the relation

$$\varepsilon^2[d_t - d_{xx}] = -f(x,t,\alpha) + f(x,t,\beta) \ge$$

$$\geq -K(x,t)[\lim_{k\to\infty}\beta^{(k)}(x,t) - \lim_{k\to\infty}\alpha^{(k)}(x,t)] = K(x,t)d$$

and the boundary and initial conditions

$$\frac{\partial d}{\partial x}(0,t) = 0, \quad \frac{\partial d}{\partial x}(1,t) = 0 \quad \text{for } t \in [0,T]; \qquad d(x,0) = 0 \quad \text{for } x \in [0,1].$$

By the maximum principle $d \geq 0$ in $[0,1] \times [0,T]$, yielding

$$\lim_{k\to\infty}\beta^{(k)}(x,t) = \lim_{k\to\infty}\alpha^{(k)}(x,t).$$

□

# 3 Asymptotic analysis

We construct the sub-solutions and super-solutions by perturbing the asymptotic expansion from [11]. Assume that

$$\frac{\partial u_0}{\partial x}(1,t) = 0, \tag{30}$$

so there is no boundary layer at $x = 1$. We shall see that there is a weak boundary layer at $x = 0$. Performing a variable stretching

$$\xi = x/\varepsilon \quad \text{and} \quad \tau = t/\varepsilon^2$$

and denoting by $v_0$ and $v_1$ the terms of the boundary layer function, $q_0$ and $q_1$ the terms of the corner function and $w_0$ the initial layer function, we write an asymptotic expansion of order one of the solution:

$$u_{as} = u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t) + w_0(x,\tau) + q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau). \tag{31}$$

We define the functions

$$F(x,t,s) := f(x,t,u_0(x,t)+s), \quad \tilde{F}(x,t,s;p) = f(x,t,u_0(x,t)+s) - ps. \tag{32}$$

The perturbed function $\tilde{F}$, with $|p| < \varepsilon$, is used in the setting of perturbed zero-order terms of the boundary-layer and initial-layer functions, which will be included in the construction of sub-solutions - for negative small values $-|p|$ - and super-solutions - for positive small values $|p|$. The following properties of $F$ yield from $u_0$ being a reduced solution:

$$F(x,t,s) = F(x,t,0) + O(s) = O(s), \tag{33}$$

$$F_x(x,t,0) = F_{xx}(x,t,0) = F_t(x,t,0) = 0, \tag{34}$$

giving

$$|F_x(x,t,s)| \leq C|s|, \quad |F_{xx}(x,t,s)| \leq C|s|, \quad |F_t(x,t,s)| \leq C|s|. \tag{35}$$

with $C$ a positive constant. Analogously, $\tilde{F}(x,t,0) = 0$ implies $\tilde{F}_x(x,t,0) = 0$ and $\tilde{F}_t(x,t,0) = 0$, so $\tilde{F}_x(x,t,s) - \tilde{F}_x(x,t,0) = s\tilde{F}_{xs}(x,t,\hat{s})$ and

$$|\tilde{F}_x(x,t,s)| \leq C|s|, \quad |\tilde{F}_{xx}(x,t,s)| \leq C|s|, \quad |\tilde{F}_t(x,t,s)| \leq C|s|. \tag{36}$$

For any differentiable function $g$, with the notations

$$g\,\big|_a^b = g(b) - g(a), \quad g\big|_{a;b}^{c} = g(c) - g(b) - g(a), \tag{37}$$

since

$$g(a+b) - g(a) - g(b) + g(0) = a\,b\,g''(\theta),$$

if $g(0) = 0$, then

$$g\big|_{a;b}^{a+b} = O(|ab|).$$

Therefore, $\tilde{F}(x,t,0) = 0$ implies

$$\tilde{F}(x,t,\cdot)\big|_{a;b}^{a+b} = O(|ab|). \tag{38}$$

## 3.1 Boundary-layer function

From (31) and the left boundary condition we obtain

$$\frac{\partial u_0}{\partial x}(0,t) + \frac{1}{\varepsilon}\frac{\partial(v_0+\varepsilon v_1)}{\partial \xi}(0,t) = 0,$$

i.e.,

$$\frac{1}{\varepsilon}\frac{\partial v_0}{\partial \xi}(0,t) + \frac{\partial v_1}{\partial \xi}(0,t) = -\frac{\partial u_0}{\partial x}(0,t) \tag{39}$$

and equalize the terms in (39) containing same powers of $\varepsilon$ :

$$\begin{aligned} \varepsilon^{-1} \text{ terms :} \qquad & \frac{\partial v_0}{\partial \xi}(0,t) = 0 \\ \varepsilon^{0} \text{ terms :} \qquad & \frac{\partial v_1}{\partial \xi}(0,t) = -\frac{\partial u_0}{\partial x}(0,t). \end{aligned} \tag{40}$$

Set the zero-order term of the boundary-layer function $v_0(\xi,t)$ as the solution of

$$\begin{cases} \dfrac{\partial^2 v_0}{\partial \xi^2} & = F(x,t,v_0) \\ \dfrac{\partial v_0}{\partial \xi}\big|_{\xi=0} & = 0 \\ v_0|_{\xi\to\infty} & = 0. \end{cases} \tag{41}$$

Homogeneous boundary conditions in (41) and $F(x,t,0) = 0$ imply that $\underline{v_0 = 0}$ is solution of (41). This is not the case for a problem with Dirichlet boundary conditions [9, 2] . Therefore, the boundary-layer component of the asymptotic expansion is $v_0 + \varepsilon v_1 = \varepsilon v_1$, so we deal with a weak boundary layer.
Now set the boundary layer term of order one $v_1(\xi,t)$ as solution of

$$\begin{cases} -\dfrac{\partial^2 v_1}{\partial \xi^2} + v_1 F_s(0,t,v_0(x,t)) + \xi F_x(0,t,v_0(0,t)) & = 0 \\ \dfrac{\partial v_1}{\partial \xi}\big|_{\xi=0} & = -\dfrac{\partial u_0}{\partial x} \\ v_1|_{\xi\to\infty} & = 0. \end{cases} \tag{42}$$

With (30) and $v_0$ vanishing, (42) becomes

$$\begin{cases} -\dfrac{\partial^2 v_1}{\partial \xi^2} + v_1 F_s(0,t,0) & = 0 \\ \dfrac{\partial v_1}{\partial \xi}|_{\xi=0} & = -\dfrac{\partial u_0}{\partial x}(0,t) \\ v_1|_{\xi\to\infty} & = 0. \end{cases} \tag{43}$$

For $v_0$ we have a nonlinear autonomous ordinary differential equation, whereas for $v_1$, a linear ordinary differential equation. Assumption $A1$ ensures existence of $v_0$ and existence and asymptotic properties of $v_1$. Using assumption $A1$ to set

$$\gamma_L^2 := \min_{t\geq 0} f_u(0,t,u_0(0,t)) > \gamma^2, \tag{44}$$

for some positive $\delta$ and $\gamma_L$ we have the upper bounds

$$|v_1^{(k)}| \leq C_\delta e^{-(\gamma_L-\delta)\xi} \quad \text{for } \xi \in [0,\infty),\ k = \overline{0,4}. \tag{45}$$

A proof for this estimate features in [2] for the problem with Dirichlet boundary conditions, while the property $F_s(0,t,0) \geq \gamma_L^2$ in our case simplifies the argument therein. Also, the following estimates hold true:

$$\left|\frac{\partial^k \tilde{v}_0}{\partial \xi^k}\right| + \left|\frac{\partial^l \tilde{v}_0}{\partial t^l}\right| + \frac{\partial \tilde{v}_0}{\partial p} \leq C_\delta e^{-(\gamma_L-\sqrt{p_0}-\delta)\xi} \tag{46}$$

for $\xi, t \geq 0,\ k = 0, ..., 4,\ l = 0, 1, 2;$

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](\varepsilon v_1) + F(x,t,\varepsilon v_1) = O(\varepsilon^2). \tag{47}$$

### 3.2 Initial-layer function

Using the stretched variable $\tau = t/\varepsilon^2$, we construct a initial-layer function and its perturbation to describe the solution near $t = 0$. We define $w_0(x,\tau)$ as the solution of the initial-value problem

$$\frac{\partial w_0}{\partial \tau} = -F(x,0,w_0), \quad \tau > 0, \quad w_0(x,0) = \varphi(x) - u_0(x,0) \tag{48}$$

and set the perturbed function $\tilde{w}_0(x,\tau;p)$ with $\tilde{w}_0(x,\tau;0) = w_0(x,\tau)$ as solution of the initial-value problem

$$\frac{\partial \tilde{w}_0}{\partial \tau} = -\tilde{F}(x,0,\tilde{w}_0;p) \quad \tau > 0, \quad \tilde{w}_0(x,0;p) = \varphi(x) - u_0(x,0). \tag{49}$$

The problem for $w_0$

$$\begin{cases} w_{0,\tau}(x,\tau) & = -F(x,0,w_0(x,\tau)) \\ w_0(x,0) & = \varphi(x) - u_0(x,0) \end{cases} \tag{50}$$

has a solution satisfying $w_0(x,\infty) = 0$ and the problem for the perturbed initial layer function

$$\begin{cases} \tilde{w}_{0,\tau}(x,\tau,p) & = -F(x,0,\tilde{w}_0(x,\tau,p)) + p\tilde{w}_0(x,\tau,p) \\ \tilde{w}_0(x,0,p) & = \varphi(x) - u_0(x,0) \end{cases} \tag{51}$$

also has a solution satisfying $\lim_{\tau\to\infty}\tilde{w}_0(x,\tau,p)=0$.
In comparison with a problem with Dirichlet boundary conditions where the zero-order term of the initial-layer function in $x=0$ is 0, for our problem the initial-layer function in 0 does not vanish. Let

$$\gamma_0^2 := \min_{x\in[0,1]} f_u(x,0,u_0(x,0)) > \gamma^2, \tag{52}$$

where $\gamma$ is the positive constant from assumption $A1$. The non-negative value $\gamma_0$ will feature in the construction of layer-adapted meshes and in the following

**Proposition 3.1.** *(i) There exists $p_0\in(0,\gamma_0^2)$ such that for all $p$ with $|p|\le p_0$, problem (51) has a solution.*
*(ii) The initial-layer function $w_0$ and the perturbed initial-layer function $\tilde{w}_0$ satisfy*

$$w_0(x,\tau)\ge 0, \qquad \frac{\partial\tilde{w}_0}{\partial p}\ge 0 \qquad \forall\, x\in[0,1]\ \ \forall\,\tau\ge 0. \tag{53}$$

