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Asymptotics for the spectrum of a thin film equation in a singular limit

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Abstract

In this paper the linear stability properties of the steady states of a no-slip lubrication equation are studied. The steady states are configurations of droplets and arise during the late-phase dewetting process under the influence of both destabilizing van der Waals and stabilizing Born intermolecular forces, which in turn give rise to the minimum thickness ε of the remaining film connecting the droplets. The goal of this paper is to give an asymptotic description of the eigenvalues and eigenfunctions of the problem, linearized about the one-droplet solutions, as $\varepsilon \to 0$.

For this purpose, corresponding asymptotic eigenvalue problems with piecewise constant coefficients are constructed, such that their eigenvalue asymptotics can be determined analytically. A comparison with numerically computed eigenvalues and eigenfunctions shows good agreement with the asymptotic results and the existence of a spectrum gap to a single exponentially small eigenvalue for sufficiently small ε.

1 Introduction

Thin liquid polymer films of nanometer thickness typically destabilize and dewet from a substrate due to intermolecular forces between the film and the substrate. The late phases of the ensuing complex dewetting process are found to be configurations of droplets. These droplets tend to assume their own slow dynamics via mass exchange through an even thinner film connecting the droplets, which is a result of competing long-range van der Waals attraction and short-range Born repulsion forces [1, 2]. Using the scale separation between the height of the thin film and the lateral extend of the evolving patterns, the free boundary boundary problem for the Navier-Stokes equations can be reduced to a single equation for the profile of the film, see for example the reviews [3, 4]. For the two-dimensional case, i.e. considering the cross-section of the film, the evolution of the profile is described by the corresponding one-dimensional thin film equation

$$\partial_t h = -\partial_x \Big(h^3 \partial_x \left(\partial_{xx} h - \Pi_\varepsilon \left(h \right) \right) \Big), \tag{1.1}$$

considered on a fixed interval $[-L,\, L]$ with boundary conditions

$$\partial_{xxx} h = 0, \quad \text{and} \quad \partial_x h = 0 \quad \text{at} \quad x = \pm L, \tag{1.2}$$

which incorporate zero flux at the boundary and imply the conservation of mass law

$$h_c \equiv \frac{1}{2L} \int_{-L}^{+L} h(x,t)\, dx \text{ for all } t > 0, \tag{1.3}$$

and the potential function

$$\Pi_\varepsilon(h) = \frac{\varepsilon^2}{h^3} - \frac{\varepsilon^3}{h^4}. \qquad (1.4)$$

It can be used to study the dewetting dynamics from rupture of the thin film to the late-phase process of coarsening. Interestingly, this late phase behaviour parallels those found in various phase separation phenomena modelled by Cahn-Hilliard type equations, the analysis of which has been developed very intensively during the last decades. For these equations approximate models, consisting of sets of coupled ordinary differential equations, have been derived in [5, 6, 7] in order to efficiently capture the late time Ostwald ripening and specifically properties such as coarsening rates.

For dewetting thin films a very effective method to obtain the long-time behaviour for large arrays of droplets during the late phase dewetting has been developed by Glasner and Witelski [1, 8], who used the quasi-stationary shape of the droplets to approximate the thin film model via a formal singular perturbation argument to a set of ordinary differential equations for the pressure and location of the droplets. Their method has been extended further to higher dimensions and by including other effects such as interfacial slippage or gravity [9, 10, 11, 12, 13]. Moreover, in the studies by Otto et al. [9] and Glasner et al. [10] the analysis of this asymptotic limit on the basis of the corresponding gradient flow structure was developed. Nevertheless, the full rigorous justification of this question is still open.

Our analysis intends to explore an alternative approach. In a companion paper [14] we show that it is possible to extend a center-manifold construction, that has been developed for the class of semi-linear partial differential equations by Mielke and Zelig [15], to the type of quaslilinear thin film equation considered here. It turns out, that one of the main assumptions for an existence proof of the center-manifold concerns the asymptotics of the spectrum of the corresponding thin film equation as $\varepsilon \to 0$, linearized about the steady state solution $h_{0,\varepsilon}$, which describes a droplet on a bounded interval. This will be the focus of the analysis presented in this paper.

There is already a body of work on steady state solutions and their linear stability for a family of thin film equations including (1.1) considered with fixed $\varepsilon > 0$. For Neumann boundary conditions for h and $\partial_{xx}h$ Bertozzi et al. [16] derived in the one-dimensional case the global structure of the bifurcation diagramm for the steady state solutions and proved existence of smooth solutions for $\varepsilon > 0$. Linear stability of smooth steady states with periodic and Neumann boundary conditions under mass conserving perturbations was investigated in the work by Laugesen and Pugh [17] . The linear stability of droplet steady states states for related problems has been considered numerically by Goldstein et al. [18] and briefly discussed [17] . Hence, apart from the overall goal of a rigorous foundation for the approach leading to the reduced ODE model for the long-time behaviour, establishing the spectrum asymptotics as $\varepsilon \to 0$ is of interest of its own.

As has been first suggested in Glasner and Witelski [11], it is useful to rescale the problem such that the small parameter ε is eliminated from the problem (1.1) with (1.4) and appears instead in the boundary conditions (1.2). For this purpose we introduce scalings

$$\bar{x} = \frac{x}{\varepsilon}, \quad \bar{h} = \frac{h}{\epsilon}, \quad \bar{t} = \frac{t}{\varepsilon}, \tag{1.5}$$

so that

$$\bar{\Pi}(\bar{h}) = \varepsilon \Pi_\varepsilon(\bar{h}\,\varepsilon) = \frac{1}{\bar{h}^3} - \frac{1}{\bar{h}^4}. \tag{1.6}$$

The analysis in remainder of the paper will refer to the problem in this scaling, so that for ease of notation we will drop the "$^-$" from now on and consider the rescaled problem

$$\partial_t h = -\partial_x \Big(h^3 \partial_x \left(\partial_{xx} h - \Pi(h) \right) \Big), \tag{1.7}$$

with boundary conditions

$$\partial_{xxx} h = 0, \quad \text{and} \quad \partial_x h = 0 \quad \text{at} \quad x = \pm L/\varepsilon, \tag{1.8}$$

which imply conservation of mass

$$h_c \equiv \frac{\varepsilon}{2L} \int_{-L/\varepsilon}^{L/\varepsilon} h(x,t)\, dx, \ \forall t > 0. \tag{1.9}$$

We first summarize some results from [8, 16] that we will use for our analysis.

Proposition 1.1. *For each $\varepsilon > 0$ equation (1.7) with boundary conditions (1.8) has a family of positive nonconstant steady state solutions $h_{0,\,\varepsilon}(x;P)$ parameterized by a constant ("pressure") $P \in (0,\ P_{max}(\varepsilon))$, where*

$$P_{max}(\varepsilon) = \frac{27}{256\varepsilon}, \tag{1.10}$$

which satisfy

$$\partial_{xx} h_{0,\,\varepsilon}(x;P) = \Pi\left(h_{0,\,\varepsilon}(x;P)\right) - \varepsilon P, \tag{1.11a}$$

$$h_{0,\,\varepsilon}(x;P) = h_{0,\,\varepsilon}(-x;P), \tag{1.11b}$$

$$\partial_x h_{0,\,\varepsilon}(0;P) = 0 \quad \textit{and} \quad \partial_x h_{0,\,\varepsilon}(x;P) < 0 \ \textit{for}\ x \in (0,\ L/\varepsilon). \tag{1.11c}$$

We note that it is easy to check that a solution to (1.7) with (1.8) is a stationary if and only if it satisfies (1.11a) with (1.8). The rest of the proof can be done via a phase plane analysis as described in [16]. It shows that for each $\varepsilon > 0$ and a fixed $P \in (0,\ P_{max}(\varepsilon))$ there exists a family of periodic orbits to the equation (1.11a) nested into a homoclinic loop. For any orbit there exists a phase shift such that the corresponding periodic solution restricted to the interval $[-L/\varepsilon,\ L/\varepsilon]$ gives a smooth steady state solution $h_{0,\varepsilon}(x;P)$ to (1.7)–(1.8) satisfying (1.11b)–(1.11c). Everywhere below we consider P fixed and therefore omit the dependence on it in the notation for the stationary solutions writing $h_{0,\varepsilon}(x)$.

The linear operator that is obtained by linearizing the right-hand side of (1.7) about the steady state $h_{0,\varepsilon}$ is given by

$$\mathcal{L}_\varepsilon = -\frac{d}{dx} \left[h_{0,\varepsilon}^3 \frac{d}{dx} \left(\frac{d^2}{dx^2} \cdot - \frac{d\Pi}{dh}(h_{0,\varepsilon}) \cdot \right) \right], \tag{1.12}$$

where

$$D\left(\mathcal{L}_\varepsilon\right)=\left\{\eta\in W^{4,2}(-L/\varepsilon,L/\varepsilon):\eta'''(\pm L/\varepsilon)=\eta'(\pm L/\varepsilon)=\int_{-L/\varepsilon}^{L/\varepsilon}\eta\,dx=0\right\}. \tag{1.13}$$

Here, $'=d/dx$. The eigenvalue problem associated with the operator $\mathcal{L}_\varepsilon$ can be written as

$$\mathcal{L}_\varepsilon\eta=-\lambda\eta,\ \eta\in D\left(\mathcal{L}_\varepsilon\right). \tag{1.14}$$

One can check that the set of eigenvalues to (1.14) divided by ε gives the set of eigenvalues for the linearized eigenvalue problem corresponding to the unscaled equation (1.1) with (1.4) and (1.2). For a fixed $\varepsilon>0$ the operator $\mathcal{L}_\varepsilon$ is a particular case of a general class of linear operators associated with the linearized thin film type equations. For such operators qualitative properties of their spectra have been investigated by [17]. For our subsequent purposes we summarize them here, applied to $\mathcal{L}_\varepsilon$. Firstly, we use a transformation of the eigenvalue problem (1.14) to a symmetric one. We define the functions

$$r_\varepsilon(x) = -\frac{d\Pi}{dh}(h_{0,\varepsilon}), \tag{1.15a}$$

$$f_\varepsilon(x) = \frac{1}{h_{0,\varepsilon}^3}. \tag{1.15b}$$

In Appendix B of [17] it was shown that if a pair $[\eta,\,\lambda]$ is a solution to the original eigenvalue problem (1.14), then a pair $[h,\,\lambda]$ with

$$h(x)=\int_{-L/\varepsilon}^{x}\eta(s)\,ds$$

satisfies

$$h^{(4)}(x)+(r_\varepsilon(x)h'(x))' = \lambda f_\varepsilon(x)h(x), \tag{1.16a}$$

$$h''(\pm L/\varepsilon)=h(\pm L/\varepsilon) = 0. \tag{1.16b}$$

Vice versa any solution $[h,\,\lambda]$ to (1.16a)–(1.16b) gives a solution $[\eta,\,\lambda]$ to (1.14) with $\eta(x)=h'(x)$.

We define next Hilbert spaces

$$W_\varepsilon:=H^2(-L/\varepsilon,\,L/\varepsilon)\cap H_0^1(-L/\varepsilon,\,L/\varepsilon) \tag{1.17}$$

equipped with the standard $H^2(-L/\varepsilon,\,L/\varepsilon)$ inner product, and $L^2(-L/\varepsilon,\,L/\varepsilon)$ with the weighted one

$$(h,\,w)_\varepsilon:=\int_{-L/\varepsilon}^{L/\varepsilon}h\,w\,f_\varepsilon\,dx. \tag{1.18}$$

The next theorem is a reformulation of Theorem 23 of [17] for our case.

Theorem 1.2. *Consider a symmetric eigenvalue problem*

$$h \in W_\varepsilon,\, \lambda \in \mathbb{R} \;:\quad \int_{-L/\varepsilon}^{L/\varepsilon} (h''w'' - r_\varepsilon h'w' - \lambda f_\varepsilon h\, w)\, dx = 0,\, \forall w \in W_\varepsilon. \tag{1.19}$$

For a fixed $\varepsilon > 0$ *there exist sequences* $\{\lambda_\varepsilon^*,\, \lambda_\varepsilon^0,\, \lambda_\varepsilon^1,\, \lambda_\varepsilon^2, ...\}$, $\{h_\varepsilon^*,\, h_\varepsilon^0,\, h_\varepsilon^1,\, h_\varepsilon^2, ...\}$ *such that:*

(i) for each $j \in \{*,\, 0,\, 1,\, 2, ...\}$ *the pair* $[h_\varepsilon^j,\, \lambda_\varepsilon^j]$ *is a solution to (1.19);*

(ii) $\lambda_\varepsilon^* \le \lambda_\varepsilon^0 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le ... \to \infty$;

(iii) the set of eigenfunctions h_ε^j, $j \in \{*,\, 0,\, 1,\, 2, ...\}$ *forms an orthonormal basis in* $L^2(-L/\varepsilon,\, L/\varepsilon)$ *with respect to the weighted inner product (1.18). Moreover* h_ε^j *are* C^4 *smooth on* $[-L/\varepsilon, L/\varepsilon]$ *and the corresponding pair* $[h_\varepsilon^j,\, \lambda_\varepsilon^j]$ *satisfies (1.16a)–(1.16b).*

Application of Theorem 4 of [17] and Proposition 1.1 above to the eigenvalue problem (1.14) states that for any $\varepsilon > 0$ its largest eigenvalue is equal to $-\lambda_\varepsilon^*$ and positive, where λ_ε^* is defined in Theorem 1.2. Therefore, the equation (1.7) with (1.8) is linearly unstable at $h_{0,\varepsilon}$.

In addition, we note that as in [17] the transformation procedure stated above and the last theorem make it natural for us to investigate the equivalent symmetric eigenvalue problem (1.19) instead of the original one (1.14). For the thin film equation in the form (1.1)–(1.2) it is known that it is not uniformly elliptic as $h \to 0$ and degenerates in this limit. As a consequence the rescaled system (1.7)–(1.8) and the corresponding eigenvalue problems (1.14) and (1.19) have a singularity at $\varepsilon = 0$. When $\varepsilon = 0$ the assertions of the Proposition 1.1 are not valid anymore and one can not define a linearization of (1.7)–(1.8) at $h_{0,\varepsilon}$, because the latter even does not exist. This implies that the eigenvalue problems (1.14) and (1.19) are essentially singularly perturbed ones.