*For an arbitrarily small but fixed $\delta\in(0,\gamma_0^2-p_0)$, there exists a positive constant $C_\delta$ such that for $k=\overline{0,4}$ and $l=\overline{0,2}$ the following estimate holds true:*

$$\left|\frac{\partial^l\tilde{w}_0}{\partial\tau^l}\right|+\left|\frac{\partial\tilde{w}_0}{\partial x^k}\right|+\left|\frac{\partial\tilde{w}_0}{\partial p}\right|\le C_\delta e^{-(\gamma_0^2-|p_0|-\delta)\tau} \qquad \forall\, x\in[0,1]\,;\forall\,\tau\ge 0. \tag{54}$$

The proof from [2] for the initial-layer function of the problem with Dirichlet boundary conditions is fully valid here and is based on the following result which we shall also use in the next subsection to estimate the derivatives of the corner function:

**Lemma 1.** *Consider the initial value problem*

$$\frac{d}{d\tau}\omega=-\phi(\omega)\ \ \text{for}\ \ \tau>0,\ \ \omega(0)=\omega_0\ge 0,\ \ \omega(\infty)=0, \tag{55}$$

*Let a sufficiently smooth function $\phi$ satisfy*

$$\phi(0)=0,\quad \phi'(0)>0,\quad \phi(s)>0,\ \ \forall\, s\in(0,\omega_0]. \tag{56}$$

1. *Then the problem (55) has a solution $0\le\omega\le\omega_0$ and for any arbitrarily small, but fixed $\delta\in(0,\phi'(0))$, there exists a constant $C_\delta$ such that*

$$|\omega|+|\omega'|+|\omega''|\le\omega_0 C_\delta\exp[-(\phi'(0)-\delta)\tau]\ \ \text{for}\ \ \tau\ge 0. \tag{57}$$

2. *Set*

$$\hat{\omega}:=\begin{cases}\dfrac{\omega}{\omega_0} & \text{if}\ \ \omega_0>0\\ e^{-\phi'(0)\tau} & \text{if}\ \ \omega_0=0.\end{cases} \tag{58}$$

*Then the linear problem*

$$\frac{d}{d\tau}\chi+\chi\phi'(\omega)=\psi(\tau)\ \ \text{for}\,\tau>0;\ \ \chi(0)=\chi_0;\ \ \chi(\infty)=0, \tag{59}$$

*where $|\chi_0|\le C$ and*

$$|\psi(\tau)|\le C(1+\tau^m)\hat{\omega}(x,\tau)\ \ \text{for some}\ \ m\ge 0, \tag{60}$$

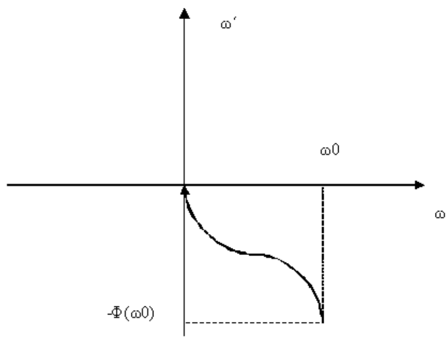


Figure 1: Phase plane: the horizontal axis is $\omega(x,\tau)$; the vertical axis is $\omega'(\tau)$. There exists a trajectory that leaves the point $(\omega_0, -\phi(\omega_0))$ and enters the point $(0,0)$. The fact that $\omega > 0$ and $\omega' < 0$ means that the plot of $\omega'$ versus $\omega$ enters the origin from quadrant IV. Hence $\omega$ becomes zero at infinity

*has a solution that satisfies*

$$|\chi(\tau)| \leq C(\chi_0 + 1 + \tau^{m+1})\hat{\omega}(x,\tau). \tag{61}$$

*If we also have* $\chi_0 = 0$ *and* $\psi \geq 0$, *then* $\chi \geq 0$ *for all* $\tau \geq 0$.

We only sketch the part of the proof which is going to be used in the corner function; for the remaining part of the proof we refer the reader to [2].

*Proof.* 1. If $\omega_0 = 0$, then $\omega(x,\tau) = 0$ for all $\tau$ and the assertion follows. If $\omega_0 > 0$, from (56) this gives $\phi(s) > 0,\ s \in (0,\omega_0]$. Consider the phase plane $(\omega,\omega')$ for the equation $\omega' = -\phi(\omega)$. By (56), there exists a trajectory that leaves the point $(\omega_0, -\phi(\omega_0))$ and enters the point $(0,0)$, as shown by Figure 1. Furthermore, since $\phi(\omega) > 0$ for all $\omega \in (0,\omega_0]$, this entire trajectory lies in the quarter plane $\{\omega > 0, \omega' < 0\}$. Therefore the corresponding solution $\omega(x,\tau)$ is positive and decreasing to zero. It remains to show that the solution trajectory enters $(0,0)$ as $\tau \to \infty$ and also the exponential decay estimates (57). By the definition of the derivative as a limit,

$$\phi'(0) = \lim_{s\to 0} \frac{\phi(s) - \phi(0)}{s - 0},$$

where $\phi(0) = 0$, so

$$\lim_{s\to 0} \frac{\phi(s)}{s} = \phi'(0),$$

i.e. that for any $\delta \in (0, \phi'(0))$, there exists $s_\delta \in (0,\omega_0)$ such that

$$\left| \frac{\phi(s)}{s} - \phi'(0) \right| \leq \delta \ \ \forall s \in [0, s_\delta]$$

which is

$$\phi'(0) - \delta \leq \frac{\phi(s)}{s} \leq \phi'(0) + \delta, \ \ \forall\ s \in [0, s_\delta]. \tag{62}$$

Furthermore, there exists $\tau_\delta > 0$ such that

$$\omega(x, \tau_\delta) = s_\delta. \tag{63}$$

Otherwise, if $\omega$ is bounded away from zero, $\omega(x, \tau) > s_\delta$ for all $\tau$, let

$$m = \min_{\omega \in [s_\delta, \omega_0]} \phi(s).$$

We will use the function $\phi$ in its particular form $\phi(s) = f(x, 0, u_0(x, 0) + s)$. Assumption $A2$ implies $m > 0$ and

$$\phi(\omega) \geq m \quad \forall \tau$$

yields

$$\omega' = -\phi(\omega) \leq -m \qquad \forall\, \tau.$$

We have

$$\phi(\omega) \geq \begin{cases} \frac{m}{\omega_0}\omega & \text{for } \omega \in [s_\delta, \omega_0] \\ (\phi'(0) - \delta)\omega & \text{for } \omega \in [0, s_\delta], \end{cases} \tag{64}$$

i.e. $\phi(\omega) \geq C\omega \;\; \forall \omega \in [0, \omega_0]$, so

$$-\phi(\omega) = \omega'(\tau) \leq -m,$$

which yields $\omega(x, \infty) = -\infty$, in contradiction with (55). Hence (63) is established.

- Case $\tau \geq \tau_\delta$
  The function $\omega(x, \tau)$ is decreasing, so $\omega(x, \tau) < \omega(x, \tau_\delta) = s_\delta$, thus $\omega(x, \tau) \in (0, s_\delta)$ which implies that (62) holds true

$$\phi'(0) - \delta \leq \frac{\omega'}{\omega} \leq \phi'(0) + \delta. \tag{65}$$

  Integrating (65) from $\tau_\delta$ to $\tau$ we obtain

$$[\phi'(0) - \delta](\tau - \tau_\delta) \leq (\ln \omega)|_{\tau_\delta}^{\tau} \leq [\phi'(0) + \delta](\tau - \tau_\delta)$$

$$e^{-[\phi'(0)+\delta](\tau - \tau_\delta)} \leq \frac{\omega(x, \tau)}{\omega(x, \tau_\delta)} \leq e^{-[\phi'(0)-\delta](\tau - \tau_\delta)}, \;\; \forall \tau \geq \tau_\delta. \tag{66}$$

  As $\omega(x, \tau_\delta) \leq \omega_0$, the estimates for $\omega$ and $\omega'$ in (57) follow from (66) and (65).

- Case $0 \leq \tau \leq \tau_\delta$
  As a decreasing function, $\omega \leq \omega_0$ and for $C_* := e^{[\phi'(0)-\delta]\tau_\delta}$ we have

$$|\omega| \leq C_* \omega_0 e^{-[\phi'(0)-\delta]\tau_\delta} \leq C_* \omega_0 e^{-[\phi'(0)-\delta]\tau}$$

  yielding the estimate for $\omega$ and from (64), where $\phi(\omega) = -\omega'$, also the estimate for $\omega'$.

□

### 3.3 Corner-layer function

The terms of order zero and one of the corner layer function $q_0 + \varepsilon q_1$ are characterized by

$$\lim_{\xi\to\infty} q_0(\xi,\tau) = 0, \qquad \lim_{\tau\to\infty} q_0(\xi,\tau) = 0, \tag{67}$$

$$\lim_{\xi\to\infty} q_1(\xi,\tau) = 0, \qquad \lim_{\tau\to\infty} q_1(\xi,\tau) = 0. \tag{68}$$

Introducing (31) in (2) yields the compatibility condition

$$v_0(\xi,0) + \varepsilon v_1(\xi,0) + q_0(\xi,0) + \varepsilon q_1(\xi,0) = 0.$$

Equalizing the terms containing the power $-1$ of $\varepsilon$, we obtain

$$q_0(\xi,0) = -v_0(\xi,0) \tag{69}$$

and the terms containing the power 0 of $\varepsilon$ give

$$q_1(\xi,0) = -v_1(\xi,0). \tag{70}$$

Introducing (31) in (3) yields the compatibility condition

$$\frac{\partial w_0}{\partial x}(0,\tau) + \frac{1}{\varepsilon}\frac{q_0 + \varepsilon q_1}{\partial \xi}(0,\tau) = 0. \tag{71}$$

Equalizing the terms with the same powers of $\varepsilon$ in (71), we have

$$\frac{\partial q_0}{\partial \xi}(0,\tau) = 0 \tag{72}$$

and

$$\frac{\partial q_1}{\partial \xi}(0,\tau) = -\frac{\partial w_0}{\partial x}(0,\tau). \tag{73}$$

Using the equation from (41)

$$-\frac{\partial^2 v_0}{\partial \xi^2} + F(0,t,v_0) = 0$$

and (48), equation

$$-\frac{\partial^2}{\partial \xi^2} v_0(\xi,0) + \frac{\partial}{\partial \tau} w_0(0,\tau) + \left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] q_0 +$$

$$+ f(0,0,u_0(0,0) + v_0(\xi,0) + w_0(0,\tau) + q_0) = 0$$

becomes

$$\left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] q_0 =$$

$$-F(0,0,v_0(\xi,0) + w_0(0,\tau) + q_0) + F(0,0,w_0(0,\tau)) - F(0,0,v_0(\xi,0)). \tag{74}$$

By $v_0 = 0$ and $F(0,0,0) = 0$, the right-hand side of equation (74) becomes

$$-F(0,0,v_0(\xi,0) + w_0(0,\tau) + q_0) + F(0,0,w_0(0,\tau)) - F(0,0,v_0(\xi,0)) =$$

$$= -F(0,0,w_0(0,\tau) + q_0) + F(0,0,w_0(0,\tau)).$$

In the horizontal part of the corner layer, along $\xi$, where $\tau = 0$, we have

$$u_0(x,0) + v_0(\xi,0) + \varepsilon v_1(\xi,0) + w_0(\xi,0) + q_0(\xi,0) + \varepsilon q_1(\xi,0) = \varphi(x), \quad (75)$$

which yields the initial condition for $q_1$ :

$$q_1(\xi,0) = -v_1(\xi,0). \quad (76)$$

Due to the initial value $w_0(x,0) = \varphi(x) - u_0(x,0)$, (75) yields the initial condition for $q_0$

$$q_0(\xi,0) = -v_0(\xi,0). \quad (77)$$

A boundary condition for $q_0$ is given by (72) and the initial condition is given by (69). Therefore, the problem for the zero-order term of the corner layer function $q_0(\xi,\tau)$ is

$$\begin{cases} \frac{\partial q_0}{\partial \tau} - \frac{\partial^2 q_0}{\partial \xi^2} + F(0,0,w_0(0,\tau)+q_0) - F(0,0,w_0(0,\tau)) = 0 \\ q_0(\xi,0) = -v_0(\xi,0) = 0 \\ \frac{\partial q_0}{\partial \xi}(0,\tau) = 0. \end{cases} \quad (78)$$

Having homogeneous initial and boundary condition, a solution of (78) is $\underline{q_0 = 0}$.

The first-order term of the corner layer function, $q_1(\xi,\tau)$ is obtained by formally introducing

$$u_{as} = u_0(\varepsilon\xi, \varepsilon^2\tau) + v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)$$

in the original equation (1):

$$\varepsilon^2 \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) [u_0(\varepsilon\xi,\varepsilon^2\tau) + v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)] +$$

$$+ F(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)) = 0.$$

Here we have

$$\varepsilon^2 \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u_0 = O(\varepsilon^2).$$

We use the estimates from [2]

$$\varepsilon^2 \left[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right] (v_0 + \varepsilon v_1) + F(x,t,v_0+\varepsilon v_1) = O(\varepsilon^2). \quad (79)$$

and

$$\varepsilon^2 \left[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right] w_0 + F(x,t,w_0) = O(\varepsilon^2). \quad (80)$$

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) - F(\varepsilon\xi,\varepsilon^2\tau, w_0(\varepsilon\xi,\tau)) - F(\varepsilon\xi,\varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau)) +$$

$$+ F(\varepsilon\xi,\varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + \varepsilon q_1(\xi,\tau)) + O(\varepsilon^2) = 0.$$

By notation (37), this can be written as

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) + F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|^{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]+w_0(\varepsilon\xi,\tau)}_{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)} +$$

$$+F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} = O(\varepsilon^2) \tag{81}$$

and becomes

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) + F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{[\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)}^{[\varepsilon v_1(\xi,\varepsilon^2\tau)]+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} = O(\varepsilon^2). \tag{82}$$

Now set the function $G(\varepsilon)$ as

$$G(\varepsilon) = F(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{[\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)}^{[\varepsilon v_1(\xi,\varepsilon^2\tau)]+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)}.$$

Then

$$\begin{aligned}
&G'(\varepsilon) = \\
&= \xi F_x(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{\varepsilon v_1(\xi,\varepsilon^2\tau)\,;\,w_0(\varepsilon\xi,\tau)}^{\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} + 2\varepsilon\tau F_t(\varepsilon\xi,\varepsilon^2\tau,\cdot)\Big|_{\varepsilon v_1(\xi,\varepsilon^2\tau)\,;\,w_0(\varepsilon\xi,\tau)}^{\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} + \\
&+[2\varepsilon^2\tau v_{1,t}(\xi,\varepsilon^2\tau) + v_1(\xi,\varepsilon^2\tau) + \xi w_{0,x}(\varepsilon\xi,\tau) + q_1(\xi,\tau)]\cdot \\
&\cdot F_s(\varepsilon\xi,\varepsilon^2\tau,\varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)) - \\
&-[2\varepsilon^2\tau v_{1,t}(\xi,\varepsilon^2\tau) + v_1(\xi,\varepsilon^2\tau)]\cdot F_s(\varepsilon\xi,\varepsilon^2\tau,\varepsilon v_1(\xi,\varepsilon^2\tau)) - \\
&-\xi w_{0,x}(\varepsilon\xi,\tau)\cdot F_s(\varepsilon\xi,\varepsilon^2\tau,w_0(\varepsilon\xi,\tau))
\end{aligned} \tag{83}$$

and

$$G(0) = F(0,0,\cdot)\Big|_{0;\,w_0(\varepsilon\xi,\tau)}^{w_0(\varepsilon\xi,\tau)} = 0.$$

A calculation shows that

$$G'(0) = [v_1(\xi,0) + q_1]F_s(0,0,w_0(0,\tau)) - v_1(\xi,0)F_s(0,0,0).$$

Using the conditions (70) and (73), $q_1$ is given by the linear problem

$$\begin{cases} \frac{\partial q_1}{\partial \tau} - \frac{\partial^2 q_1}{\partial \xi^2} + q_1 F_s(0,0,w_0(0,\tau)) = -v_1(\xi,0)F_s(0,0,\cdot)\Big|_0^{w_0(0,\tau)} \\ q_1(\xi,0) = -v_1(\xi,0) \\ \frac{\partial q_1}{\partial \xi}(0,\tau) = -\frac{\partial w_0}{\partial x}(0,\tau). \end{cases} \tag{84}$$

Note that, by (46) and (54)

$$\left| -v_1(\xi,0)F_s(0,0,\cdot)\Big|_0^{w_0(0,\tau)} \right| \le C|v_1(\xi,0)||w_0(0,\tau)| \le C_\delta e^{-(\gamma_L-\delta)\xi-(\gamma_0^2-\delta)\tau}, \tag{85}$$

i.e., the right-hand side of the first equation in (84) is exponentially decaying. From the initial condition in (50) for $x = 0$, we obtain

$$w_{0,x}(0,0) = \varphi_x(0) - u_{0,x}(0,0).$$

From (40) for $t = 0$ we have

$$v_{1,\xi}(0,0) = -u_{0,x}(0,0).$$

Thus, by (10), i.e. $\varphi_x(0) = 0$, we have

$$v_{1,\xi}(0,0) = w_{0,x}(0,0), \tag{86}$$

i.e., compatibility at the corner $(0,0)$. We are going to investigate solutions of the problem (84).

#### 3.3.1 Fundamental solution for the term of order one of the corner-layer function

We now construct the operator

$$L = -\frac{\partial^2}{\partial \xi^2} + \lambda(\tau), \tag{87}$$

where

$$\lambda(\tau) = F_s(0, 0, w_0(0, \tau)) \tag{88}$$

and in the half plane we consider the problem

$$\begin{cases} z_t + Lz = b(\xi, \tau), & -\infty < \xi < \infty,\ \tau > 0 \\ z(0, \xi) = 0. \end{cases} \tag{89}$$

The solution of (89) is

$$\nu(\xi, \tau) = \int_0^{\tau} d\tau_0 \int_{-\infty}^{\infty} d\xi_0 G(\xi - \xi_0, \tau - \tau_0) b(\xi_0, \tau_0), \tag{90}$$

where $G$ is the Green's function of problem (89), which will be explicitly defined by

$$G(\xi, \tau - \tau_0) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{\tau - \tau_0}} e^{-\frac{\xi^2}{4(\tau-\tau_0)}} = \frac{1}{\sqrt{2}\sqrt{\tau - \tau_0}} e^{-\frac{\xi^2}{4(\tau-\tau_0)}}. \tag{91}$$

Note that we do not enjoy $\lambda > 0$, but $\lambda = \lambda(\tau)$ is a function of one variable, $\tau$, which will facilitate our analysis. In the quarter plane we have

$$\begin{cases} \nu_\tau + Lu = b(\xi, t), & \xi > 0,\ \tau > 0 \\ \nu(0, \xi) = 0 \\ \nu_\xi(0, \tau) = 0. \end{cases} \tag{92}$$

We shall define a solution of (92) in terms of the Green's function $G$ of (89). We extend the function $b(\xi, \tau)$ to the left quarter plane and denote it by $b^*$:

$$b^*(\xi, \tau) = \begin{cases} b(-\xi, \tau) & \text{if } \xi_0 < 0 \\ b(\xi, \tau) & \text{if } \xi_0 > 0. \end{cases} \tag{93}$$

This is an even continuation. Let $\nu^*(\xi, \tau)$ satisfy

$$\begin{cases} \nu^*_\tau + Lu^* = b^*, & \xi \in \mathbb{R},\ \tau > 0 \\ \nu^*(\xi, 0) = 0. \end{cases} \tag{94}$$

Function $b^*$ being even , $\nu^*$ will also be even with respect to $\xi$, so $\nu^*(\xi, \tau) = \nu^*(-\xi, \tau)$. By (90), the solution of problem is (94), we have

$$\nu^*(\xi, \tau) = \int_0^{\tau} d\tau_0 \int_{-\infty}^{\infty} d\xi_0\, G(\xi - \xi_0, \tau - \tau_0) b^*(\xi_0, \tau_0) =$$

$$\int_0^{\tau_0} d\tau_0 \int_{-\infty}^{0} d\xi_0\, G(\xi - \xi_0, \tau - \tau_0) b^*(\xi_0, \tau_0) + \int_0^{\tau_0} d\tau_0 \int_0^{\infty} d\xi_0 G(\xi - \xi_0, \tau - \tau_0) b^*(\xi_0, \tau_0), \tag{95}$$

where $b^*$ is from (93). We make the variable change $\xi_0' := -\xi_0$ in the first integral of (95):

$$\int_{-\infty}^{0} d\xi_0 G(\xi - \xi_0, \tau - \tau_0) b^*(\xi_0, \tau_0) = \int_0^\infty d\xi_0' G(\xi + \xi_0', \tau - \tau_0) b(\xi_0', \tau_0)$$

$$= \int_0^\infty d\xi_0 G(\xi + \xi_0, \tau - \tau_0) b(\xi_0, \tau_0).$$

Hence

$$\nu(\xi,\tau) = \int_0^\tau d\tau_0 \int_0^\infty d\xi_0 [G(\xi + \xi_0, \tau - \tau_0) + G(\xi - \xi_0, \tau - \tau_0)] b(\xi_0, \tau_0) \quad (96)$$

and

$$\nu(\xi,\tau) = \int_0^\tau d\tau_0 \int_0^\infty d\xi_0 g(\xi_0, \xi, \tau_0, \tau) b(\xi_0, \tau_0), \quad (97)$$

with

$$g := G(\xi + \xi_0, \tau - \tau_0) + G(\xi - \xi_0, \tau - \tau_0). \quad (98)$$