In the following section 2 we will first give a summary of our results. We begin our analysis in section 3, where we first set up the problem for half-droplets under Dirichlet and Neumann conditions and derive their approximations. The spectrum asymptotics for these as $\varepsilon \to 0$ .is then derived in section 4. Finally, we compare it to numerical solutions of the initial eigenvalue problem (1.14) in section 5.

2 Summary of the main results

To fix notation and for the convenience of the reader, we start with the definition of the asymptotic symbols we use throughout the paper (see also e.g. Kevorkian and Cole [19]).

Definition 2.1. Let $\varepsilon_0 > 0$, the functions $f,\, g : (0,\, \varepsilon_0) \times \mathbb{R}^m \to \mathbb{R}$ with $m \ge 0$ and a set $D \subset \mathbb{R}^m$ be given.

(i) We write $f = O(g)$ for all x in D if and only if there exist numbers $M > 0$ and $\varepsilon_1 \in (0,\, \varepsilon_0)$ such that

$$|f(\varepsilon,\, x)| \leq M\, |g(\varepsilon,\, x)| \text{ for all } x \in D \text{ and } \varepsilon \in (0,\, \varepsilon_1). \tag{2.1}$$

(ii) We write $f = o(g)$ for all x in D if and only if for any given $\delta > 0$ there exists $\varepsilon_1(\delta) \in (0,\, \varepsilon_0)$ such that

$$|f(\varepsilon,\, x)| \leq \delta\, |g(\varepsilon,\, x)| \text{ for all } x \in D \text{ and } \varepsilon \in (0,\, \varepsilon_1). \tag{2.2}$$

(iii) We write $f \sim g$ for all x in D if and only if $f - g = o(g)$.

Next, let us fix P and L so that

$$L - A/P > 0 \text{ with } A = \frac{1}{\sqrt{3}}. \tag{2.3}$$

Assumption (2.3) allows us to distinguish three different asymptotic regions for the steady state solutions as $\varepsilon \to 0$ and will be important for the results that follow. We call these regions $(0,\ a_\varepsilon/\varepsilon)$, $(a_\varepsilon/\varepsilon, b_\varepsilon/\varepsilon)$, $(b_\varepsilon/\varepsilon,\ L/\varepsilon)$ the *droplet core*, *contact line* and the *outer layer* region, respectively. This notation formally corresponds to the one used in [1] and reflects the fact that on the steady state solutions $h_{0,\varepsilon}(x)$ defined on the droplet core and the outer layer is given to leading order as $\varepsilon \to 0$ by a parabola and the constant 1, respectively. The maximum of $h_{0,\varepsilon}(x)$ is $O(1/\varepsilon)$. It is attained at $x = 0$ and gives a so called "peak" of the droplet and is illustrated in Figure 1. As in [1] we call the value A/P the droplet half-width.

Our result concerning the asymptotics of steady state solutions $h_{0,\varepsilon}(x)$ as $\varepsilon \to 0$ is formulated in the following Lemma, whose proof we give in the appendix.

Lemma 2.2. *There exist $\tilde{\varepsilon} > 0$ and functions $a_\varepsilon, b_\varepsilon : (0,\, \tilde{\varepsilon}) \to \mathbb{R}$ such that the following assertions hold:*

(i) for all $\varepsilon \in (0,\, \tilde{\varepsilon})$ one has $0 < a_\varepsilon < b_\varepsilon < L$;

(ii) $a_\varepsilon, b_\varepsilon \to A/P$ and $b_\varepsilon - a_\varepsilon = O(\varepsilon^{1/4}) \to 0$ as $\varepsilon \to 0$, where A is defined in (2.3);

(iii) for all $x \in [0,\ a_\varepsilon/\varepsilon]$ it holds: $\varepsilon^{-3/4} \leq h_{0,\varepsilon}(x) = O(1/\varepsilon)$ and

$$h_{0,\varepsilon}(x) \sim \frac{P\varepsilon}{2}\left(\left(\frac{A}{P\,\varepsilon}\right)^2 - x^2\right);$$

(iv) for all $x \in [b_\varepsilon/\varepsilon,\ L/\varepsilon]$ it holds: $1 \leq h_{0,\varepsilon}(x) = 1 + O(\varepsilon^{1/6})$.

The main result of the lemma given by assertion (ii) is that the ratio of the length of contact line region defined by $(a_\varepsilon/\varepsilon,\ b_\varepsilon/\varepsilon)$ to the one of whole interval $[-L/\varepsilon, L/\varepsilon]$ tends to 0 as $\varepsilon \to 0$. For the problem formulation (1.1) with (1.2) this means that the length of the contact line region $(a_\varepsilon,\ b_\varepsilon)$ tends to zero as $\varepsilon \to 0$. From the proof of the lemma in the appendix it becomes clear that the three intervals $(0,\ a_\varepsilon/\varepsilon)$, $(a_\varepsilon/\varepsilon,\ b_\varepsilon/\varepsilon)$, $(b_\varepsilon/\varepsilon,\ L/\varepsilon)$ with above asymptotical properties are not uniquely defined. Hence, in order to avoid unnecessary technicalities we fix one possible definition for a_ε and b_ε. Once it is fixed then the asymptotic bounds on the steady state solutions $h_{0,\varepsilon}(x)$, stated in assertions (iii) and (iv) of the above lemma are determined uniquely. This in turn determines the asymptotic bounds for the functions $r_\varepsilon(x)$and $f_\varepsilon(x)$. In section

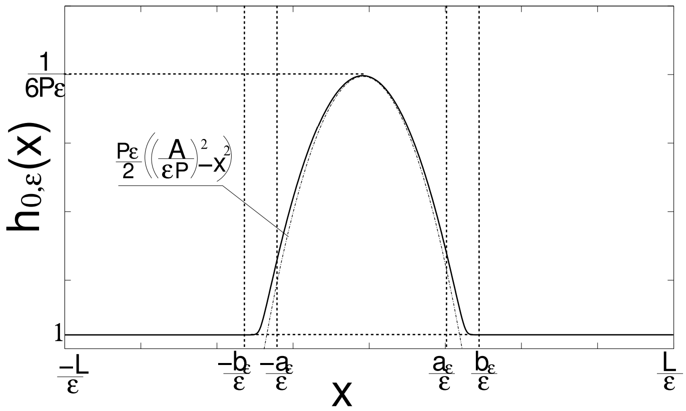

Figure 1: Stationary solution $h_{0,\varepsilon}(x)$ obtained numerically for $\varepsilon = 0.1,\ P = 0.1,\ L = 20.$

3 we decompose the eigenvalue problem (1.19) to two eigenvalue problems on the half-interval $[0,\ L/\varepsilon]$, which we call Dirichlet and Neumann half-droplet problems, respectively, such that the spectrum of the former problem is given by the joint union of the spectra for the latter ones. Next, using the derived asymptotics for $h_{0,\varepsilon}(x)$ and the corresponding ones for $r_\varepsilon(x)$ and $f_\varepsilon(x)$ as $\varepsilon \to 0$ we define for each of the half-droplet eigenvalue problems two approximating problems. Their coefficients are constant and have the same leading order as $r_\varepsilon(x)$ and $f_\varepsilon(x)$ as $\varepsilon \to 0$. The corresponding eigenvalues provide us lower and upper approximations for the eigenvalues of the initial half-droplet problems.

As our main result (Theorem 4.1) we present a rigorous derivation of the explicit asymptotics of the spectra of the approximating eigenvalue problems as $\varepsilon \to 0$. Namely, if one defines a discrete countable set

$$M = \left\{ \left(\frac{\pi(2j+1)}{2(L - A/P)} \right)^2,\ j \in \mathbb{N}_0 \right\} \tag{2.4}$$

with constant A given in (2.3) then by Theorem 4.1 the eigenvalues of the approximating problems divided by ε^2 either tend to ∞ or converge to the set M as $\varepsilon \to 0$. Moreover, for all sufficiently small ε there exists a spectrum gap given in (4.2) and the spectra of the approximating problems divided by ε^2 are separated by it from 0.

In (4.29) we state that the obtained leading order eigenvalues of the approximating problems present the asymptotic approximations for the eigenvalues of (1.19) as $\varepsilon \to 0$. Moreover, direct numerical solutions of the initial eigenvalue problem (1.19) in section 5 show that the leading orders for its positive eigenvalues coincide with the asymptotically obtained approximations (4.29). The only difference appears in the existence of one negative eigenvalue for the numerical solutions of the initial eigenvalue problem (1.19), whereas the spectra of the approximating problems by Theorem 4.1 is strictly positive. Based on our numerical results and those of [20] we conjecture here the existence of a unique negative exponentially small eigenvalue to (1.19) as $\varepsilon \to 0$.

For each $\varepsilon > 0$ let us denote the minimum of the stationary solution $h_{0,\varepsilon}$ by h_ε^-, which by (1.11b)–(1.11c) is attained in points $x = \pm L/\varepsilon$. From assertion (iii) of Theorem 1.2 it follows that any solution to the eigenvalue problem (1.19) is a strong solution of (1.16a)–(1.16b). Vice versa any solution to (1.16a)–(1.16b) gives a solution to (1.19). Using (1.11a) and the definitions (1.15a)–(1.15b) one can easily deduce that for each $\varepsilon > 0$ the pair $[h_{0,\varepsilon}(x) - h_\varepsilon^-,\, 0]$ satisfies (1.16a), but not (1.16b) because

$$h''_{0,\varepsilon}(\pm L/\varepsilon) \neq 0.$$

Indeed, if we suppose that. $h''_{0,\varepsilon}(L/\varepsilon) = 0$, then from $h_{0,\varepsilon}(L/\varepsilon) = h_\varepsilon^-$ and the fact that the stationary solution $h_{0,\varepsilon}(x)$ satisfies the boundary conditions (1.8), it follows that at $x = L/\varepsilon$ the function $h_{0,\varepsilon}(x) - h_\varepsilon^-$ and its first three derivatives should be zero. Next, by uniqueness of the solution to (1.16a) with $\lambda = 0$ and given the initial condition $h^{(k)}(L/\varepsilon) = 0$ for $k = 0, 1, 2, 3$, it follows that $h_{0,\varepsilon}(x) - h_\varepsilon^- \equiv 0$. However, this contradicts the fact that for each $\varepsilon > 0$ the stationary solution $h_{0,\varepsilon}(x)$ by its very definition is not a constant. Consequently, using (1.11b) one concludes that $h''_{0,\varepsilon}(-L/\varepsilon) = h''_{0,\varepsilon}(L/\varepsilon) \neq 0$. Nevertheless, in Lemma A.2 we show that $h''_{0,\varepsilon}(\pm L/\varepsilon)$ tends to zero exponentially as $\varepsilon \to 0$.

In view of above observations, the question arises if there exists an eigenvalue of eigenvalue problem (1.19) which exponentially tends to zero as $\varepsilon \to 0$. basing on the discussion above and estimates of Lemma A.2 we conjecture that the smallest negative eigenvalue to (1.19) is of the form

$$\lambda_\varepsilon^* = O\left(\exp\left(-\frac{\alpha}{\varepsilon^{2/3}}\right)\right). \tag{2.5}$$

Our results also suggest that the approximations of the coefficients $r_\varepsilon(x)$ and $f_\varepsilon(x)$ should be exponentially fine in ε in order that the approximating eigenvalue problems posses the corresponding negative eigenvalue.

In summary, our results suggest the following picture for the asymptotics of the spectrum of the eigenvalue problem (1.19) as $\varepsilon \to 0$. In the spectrum of the eigenvalue problem (1.19) a set of positive eigenvalues $\mathcal{R}_\varepsilon := \{\lambda_D^j(\varepsilon),\, \lambda_N^j(\varepsilon)\ j \in \mathbb{N}_0\}$ is separated from exactly one exponentially small negative eigenvalue $\lambda^*(\varepsilon)$ by a spectrum gap given in (4.2).

Note, that the right end of it we choose as $K^1\varepsilon^2/4$ where K^1 is the smallest positive element of the set M from (2.4). Elements of the above set $\mathcal{R}_\varepsilon$ have asymptotics of $O(\varepsilon^2)$. Here, we do not state any results on the existence of eigenvalues that are much larger than $O(\varepsilon^2)$, but their possible presence by no-means influences the spectral gap property described above.

3 Half-droplet problems and their approximations

We observe, that by (1.11b) for each $\varepsilon > 0$ the steady state solution $h_{0,\,\varepsilon}(x)$ is an even function. Hence the functions (1.15a)–(1.15b) are also even. Therefore, if $[h(x),\,\lambda]$ is an eigenpair of the eigenvalue problem (1.19), then $[h(-x),\,\lambda]$ is also an eigenpair of it. If $h(x)$ is not an even or odd function, then the multiplicity of λ is at least two. Indeed, numerical solutions in section 5 give us pairs of very close eigenvalues, which indicate that there could be eigenvalues of (1.19) with multiplicity two.

To simplify subsequent calculations we introduce a decomposition of (1.19) to two eigenvalue problems on the half-interval $[0,\,L/\varepsilon]$. Every eigenfunction $h(x)$ of (1.19) defines an eigensubspace which is spanned by an even eigenfunction $h^e(x) := (h(x)+h(-x))/2$ and an odd one $h^o(x) := (h(x) - h(-x))/2$ (one of them may be identically zero). This decomposition one can actually apply to any function in the Hilbert space W_ε defined in (1.17). Therefore, one can decompose W_ε into a direct sum of the closed subspace of even functions W_ε^e and the closed subspace of odd functions W_ε^o:

$$W_\varepsilon = W_\varepsilon^e \oplus W_\varepsilon^o.$$

Analogously, any eigensubspace for the eigenvalue problem (1.19) can be decomposed in two, one of which belongs to W_ε^e and another to W_ε^o. Using this and again the property that the functions $r_\varepsilon(x),\ f_\varepsilon(x)$ are even, one obtains that the set of solutions to eigenvalue problem (1.19) is the union of the sets of solutions of two symmetric eigenvalue problems:

$$h \in W_\varepsilon^o,\ \lambda \in \mathbb{R}\ :\ \int_{-L/\varepsilon}^{L/\varepsilon} (h''w'' - r_\varepsilon h'w' - \lambda f_\varepsilon hw)\,dx = 0 \ \text{ for all }\ w \in W_\varepsilon^o,$$

and

$$h \in W_\varepsilon^e,\ \lambda \in \mathbb{R}\ :\ \int_{-L/\varepsilon}^{L/\varepsilon} (h''w'' - r_\varepsilon h'w' - \lambda f_\varepsilon hw)\,dx = 0 \ \text{ for all }\ w \in W_\varepsilon^e.$$

Moreover, it is easy to see that the first eigenvalue problem above is equivalent to the one we call the Dirichlet *half-droplet problem*:

$$h \in V_\varepsilon,\ \lambda \in \mathbb{R}\ :\ \int_0^{L/\varepsilon} (h''w'' - r_\varepsilon h'w' - \lambda f_\varepsilon hw)\,dx = 0 \ \text{ for all }\ w \in V_\varepsilon; \tag{3.1}$$

and the second eigenvalue problem to the one we call the Neumann *half-droplet problem*

$$h \in Q_\varepsilon,\ \lambda \in \mathbb{R}\ :\ \int_0^{L/\varepsilon} (h''w'' - r_\varepsilon h'w' - \lambda f_\varepsilon hw)\,dx = 0 \ \text{ for all }\ w \in Q_\varepsilon, \tag{3.2}$$

where the Hilbert spaces V_ε and Q_ε are defined as

$$\begin{aligned} V_\varepsilon &= H^2(0,\,L/\varepsilon) \cap H_0^1(0,\,L/\varepsilon), \\ Q_\varepsilon &= \left\{ h \in H^2(0,\,L/\varepsilon) :\ h'(0) = h(L/\varepsilon) = 0 \right\}, \end{aligned} \tag{3.3}$$

and both are equipped with the standard inner product of $H^2(0,\ L/\varepsilon)$.

Below we introduce two approximating eigenvalue problems for the Dirichlet half-droplet problem (3.1) and prove several results about their solutions. Note, that analogous approximating problems can be defined for the corresponding Neumann half-droplet problem (3.2) and statements of Propositions 3.2, 3.3, Theorem 4.1 and Lemma 4.2 can be verified for them in the exact same manner.

In the next Lemma 3.1, the proof of which is given in the appendix, we derive the asymptotics for the functions (1.15a) and (1.15b) as $\varepsilon \to 0$, using the asymptotics for $h_{0,\varepsilon}(x)$ stated in Lemma 2.2. In particularly, there we show that $f_\varepsilon(x)$ is positive and bounded from below away from zero, and that $r_\varepsilon(x)$ is bounded, as functions of x uniformly in $\varepsilon > 0$. We note here, that an advantage of the rescaled version (1.7) with (1.8) and the corresponding eigenvalue problems is also that L_∞ bounds hold uniformly in ε, for the coefficients of the symmetric eigenvalue problem (1.19).

Lemma 3.1. *For sufficiently small $\varepsilon > 0$ the following holds:*

$$(i)\ 0 \le r_\varepsilon(x) \le 2\varepsilon^3,\ \textit{for}\ x \le a_\varepsilon/\varepsilon,\ \ -1 \le r_\varepsilon(x) = -1 + O\left(\varepsilon^{1/6}\right),\ \textit{for}\ x \ge b_\varepsilon/\varepsilon;$$
$$\varepsilon^3 \le f_\varepsilon(x) \le \varepsilon^{9/4},\ \textit{for}\ x \le a_\varepsilon/\varepsilon,\quad 1 - O\left(\varepsilon^{1/6}\right) \le f_\varepsilon(x) \le 1,\ \textit{for}\ x \ge b_\varepsilon/\varepsilon. \tag{3.4}$$

(ii) There exists a unique point x_ε^m and a number $k_1 > 0$ such that $a_\varepsilon/\varepsilon < x_\varepsilon^m < b_\varepsilon/\varepsilon$ and $r_\varepsilon(x_\varepsilon^m) = k_1$ gives the maximum of $r_\varepsilon(x)$ on $[0,\ L/\varepsilon]$.

(iii) The function $f_\varepsilon(x)$ monotonically increases on $[0,\ L/\varepsilon]$, and the function $r_\varepsilon(x)$ is monotonically increasing on $[0,\ x_\varepsilon^m]$ and decreases on $[x_\varepsilon^m,\ L/\varepsilon]$.

Next, we define four functions (see also Figure 2):

$$r_\varepsilon^1(x) = \begin{cases} 2\varepsilon^3, & 0 \le x \le a_\varepsilon/\varepsilon \\ k_1, & a_\varepsilon/\varepsilon < x \le b_\varepsilon/\varepsilon \\ -1+\varepsilon^{1/12}, & b_\varepsilon/\varepsilon < x \le L/\varepsilon \end{cases} \quad r_\varepsilon^2(x) = \begin{cases} 0, & 0 \le x \le a_\varepsilon/\varepsilon \\ -1, & a_\varepsilon/\varepsilon < x \le L/\varepsilon \end{cases} \tag{3;5a}$$

$$f_\varepsilon^1(x) = \begin{cases} \varepsilon^{9/4}, & 0 \le x \le a_\varepsilon/\varepsilon \\ 1, & a_\varepsilon/\varepsilon < x \le L/\varepsilon \end{cases}, \quad f_\varepsilon^2(x) = \begin{cases} \varepsilon^4, & 0 \le x \le b_\varepsilon/\varepsilon \\ 1-\varepsilon^{1/12}, & b_\varepsilon/\varepsilon < x \le L/\varepsilon \end{cases} \tag{3;5b}$$

where $a_\varepsilon, b_\varepsilon$ and k_1 are defined in Lemmata 2.2, 3.1. Using (3.5a)–(3.5b) one can define for the Dirichlet half-droplet problem two approximating eigenvalue problems, replacing the functions (1.15a)-(1.15b) in (3.1) by their approximations $r_\varepsilon^i(x),\ f_\varepsilon^i(x)$ with either $i = 1$ or $i = 2$. Define now the Hilbert space

$$H_\varepsilon = L^2(0,\ L/\varepsilon), \tag{3.6}$$

with an inner product

$$(h,\ \tilde{h})_{H_\varepsilon} := \int_0^{L/\varepsilon} h\tilde{h} f_\varepsilon^2\, dx. \tag{3.7}$$

The next proposition is an analog of Theorem 1.2 for the approximating eigenvalue problems.

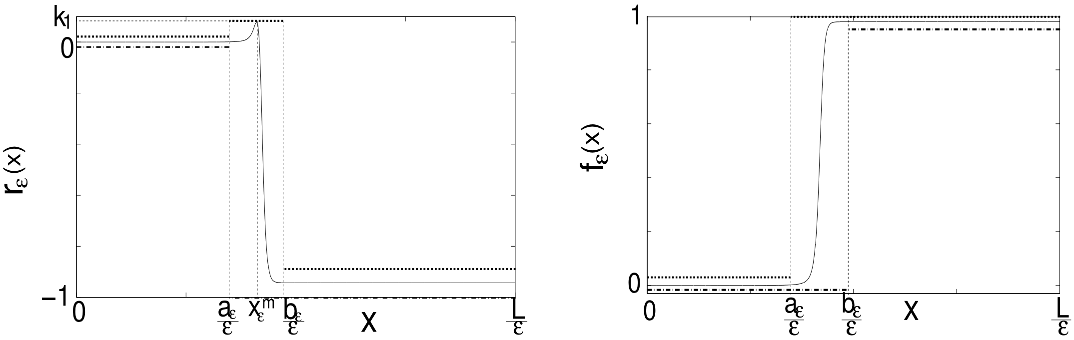

Figure 2: Function $r_\varepsilon(x)$ and its approximations (left), function $f_\varepsilon(x)$ and its approximations (right), obtained numerically for $L = 20$, $P = 0.1$ and $\varepsilon = 0.1$. Approximations $r^1_\varepsilon(x)$, $f^1_\varepsilon(x)$ are denoted by short dashes and $r^2_\varepsilon(x)$, $f^2_\varepsilon(x)$ are denoted by dash-dot lines.

.

Proposition 3.2. *Consider the two approximating eigenvalue problems,*

$$h \in V_\varepsilon,\ \lambda \in \mathbb{R}\ :\ \int_{-L/\varepsilon}^{L/\varepsilon} (h''w'' - r^i_\varepsilon h'w' - \lambda f^i_\varepsilon h\, w)\, dx = 0,\ \forall w \in V_\varepsilon. \tag{3.8}$$

with $i = 1$ or $i = 2$. For fixed i and $\varepsilon > 0$ there exist sequences $\{\lambda^{i,j}_\varepsilon\}$, $\{h^{i,j}_\varepsilon\}$, where $j \in \mathbb{N}_0$ such that:

(i) for each $j \in \mathbb{N}_0$ the pair $[h^{i,j}_\varepsilon,\ \lambda^{i,j}_\varepsilon]$ is a solution to (3.8);

(ii)

$$\lambda^{i,0}_\varepsilon \le \lambda^{i,1}_\varepsilon \le \lambda^{i,2}_\varepsilon \le ... \to \infty; \tag{3.9}$$

(iii) the set $\{h^{i,j}_\varepsilon,\ j \in \mathbb{N}_0\}$ forms an orthonormal basis in (3.6) with respect to inner product (3.7).

The next proposition describes regularity for the solutions of (3.8) and introduces an important property of them, namely the connection conditions (3.10)–(3.11b).

Proposition 3.3. *Let $\varepsilon > 0$ be fixed and $[h, \lambda]$ be a solution to (3.8) for $i = 1$, $[\tilde{h}, \tilde{\lambda}]$ be a solution to (3.8) for $i = 2$. Then the following properties hold:*

(i) On each of the three intervals $(0,\ a_\varepsilon/\varepsilon)$, $(a_\varepsilon/\varepsilon,\ b_\varepsilon/\varepsilon)$ and $(b_\varepsilon/\varepsilon,\ L/\varepsilon)$

$$\tilde{h}^{(4)}(x) + \Big(r^2_\varepsilon(x)\tilde{h}'(x)\Big)' = \tilde{\lambda} f^2_\varepsilon(x)\tilde{h}(x) \ \text{ and } \ h^{(4)}(x) + \big(r^1_\varepsilon(x)h'(x)\big)' = \lambda f^1_\varepsilon(x)h(x).$$

(ii) At the point $x = b_\varepsilon/\varepsilon$ the function $\tilde{h}(x)$ is smooth, $h(x)$ is twice continuously differentiable and satisfies

$$h'''(b_\varepsilon/\varepsilon - 0) + k\, h'(b_{eps}/\varepsilon) = h'''(b_\varepsilon/\varepsilon + 0), \tag{3.10}$$

where $k := k_1 + 1 - \varepsilon^{1/12}$ and k_1 is defined in assertion (ii) of Lemma 3.1.

(iii) At the point $x = a_\varepsilon/\varepsilon$ both h and $\tilde{h}$ are twice continuously differentiable and satisfy:

$$\tilde{h}'''(a_\varepsilon/\varepsilon - 0) + \tilde{h}'(a_\varepsilon/\varepsilon) = \tilde{h}'''(a_\varepsilon/\varepsilon + 0). \tag{3.11a}$$

$$h'''(a_\varepsilon/\varepsilon - 0) - k_1\, h'(a_\varepsilon/\varepsilon) = h'''(a_\varepsilon/\varepsilon + 0). \tag{3.11b}$$

(iv) Both functions h and $\tilde{h}$ satisfy Dirichlet boundary conditions, namely

$$h''(0) = h(0) = h''(L/\varepsilon) = h(L/\varepsilon) = \tilde{h}''(0) = \tilde{h}(0) = \tilde{h}''(L/\varepsilon) = \tilde{h}(L/\varepsilon) = 0.$$

Proof. We prove assertions (i)–(iv) only for solutions $[\tilde{h}, \tilde{\lambda}]$. In the exact same way the remaining assertions for $[h, \lambda]$ can be proved. From (3.8) with $i = 2$ it follows

$$\int_0^{a_\varepsilon/\varepsilon} (\tilde{h}''w'' - r_\varepsilon^2 \tilde{h}'w' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}w)\, dx + \\ + \int_{a_\varepsilon/\varepsilon}^{L/\varepsilon} (\tilde{h}''w'' - r_\varepsilon^2 \tilde{h}'w' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}w)\, dx = 0,\ \forall w \in C_c^\infty(0,\, L/\varepsilon). \tag{3.12}$$

Integrating each integral in the last expression separately two times by parts and using definitions (3.5a)–(3.5b) gives

$$\int_0^{a_\varepsilon/\varepsilon} \left(\tilde{h}^{(4)} + \left(r_\varepsilon^2 \tilde{h}'\right)' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}\right) w\, dx + \int_{a_\varepsilon/\varepsilon}^{L/\varepsilon} \left(\tilde{h}^{(4)} + \left(r_\varepsilon^2 \tilde{h}'\right)' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}\right) w\ dx \\ + \left(\tilde{h}''w'\right)\Big|_{a_\varepsilon/\varepsilon - 0}^{a_\varepsilon/\varepsilon + 0} - \left(\tilde{h}''' + r_\varepsilon^2 \tilde{h}'\right) w\Big|_{a_\varepsilon/\varepsilon - 0}^{a_\varepsilon/\varepsilon + 0} = 0,\ \forall w \in C_c^\infty(0,\, L/\varepsilon), \tag{3.13}$$

From this it follows that

$$\begin{aligned} \int_0^{a_\varepsilon/\varepsilon} \left(\tilde{h}^{(4)} + \left(r_\varepsilon^2 \tilde{h}'\right)' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}\right) w\, dx &= 0,\ \forall w \in C_c^\infty(0,\, a_\varepsilon/\varepsilon) \\ \int_{a_\varepsilon/\varepsilon}^{L/\varepsilon} \left(\tilde{h}^{(4)} + \left(r_\varepsilon^2 \tilde{h}'\right)' - \tilde{\lambda} f_\varepsilon^2 \tilde{h}\right) w\ dx &= 0,\ \forall w \in C_c^\infty(a_\varepsilon/\varepsilon,\, L/\varepsilon). \end{aligned} \tag{3.14}$$

Hence assertion (i) for $[\tilde{h},\, \tilde{\lambda}]$ is true and

$$\left(\tilde{h}''w'\right)\Big|_{a_\varepsilon/\varepsilon - 0}^{a_\varepsilon/\varepsilon + 0} - \left(\tilde{h}''' + r_\varepsilon^2 \tilde{h}'\right) w\Big|_{a_\varepsilon/\varepsilon - 0}^{a_\varepsilon/\varepsilon + 0} = 0,\ \forall w \in C_c^\infty(0,\, L/\varepsilon),$$

Taking in the last expression test functions $w(x)$ such that $w'(a/\varepsilon) = 0$ or $w(a/\varepsilon) = 0$ the connection condition (3.11a) follows. The proof of assertion (iv) is completely analogous to that for the natural boundary conditions in Theorem 23 of [17]. ☐

Remark By Lemma 3.1 for sufficiently small ε and all $x \in [0,\ L/\varepsilon]$ we have

$$r_\varepsilon^2(x) \le r_\varepsilon(x) \le r_\varepsilon^1(x), \ \ f_\varepsilon^2(x) \le f_\varepsilon(x) \le f_\varepsilon^1(x).$$

From this and the fact that the Rayleigh quotient

$$\frac{\int_0^{L/\varepsilon} (h'')^2 - (h')^2\, r_\varepsilon\, dx}{\int_0^{L/\varepsilon} h^2 f_\varepsilon\, dx}$$

for the Dirichlet half-droplet problem (3.1) is bounded from below and from above by the Raleigh quotients of the approximating problems, we conclude that the eigenvalues of the approximating eigenvalue problems (3.8) for $i = 1$ and $i = 2$ give the approximations from below and above for the corresponding eigenvalues of the Dirichlet eigenvalue problem (3.1), respectively. Below we call eigenvalue problems (3.8) for $i = 1$ and $i = 2$ the approximating problems “from below” and “from above”, respectively.