Next, we focus on the problem (89) in the half plane, which we rewrite as

$$\begin{cases} \left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] z + \lambda(\tau) z = b, \ \ \xi \in \mathbb{R}, \ \tau > 0 \\ z(\xi, 0) = 0. \end{cases} \quad (99)$$

An estimate of the solution of problem (99) features in [3, chapter IV, p. 320, 352]. We derive an estimate consistent to the one in [3], but with the precise specification of the exponential decay constant. Apply a Fourier transform to $z:$

$$\hat{z} = \zeta(s,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(\xi,\tau) e^{-is\xi} d\xi.$$

Introducing it in the original equation, we obtain

$$\frac{\partial \zeta}{\partial \tau}(s,\tau) + (\lambda(\tau) + s^2)\zeta(s,\tau) = \hat{b}(s,\tau).$$

The transformation $\tilde{\zeta}(s,\tau)$ defined by

$$\zeta(s,\tau) := \tilde{\zeta}(s,\tau) e^{-s^2\tau} \quad (100)$$

yields

$$\tilde{\zeta}_\tau(s,\tau) + \lambda(\tau)\tilde{\zeta}(s,\tau) = \hat{b}(s,\tau) e^{s^2\tau}, \quad (101)$$

$$\tilde{\zeta}(s,0) = \zeta(s,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(\xi,0) e^{-is\xi} d\xi = 0. \quad (102)$$

As $w_{0,\tau} < 0$ is a particular solution of the homogeneous problem

$$\begin{cases} \left[\frac{\partial}{\partial \tau} + \lambda(\tau)\right] w_{0,\tau} = 0 \\ w_{0,\tau}(0) = 0, \end{cases}$$

which follows by differentiating the equation and the initial condition in (50) and $\lambda$ is defined in (88), we obtain

$$\tilde{\zeta}(s,\tau) = w_{0,\tau}(\tau) \int_0^\tau \frac{\hat{b}(\tau_0) e^{s^2\tau_0}}{w_{0,\tau}(\tau_0)} d\tau_0. \quad (103)$$

Denote the inverse Fourier transform by $\mathcal{F}^{-1}$. Then

$$z(\xi,\tau) = \mathcal{F}^{-1}\zeta = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\zeta(s,\tau)e^{is\xi}ds,$$

where

$$\zeta = e^{-s^2\tau}\left[w_{0,\tau}(\tau)\int_0^\tau w_{0,\tau}^{-1}(\tau_0)\hat{b}(s,\tau_0)e^{s^2\tau_0}d\tau_0\right].$$

Then

$$z(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{-s^2\tau}w_{0,\tau}(\tau)\int_0^\tau w_{0,\tau}^{-1}(\tau_0)\hat{b}(s,\tau_0)e^{s^2\tau_0}d\tau_0\right]e^{is\xi}ds.$$

We have

$$\begin{aligned}
z(\xi,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-s^2\tau}w_{0,\tau}(\tau)\int_0^\tau w_{0,\tau}^{-1}(\tau_0)\hat{b}(s,\tau_0)e^{s^2\tau_0}d\tau_0 e^{is\xi}ds = \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\tau w_{0,\tau}(\tau)w_{0,\tau}^{-1}(\tau_0)d\tau_0\int_{-\infty}^{\infty}\left[\hat{b}(s,\tau_0)e^{-s^2(\tau-\tau_0)}\right]e^{is\xi}ds = \\
&= \int_0^\tau w_{0,\tau}(\tau)w_{0,\tau}^{-1}d\tau_0\cdot\mathcal{F}^{-1}[\hat{b}(s,\tau_0)\cdot\hat{G}(s,\tau-\tau_0)],
\end{aligned}\tag{104}$$

where $\hat{G}(s,\tau-\tau_0) = e^{-s^2(\tau-\tau_0)}$. By the convolution theorem,

$$\mathcal{F}^{-1}[\hat{b}(s,\tau_0)\hat{G}(s,\tau-\tau_0)] = \frac{b(\xi,\tau_0)*G(\xi,\tau-\tau_0)}{\sqrt{2\pi}},$$

$$\mathcal{F}^{-1}[\hat{b}(s,\tau_0)\hat{G}(s,\tau-\tau_0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}b(\xi_0,\tau_0)G(\xi-\xi_0,\tau-\tau_0)d\xi_0, \tag{105}$$

where

$$G(\xi,\tau-\tau_0) = \mathcal{F}^{-1}[\hat{G}(s,\tau-\tau_0)] = \mathcal{F}^{-1}[e^{-s^2(\tau-\tau_0)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ds[e^{i\xi s-s^2(\tau-\tau_0)}].$$

Using [1, §4.3b], we obtain

$$G(\xi,\tau-\tau_0) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{\tau-\tau_0}}e^{-\frac{\xi^2}{4(\tau-\tau_0)}} = \frac{1}{\sqrt{2}\sqrt{\tau-\tau_0}}e^{-\frac{\xi^2}{4(\tau-\tau_0)}} \tag{106}$$

and (104) becomes

$$z(\xi,\tau) = \frac{w_{0,\tau}(\tau)}{\sqrt{2\pi}}\int_0^\tau d\tau_0 w_{0,\tau}^{-1}(\tau_0)\int_{-\infty}^{\infty}G(\xi-\xi_0,\tau-\tau_0)b(\xi_0,\tau_0)d\xi_0.$$

Therefore, the solution of problem (99) is

$$z(\xi,\tau) = \int_0^\tau\int_{-\infty}^{\infty}w_{0,\tau}(\tau)w_{0,\tau}^{-1}(\tau_0)\frac{1}{2\sqrt{\pi(\tau-\tau_0)}}e^{-\frac{(\xi-\xi_0)^2}{4(\tau-\tau_0)}}b(\xi_0,\tau_0)d\xi_0 d\tau_0. \tag{107}$$

and in the quarter plane the solution of (92) is

$$\begin{aligned}
\nu(\xi,\tau) &= \frac{1}{2\sqrt{\pi}}\int_0^\tau d\tau_0\int_0^\infty d\xi_0[G(\xi+\xi_0,\tau-\tau_0)+G(\xi-\xi_0,\tau-\tau_0)]b(\xi_0,\tau_0) \\
&= \frac{1}{2\sqrt{\pi}}\int_0^\tau d\tau_0\int_0^\infty d\xi_0\, w_{0,\tau}(\tau)w_{0,\tau}^{-1}(\tau_0)\frac{1}{\sqrt{(\tau-\tau_0)}}\left[e^{-\frac{(\xi-\xi_0)^2}{4(\tau-\tau_0)}}+e^{-\frac{(\xi+\xi_0)^2}{4(\tau-\tau_0)}}\right]b(\xi_0,\tau_0).
\end{aligned}\tag{108}$$

### 3.3.2 Upper bounds for $q_1$ and its derivatives

In the previous subsection we solved the problem (92) with homogeneous boundary conditions. However, the problem for the first-order term of the corner layer function satisfies nonhomogeneous boundary conditions. Consider thus the linear initial value problem with Neumann boundary condition

$$\left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] \Phi(\xi,\tau) + \lambda(\tau)\Phi(\xi,\tau) = b(\xi,\tau), \tag{109}$$

$$\Phi_\xi(0,\tau) = \eta(\tau), \tag{110}$$

$$\Phi(\xi,0) = \Phi_0(\xi) \tag{111}$$

with

$$\eta(0) = \Phi_0'(0), \tag{112}$$

where $\lambda(\tau)$ is given by (88).

$$\lambda(\tau) := F_s(0,0,w_0(0,\tau)).$$

In the quarter plane $(x,t) \in (0,1) \times (0,T]$ equation (109) is

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] \Phi(x,t) + \lambda(\tau)\Phi(x,t) = b(x,t). \tag{113}$$

A comparison principle and un upper bound for the solution of this problem feature in [6]. The bound is in the $L^2-$norm and depends on the space derivative of the solution and on $f_u(0,0,w_0(0,\tau))$. The following result from [4] is also worth mentioning.

**Theorem 3.2.** *Let $D$ be a bounded domain in $\mathbb{R}$ with a piecewise smooth boundary. A solution of the problem (109), (111) exists, is unique and satisfies the following estimate*

$$||\Phi||_{W_2^{1,0}(D\times(0,T))} \le c(T)[||\Phi_0(\xi)||_{L^2(D)} + ||f||_{L^2(D\times(0,T))}]. \tag{114}$$

We shall prove

**Lemma 2.** *The solution of the problem (109), (110), (111) is invariant in the maximum norm, i.e., the upper bound satisfied by the solution is satisfied by its derivatives, too.*

More precisely,

**Lemma 3.** *Let $\Phi(\xi,\tau)$ be the solution of the problem (109), (110), (111) and satisfying (112). Define*

$$\gamma_*^2 = \min\{\min_{t\in[0,T]} f_u(0,t,u(0,t)), \min_{x\in[0,1]} f_u(x,0,u(x,0))\}. \tag{115}$$

*Assume that for some positive constant $C$ and some small $\delta > 0$ , we have*

$$|\eta(\tau)| + |\eta'(\tau)| \le Ce^{-(\gamma_*^2-\delta)\tau}, \tag{116}$$

$$|\Phi_0(\xi)| + |\Phi_0''(\xi)| \le Ce^{-(\gamma_*-\delta)\xi}, \tag{117}$$

$$|b| \leq C \min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi}\}. \tag{118}$$