4 Spectrum asymptotics for the approximating problems

In this section we derive the asymptotics for the spectra of the two approximating problems of the Dirichlet half-droplet problem as $\varepsilon \to 0$. The same asymptotics can be verified in the same manner for the spectra of the approximating problems for the corresponding Neumann half-droplet problem. Recall also definition (2.4) of the set M. We can now state and prove one of our key main results.

Theorem 4.1. *(i) If $\{\lambda_l\}$, $l = 1, 2, ...,$ is a sequence of eigenvalues to the eigenvalue problem (3.8) considered for either $i = 1$ or $i = 2$ corresponding to a sequence $\{\varepsilon_l\} \to 0$ and if there exists a number $K^* > 0$ such that*

$$\left|\frac{\lambda_l}{\varepsilon_l^2}\right| \le K^* \ \text{ for all } \ l \in \mathbb{N}\,, \tag{4.1}$$

then

$$\operatorname{dist}\left(\frac{\lambda_l}{\varepsilon_l^2}, M\right) := \inf_{K\in M}\left|K - \frac{\lambda_l}{\varepsilon_l^2}\right| \to 0.$$

(ii) Moreover, for sufficiently small $\varepsilon > 0$ and any eigenvalue λ of the eigenvalue problem (3.8) considered for either $i = 1$ or $i = 2$, one has

$$\lambda \notin \left(0,\ \left[\frac{\pi}{4(L - A/P)}\varepsilon\right]^2\right)\,. \tag{4.2}$$

Proof. We show the proof only for the approximating problem "from below". In exactly the same way one can show it for the approximating problem "from above". Let us fix $\tilde{\varepsilon} > 0$ and a family $[h_\varepsilon, \lambda_\varepsilon]$ of solutions to (3.8) with $i = 1$ for $\varepsilon \in (0, \tilde{\varepsilon})$. Then by assertion (i) of Proposition 3.3 and definitions (3.5a)–(3.5b) in the droplet core region one has

$$h_\varepsilon^{(4)}(x) + 2\varepsilon^3 h_\varepsilon''(x) = \varepsilon^{9/4}\lambda_\varepsilon h_\varepsilon(x) \quad \text{for} \quad x \in (0, a_\varepsilon/\varepsilon), \tag{4.3}$$

Let us denote by $\phi_i(x, \lambda_\varepsilon)$ a fundamental system to (4.3) such that

$$\phi_i^{(k)}(0, \lambda_\varepsilon) = \varepsilon^k \delta_{i,k};\ i,\ k = 0,\ 1,\ 2,\ 3.$$

One can easily deduce that

$$\begin{aligned}
\phi_1(x, \lambda_\varepsilon) &= \varepsilon\left(\frac{z_{1,-}^2 \sinh(z_{1,+}x)}{z_{1,+}(z_{1,-}^2 + z_{1,+}^2)} + \frac{z_{1,+}^2 \sin(z_{1,-}x)}{z_{1,-}(z_{1,-}^2 + z_{1,+}^2)}\right), \\
\phi_3(x, \lambda_\varepsilon) &= \varepsilon^3\left(\frac{\sinh(z_{1,+}x)}{z_{1,+}(z_{1,-}^2 + z_{1,+}^2)} - \frac{\sin(z_{1,-}x)}{z_{1,-}(z_{1,-}^2 + z_{1,+}^2)}\right),
\end{aligned} \tag{4.4}$$

where we denote

$$z_{1,-} := \frac{1}{2}\sqrt{4\varepsilon^3 + 2\sqrt{4\varepsilon^6 + 4\lambda_\varepsilon \varepsilon^{9/4}}},\ z_{1,+} := \frac{1}{2}\sqrt{-4\varepsilon^3 + 2\sqrt{4\varepsilon^6 + 4\lambda_\varepsilon \varepsilon^{9/4}}}.$$

By assertion (iv) of Proposition 3.3 $h_\varepsilon(0) = h_\varepsilon''(0) = 0$, and therefore

$$h_\varepsilon(x) = C_\varepsilon^1 \phi_1(x, \lambda_\varepsilon) + C_\varepsilon^2 \phi_3(x, \lambda_\varepsilon) \text{ for } \quad x \in (0, a_\varepsilon/\varepsilon), \tag{4.5}$$

where $C_\varepsilon^p,\ p = 1, 2$ do not depend on x. By Proposition 3.3 and definitions (3.5a)–(3.5b) in the contact line region one has

$$h_\varepsilon^{(4)}(x) + k_1 h_\varepsilon''(x) = \lambda_\varepsilon h_\varepsilon(x) \text{ for } x \in (a_\varepsilon/\varepsilon, b_\varepsilon\varepsilon).$$

If one denotes by $\psi_i(x, \lambda_\varepsilon)$ a fundamental system to the last equation such that

$$\psi_i^{(k)}(a_\varepsilon/\varepsilon, \lambda_\varepsilon) = \delta_{i,k};\ i,\ k = 0,\ 1,\ 2,\ 3,$$

then one has

$$h_\varepsilon(x) = \sum_{i=0}^{3} h_\varepsilon^{(i)}(a_\varepsilon/\varepsilon + 0)\psi_i(x, \lambda_\varepsilon) \text{ for } x \in (a_\varepsilon/\varepsilon, b_\varepsilon/\varepsilon). \tag{4.6}$$

It is easy to check that

$$\begin{aligned}
\psi_0(x, \lambda_\varepsilon) &= \frac{z_{2,-}^2 \cosh(z_{2,+}(x - a_\varepsilon/\varepsilon))}{(z_{2,-}^2 + z_{2,+}^2)} + \frac{z_{2,+}^2 \cos(z_{2,-}(x - a_\varepsilon/\varepsilon))}{(z_{2,-}^2 + z_{2,+}^2)}, \\
\psi_1(x, \lambda_\varepsilon) &= \frac{z_{2,-}^2 \sinh(z_{2,+}(x - a_\varepsilon/\varepsilon))}{z_{2,+}(z_{2,-}^2 + z_{2,+}^2)} + \frac{z_{2,+}^2 \sin(z_{2,-}(x - a_\varepsilon/\varepsilon))}{z_{2,-}(z_{2,-}^2 + z_{2,+}^2)}, \\
\psi_2(x, \lambda_\varepsilon) &= \frac{\cosh(z_{2,+}(x - a_\varepsilon/\varepsilon))}{(z_{2,-}^2 + z_{2,+}^2)} - \frac{\cos(z_{2,-}(x - a_\varepsilon/\varepsilon))}{(z_{2,-}^2 + z_{2,+}^2)}, \\
\psi_3(x, \lambda_\varepsilon) &= \frac{\sinh(z_{2,+}(x - a_\varepsilon/\varepsilon))}{z_{2,+}(z_{2,-}^2 + z_{2,+}^2)} - \frac{\sin(z_{2,-}(x - a_\varepsilon/\varepsilon))}{z_{2,-}(z_{2,-}^2 + z_{2,+}^2)},
\end{aligned} \tag{4.7}$$

where we denote

$$z_{2,-} := \frac{1}{2}\sqrt{2k_1 + 2\sqrt{k_1^2 + 4\lambda_\varepsilon}},\ z_{2,+} := \frac{1}{2}\sqrt{-2k_1 + 2\sqrt{k_1^2 + 4\lambda_\varepsilon}}.$$

Finally, in the outer interval $[h_\varepsilon,\ \lambda_\varepsilon]$ satisfies

$$h_\varepsilon^{(4)}(x) - (1 - \varepsilon^{1/12})h_\varepsilon''(x) = \lambda_\varepsilon h_\varepsilon(x) \text{ for } \ x \in (b_\varepsilon/\varepsilon,\ L/\varepsilon).$$

Using this and $h_\varepsilon(L/\varepsilon) = h_\varepsilon''(L/\varepsilon) = 0$ one can write

$$h_\varepsilon(x) = -C_\varepsilon^3 \sin(z_{3,-}(x - L/\varepsilon)) - C_\varepsilon^4 \sinh(z_{3,+}(x - L/\varepsilon)) \text{ for } x \in (b_\varepsilon/\varepsilon,\ L/\varepsilon), \quad (4.8)$$

where $C_\varepsilon^p,\ p = 3, 4$ do not depend on x and

$$\begin{aligned} z_{3,-} &= \frac{1}{2}\sqrt{-2(1 - \varepsilon^{1/12}) + 2\sqrt{(1 - \varepsilon^{1/12})^2 + 4\lambda_\varepsilon}}, \\ z_{3,+} &= \frac{1}{2}\sqrt{2(1 - \varepsilon^{1/12}) + 2\sqrt{(1 - \varepsilon^{1/12})^2 + 4\lambda_\varepsilon}}. \end{aligned} \quad (4.9)$$

Let us denote $\psi_{i,b_\varepsilon}^{(k)} := \psi_i^{(k)}(b_\varepsilon/\varepsilon, \lambda_\varepsilon)$ and $\phi_{i,a_\varepsilon}^{(k)} := \phi_i^{(k)}(a_\varepsilon/\varepsilon, \lambda_\varepsilon)$ for $i,\ k = 0, 1, 2, 3$. Then, using representations (4.5), (4.6), (4.8) as well as the connection conditions (3.10), (3.11b) for $h_\varepsilon(x)$ at the points $x = a_\varepsilon/\varepsilon$ and $x = b_\varepsilon/\varepsilon$, one can construct a system of linear algebraic equations imposed for each $\varepsilon \in (0, \tilde{\varepsilon})$ on $C_\varepsilon^p,\ p = 1...4$ in the following form:

$$\begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \sin(z_{3,-}(a_\varepsilon - L)/\varepsilon) & \sinh(z_{3,+}(a_\varepsilon - L)/\varepsilon) \\ \gamma_{2,1} & \gamma_{2,2} & z_{3,-}\cos(z_{3,-}(a_\varepsilon - L)/\varepsilon) & z_{3,+}\cosh(z_{3,+}(a_\varepsilon - L)/\varepsilon) \\ \gamma_{3,1} & \gamma_{3,2} & -z_{3,-}^2\sin(z_{3,-}(a_\varepsilon - L)/\varepsilon) & z_{3,+}^2\sinh(z_{3,+}(a_\varepsilon - L)/\varepsilon) \\ \gamma_{4,1} & \gamma_{4,2} & -z_{3,-}^3\cos(z_{3,-}(a_\varepsilon - L)/\varepsilon) & z_{3,+}^3\cosh(z_{3,+}(a_\varepsilon - L)/\varepsilon) \end{pmatrix} \begin{bmatrix} C_\varepsilon^1 \\ C_\varepsilon^2 \\ C_\varepsilon^3 \\ C_\varepsilon^4 \end{bmatrix} = 0, \quad (4.10)$$

where we denoted

$$\Gamma_\varepsilon = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \\ \gamma_{3,1} & \gamma_{3,2} \\ \gamma_{4,1} & \gamma_{4,2} \end{pmatrix} = \Psi_\varepsilon \cdot \Phi_\varepsilon, \quad (4.11a)$$

$$\Psi_\varepsilon = \begin{pmatrix} \psi_{0,b_\varepsilon}^{(0)} & \psi_{1,b_\varepsilon}^{(0)} & \psi_{2,b_\varepsilon}^{(0)} & \psi_{3,b_\varepsilon}^{(0)} \\ \psi_{0,b_\varepsilon}^{(1)} & \psi_{1,b_\varepsilon}^{(1)} & \psi_{2,b_\varepsilon}^{(1)} & \psi_{3,b_\varepsilon}^{(1)} \\ \psi_{0,b_\varepsilon}^{(2)} & \psi_{1,b_\varepsilon}^{(2)} & \psi_{2,b_\varepsilon}^{(2)} & \psi_{3,b_\varepsilon}^{(2)} \\ \psi_{0,b_\varepsilon}^{(3)} + k\psi_{0,b_\varepsilon}^{(1)} & \psi_{1,b_\varepsilon}^{(3)} + k\psi_{1,b_\varepsilon}^{(1)} & \psi_{2,b_\varepsilon}^{(3)} + k\psi_{2,b_\varepsilon}^{(1)} & \psi_{3,b_\varepsilon}^{(3)} + k\psi_{3,b_\varepsilon}^{(1)} \end{pmatrix}, \quad (4.11b)$$

$$\Phi_\varepsilon = \begin{pmatrix} \phi_{1,a_\varepsilon}^{(0)} & \phi_{3,a_\varepsilon}^{(0)} \\ \phi_{1,a_\varepsilon}^{(1)} & \phi_{3,a_\varepsilon}^{(1)} \\ \phi_{1,a_\varepsilon}^{(2)} & \phi_{3,a_\varepsilon}^{(2)} \\ \phi_{1,a_\varepsilon}^{(3)} - k_1\phi_{1,a_\varepsilon}^{(1)} & \phi_{3,a_\varepsilon}^{(3)} - k_1\phi_{3,a_\varepsilon}^{(1)} \end{pmatrix}, \quad (4.11c)$$

and $k = k_1 + 1 - \varepsilon^{1/12}$ is defined in (3.10). The homogeneous linear system of equations (4.10) has a nontrivial solution for each $\varepsilon \in (0,\, \tilde{\varepsilon})$ if and only if its determinant is identically zero in ε. Expanding its determinant in the third column this implies

$$0 \equiv \sin\left(z_{3,-}(a_\varepsilon - L)/\varepsilon\right) N_\varepsilon - z_{3,-} \cos\left(z_{3,-}(a_\varepsilon - L)/\varepsilon\right) D_\varepsilon,$$

where we denoted the two minors as

$$N_\varepsilon = \begin{vmatrix} \gamma_{2,1} & \gamma_{2,2} & z_{3,+}\cosh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \\ \gamma_{3,1} + z_{3,-}^2\gamma_{1,1} & \gamma_{3,2} + z_{3,-}^2\gamma_{1,2} & \left(z_{3,+}^2 + z_{3,-}^2\right)\sinh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \\ \gamma_{4,1} + z_{3,-}^2\gamma_{2,1} & \gamma_{4,2} + z_{3,-}^2\gamma_{2,2} & z_{3,+}\left(z_{3,+}^2 + z_{3,-}^2\right)\cosh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \end{vmatrix},$$

$$D_\varepsilon = \begin{vmatrix} \gamma_{1,1} & \gamma_{1,2} & \sinh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \\ \gamma_{3,1} + z_{3,-}^2\gamma_{1,1} & \gamma_{3,2} + z_{3,-}^2\gamma_{1,2} & \left(z_{3,+}^2 + z_{3,-}^2\right)\sinh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \\ \gamma_{4,1} + z_{3,-}^2\gamma_{2,1} & \gamma_{4,2} + z_{3,-}^2\gamma_{2,2} & z_{3,+}\left(z_{3,+}^2 + z_{3,-}^2\right)\cosh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \end{vmatrix}.$$

Therefore, one obtains

$$\cot\left(z_{3,-}(a_\varepsilon - L)/\varepsilon\right) \equiv \frac{N_\varepsilon}{z_{3,-}D_\varepsilon} \tag{4.12}$$

Next, we denote $K_\varepsilon := \lambda_\varepsilon/\varepsilon^2$. Let us first describe the case $K^* \geq K_\varepsilon > 0$ for all $\varepsilon \in (0, \tilde{\varepsilon})$, where the constant K^* does not depend on ε. Using this and the definition of $z_{3,\pm}$ one obtains

$$z_{3,-} \sim \sqrt{K_\varepsilon}\varepsilon \text{ and } \quad z_{3,+} \sim 1. \tag{4.13}$$

Using this and the assertion (ii) of Lemma 2.2 one obtains

$$\begin{aligned} \cot\left(z_{3,-}(a_\varepsilon - L)/\varepsilon\right) &\quad \sim \cot\left(\sqrt{K_\varepsilon}(a_\varepsilon - L)\right) \\ \coth\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) &\quad \sim \tanh\left(z_{3,+}(a_\varepsilon - L)/\varepsilon\right) \sim 1. \end{aligned} \tag{4.14}$$

Applying last three asymptotic relationships to (4.12) results in

$$\cot\left(\sqrt{K_\varepsilon}(a_\varepsilon - L)\right) \sim \frac{\begin{vmatrix} \gamma_{2,1} & \gamma_{2,2} & 1 \\ \gamma_{3,1} + K_\varepsilon\varepsilon^2\gamma_{1,1} & \gamma_{3,2} + K_\varepsilon\varepsilon^2\gamma_{1,2} & 1 \\ \gamma_{4,1} + K_\varepsilon\varepsilon^2\gamma_{2,1} & \gamma_{4,2} + K_\varepsilon\varepsilon^2\gamma_{2,2} & 1 \end{vmatrix}}{\sqrt{K_\varepsilon}\varepsilon \begin{vmatrix} \gamma_{1,1} & \gamma_{1,2} & 1 \\ \gamma_{3,1} + K_\varepsilon\varepsilon^2\gamma_{1,1} & \gamma_{3,2} + K_\varepsilon\varepsilon^2\gamma_{1,2} & 1 \\ \gamma_{4,1} + K_\varepsilon\varepsilon^2\gamma_{2,1} & \gamma_{4,2} + K_\varepsilon\varepsilon^2\gamma_{2,2} & 1 \end{vmatrix}}. \tag{4.15}$$

Let us now derive the asymptotics for the matrix Γ_ε as $\varepsilon \to 0$. Below we apply symbol '$\sim$' for matrices to denote their element-wise asymptotic equivalence in the sense of Definition 4.1. Analogously to (4.13) by definitions of $z_{1,\pm},\, z_{2,\pm}$ one obtains

$$z_{1,-} \sim z_{1,+} \sim K_\varepsilon^{1/4}\varepsilon \text{ and } \; z_{2,-} \sim \sqrt{k1}, \quad z_{2,+} \sim \sqrt{\frac{K_\varepsilon}{k_1}}\varepsilon.$$

This and definition of $\phi_i,\, i=1,3$ imply that for all $x\in(0,\, a_\varepsilon/\varepsilon)$

$$\begin{aligned}
\phi_1(x,\,\lambda_\varepsilon) &\sim \varepsilon x, \qquad \phi_3(x,\,\lambda_\varepsilon)\sim\varepsilon^3x^3/6;\\
\phi_1'(x,\,\lambda_\varepsilon) &\sim \varepsilon, \qquad \phi_3'(x,\,\lambda_\varepsilon)\sim\varepsilon^3x^2/2;\\
\phi_1''(x,\,\lambda_\varepsilon) &\sim \varepsilon^6\mathrm{const}x^3/6,\ \phi_3''(x,\,\lambda_\varepsilon)\sim\varepsilon^3x;\\
\phi_1'''(x,\,\lambda_\varepsilon) &\sim \varepsilon^5K_\varepsilon, \qquad \phi_3'''(x,\,\lambda_\varepsilon)\sim\varepsilon^3.
\end{aligned}\tag{4.16}$$

Therefore, by definition (4.11c) one has

$$\Phi_\varepsilon\sim\begin{pmatrix} a_\varepsilon & \frac{a_\varepsilon^3}{6}\\ \varepsilon & \varepsilon\frac{a_\varepsilon^2}{2}\\ \varepsilon^3\mathrm{const} & \varepsilon^2a_\varepsilon\\ -\varepsilon k_1 & -\varepsilon k_1\frac{a_\varepsilon^2}{2}\end{pmatrix}$$

Similarly using definition (4.11b) and assertion (ii) of Lemma 2.2 one obtains

$$\Psi_\varepsilon\sim\begin{pmatrix} 1 & \frac{\sin\rho_\varepsilon}{\sqrt{k_1}} & \frac{1-\cos\rho_\varepsilon}{k_1} & \frac{\sqrt{K_\varepsilon}\rho_\varepsilon}{k_1^{5/2}}\varepsilon\\ \frac{K_\varepsilon\rho_\varepsilon}{k_1^{3/2}}\varepsilon^2 & \cos\rho_\varepsilon & \frac{\sin\rho_\varepsilon}{\sqrt{k_1}} & \frac{\sqrt{K_\varepsilon}(1-\cos\rho_\varepsilon)}{k_1^2}\varepsilon\\ \frac{K_\varepsilon(1+\cos\rho_\varepsilon)}{k_1}\varepsilon^2 & -\sqrt{k_1}\sin\rho_\varepsilon & -\cos\rho_\varepsilon & \frac{\sqrt{K_\varepsilon}\sin\rho_\varepsilon}{k_1^{3/2}}\varepsilon\\ \frac{K_\varepsilon\rho_\varepsilon(1+k_1)}{k_1^{3/2}}\varepsilon^2 & \cos\rho_\varepsilon & \frac{\sin\rho_\varepsilon}{\sqrt{k_1}} & \frac{\sqrt{K_\varepsilon}(1+k_1-\cos\rho_\varepsilon)}{k_1}\varepsilon\end{pmatrix},$$

where we denoted

$$\rho_\varepsilon:=\sqrt{k_1}(b_\varepsilon-a_\varepsilon)/\varepsilon.\tag{4.17}$$

Using the simple rule

$$f_1(\varepsilon)\sim f_2(\varepsilon),\ g_1(\varepsilon)\sim g_2(\varepsilon)\ \Rightarrow\ f_1(\varepsilon)\cdot g_1(\varepsilon)\sim f_2(\varepsilon)\cdot g_2(\varepsilon),$$

definition (4.11a) together with the asymptotics for $\Phi_\varepsilon,\,\Psi_\varepsilon$ and the fact $b_\varepsilon-a_\varepsilon=O(\varepsilon^{1/4})$ one obtains

$$\Gamma_\varepsilon\sim\begin{pmatrix} a_\varepsilon & \frac{a_\varepsilon^3}{6}\\ \varepsilon\cos\rho_\varepsilon & \varepsilon\frac{a_\varepsilon^2}{2}\cos\rho_\varepsilon\\ -\varepsilon\sqrt{k_1}\sin\rho_\varepsilon & -\varepsilon\sqrt{k_1}\frac{a_\varepsilon^2}{2}\sin\rho_\varepsilon\\ \varepsilon\cos\rho_\varepsilon & \varepsilon\frac{a_\varepsilon^2}{2}\cos\rho_\varepsilon\end{pmatrix}.\tag{4.18}$$

Finally, from this and (4.11a), (4.15) one gets:

$$\cot\left(\sqrt{K_\varepsilon}(a_\varepsilon-L)\right)\sim\frac{O(\varepsilon)}{\sqrt{K_\varepsilon}(\cos\rho_\varepsilon+\sqrt{k_1}\sin\rho_\varepsilon)}.\tag{4.19}$$

The last asymptotic relationship prohibits sequences $\{\varepsilon_l\}\to0$ and $\{K_{\varepsilon_l}\}$ such that $K_{\varepsilon_l}\to0$ and $K^*>K_{\varepsilon_l}>0$ for all $l\in\mathbb{N}$, because in such cases $\cot\sqrt{K_{\varepsilon_l}}(a_{\varepsilon_l}-L)\sim 1/\sqrt{K_{\varepsilon_l}}$ and this would contradict to (4.19). Therefore, without loss of generality (see also Remark 26 below) one obtains that $\cot\sqrt{K_\varepsilon}(a_\varepsilon - L) \to 0$ as $\varepsilon \to 0$. From this one concludes that $K_\varepsilon \to M$, where set M is defined in (2.4).

Next, we consider the case $-K^* \le K_\varepsilon < 0$ for all $\varepsilon \in (0,\, \tilde{\varepsilon})$ and substitute it again in expression (4.12). As before after derivation of the leading order asymptotics for (4.12) for this case one obtains that the asymptotic balance (4.19) transforms to

$$\coth\left(\sqrt{-K_\varepsilon}(a_\varepsilon - L)\right) \sim \frac{O(\varepsilon)}{\sqrt{-K_\varepsilon}(\cos\rho_\varepsilon + \sqrt{k_1}\sin\rho_\varepsilon)}. \tag{4.20}$$

Again firstly we obtain from it that sequences $\{\varepsilon_l\} \to 0$ and $\{K_{\varepsilon_l}\}$ such that $K_{\varepsilon_l} \to 0$ and $-K^* < K_{\varepsilon_l} < 0$ for all $l \in \mathbb{N}$ are not possible. But then the right-hand side of (4.20) tends to zero as $\varepsilon \to 0$ and we arrive to a contradiction because the function $\coth\left(\sqrt{-K_\varepsilon}(a_\varepsilon - L)\right)$ is bounded away from 0. Therefore, the case $-K^* \le K_\varepsilon < 0$ for all $\varepsilon \in (0,\, \tilde{\varepsilon})$ is not possible. Proceeding similarly one can show that the case $K_\varepsilon \equiv 0$ for all $\varepsilon \in (0,\, \tilde{\varepsilon})$ is not possible as well.

We conclude that if there exists a constant $K^* > 0$ such that $|K_\varepsilon| \le K^*$ for all $\varepsilon \in (0,\, \tilde{\varepsilon})$ then one has $K_\varepsilon > 0$ for sufficiently small $\varepsilon > 0$ and $K_\varepsilon \to M$ as $\varepsilon \to 0$. This in fact implies both assertions of the theorem. ☐

Remark One may ask what happens with the relations (4.19)–(4.20) if one takes a sequence $\{\varepsilon_l\} \to 0$, such that $\sin(\rho_{\varepsilon_l} + \varphi) \to 0$ as $l \to \infty$, where we let

$$\varphi = \arcsin 1/\sqrt{1 + k_1} \in (0,\, \pi/2).$$

Clearly, in this case it may happen that $\cot\sqrt{K_{\varepsilon_l}}(a_{\varepsilon_l} - L)$ does not tend to zero and the assertions of Theorem 4.1 become unclear. To avoid this situation one should recall definition (4.17) and the fact that there is a certain freedom in defining the functions a_ε and b_ε with the properties stated in Lemma 2.2. One can in fact redefine the contact line region $(a_\varepsilon/\varepsilon,\, b_\varepsilon/\varepsilon)$ for all ε belonging to a special set $\mathcal{O} \subset \mathbb{R}$, so that for any sequence $\{\varepsilon_l\} \to 0$ one would have

$$|\sin(\sqrt{k_1}(b_{\varepsilon_l} - a_{\varepsilon_l})/\varepsilon_l + \varphi)| \ge \text{const} > 0 \text{ for all } l \in \mathbb{N}\,. \tag{4.21}$$

and all the assertions of Lemmata 2.2, 3.1 would hold with the redefined $a_\varepsilon,\, b_\varepsilon$ as well without any changes to the results of this chapter. Namely, let $a_\varepsilon = a(\varepsilon)$ and $b_\varepsilon = b(\varepsilon)$ satisfy the assertions of Lemma 2.2. The set $\mathcal{O}$ can the be defined for example as

$$\mathcal{O} = \left\{\varepsilon > 0 :\ \exists n \in \mathbb{N}\,,\ \frac{\sqrt{k_1}(b(\varepsilon) - a(\varepsilon))}{\varepsilon} \in (-7/6\varphi + \pi n, -5/6\varphi + \pi n)\right\}. \tag{4.22}$$

Next, redefine the functions $a_\varepsilon, b_\varepsilon$ as

$$a_\varepsilon = \begin{cases} a(\varepsilon), & \varepsilon \notin \mathcal{O} \\ a(\varepsilon) - \dfrac{b(\varepsilon) - a(\varepsilon)}{2}, & \varepsilon \in \mathcal{O} \end{cases},\ b_\varepsilon = \begin{cases} b(\varepsilon), & \varepsilon \notin \mathcal{O} \\ b(\varepsilon) + \dfrac{b(\varepsilon) - a(\varepsilon)}{2}, & \varepsilon \in \mathcal{O} \end{cases} \tag{4.23}$$

and fix any sequence $\{\varepsilon_l\} \to 0$. It can be decomposed into two subsequences $\{\varepsilon_{l_k}\}, \{\varepsilon_{l_m}\} \to 0$ (one of which may be empty or finite), such that $\varepsilon_{l_k} \in \mathcal{O}$ for all $k \in \mathbb{N}$ and $\varepsilon_{l_m} \notin \mathcal{O}$ for all $m \in \mathbb{N}$. Then by definitions (4.22) and (4.23) one obtains

$$|\sin(\sqrt{k_1}(b_{\varepsilon_{l_k}} - a_{\varepsilon_{l_k}})/\varepsilon_{l_k} + \varphi)| = |\sin(2\sqrt{k_1}(b(\varepsilon_{l_k}) - a(\varepsilon_{l_k}))/\varepsilon_{l_k} + \varphi)| \geq \sin(2/3\varphi) > 0,$$
$$|\sin(\sqrt{k_1}(b_{\varepsilon_{l_m}} - a_{\varepsilon_{l_m}})/\varepsilon_{l_m} + \varphi)| = |\sin(\sqrt{k_1}(b(\varepsilon_{l_m}) - a(\varepsilon_{l_m}))/\varepsilon_{l_m} + \varphi)| \geq \sin(1/6\varphi) > 0$$

for all $k,\, m \in \mathbb{N}$. We conclude that the definition (4.23) makes all assertions of Lemmata 2.2, 3.1 to be fulfilled again and (4.21) holds for any sequence $\{\varepsilon_l\} \to 0$. Therefore, (4.19)–(4.20) imply all assertions of Theorem 4.1.