*Then the pointwise estimate*

$$|\Phi| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\} \tag{119}$$

*holds true.*

*Proof.* Set

$$\tilde{\Phi} = \Phi_0(\xi) e^{-(\gamma_*^2-\delta)\tau} + [\eta(\tau) - \eta(0) e^{-(\gamma_*^2-\delta)\tau}] \left( -\frac{e^{-(\gamma_*-\delta)\xi}}{\gamma_* - \delta} \right), \tag{120}$$

so that, using (112), we obtain

$$(\Phi - \tilde{\Phi})\big|_{\tau=0} = 0,$$

$$\frac{\partial}{\partial \xi} (\Phi - \tilde{\Phi})\big|_{\xi=0} = 0$$

and

$$\tilde{b} = \left( \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2} + \lambda(\tau) \right) \tilde{\Phi} =$$

$$= -\Phi_0(\xi) \frac{e^{-(\gamma_*^2-\delta)\tau}}{\gamma_*^2 - \delta} - \frac{e^{-(\gamma_*-\delta)\xi}}{\gamma_* - \delta} \left[ \eta'(\tau) + \eta(0) \frac{e^{-(\gamma_*^2-\delta)\tau}}{\gamma_*^2 - \delta} \right] -$$

$$-\Phi_0''(\xi) e^{-(\gamma_*^2-\delta)\tau} - \frac{\eta(\tau) - \eta(0) e^{-(\gamma_*^2-\delta)\tau}}{(\gamma_* - \delta)^3} e^{-(\gamma_*-\delta)\xi} +$$

$$+\lambda(\tau) \left[ \Phi_0(\xi) e^{-(\gamma_*^2-\delta)\tau} - [\eta(\tau) - \eta(0) e^{-(\gamma_*^2-\delta)\tau}] \frac{e^{-(\gamma_*-\delta)\xi}}{\gamma_* - \delta} \right]. \tag{121}$$

We have

$$|b - \tilde{b}| \leq C \min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi}\}. \tag{122}$$

Then $\Phi - \tilde{\Phi}$ satisfies

$$\left( \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2} + \lambda(\tau) \right) [\Phi - \tilde{\Phi}] = b - \tilde{b} = \hat{b}. \tag{123}$$

We have

$$\Phi - \tilde{\Phi} =$$

$$\frac{1}{2\sqrt{\pi}} \int_0^\tau d\tau_0 \int_0^\infty d\xi_0 \frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau_0)} \frac{1}{\sqrt{\tau - \tau_0}} \left[ e^{-\frac{(\xi-\xi_0)^2}{4(\tau-\tau_0)}} + e^{-\frac{(\xi+\xi_0)^2}{4(\tau-\tau_0)}} \right] \hat{b}(\xi_0, \tau_0). \tag{124}$$

Using the proof of Lemma 1, for all $\tau \geq \tau_0$

$$\frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau_0)} \leq C e^{-(\gamma_*^2-\delta)(\tau-\tau_0)}. \tag{125}$$

We introduce the function

$$\hat{\Psi}(\xi_0, \tau, \hat{\tau}) = \frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau - \hat{\tau})} \hat{b}(\xi_0, \tau - \hat{\tau}), \tag{126}$$

where

$$\hat{\tau} = \tau - \tau_0.$$

Thus we rewrite (124) in the new variable $\hat{\tau}$ as

$$\Phi - \tilde{\Phi} = \frac{1}{2\sqrt{\pi}} \int_0^\tau d\tau_0 \int_0^\infty d\xi_0 \frac{1}{\sqrt{\hat{\tau}}} \left[ e^{-\frac{(\xi-\xi_0)^2}{4(\hat{\tau})}} + e^{-\frac{(\xi+\xi_0)^2}{4(\hat{\tau})}} \right] \hat{\Psi}(\xi_0, \tau, \hat{\tau}). \tag{127}$$

In (127) we have

$$e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} + e^{-\frac{(\xi+\xi_0)^2}{4\hat{\tau}}} \leq 2e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}}. \tag{128}$$

By (122) and (125),

$$|\hat{\Psi}| = \left| \hat{b}(\xi_0, \tau_0) \frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau_0)} \right| \leq C \min\{e^{-(\gamma_*^2-\delta)\tau_0}, e^{-(\gamma_*-\delta)\xi_0}\} \cdot e^{-(\gamma_*^2-\delta)(\tau-\tau_0)} \leq$$

$$\leq C \min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi_0-(\gamma_*^2-\delta)\hat{\tau}}\}. \tag{129}$$

Now (124) implies

$$|\Phi - \tilde{\Phi}| \leq$$

$$C \min\Big\{e^{-(\gamma_*^2-\delta)\tau} \int_0^\tau d\hat{\tau} \int_0^\infty d\xi_0 \frac{1}{\sqrt{\hat{\tau}}} e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}},$$

$$\int_0^\tau d\hat{\tau} e^{-(\gamma_*^2-\delta)\hat{\tau}} \int_0^\infty d\xi_0 \frac{1}{\sqrt{\hat{\tau}}} 2e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}-(\gamma_*-\delta)\xi_0}\Big\}. \tag{130}$$

Write

$$\int_0^\infty d\xi_0 e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} e^{-(\gamma_*-\delta)\xi_0} =$$

$$= \int_0^\infty d\xi_0 e^{-\left[\frac{\xi_0^2}{4\hat{\tau}} + \left(\frac{-2\xi}{4\hat{\tau}} + \gamma_* - \delta\right)\xi_0 + \frac{\xi^2}{4\hat{\tau}}\right]} = \int_0^\infty d\xi_0\, e^{-[a_1\xi_0^2 + 2a_2\xi_0 + a_3]}, \tag{131}$$

where $a_1 = \frac{1}{4\hat{\tau}}$, $a_2 = -\frac{\xi}{4\hat{\tau}} + (\gamma_* - \delta)/2$, $a_3 = \frac{\xi^2}{4\hat{\tau}}$. The right-hand side of (131) becomes

$$\int_0^\infty d\xi_0 e^{-\left[a_1\left(\xi_0 + \frac{a_2}{a_1}\right)^2 + a_3 - \frac{a_2^2}{a_1}\right]} = e^{-a_3 + \frac{a_2^2}{a_1}} \int_0^\infty e^{-\left[\sqrt{a_1}\left(\xi_0 + \frac{a_2}{a_1}\right)\right]^2} d\xi_0. \tag{132}$$

A change of variable

$$\hat{\xi} = \sqrt{a_1}\left(\xi_0 + \frac{a_2}{a_1}\right), \qquad d\hat{\xi} = \sqrt{a_1}\, d\xi_0$$

yields in (132)

$$\int_0^\infty d\xi_0 e^{-\left[a_1\left(\xi_0 + \frac{a_2}{a_1}\right)^2 + a_3 - \frac{a_2^2}{a_1}\right]} = \frac{C}{a_1} e^{-a_3 + \frac{a_2^2}{a_1}} \int_{\frac{a_2}{\sqrt{a_1}}}^\infty e^{-\hat{\xi}^2} d\hat{\xi} \leq \frac{C}{\sqrt{a_1}} e^{-a_3 + \frac{a_2^2}{a_1}}, \tag{133}$$

where

$$-a_3 + \frac{a_2^2}{a_1} = -\frac{\xi^2}{4\hat{\tau}} + 4\hat{\tau}\left[-\frac{\xi}{4\hat{\tau}} + (\gamma_* - \delta)/2\right]^2$$

$$= -\frac{\xi^2}{4\hat\tau} + 4\hat\tau\left[\frac{\xi^2}{16\hat\tau^2} + (\gamma_* - \delta)^2/4 - (\gamma_* - \delta)\frac{\xi}{4\hat\tau}\right]$$
$$= -\xi(\gamma_* - \delta) + (\gamma_* - \delta)^2\hat\tau \le -\xi(\gamma_* - \delta) + (\gamma_*^2 - \delta)\hat\tau,$$

as $\delta$ is sufficiently small. Therefore

$$\int_0^\infty d\xi_0 e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}} e^{-(\gamma_*-\delta)\xi_0} \le C\frac{e^{-\xi(\gamma_*-\delta)+(\gamma_*^2-\delta)\hat\tau}}{\frac{1}{2\sqrt{\hat\tau}}}. \tag{134}$$

In (130) we have

$$|\Phi - \tilde\Phi| \le$$
$$C\min\left\{e^{-(\gamma_*^2-\delta)\tau}\int_0^\tau d\hat\tau\int_0^\infty d\xi_0\frac{e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}}}{\sqrt{\hat\tau}}, \int_0^\tau d\hat\tau\frac{e^{-(\gamma_*^2-\delta)\hat\tau}}{\sqrt{\hat\tau}}\frac{e^{-\xi(\gamma_*-\delta)+(\gamma_*^2-\delta)\hat\tau}}{\frac{1}{2\sqrt{\hat\tau}}}\right\}. \tag{135}$$

1. We analyse the following term from (135)

$$\int_0^\tau d\hat\tau\frac{e^{-(\gamma_*^2-\delta)\hat\tau}}{\sqrt{\hat\tau}}\frac{e^{-\xi(\gamma_*-\delta)+(\gamma_*^2-\delta)\hat\tau}}{\frac{1}{2\sqrt{\hat\tau}}} \le C\tau e^{-(\gamma_*-\delta)\xi}. \tag{136}$$

2. Next we analyse $e^{-(\gamma_*^2-\delta)\tau}\int_0^\tau d\hat\tau\int_0^\infty d\xi_0\frac{e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}}}{\sqrt{\hat\tau}}$. We claim that

$$\int_0^\tau d\hat\tau\int_0^\infty d\xi_0\frac{e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}}}{\sqrt{\hat\tau}} \le \tau.$$

Indeed, using $\xi' = \frac{\xi-\xi_0}{2\sqrt{\hat\tau}}$, we obtain

$$\int_0^\infty d\xi_0 e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}} = 2\sqrt{\hat\tau}\int_{-\frac{\xi_0}{2\sqrt{\hat\tau}}}^\infty e^{-\xi'^2}d\xi' \le C\sqrt{\hat\tau}. \tag{137}$$

Then

$$\int_0^\tau d\hat\tau\int_0^\infty d\xi_0\frac{e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}}}{\sqrt{\hat\tau}} \le C\int_0^\tau d\hat\tau\frac{\hat\tau}{\sqrt{\hat\tau}} = C\tau. \tag{138}$$
$$C\tau e^{-(\gamma_*^2-\delta)\tau} \le Ce^{-(\gamma_*^2-2\delta)\tau}.$$

This gives

$$e^{-(\gamma_*^2-\delta)\tau}\int_0^\tau d\hat\tau\int_0^\infty d\xi_0\frac{e^{-\frac{(\xi-\xi_0)^2}{4\hat\tau}}}{\sqrt{\hat\tau}} \le Ce^{-(\gamma_*^2-2\delta)\tau}.$$

In view of (136), the estimate (135) becomes

$$|\Phi - \tilde\Phi| \le C\min\{\tau e^{-(\gamma_*-\delta)\xi}, \tau e^{-(\gamma_*^2-\delta)\tau}\}.$$

If $(\gamma_*^2 - 2\delta)\tau \ge (\gamma_* - 2\delta)\xi$, then

$$|\Phi - \tilde\Phi| \le C\tau e^{-(\gamma_*^2-\delta)\tau} \le Ce^{(-\gamma_*^2-2\delta)\tau}.$$