The next lemma shows that eigenvalues of approximating problems are asymptotically simple. Besides, it allows for the derivation of leading orders as $\varepsilon \to 0$ for eigenfunctions of approximating problems.

Lemma 4.2. *Let $k,\, m \in \mathbb{N}_0$ and the corresponding $\lambda_\varepsilon^{i,k}$ and $\lambda_\varepsilon^{i,m}$ be the k-th and m-th eigenvalues, respectively, from the ordering (3.9) for the approximating eigenvalue problem (3.8) with fixed $i = 1$ or $i = 2$. If there exists a number $K^* > 0$ such that*

$$\left|\frac{\lambda_\varepsilon^{i,k}}{\varepsilon^2}\right| \leq K^* \text{ and } \left|\frac{\lambda_\varepsilon^{i,m}}{\varepsilon^2}\right| \leq K^*$$

for all sufficiently small $\varepsilon > 0$, then there exist positive numbers K^{**} and $\tilde{\varepsilon}$ such that

$$\left|\lambda_\varepsilon^{i,k} - \lambda_\varepsilon^{i,m}\right| \geq K^{**}\varepsilon^2$$

for all $\varepsilon \in (0,\, \tilde{\varepsilon})$.

Proof. Let us prove the lemma using a contradiction argument. We do it again only for the case (3.8) with $i = 1$. For the case $i = 2$ the proof is analogous. Suppose that the assertion of the lemma is not true. Then by assertion (i) of Theorem 4.1 it follows that there exists a positive number $K \in M$ and sequences $\{\varepsilon_l\} \to 0,\, \{\lambda_{\varepsilon_l}\},\, \{\widetilde{\lambda}_{\varepsilon_l}\}$ such that

$$\lambda_{\varepsilon_l} = \lambda_{\varepsilon_l}^{1,k},\ \ \widetilde{\lambda}_{\varepsilon_l} = \lambda_{\varepsilon_l}^{1,m} \text{ for each } l \in \mathbb{N},$$
$$\frac{\lambda_{\varepsilon_l}}{\varepsilon_l^2} \to K,\ \frac{\widetilde{\lambda}_{\varepsilon_l}}{\varepsilon_l^2} \to K \text{ as } \to \infty. \tag{4.24}$$

Let $\{h_{\varepsilon_l}\}$ be the sequence of eigenfunctions h_{ε_l} corresponding to λ_{ε_l} for each $l \in \mathbb{N}$. Following the lines of the proof for Theorem 4.1 one obtains that there exist $C_{\varepsilon_l}^p,\ p = 1, 2, 3, 4$ such that the representations (4.5), (4.6), (4.8) hold for h_{ε_l} on the droplet core, contact line and outer layer intervals, respectively. Moreover, $C_{\varepsilon_l}^p$ are solutions of the homogeneous linear system (4.10) with the notation defined in the proof of Theorem 4.1. Let

$$\mathbb{M}_{\varepsilon_l} = \begin{pmatrix} \gamma_{1,2} & \sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & \sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{2,2} & z_{3,-}\cos(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}\cosh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,2} & -z_{3,-}^2\sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}^2\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{pmatrix}. \tag{4.25}$$

We claim that there exists a subsequence $\{\lambda_{\varepsilon_l}\}$ (to avoid cumbersome notation we denote all subsequences below also by $\{\lambda_{\varepsilon_l}\}$) such that $\det \mathbb{M}_{\varepsilon_l} \neq 0$ for all $l \in \mathbb{N}$. Suppose the inverse is true, then one can fix a subsequence $\{\lambda_{\varepsilon_l}\}$, such that $\det \mathbb{M}_{\varepsilon_l} \equiv 0$ for all $l \in \mathbb{N}$. Expanding the determinant of $\mathbb{M}_{\varepsilon_l}$ in the second column and dividing the resulting expression by $\sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l)$ one obtains:

$$\cot(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) = \frac{\begin{vmatrix} \gamma_{2,2} & z_{3,+}\cosh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,2} + z_{3,-}^2\gamma_{1,2} & \left(z_{3,+}^2 + z_{3,-}^2\right)\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{vmatrix}}{z_{3,-}\begin{vmatrix} \gamma_{1,2} & \sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,2} + z_{3,-}^2\gamma_{1,2} & \left(z_{3,+}^2 + z_{3,-}^2\right)\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{vmatrix}}.$$

One can check that relations (4.13), (4.14) and (4.18) hold in the current case with $K_{\varepsilon_l} \to K$ as $l \to \infty$ by (4.24). From these it follows that the right hand side of the last expression is $O(1)$ and the left hand side tends to zero as $l \to \infty$. This gives a contradiction, and therefore we can fix a subsequence $\{\lambda_{\varepsilon_l}\}$ such that $\det \mathbb{M}_{\varepsilon_l} \neq 0$ for all $l \in \mathbb{N}$. For such a subsequence the matrix of the linear system of algebraic equations (4.10) has rank three. Therefore, using Cramer's rule and fixing $C^1_{\varepsilon_l}$ for each $l \in \mathbb{N}$ one gets uniquely $C^p_{\varepsilon_l},\ p = 2, 3, 4$ as the solution of a linear system:

$$\begin{pmatrix} \gamma_{1,2} & \sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & \sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{2,2} & z_{3,-}\cos(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}\cosh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,2} & -z_{3,-}^2\sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}^2\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{pmatrix} \begin{bmatrix} C^2_{\varepsilon_l} \\ C^3_{\varepsilon_l} \\ C^4_{\varepsilon_l} \end{bmatrix} = -C^1_{\varepsilon_l} \begin{bmatrix} \gamma_{1,1} \\ \gamma_{2,1} \\ \gamma_{3,1} \end{bmatrix}.$$

This allows us to obtain the asymptotics for $C^p_{\varepsilon_l},\ p = 2, 3, 4$ as $l \to \infty$. For example, due to Cramer's rule

$$\frac{C^2_{\varepsilon_l}}{C^1_{\varepsilon_l}} = -\frac{\begin{vmatrix} \gamma_{1,1} & \sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & \sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{2,1} & z_{3,-}\cos(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}\cosh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,1} & -z_{3,-}^2\sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}^2\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{vmatrix}}{\begin{vmatrix} \gamma_{1,2} & \sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & \sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{2,2} & z_{3,-}\cos(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}\cosh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \\ \gamma_{3,2} & -z_{3,-}^2\sin(z_{3,-}(a_{\varepsilon_l} - L)/\varepsilon_l) & z_{3,+}^2\sinh(z_{3,+}(a_{\varepsilon_l} - L)/\varepsilon_l) \end{vmatrix}}. \tag{4.26}$$

Expanding the nominator and the denominator of the last expression and again using the asymptotics (4.13)–(4.14), (4.18) one obtains $C^2_{\varepsilon_l}/C^1_{\varepsilon_l} = -2/a^2_{\varepsilon_l} + o(1)$. Analogously, one can obtain that $C^3_{\varepsilon_l}/C^1_{\varepsilon_l} = -2a_{\varepsilon_l}/3 + o(1)$ and $C^4_{\varepsilon_l}/C^1_{\varepsilon_l}$ is exponentially small, namely $C^4_{\varepsilon_l}/C^1_{\varepsilon_l} \sim o(\varepsilon_l)\exp((a_{\varepsilon_l} - L)/\varepsilon_l)$.

Next, let $\{\widetilde{h}_{\varepsilon_l}\}$ be the sequence of eigenfunctions $\widetilde{h}_{\varepsilon_l}$ corresponding to $\widetilde{\lambda}_{\varepsilon_l}$ from (4.24) for each $l \in \mathbb{N}$. By assertion (iii) of Proposition 3.2 one has

$$(h_{\varepsilon_l}, \widetilde{h}_{\varepsilon_l})_{H_{\varepsilon_l}} \equiv 0 \text{ for all } l \in \mathbb{N}, \tag{4.27}$$

where we use the inner product (3.7). Denote for each $l \in \mathbb{N}$ by $\widetilde{C}^p_{\varepsilon_l}$ for $p = 1, 2, 3, 4$ the solutions of the linear system corresponding to the eigenfunction $\widetilde{h}_{\varepsilon_l}$. We can fix $C^1_{\varepsilon_l} \equiv \widetilde{C}^1_{\varepsilon_l}$ for all $l \in \mathbb{N}$ so that (4.27) still holds. Then by the considerations above and

(4.24) it follows that $C^p_{\varepsilon_l} \sim \widetilde{C}^p_{\varepsilon_l}$ for $p = 2, 3, 4$ as $l \to \infty$. From this, again (4.24) and the representations (4.5), (4.6), (4.8) for $h_{\varepsilon_l}(x)$ and $\widetilde{h}_{\varepsilon_l}(x)$ yields

$$(h_{\varepsilon_l}, \widetilde{h}_{\varepsilon_l})_{H_{\varepsilon_l}} \to 1 \text{ as } l \to \infty.$$

However, the last relation contradicts (4.27). Therefore, we arrive to a contradiction and the assertion of the lemma is true. □

Leading order for eigenvalues and eigenfunctions The results of Theorem 4.1 and Lemma 4.2 allow us the derivation of the leading order eigenvalues and eigenfunctions of the approximating eigenvalue problems (3.8) as $\varepsilon \to 0$, which in turn provide an asymptotic approximation for the eigenvalues of the corresponding Dirichlet half-droplet problem (3.1). Indeed, Theorem 4.1 allows for eigenvalues $\lambda_\varepsilon \sim K\varepsilon^2$, where $K \in M$.

Suppose such an eigenvalue exists for the eigenvalue problem "from below". Then for the corresponding eigenfunction h_ε the representations (4.5), (4.6), (4.8) should hold. Moreover, from the proof of Lemma 4.2 it follows that h_ε can be normalized so that

$$C^1_\varepsilon \equiv 1, \;\; C^2_\varepsilon \sim -2/a^2_\varepsilon, \;\; C^3_\varepsilon \sim -2a_\varepsilon/3, \;\; C^4_\varepsilon \sim o(\varepsilon)\exp\left((a_\varepsilon - L)/\varepsilon\right) \tag{4.28}$$

From this, the representation (4.5) and asymptotics (4.16), it follows that on the droplet core interval $(0, \; a_\varepsilon/\varepsilon)$ to leading order $h_\varepsilon(x)$ is a linear combination of polynomials and does not depend on K. One can explain this fact by looking at equation (4.3) for h_ε on the droplet core. The term $2\varepsilon^3 h''_\varepsilon(x) - \varepsilon^{9/4}\lambda_\varepsilon h_\varepsilon(x)$ is small enough, so that the leading orders for the fundamental system on this interval are given by the solutions of the equation $h^{(4)}_\varepsilon(x) = 0$, i.e. by polynomials. This property of (4.3) in turn comes about because defining the approximating problems (3.8) we explored the asymptotics derived in Lemma 3.1 for the coefficients $r_\varepsilon(x), \; f_\varepsilon(x)$ of the symmetric eigenvalue problem (1.19).

Taking next the leading order as $\varepsilon \to 0$ in the representation (4.6) one can see that on the contact line region $(a_\varepsilon/\varepsilon, \; b_\varepsilon/\varepsilon)$ to leading order $h_\varepsilon(x)$ is constant and its derivatives actually oscillate with a high frequency proportional to $(b_\varepsilon - a_\varepsilon)/\varepsilon$. Such oscillations needs some work to resolve numerically the derivatives of the eigenfunctions in this region (see details in the next section). Finally, on the outer layer $(b_\varepsilon/\varepsilon, \; L/\varepsilon)$ due to (4.8) and asymptotics (4.13), (4.28) holding with $K_\varepsilon \to K$ as $\varepsilon \to 0$ one obtains that

$$h_\varepsilon(x) \sim C^3_\varepsilon \sin\left(\sqrt{K}(\varepsilon x - L)\right)$$

and essentially depends on K. If we consider instead of the approximating problem "from below" the one "from above" we end up with the same leading orders on the droplet core and the outer layer for the eigenfunctions h_ε corresponding to $\lambda_\varepsilon \sim K\varepsilon^2$ with $K \in M$.