Otherwise, if $(\gamma_*^2 - 2\delta)\tau \le (\gamma_* - 2\delta)\xi$, then

$$|\Phi - \tilde\Phi| \le C\tau e^{-(\gamma_*-\delta)\xi} \le C\xi e^{-(\gamma_*-\delta)\xi} \le Ce^{-(\gamma_*-2\delta)\xi}.$$

These inequalities and (120) imply (119). □

**Lemma 4.** *Under the conditions of Lemma 3, if for some positive constant C and some small $\delta > 0$*

$$|\eta''(\tau)| \le Ce^{-(\gamma_*^2-\delta)\tau} \tag{139}$$

*and*

$$|b_\tau| \le C\min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi}\}, \tag{140}$$

*then the pointwise estimate*

$$|\Phi_\tau| \le C\min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\} \tag{141}$$

*holds true.*

*Proof.* We differentiate with respect to $\tau$ the solution $\Phi = \tilde{\Phi} + (\Phi - \tilde{\Phi})$, where $\tilde{\Phi}$ is from (120), so

$$\tilde{\Phi}_\tau = -\Phi_0(\xi)(\gamma_*^2-\delta)e^{-(\gamma_*^2-\delta)\tau} + [\eta'(\tau) + \eta(0)(\gamma_*^2-\delta)e^{-(\gamma_*^2-\delta)\tau}]\left(-\frac{e^{-(\gamma_*-\delta)\xi}}{\gamma_*-\delta}\right).$$

We use (117) and (116) to obtain

$$|\tilde{\Phi}_\tau| \le C\min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}. \tag{142}$$

and $\Phi - \tilde{\Phi}$ is from (127), so

$$\Phi - \tilde{\Phi} = \\ = \frac{1}{2\sqrt{\pi}}\frac{d}{d\tau}\int_0^\tau d\hat{\tau}\int_0^\infty d\xi_0 \frac{1}{\sqrt{\hat{\tau}}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} + e^{-\frac{(\xi+\xi_0)^2}{4\hat{\tau}}}\right]\hat{\Psi}(\xi_0,\tau,\hat{\tau}). \tag{143}$$

In (143) we have

$$2\sqrt{\pi}(\Phi - \tilde{\Phi})_\tau = \\ = \int_0^\infty d\xi_0 \frac{1}{\sqrt{\tau}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\tau}} + e^{-\frac{(\xi+\xi_0)^2}{4\tau}}\right]\hat{\Psi}(\xi_0,\tau,\tau) + \\ + \int_0^\infty d\xi_0 \int_0^\tau \frac{d\hat{\tau}}{\sqrt{\hat{\tau}}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} + e^{-\frac{(\xi+\xi_0)^2}{4\hat{\tau}}}\right]\frac{d}{d\tau}\hat{\Psi}(\xi_0,\tau,\hat{\tau}), \tag{144}$$

where, by (126), we have

$$\hat{\Psi}(\xi_0,\tau,\hat{\tau}) = \frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau-\hat{\tau})}\hat{b}(\xi_0,\tau-\hat{\tau}),$$

$$\hat{\Psi}(\xi_0,\tau,\tau) = \frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,0)}\hat{b}(\xi_0,0).$$

By (129), we have

$$|\hat{\Psi}(\xi_0,\tau,\tau)| \le C\min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi_0-(\gamma_*^2-\delta)\tau}\} \le Ce^{-(\gamma_*-\delta)\xi_0-(\gamma_*^2-\delta)\tau}. \tag{145}$$

We combine (145) with (128) and (134) with $\hat{\tau}$ replaced by $\tau$, which gives

$$\left|\int_0^\infty d\xi_0 \frac{1}{\sqrt{\tau}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\tau}} + e^{-\frac{(\xi+\xi_0)^2}{4\tau}}\right]\hat{\Psi}(\xi_0,\tau,\tau)\right| \le$$

$$\le Ce^{-(\gamma_*^2-\delta)\tau}\frac{2}{\sqrt{\tau}}\int_0^\infty d\xi_0 e^{-\frac{(\xi-\xi_0)^2}{4\tau}-(\gamma_*-\delta)\xi_0} \le$$

$$\leq Ce^{-(\gamma_*^2-\delta)\tau}\frac{2}{\sqrt{\tau}}\frac{e^{-(\gamma_*-\delta)\xi_0+(\gamma_*^2-\delta)\tau}}{\frac{1}{2\sqrt{\tau}}} = Ce^{-(\gamma_*-\delta)\xi_0}. \tag{146}$$

On the other hand, by (137), we have

$$\left|\int_0^\infty d\xi_0 \frac{1}{\sqrt{\tau}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\tau}} + e^{-\frac{(\xi+\xi_0)^2}{4\tau}}\right]\hat{\Psi}(\xi_0,\tau,\tau)\right| \leq Ce^{-(\gamma_*^2-\delta)\tau}\frac{2}{\sqrt{\tau}}\int_0^\infty d\xi_0 e^{-\frac{(\xi-\xi_0)^2}{\tau}} \leq$$

$$\leq Ce^{-(\gamma_*-\delta)\tau}. \tag{147}$$

Combining (146) and (147), we obtain

$$\left|\int_0^\infty d\xi_0 \frac{1}{\sqrt{\tau}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\tau}} + e^{-\frac{(\xi+\xi_0)^2}{4\tau}}\right]\hat{\Psi}(\xi_0,\tau,\tau)\right| \leq C\min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi}\} \leq$$

$$\leq C\min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}. \tag{148}$$

Next, we estimate

$$\frac{d}{d\tau}\hat{\Psi}(\xi_0,\tau,\hat{\tau}) = \hat{b}(\xi_0,\tau-\hat{\tau})\frac{d}{d\tau}\left(\frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}^2(0,\tau-\hat{\tau})}\right) +$$

$$+\frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau-\hat{\tau})}\hat{b}_\tau(\xi_0,\tau-\hat{\tau}).$$

Recall (125) and that the proof of Lemma 1 yields a similar estimate for $\frac{d}{d\tau}\left(\frac{w_{0,\tau}(0,\tau)}{w_{0,\tau}(0,\tau-\hat{\tau})}\right)$. Combining this with (118), (140), then differentiating (121) with respect to $\tau$ and using (116) and (139), we obtain

$$\left|\frac{d}{d\tau}\hat{\Psi}(\xi_0,\tau,\hat{\tau})\right| \leq C\min\{e^{-(\gamma_*^2-\delta)\tau}, e^{-(\gamma_*-\delta)\xi}\}.$$

In view of this estimate, comparing the term featuring in (144)

$$\left|\int_0^\infty d\xi_0\int_0^\tau \frac{d\hat{\tau}}{\sqrt{\hat{\tau}}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} + e^{-\frac{(\xi+\xi_0)^2}{4\hat{\tau}}}\right]\frac{d}{d\tau}\hat{\Psi}(\xi_0,\tau,\hat{\tau})\right|$$

with (127) and (129), in a similar manner as in the proof of Lemma 3 we obtain

$$\left|\int_0^\infty d\xi_0\int_0^\tau \frac{d\hat{\tau}}{\sqrt{\hat{\tau}}}\left[e^{-\frac{(\xi-\xi_0)^2}{4\hat{\tau}}} + e^{-\frac{(\xi+\xi_0)^2}{4\hat{\tau}}}\right]\frac{d}{d\tau}\hat{\Psi}(\xi_0,\tau,\hat{\tau})\right| \leq$$

$$\leq C\min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}. \tag{149}$$

Combining the estimates from (148) and (149) with (144) and (142), the proof is completed. □

**Corollary 3.3.** *For a positive constant $C$ and sufficiently small $\delta$, the solution of (84) satisfies the pointwise estimate*

$$|q_1| \leq C\min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}. \tag{150}$$

*Proof.* By (45) for $v_1$, we have that $q_1(\xi, 0)$ satisfies (117) of Lemma 3. By (54) for $w_0$, we have that $\frac{\partial q_1}{\partial \xi}(0, \tau)$ satisfies (116) of the Lemma 3. By (85), the right-hand side of the equation for $q_1$ satisfies (118) of the Lemma 3. Now we check (112) for the initial and boundary conditions of (84)

$$-\frac{\partial w_0}{\partial x}(0,0) = -\frac{\partial v_1}{\partial \xi}(0,0). \tag{151}$$

This is the compatibility condition at the corner $(0, 0)$ derived as (86). Therefore, we have (150). □

**Corollary 3.4.** *The following estimate holds true for the first order derivative of the corner layer function term* $q_1$ :

$$\left|\frac{\partial q_1}{\partial \xi}(\xi,\tau)\right| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}, \tag{152}$$

*where* $C$ *is some positive constant and* $\delta > 0$ *is sufficiently small.*

*Proof.* The first derivative $q_{1,\xi}$ is the solution of the problem

$$\left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2} + \lambda(\tau)\right] q_{1,\xi} = -v_{1,\xi}(\xi, 0) F_s(0,0,\cdot)\big|_0^{w_0(0,\tau)}, \tag{153}$$

where $|v_{1,\xi}(\xi, 0)|$ is exponentially decaying in $\xi$ and $\left|F_s(0, 0, \cdot)\big|_0^{w_0(0,\tau)}\right|$ is exponentially decaying in $\tau$, thus satisfying (118), subject to the initial condition

$$q_{1,\xi}(\xi,\tau)|_{\tau=0} = -v_{1,\xi}(\xi,t)|_{t=0}, \tag{154}$$

where $|v_{1,\xi}(\xi, 0)| + |v_{1,\xi\xi}(\xi, 0)| \leq Ce^{-(\gamma_*-\delta)\xi}$, thus it satisfies (117), and the Dirichlet boundary condition

$$q_{1,\xi}(\xi,\tau)|_{\xi=0} = -w_{0,x}(x,\tau)|_{x=0}, \tag{155}$$

which satisfies $| - w_{0,x}(0, \tau)| + | - w_{0,x\tau}(x, \tau)| \leq Ce^{-(\gamma_*^2-\delta)\tau}$. Thus (116) is satisfied. By (151), the problem for $q_{1,\xi}$ satisfies the zero-order compatibility condition at $(0, 0)$. The analogue of Lemma 3 for the problem with Dirichlet boundary conditions is then applied to the problem for $q_{1,\xi}$. □

**Lemma 5.** *The following derivative estimates for the term* $q_1$ *of the corner layer function hold true:*

*a)* $|q_{1,\xi\xi}(\xi,\tau)| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}$,

*b)* $|q_{1,\tau}(\xi,\tau)| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}$,

*c)* $|q_{1,\tau\tau}(\xi,\tau)| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}$,

*d)* $|q_{1,\xi\xi\xi}(\xi,\tau)| \leq C \min\{e^{-(\gamma_*^2-2\delta)\tau}, e^{-(\gamma_*-2\delta)\xi}\}$.