Motivated by this, let us construct for each $j \in \mathbb{N}_0$ approximations to the solutions of the Dirichlet half-droplet problem (3.1) as

$$h_\varepsilon^j(x) \approx C_\varepsilon \begin{cases} \varepsilon x - \dfrac{\varepsilon^3}{3a_\varepsilon^2} x^3 + p_\varepsilon^j(x), & x \le a_\varepsilon/\varepsilon \\ \dfrac{2a_\varepsilon}{3} \sin\left(\sqrt{K^j}(\varepsilon x - L)\right), & x \ge a_\varepsilon/\varepsilon \end{cases}, \qquad \lambda_\varepsilon^j \approx K^j \varepsilon^2, \tag{4.29}$$

where we denote

$$K^j = \left(\frac{\pi(2j+1)}{2(L-b_\varepsilon)}\right)^2.$$

We note, that one can show that the leading orders for the eigenvalues of the approximating problems for the Neumann half-droplet problem (3.2) are also given by $\lambda_\varepsilon^j \approx K^j \varepsilon^2$. This suggests that initial eigenvalue problem (1.19) posses a countable set $\mathcal{R}_\varepsilon := \{\lambda_D^j(\varepsilon),\, \lambda_N^j(\varepsilon)\ j \in \mathbb{N}_0\}$ of positive eigenvalues. Elements of $\mathcal{R}_\varepsilon$ have asymptotics of $O(\varepsilon^2)$. Our discussion at the end of section 2 suggest that (1.19) posses additionally a solution with the leading order asymptotics of the following form

$$h_\varepsilon^*(x) \approx C_\varepsilon(h_{0,\varepsilon}(x) - h_\varepsilon^-) \text{ and } \lambda_\varepsilon^* \approx 0, \tag{4.30}$$

where λ_ε^* is negative and tends to zero exponentially fast as $\varepsilon \to 0$. Finally, for sufficiently small $\varepsilon > 0$ the set $\mathcal{R}_\varepsilon$ is separated from exactly one exponentially small negative eigenvalue $\lambda^*(\varepsilon)$ by a spectrum gap given in (4.2). In the next section direct numerical solutions confirm our asymptotically derived results.

5 Numerical solutions and comparisons

Here we describe the numerical solution of the eigenvalue problem (1.14) and compare it with the leading order approximations (4.29), (4.30) for the set of eigenvalues of the symmetric EVP (1.19).

We proceed in three steps. Firstly, for fixed P, L and sufficiently small $\varepsilon > 0$ we solve (1.11a) with boundary conditions

$$h'_{0,\,\varepsilon}(\pm L/\varepsilon, P) = 0$$

numerically and calculate the stationary solution $h_{0,\,\varepsilon}(x)$. Using $h_{0,\,\varepsilon}(x)$ we then calculate the coefficient functions for the linear operator $\mathcal{L}_\varepsilon$. Secondly, we apply a finite difference discretization on a uniform mesh on the interval $[-L/\varepsilon,\, L/\varepsilon]$ to the linear operator $\mathcal{L}_\varepsilon$ including also the boundary conditions (1.8). The resulting approximation of our finite-difference scheme is $O(1/N^2)$, where N is the mesh size. Finally, the problem transforms to one of finding the eigenvalues and eigenfunction of the matrix $A \in \mathbb{M}(N \times N)$ corresponding to the discretized operator $\mathcal{L}_\varepsilon$. We calculate them using an Implicitly Restarted Arnoldi Method, which was developed in [21] for the cases of large sparse matrices and implemented in the Fortran library ARPACK. The set of eigenpairs of matrix M give us a numerical approximation for the smallest eigenpairs

j	0	1	2	3	4	5
$\lambda_{appr}/\varepsilon^2$	0.1381	1.2431	3.4532	6.7682	11.1883	16.7134
$\lambda_{num}/\varepsilon^2$	0.1437	1.2926	3.5843	7.0068	11.5425	17.1675

Table 1: Comparison of the first 6 eigenvalues for the Dirichlet eigenvalue problem (3.1) with $P = 0.1, L = 10, \epsilon = 10^{-6}$.

of EVP (1.14). In Table 1 we compare the first six eigenvalues calculated numerically (second row) for the Dirichlet half-droplet problem (3.1) and using the analytical approximations (4.29) (first row). Similar agreement between the numerical results and the analytical approximations (4.29), (4.30) are obtained for the eigenvalues of the Neumann half-droplet problem (3.2).

Our numerical results also show that for fixed $\varepsilon > 0$ and $j \in N_0$ the corresponding $\lambda^j_{D,\varepsilon}$ and $\lambda^j_{N,\varepsilon}$ are very close. In Figure 3 two numerically obtained eigenfunctions corresponding to the eigenvalues $-\lambda^2_{D,\varepsilon}$ and $-\lambda^2_{N,\varepsilon}$ of the initial eigenvalue problem (1.14) are presented. As was stated in the introduction, the eigenfunctions of the eigenvalue problem (1.14) are the derivatives of the corresponding eigenfunctions of the symmetric eigenvalue problem (1.19). According to this and that the set of solutions to the latter problem is the union of solutions to Dirichlet and Neumann half-droplet eigenvalue problems, Figure 3 shows that the left numerical eigenfunction is an even function and corresponds to the derivative of the eigenfunction for the Dirichlet half-droplet problem. The right one is an odd function and corresponds to the derivative of the eigenfunction for the Neumann half-droplet problem.

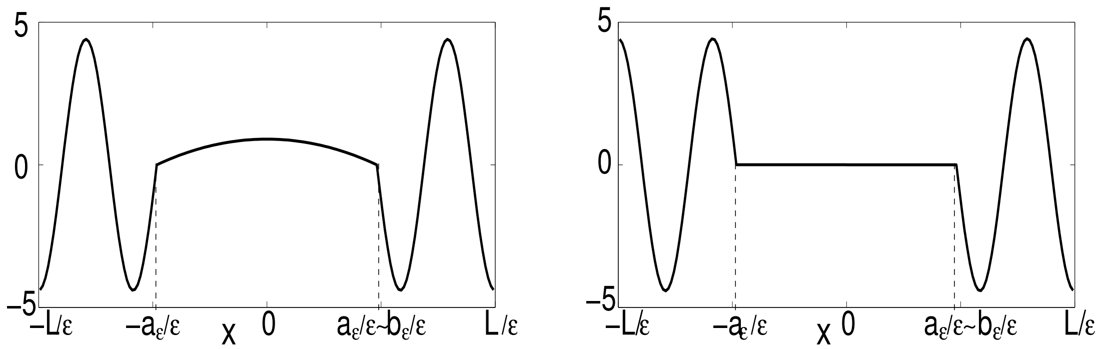

Figure 3: Eigenfunctions corresponding to eigenvalues $-\lambda^2_{D,\varepsilon}$ (left) and $-\lambda^2_{N,\varepsilon}$ (right) of EVP (1.14), $P = 0.1,\ L = 10,\ \epsilon = 10^{-6}$.

Figure 3 clearly shows two regions in the interval $[-L/\varepsilon,\ L/\varepsilon]$. In the outer interval both eigenfunctions are represented by trigonometric functions and in the inner one, corresponding to the droplet core, by polynomials. This stays in a good correspondence with the approximation for the eigenfunctions in the last paragraph of section 4 and (4.29). Due to the chosen very small $\varepsilon = 10^{-6}$ and the fact that the relative length of the contact line interval between those regions tends to zero as $\varepsilon \to 0$ (see Lemma 2.2) numerically we observe this interval as a point on Figure 3.

Another observation concerns the smoothness of the numerical eigenfunctions. In Figure 3 one can see that the first derivative of the eigenfunctions is discontinuous at this point. This can be explained using the analytical result that we obtained for the approximating problem “from below”, namely in the contact line region the derivative of the eigenfunctions for the above problem oscillates fast and proportionally to the negative power of ε. Therefore, it is a challenging numerical problem to resolve the derivatives of eigenfunctions in this very small contact line region. Nevertheless, the numerical eigenfunctions from Figure 3 are continuous and this stays also in correspondence with our analytical result, which predicts that in the contact line region to the leading order in ε an eigenfunction itself is determined by a constant and does not possess oscillations.

Finally, Figure 4 shows the numerical eigenfunction of the eigenvalue problem (1.14) corresponding to the exponentially small eigenvalue $-\lambda_\varepsilon^*$. Comparing Figures 7 and 4 one can see that this function is close to $h'_{0,\varepsilon}(x)$. This is in a agreement with the approximation (4.30) for the corresponding eigenfunction of the the symmetric eigenvalue problem (1.19).

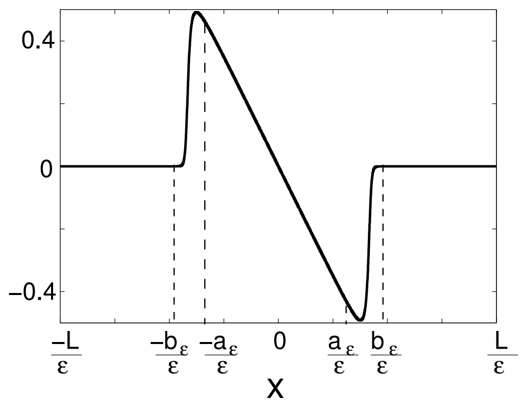

Figure 4: Eigenfunction corresponding to eigenvalue $-\lambda_\varepsilon^*$ of eigenvalue problem (1.14), $P = 0.1,\ L = 20,\ \epsilon = 10^{-2}$.

A Asymptotics for the steady state solutions

In this appendix we prove important Lemmata 2.2 and 3.1. Let us consider equation

$$h''(x) = \Pi\left(h(x)\right) - \varepsilon P. \tag{A.1}$$

By results of Appendix A of [8] (see also Figure 5) there exists a hyperbolic saddle point $\hat{h}_\varepsilon^-$ and an elliptic center point $\hat{h}_\varepsilon^c$ of equation (A.1), which are the two real roots of the algebraic equation $\Pi\left(h\right) - \varepsilon P = 0$ and have the following asymptotic form:

$$\hat{h}_\varepsilon^- = 1 + \varepsilon P + O(\varepsilon^2), \quad \hat{h}_\varepsilon^c \sim (\varepsilon P)^{-1/3}. \tag{A.2}$$

Corresponding to this there exists a homoclinic solution $\hat{h}_\varepsilon(x)$ to (A.1), the minimum of which is given by $\hat{h}_\varepsilon^-$, and its maximum $\hat{h}_\varepsilon^+$ (see [8]) has asymptotics form

$$\hat{h}_\varepsilon^+ = \frac{1}{6\varepsilon P} + 1 + O(\varepsilon). \tag{A.3}$$

One can define a first integral for $\hat{h}_\varepsilon(x)$ as

$$1/2\left(\hat{h}_\varepsilon'(x)\right)^2 + \mathcal{U}_\varepsilon\left(\hat{h}_\varepsilon(x)\right) = 0,$$

where

$$\mathcal{U}_\varepsilon(h) = -U(h) + U\left(\hat{h}_\varepsilon^-\right) + \varepsilon P(h - \hat{h}_\varepsilon^-), \tag{A.4a}$$

$$\mathcal{U}_\varepsilon\left(\hat{h}_\varepsilon^-\right) = \mathcal{U}_\varepsilon'\left(\hat{h}_\varepsilon^-\right) = \mathcal{U}_\varepsilon\left(\hat{h}_\varepsilon^+\right) = 0. \tag{A.4b}$$

The function $U(h)$ in (A.4a) (see its plot in Figure 6 is such that $dU/dh = \Pi(h)$ and

$$U(h) := \frac{1}{3h^3} - \frac{1}{2h^2}. \tag{A.5}$$

The next proposition is needed for the proof of Lemma 2.2 below.

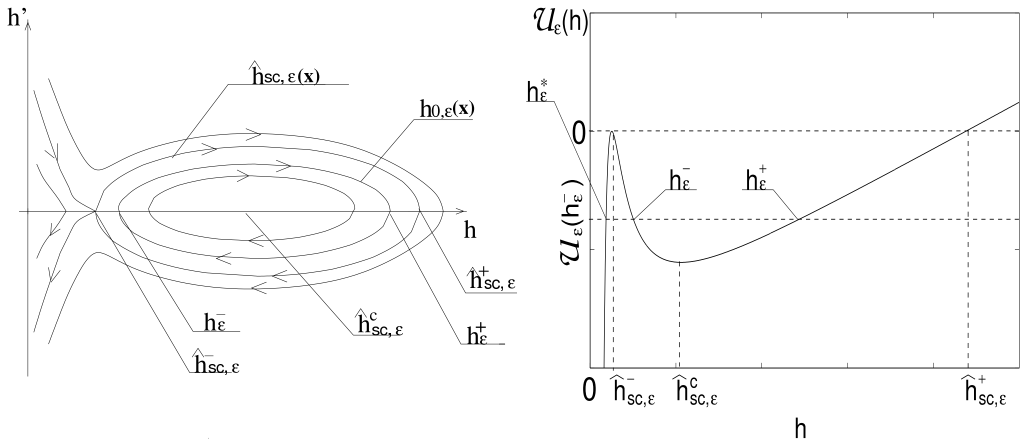

Figure 5: Phase plane portrait for the equation (A.1) (left) and plot of function $\mathcal{U}_\varepsilon(h)$ (right).