*where in each case* $C$ *is some positive constant and* $\delta > 0$ *are sufficiently small.*

*Proof.* *a)* The second- order derivative $q_{1,\xi\xi}(\xi,\tau)$ is the solution of the problem

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\xi^2}\right)q_{1,\xi\xi}(\xi,\tau)+\lambda(\tau)q_{1,\xi\xi}(\xi,\tau)=-v_{1,\xi\xi}(\xi,0)F_s(0,0,\cdot)|_0^{w_0(0,\tau)}, \quad (156)$$

with the initial condition

$$q_{1,\xi\xi}|_{\tau=0}=-v_{1,\xi\xi}|_{t=0}, \quad (157)$$

which is exponentially decaying in $\xi$. Also, $|-v_{1,\xi\xi}|_{\tau=0}|+|-v_{1,\xi\xi\xi}|_{\tau=0}|$ is exponentially decaying in $\xi$, thus it satisfies (117). By extracting the second term from equation (153), we obtain for (156) the Neumann boundary condition

$$\frac{\partial}{\partial\xi}q_{1,\xi\xi}\Big|_{\xi=0}=\left[\left(\frac{\partial q_{1,\xi}}{\partial\tau}+\lambda(\tau)q_{1,\xi}\right)+v_{1,\xi}(\xi,0)F_s(0,0,\cdot)|_0^{w_0(0,\tau)}\right]\Bigg|_{\xi=0},$$

i.e.

$$\frac{\partial}{\partial\xi}q_{1,\xi\xi}\Big|_{\xi=0}=\left[-\frac{\partial^2 w_0}{\partial\tau\partial x}-\lambda(\tau)\frac{\partial w_0}{\partial x}\right]\Bigg|_{x=0}+v_{1,\xi}(0,0)F_s(0,0,\cdot)|_0^{w_0(0,\tau)} \quad (158)$$

$$\leq Ce^{-(\gamma_*^2-\delta)\tau},$$

thus satisfying (116). The right-hand side of (156) satisfies (118). We now check whether $\eta(0)=\Phi_0'(0)$, i.e. whether

$$\left[\left(-\frac{\partial^2 w_0}{\partial\tau\partial x}-\lambda(\tau)\frac{\partial w_0}{\partial x}\right)\Big|_{x=0}+v_{1,\xi}(0,0)F_s(0,0,\cdot)|_0^{w_0(0,\tau)}\right]\Bigg|_{\tau=0}=-v_{1,\xi\xi\xi}|_{t=0}. \quad (159)$$

Remind the problems for $w_0$ :

$$\begin{cases} w_{0,\tau}+F(x,0,w_0)=0\\ w_0(x,0)=\varphi(x)-u_0(x,0)\end{cases} \quad (160)$$

and for $v_1$ :

$$\begin{cases} -v_{1,\xi\xi}+v_1F_s(0,t,0)=0\\ v_{1,\xi}(0,t)=u_{0,x}(0,t).\end{cases} \quad (161)$$

Recall also that

$$\lambda(\tau)=F_s(0,0,w_0(\tau)).$$

By (160),

$$-\frac{\partial}{\partial x}\frac{\partial w_0}{\partial\tau}\Big|_{(0,0)}=F_x(0,0,w_0(0,0))+F_s(0,0,w_0(0,0))w_{0,x}(0,0), \quad (162)$$

$$\left(-\frac{\partial^2 w_0}{\partial\tau\partial x}-\lambda(\tau)\frac{\partial w_0}{\partial x}\right)\Big|_{\substack{x=0,\\ \tau=0}}=$$

$$=F_x(0,0,w_0(0,0))+F_s(0,0,w_0(0,0))w_{0,x}(0,0)-F_s(0,0,w_0(0,0))w_{0,x}(0,0)=$$

$$=F_x(0,0,w_0(0,0)). \quad (163)$$

From (180) we obtain

$$v_{1,\xi}(0,0)F_s(0,0,\cdot)|_0^{w_0(0,0)}=-\frac{\partial u_0}{\partial x}(0,0)F_s(0,0,\cdot)|_0^{w_0(0,0)} \quad (164)$$

and

$$-v_{1,\xi\xi\xi}(0,0) = -v_{1,\xi}(0,0)F_s(0,0,0) = u_{0,x}(0,0)F_s(0,0,0). \qquad (165)$$

In view of (163), (164) and (165), we see that (159) is equivalent to

$$F_x(0,0,w_0(0,0)) - u_{0,x}(0,0)F_s\big|_0^{w_0(0,0)} = u_{0,x}(0,0)F_s(0,0,0);$$

or

$$F_x(0,0,w_0(0,0)) - u_{0,x}(0,0)F_s(0,0,w_0(0,0)) = 0;$$

or, recalling the definition $F(x,t,s) = f(x,t,u_0(x,t)+s)$ and the condition $u_0(0,0) = \varphi(0)$

$$f_x(0,0,\varphi(0)) = 0,$$

which is satisfied by assumption (11). After cancellations and using $u_{0,x}(0,0) = 0$, this is

$$f_x(0,0,u_0(0,0)+w_0(0,0)) + [u_{0,x}(0,0)+w_{0,x}(0,0)]f_u(0,0,u_0(0,0)+w_0(0,0)) = 0, \qquad (166)$$

where

$$u_0(0,0) + w_0(0,0) = \varphi(0) = 0$$

and

$$u_{0,x}(0,0) + w_{0,x}(0,0) = \varphi_x(0) = 0.$$

Hence (166) is equivalent to

$$f_x(0,0,0) = 0, \qquad (167)$$

which is satisfied by (11) and $\varphi(0) = 0$. Thus (159) is established.
From (86) we have

$$v_{1,\xi}(0,0) = w_{0,x}(0,0).$$

Thus we apply Lemma 3.
*b)* For $|q_{1,\tau}|$, we use the equation from (84)

$$|q_{1,\tau}| \leq \left|\frac{\partial^2 q_1}{\partial \xi^2}\right| + |q_1\lambda(\tau)| + \left|v_1(\xi,0)F_s(0,0,\cdot)\big|_0^{w_0(0,\tau)}\right|,$$

where $|\lambda(\tau)| \leq C$ and we use (85) and the estimates that we have obtained for $|q_1|$ in (150) and for $|q_{1,\xi\xi}|$ in *a)*.
*c)* We have

$$q_{1,\tau\tau} = q_{1,\xi\xi\tau} - \frac{d\lambda}{d\tau}q_1 - \lambda q_{1,\tau} - v_1\frac{d}{dt}F_s(0,0,\cdot)\big|_0^{w_0(0,\tau)}. \qquad (168)$$

Applying Lemma 4 for $\Phi = q_{1,\xi\xi}$, we have that

$$q_{1,\xi\xi\tau} = \frac{d\Phi}{d\tau}$$

satisfies (141). In (168) we have that $\left|\frac{d\lambda}{d\tau}\right|, |\lambda|$ and $\left|\frac{d}{dt}F_s(0,0,\cdot)\big|_0^{w_0(0,\tau)}\right|$ are bounded. For $|q_1|$, $|q_{1,\tau}|$ and $|v_1|$ we use the estimates previously obtained.
*d)* The third-order derivative satisfies the problem

$$\left[\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial\xi^2}\right]q_{1,\xi\xi\xi}(\xi,\tau) + \lambda(\tau)q_{1,\xi\xi\xi}(\xi,\tau) = -v_{1,\xi\xi\xi}(\xi,0)F_s(0,0,\cdot)\big|_0^{w_0(0,\tau)},$$

where $| - v_{1,\xi\xi\xi}(\xi, 0)|$ is exponentially decaying in $\xi$ and $|F_s(0, 0, \cdot)|_0^{w_0(0,\tau)}|$ is exponentially decaying in $\tau$, thus satisfying (118), with the initial condition

$$q_{1,\xi\xi\xi}(\xi, 0) = -v_{1,\xi\xi\xi}(\xi, 0)$$

which satisfies (117) and the Dirichlet boundary condition $q_{1,\xi\xi\xi}(0, \tau)$ given by the right-hand side of (158). This satisfies (116). Condition (112) is satisfied due to (159). □

### 3.4 Perturbed asymptotic expansion and existence of solution

Using the stretched variables

$$\xi = x/\varepsilon, \quad \tau = t/\varepsilon^2,$$

we set the super-solution as

$$\beta(x, t, p) = u_0(x, t) + \varepsilon v_1(\xi, t) + \tilde{w}_0(x, \tau, p) + \varepsilon q_1(\xi, \tau) + C_0 p[e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon} + 1]$$

$$:= u_{as} + W + C_0 p[e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon} + 1], \tag{169}$$

with $p$ a sufficiently small perturbation parameter which is positive for the super-solution and negative for the sub-solution, $C_0, c_0$ are positive constants and

$$W = W(x, \tau; p) = \tilde{w}_0(x, \tau; p) - w_0(x, \tau). \tag{170}$$

By (54),

$$|W| \leq p \frac{\partial \tilde{w}_0(x, \tau; \tilde{p})}{\partial p} = e^{-(\gamma_*^2 - \delta)\tau} O(p). \tag{171}$$

Due to

$$u_{as}(x, 0) = \varphi(x),$$

condition (15) of Definition 2.1 is satisfied:

$$\alpha(x, 0) \leq \varphi(x) \leq \beta(x, 0) \qquad \forall\, x \in [0, 1]. \tag{172}$$

Denote the last term in (169) by

$$\rho = e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon} + 1, \tag{173}$$

which has the property

$$1 \leq \rho \leq 3. \tag{174}$$

The rapidly decaying function $C_0 p\rho$ has been chosen to capture the Neumann boundary conditions. (In the case of a similar problem with Dirichlet boundary conditions [2], a term of the form $C_0 p$ would feature instead).

**Lemma 6.** *Let $\mathcal{F}$ be given by (1), $u_{as}$, $\beta$ by (169) and $\rho$ by (185). Then*

1. $\mathcal{F} u_{as} = O(\varepsilon^2)$;
2. $\mathcal{F}\beta =$
$= C_0 p\rho F_s(x, t, 0) + p w_0(1 + C_0\rho\lambda) - p c_0^2[e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}] + O(\varepsilon^2 + p^2)$,

*where* $\lambda = \lambda(x,t) = F_{ss}(x,t,\vartheta w_0)$ *for some* $\vartheta = \vartheta(x,t) \in (0,1)$.

*Proof.* 1. Write

$$\mathcal{F}u_{as} = \varepsilon^2(u_{as,t} - u_{as,xx}) + f(x,t,u_{as}) =$$

$$= \varepsilon^2(u_{0,t} + \varepsilon v_{1,t} + w_{0,t} + \varepsilon q_{1,t} - u_{0,xx} - \varepsilon v_{1,xx} - w_{0,xx} - \varepsilon q_{1,xx}) +$$
$$+ F(x,t,\varepsilon v_1(\xi,t) + w_0(x,\tau) + \varepsilon q_1(\xi,\tau)). \tag{175}$$

where

$$\mathcal{F}[u_0(x,t)] = O(\varepsilon^2). \tag{176}$$

From

$$\mathcal{F}(u_0 + w_0) = O(\varepsilon^2) - F(x,t,w_0) + F(x,t,w_0) = O(\varepsilon^2). \tag{177}$$

we have

$$\mathcal{F}[u_0(x,t) + w_0(x,\tau)] = O(\varepsilon^2). \tag{178}$$

If $v_1(\xi,\tau)$ is a solution of

$$\begin{cases} -\dfrac{\partial^2 v_1}{\partial \xi^2} + v_1 F_s(0,t,v_0) & = -\xi F_x(0,t,v_0) \\ v_1(0,t) & = 0 \\ v_1(\infty,t) & = 0. \end{cases} \tag{179}$$

then it is also a solution of

$$\begin{cases} -v_{1,\xi\xi} + v_1 F_s(0,t,0) = 0 \\ v_{1,\xi}(0,t) = u_{0,x}(0,t), \end{cases} \tag{180}$$

From (79)

$$\mathcal{F}(u_0 + v_0 + \varepsilon v_1) = O(\varepsilon^2), \tag{181}$$

where $v_0 = 0$, we obtain

$$\mathcal{F}[u_0(x,t) + \varepsilon v_1(\xi,t)] = O(\varepsilon^2). \tag{182}$$

Recall (82):

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) + F(\varepsilon\xi, \varepsilon^2\tau, \cdot)\Big|^{[\varepsilon v_1(\xi,\varepsilon^2\tau)] + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)}_{[\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)} = O(\varepsilon^2).$$

Combining this with (176), (178), (182), we obtain

$$|\mathcal{F}u_{as}| = O(\varepsilon^2).$$

2.Write

$$\mathcal{F}\beta = \mathcal{F}\Big|^{\beta}_{u_{as}} + \mathcal{F}u_{as}.$$

We analyze

$$\mathcal{F}\beta - \mathcal{F}u_{as} = \varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](W + \rho) + F(x,t,\cdot)\Big|^{\beta}_{u_{as}}, \tag{183}$$

where

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] W = -WF_s(x,t,w_0) + pw_0 + O(\varepsilon^2 + p^2) \tag{184}$$

and $\rho_t = 0$ yields

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] C_0 p\rho = \varepsilon^2 C_0 p(-\rho_{xx}) = -pC_0 c_0^2[e^{-c_0 x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}],$$

so

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] \rho \le 2pc_0^2. \tag{185}$$

We have

$$F(x,t,\cdot)\Big|_{u_{as}}^{\beta} = F(x,t,\cdot)\Big|_{u_{as}+W}^{\beta} + F(x,t,\cdot)\Big|_{u_{as}}^{u_{as}+W}, \tag{186}$$

where from $\varepsilon v_1 + W + \varepsilon q_1 = O(\varepsilon + p)$ we obtain

$$F(x,t,\cdot)\Big|_{u_{as}+W}^{\beta} = F(x,t,\cdot)\Big|_{u_{as}+W}^{u_{as}+W+C_0 p\rho} = C_0 p\rho[F_s(x,t,w_0) + O(\varepsilon+p)] =$$

$$= C_0 p\rho[F_s(x,t,0) + \lambda w_0] + O(\varepsilon^2 + p^2). \tag{187}$$

The second term in (186) is

$$F(x,t,\cdot)\Big|_{u_{as}}^{u_{as}+W} = F(x,t,\cdot)\Big|_{\varepsilon v_1+w_0+\varepsilon q_1}^{\varepsilon v_1+w_0+\varepsilon q_1+W} = WF_s(x,t,\varepsilon v_1+w_0+\varepsilon q_1)+O(p^2) =$$

$$= WF_s(x,t,w_0) + O(|W[\varepsilon(v_1+q_1)]| + p^2), \tag{188}$$

where $|W| = O(p)$ and $\varepsilon(v_1+q_1) = O(\varepsilon)$, so

$$O(|W\varepsilon(v_1+q_1)| + p^2) = O(\varepsilon^2 + p^2).$$

Combining (183), (184), (185), (186), (187) with $\mathcal{F}u_{as} = O(\varepsilon^2)$, we obtain

$$\mathcal{F}\beta = C_0 p\rho F_s(x,t,0) + pw_0(1+C_0\rho\lambda) - pc_0^2[e^{-c_0x/\varepsilon} + e^{-c_0(1-x)/\varepsilon}] + O(\varepsilon^2+p^2).$$

□

**Corollary 3.5.** *For all $|p| \le p_0$ there exist some positive constants $c_0, C_0$ and $C_1$ such that*

$$\mathcal{F}\beta \begin{cases} \ge \frac{C_0 p\gamma^2}{2} - C_1(\varepsilon^2+p^2) & \text{if } p > 0 \\ \le -\frac{C_0|p|\gamma^2}{2} + C_1(\varepsilon^2+p^2) & \text{if } p < 0. \end{cases} \tag{189}$$

*Proof.* In the term $pw_0(1+C_0\rho\lambda)$ we have $1 \le \rho \le 3$. and $w_0 \ge 0$. Choose $C_0$ sufficiently small, such that

$$1 + C_0\rho\lambda \ge 1 - 3C_0|\lambda| \ge 0,$$

where we used (174). Then for $p > 0$

$$\mathcal{F}\beta \ge C_0 p\gamma^2 - 2pc_0^2 - C_1(\varepsilon^2+p^2).$$

Choosing $c_0$ sufficiently small, such that

$$\frac{C_0\gamma^2}{2} - 2c_0^2 \ge 0,$$

we obtain (189) for $p > 0$. The case $p < 0$ is similar. □

We are now able to prove that $\beta$ set by (169) satisfies Definition 2.1 of a super-solution, leading to the final result of existence of solution.

**Theorem 3.6.** *There exist a sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, a solution $u$ of problem (1), (2), (3) exists and is unique. Furthermore, for this solution and its asymptotic expansion we have*

$$|u(x,t) - u_{as}(x,t)| \leq C\varepsilon^2 \quad \forall\, (x,t) \in [0,1] \times [0,T]. \tag{190}$$

*Proof.* We prove that $\beta$ defined by (169) is super-solution and $\alpha$ defined by $\alpha(x,t;p) = \beta(x,t;-p)$, where $0 < p < \bar{p}$, is sub-solution of problem (1), (2), (3). Set $\bar{p} = C_2\varepsilon^2$, where $C_2 \geq 4C_1/(C_0\gamma^2)$ so that $\frac{C_0\bar{p}\gamma^2}{4} \geq C_1\varepsilon^2$. We seek to have

$$\frac{C_0\bar{p}\gamma^2}{4} \geq C_1\bar{p}^2,$$

so

$$\bar{p} \leq \frac{C_0\gamma^2}{4C_1},$$

which is guaranteed by choosing $\varepsilon_0$ sufficiently small. Then, by Corollary 3.5 we obtain

$$\mathcal{F}\beta(x,t;-\bar{p}) \leq 0 \leq \mathcal{F}\beta(x,t;\bar{p}). \tag{191}$$

Condition (15) is satisfied due to (172).
We shall now deal with the boundary condition at $x = 0$. We evaluate $u_{as,x}(x,t)$ in $x = 0$ and $x = 1$ :

$$u_{as,x}(0,t) = u_{0,x}(0,t) + w_{0,x}(0,\tau) + v_{1,\xi}(0,t) + q_{1,\xi}(0,\tau) = 0, \tag{192}$$

as $u_{0,x}(0,t)+v_{1,\xi}(0,t) = 0$ and $w_{0,x}(0,\tau)+q_{1,\xi}(0,\tau) = 0$. Similarly, $u_{as,x}(1,t) = 0$. To prove (14), we note that for all $|p| \leq p_0$ we have

$$\frac{\partial\beta}{\partial x}\Big|_{x=0} = \frac{\partial}{\partial x}(u_{as} + W + C_0p\rho)\Big|_{x=0} = \frac{\partial}{\partial x}W\,|_{x=0} + C_0p\rho_x|_{x=0}\,,$$

where

$$\left|\frac{\partial}{\partial x}W\Big|_{x=0}\right| \leq C_W|p|$$

and

$$-\rho_x\Big|_{x=0} = \frac{c_0}{\varepsilon}(1 - e^{-c_0/\varepsilon}) \geq \frac{c_0}{2\varepsilon}$$

for sufficiently small $\varepsilon_0$. Choosing $\varepsilon_0$ sufficiently small, we obtain for $p > 0$

$$-\varepsilon\frac{\partial\beta(x,t;p)}{\partial x}\Big|_{x=0} \geq \frac{C_0c_0p}{2} - \varepsilon C_W p > \frac{C_0c_0p}{4} > 0. \tag{193}$$

Similarly,

$$-\varepsilon\frac{\partial\beta(x,t;-p)}{\partial x}\Big|_{x=0} \leq 0 \leq -\varepsilon\frac{\partial\beta(x,t;p)}{\partial x}\Big|_{x=0}, \qquad \forall p > 0, \tag{194}$$

and

$$\varepsilon\frac{\partial\beta(x,t;p)}{\partial x}\Big|_{x=1} \geq 0 \geq \varepsilon\frac{\partial\beta(x,t;-p)}{\partial x}\Big|_{x=1}, \qquad \forall p > 0. \tag{195}$$

Comparing (193), (194) and (195) with (14), (191) with (13), and having (172) and also (12), we obtain that $\beta(x,t;\bar{p})$ is a super-solution and $\beta(x,t;-\bar{p})$ is a sub-solution as defined in Definition 2.1; between them, applying [7, Theorem 5.2], we have existence of a solution $u$ of (1), (2), (3):

$$\beta(x,t;-\bar{p}) \le u(x,t) \le \beta(x,t;p). \tag{196}$$

Furthermore, Proposition 1.1 implies that this a unique solution. Since, by (169) and (171), we have

$$\beta(x,t;\pm\bar{p}) = u_{as} + O(\bar{p}) = u_{as} + O(\varepsilon^2),$$

then $|u - u_{as}| \le C\varepsilon^2$. □

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