Proposition A.1. *For each sufficiently small* $\varepsilon > 0$ *and* $\delta \in (0, -\mathcal{U}_\varepsilon(\hat{h}_\varepsilon^c))$ *there exists a unique number* $h_\varepsilon(\delta) \in (\hat{h}_\varepsilon^-, \hat{h}_\varepsilon^c)$ *such that*

$$\delta = -\mathcal{U}_\varepsilon(h_\varepsilon(\delta)).$$

Moreover, there exist positive numbers $\tilde{\varepsilon}, \tilde{\delta}$ *such that for all* $\varepsilon \in (0, \tilde{\varepsilon})$ *and* $\delta \in (0, \tilde{\delta})$ *one has*

$$h_\varepsilon(\delta) - \hat{h}_\varepsilon^- < 2\sqrt{\delta}. \tag{A.6}$$

Proof. The existence and uniqueness of $h_\varepsilon(\delta)$ follows from the fact that $\hat{h}_\varepsilon^-$ and $\hat{h}_\varepsilon^c$ are double zeros and the local minimum of $\mathcal{U}_\varepsilon(h)$ (see also Figure 5). Moreover, using (A.4a)–(A.4b) and Peano's formula one obtains

$$h_\varepsilon(\delta) - \hat{h}_\varepsilon^- = \sqrt{\frac{2\delta}{\Pi'(\theta_\varepsilon(\delta))}}, \tag{A.7}$$

where $\theta_\varepsilon(\delta) \in [\hat{h}_\varepsilon^-,\ h_\varepsilon(\delta)]$. From (1.6) it follows that the function $\Pi'(h)$ monotonically decreases for $h \in [1,\ 5/3]$. By (A.2) one has $\Pi'(\hat{h}_\varepsilon^-) \to 1$ as $\varepsilon \to 0$ and $\Pi'(\hat{h}_\varepsilon^c) < 0$ for sufficiently small $\varepsilon > 0$. Therefore, one can choose sufficiently small $\tilde{\varepsilon} > 0$ and $\tilde{\delta} > 0$ such that

$$\Pi'(h_{\tilde{\varepsilon}}(\tilde{\delta})) := 1/2. \tag{A.8}$$

Next, from (A.4a) and the definition of $\hat{h}_\varepsilon^-$ it follows that

$$\begin{aligned} \frac{\partial \mathcal{U}_\varepsilon(h)}{\partial \varepsilon} &= (\Pi(\hat{h}_\varepsilon^-) - \varepsilon P)\frac{\partial \hat{h}_\varepsilon^-}{\partial \varepsilon} + P(h - \hat{h}_\varepsilon^-) \\ &= P(h - \hat{h}_\varepsilon^-) > 0 \text{ for all } h > \hat{h}_{\tilde{\varepsilon}}^- \text{ and } \varepsilon \in (0,\ \tilde{\varepsilon}). \end{aligned} \tag{A.9}$$

Let us fix $0 < \varepsilon < \tilde{\varepsilon}$ and $0 < \delta < \tilde{\delta}$. Define a number $h^* > \hat{h}_{\tilde{\varepsilon}}^-$, such that $\mathcal{U}_{\tilde{\varepsilon}}(h^*) = -\delta$. If we suppose that $h^* \le h_\varepsilon(\delta)$ then we arrive at the following contradiction:

$$-\delta = \mathcal{U}_{\tilde{\varepsilon}}(h^*) \ge \mathcal{U}_{\tilde{\varepsilon}}(h_\varepsilon(\delta)) > \mathcal{U}_\varepsilon(h_\varepsilon(\delta)) = -\delta,$$

where we used (A.9) and that the function $\mathcal{U}_{\tilde{\varepsilon}}(h)$ decreases for $h \in (\hat{h}_{\tilde{\varepsilon}}^-,\ \hat{h}_{\tilde{\varepsilon}}^c)$. Therefore, $h^* > h_\varepsilon(\delta)$. On the other hand

$$\mathcal{U}_{\tilde{\varepsilon}}(h^*) = -\delta > -\tilde{\delta} = \mathcal{U}_{\tilde{\varepsilon}}(h_{\tilde{\varepsilon}}(\tilde{\delta}))$$

and therefore again by monotonicity of $\mathcal{U}_{\tilde{\varepsilon}}(h)$ one gets $h^* < h_{\tilde{\varepsilon}}(\tilde{\delta})$. Hence, one obtains $h_\varepsilon(\delta) < h_{\tilde{\varepsilon}}(\tilde{\delta})$. Finally, using again monotonicity of $\Pi'(h)$ and the definition (A.8) one obtains that $\Pi'(h_\varepsilon(\delta)) > 1/2$, and therefore from (A.7) estimate (A.6) follows. ☐

Next, we define h_ε^+ and h_ε^- as the maximum and the minimum of the steady state solution $h_{0,\varepsilon}(x)$ which are attained at $x = 0$ and $x = \pm L/\varepsilon$ by (1.11b)–(1.11c). A first integral for $h_{0,\varepsilon}(x)$ is determined by

$$1/2(h'_{0,\varepsilon}(x))^2 + \mathcal{U}_\varepsilon(h_{0,\varepsilon}(x)) - \mathcal{U}_\varepsilon(h_\varepsilon^-) = 0, \tag{A.10a}$$

$$\mathcal{U}_\varepsilon(h_\varepsilon^+) - \mathcal{U}_\varepsilon(h_\varepsilon^-) = 0 \tag{A.10b}$$

In the next lemma we state the asymptotics for h_ε^+, h_ε^- and $h''_{0,\varepsilon}(\pm L/\varepsilon)$ as $\varepsilon \to 0$. We should point out, that an appearance of the term $\alpha/\varepsilon^{2/3}$ in the estimates (A.11) and (A.14) is strongly connected with the fact that from the asymptotic form (A.2), (A.3) we obtain

$$\frac{\hat{h}_\varepsilon^+}{\hat{h}_\varepsilon^c} = O(\varepsilon^{-2/3}).$$

Lemma A.2. *There exists a positive constant α such that for all sufficiently small $\varepsilon > 0$ it holds*

(i)

$$h_\varepsilon^- - \hat{h}_\varepsilon^- \le \exp\left(-\frac{\alpha}{\varepsilon^{2/3}}\right), \tag{A.11}$$

(ii)

$$h_\varepsilon^- = 1 + \varepsilon\, P + o(\varepsilon), \tag{A.12}$$

(iii)

$$h_\varepsilon^+ \sim \frac{1}{6\varepsilon\, P}. \tag{A.13}$$

(iv)

$$\left|h''_{0,\,\varepsilon}(\pm L/\varepsilon)\right| \le \exp\left(-\frac{\alpha}{\varepsilon^{2/3}}\right), \tag{A.14}$$

Proof. a) Integrating (A.10a) with respect to x on $(0,\, L/\varepsilon)$ and using $h_{0,\varepsilon}(0) = h_\varepsilon^+$, $h_{0,\varepsilon}(L/\varepsilon) = h_\varepsilon^-$ one obtains

$$\frac{L}{\varepsilon} = \int_{h_\varepsilon^-}^{h_\varepsilon^+} \frac{dh}{\sqrt{2\,(\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h))}}. \tag{A.15}$$

From (A.4a) and (A.10b) one obtains

$$\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h) = U(h) - U(h_\varepsilon^-) - \varepsilon\, P(h - h_\varepsilon^-), \tag{A.16a}$$

$$U(h_\varepsilon^+) - U(h_\varepsilon^-) - \varepsilon\, P(h_\varepsilon^+ - h_\varepsilon^-) = 0. \tag{A.16b}$$

By (A.4a), (A.5) for a fixed $\varepsilon > 0$ the function $\mathcal{U}_\varepsilon(h)$ monotonically increases on $(0,\, \hat{h}_\varepsilon^-)$ from $-\infty$ to 0, it decreases on $(\hat{h}_\varepsilon^-,\, \hat{h}_\varepsilon^c)$ and increases on $(\hat{h}_\varepsilon^c,\, \hat{h}_\varepsilon^+)$ (see Figure 5). Using this and (A.16a) one arrives at the following representation:

$$\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h) = \frac{\varepsilon\, P(h - h_\varepsilon^-)(h - h_\varepsilon^*)(h - h_\varepsilon^{**})(h_\varepsilon^+ - h)}{h^3}, \tag{A.17}$$

where four real zeros of the function $\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h)$ for each fixed $\varepsilon > 0$ fulfill the following constraints:

$$\begin{aligned} &h_\varepsilon^{**} < 0,\; 0 < h_\varepsilon^* < \hat{h}_\varepsilon^-, \\ &\hat{h}_\varepsilon^- < h_\varepsilon^- < \hat{h}_\varepsilon^c,\; \hat{h}_\varepsilon^c < h_\varepsilon^+ < \hat{h}_\varepsilon^+. \end{aligned} \tag{A.18}$$

b) Let us prove using a contradiction argument, that there exist positive numbers ε_1 and α_1 such that

$$h_\varepsilon^- \le \alpha_1 \text{ for all } \varepsilon \in (0,\, \varepsilon_1). \tag{A.19}$$

Assume the inverse, then without loss of generality $h_\varepsilon^- \to +\infty$ as $\varepsilon \to 0$. Using (A.15), (A.17) and (A.18) one estimates

$$\frac{L}{\varepsilon} \le \frac{1}{\sqrt{2\varepsilon\, P}} \int_{h_\varepsilon^-}^{h_\varepsilon^+} \frac{\sqrt{h_\varepsilon^+}}{\sqrt{(h - h_\varepsilon^-)(h_\varepsilon^+ - h)}} \sqrt{\frac{h}{h - \hat{h}_\varepsilon^-}}\, dh.$$

From $h_\varepsilon^- \to +\infty$ and (A.2) it follows that there exists $\tilde\varepsilon > 0$ such that

$$\sqrt{\frac{h}{h - \hat{h}_\varepsilon^-}} \le \sqrt{\frac{h_{\tilde\varepsilon}^-}{h_{\tilde\varepsilon}^- - \hat{h}_{\tilde\varepsilon}^-}} \le \sqrt{2}$$

for all $h \in (h_{\tilde\varepsilon}^-, +\infty)$ and $\varepsilon \in (0, \tilde\varepsilon)$. Using the last two estimates one obtains for all $\varepsilon \in (0, \tilde\varepsilon)$

$$\frac{L}{\varepsilon} \le \frac{1}{\sqrt{\varepsilon P}} \int_{h_\varepsilon^-}^{h_\varepsilon^+} \frac{\sqrt{h_\varepsilon^+}}{\sqrt{(h - h_\varepsilon^-)(h_\varepsilon^+ - h)}}\, dh = \sqrt{\frac{h_\varepsilon^+}{\varepsilon P}}\pi.$$

On the other hand by $h_\varepsilon^- \to +\infty$ and (A.16b) one has $\varepsilon P h_\varepsilon^+ = o(1)$ as $\varepsilon \to 0$. Using this and the last estimate one obtains that

$$\frac{L}{\varepsilon} \le \frac{o(1)}{\varepsilon P}\pi,$$

which obviously gives a contradiction. Therefore, (A.19) holds with some positive numbers ε_1, α_1.

Next, let us show using a contradiction argument, that there exist positive numbers α_2 and ε_2 such that

$$h_\varepsilon^+ \ge \frac{\alpha_2}{\varepsilon} \text{ for all } \varepsilon \in (0, \varepsilon_2). \tag{A.20}$$

Assume the inverse, then without loss of generality $\varepsilon P h_\varepsilon^+ \to 0$ as $\varepsilon \to 0$. On the

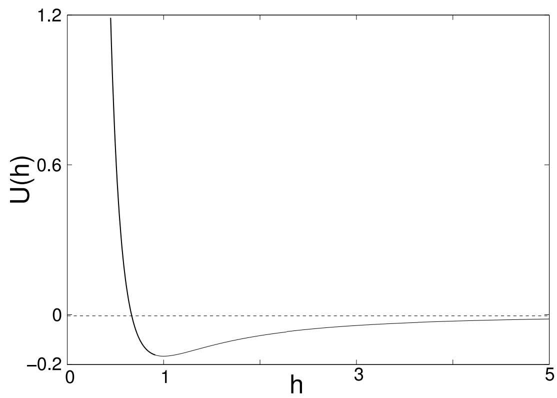

Figure 6: Plot of function $U(h)$.

other hand (A.18) and (A.2) yield $h_\varepsilon^+ \to +\infty$ as $\varepsilon \to 0$. Substituting this in (A.16b) and using (A.19), (A.5) one obtains

$$U(h_\varepsilon^-) \to 0.$$

From this, by $h_\varepsilon^- > \hat{h}_{sc,\varepsilon}^- > 1$ and (A.5) it follows that $h_\varepsilon^- \to +\infty$, which gives a contradiction to (A.19). Therefore, the estimate (A.20) holds with some positive numbers ε_2, α_2.

c) We write now the formula (A.15) as

$$\frac{L}{\varepsilon} = I_1 + I_2 := \int_{h_\varepsilon^-}^{\hat{h}_\varepsilon^c} \frac{dh}{\sqrt{2\left(\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h)\right)}} + \int_{\hat{h}_\varepsilon^c}^{h_\varepsilon^+} \frac{dh}{\sqrt{2\left(\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h)\right)}} \tag{A.21}$$

and estimate each of the integrals $I_k,\ k = 1, 2$ separately. Using again (A.18) and (A.19) one estimates

$$I_2 = \int_{\hat{h}_\varepsilon^c}^{h_\varepsilon^+} \frac{dh}{\sqrt{2\left(\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h)\right)}} \le \frac{1}{\sqrt{2\varepsilon\, P}} \int_{\hat{h}_\varepsilon^c}^{h_\varepsilon^+} \frac{h}{\sqrt{(h-\alpha_1)(h-\hat{h}_\varepsilon^-)(h_\varepsilon^+ - h)}} dh.$$

By (A.2) and (A.18) both $\hat{h}_\varepsilon^c$ and h_ε^+ tend to $+\infty$ and $\hat{h}_\varepsilon^- \to 1$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon^* > 0$ such that

$$\frac{h}{\sqrt{(h-\alpha_1)(h-\hat{h}_\varepsilon^-)}} \le (1 + C) \text{ with } C := \frac{L\,P}{2A} - 1/2 \tag{A.22}$$

holds for all $h \in (\hat{h}_{\varepsilon^*}^c,\ +\infty)$ and $\varepsilon \in (0,\ \varepsilon^*)$. Note that the number C in (A.22) by condition (2.3) is positive. Using now the last two estimates, again (A.18) and (A.3), one obtains

$$\begin{aligned} I_2 \quad &\le \frac{1+C}{\sqrt{2\varepsilon\, P}} \int_{\hat{h}_\varepsilon^c}^{h_\varepsilon^+} \frac{dh}{\sqrt{h_\varepsilon^+ - h}} \le \\ &\le \sqrt{\frac{2\hat{h}_\varepsilon^+}{\varepsilon\, P}}(1+C) = \left(\frac{L + A/P}{2}\right)\frac{1}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right). \end{aligned} \tag{A.23}$$

d) Let us now estimate the integral I_1 from (A.21). Using again (A.18) one obtains

$$\begin{aligned} I_1 \quad &= \quad \int_{h_\varepsilon^-}^{\hat{h}_\varepsilon^c} \frac{dh}{\sqrt{2\left(\mathcal{U}_\varepsilon(h_\varepsilon^-) - \mathcal{U}_\varepsilon(h)\right)}} \le \\ &\le \frac{1}{\sqrt{2\varepsilon\, P}} \frac{\hat{h}_\varepsilon^c}{\sqrt{h_\varepsilon^+ - \hat{h}_\varepsilon^c}} \int_{h_\varepsilon^-}^{\hat{h}_\varepsilon^c} \frac{dh}{\sqrt{(h-h_\varepsilon^-)(h-\hat{h}_\varepsilon^-)}}. \end{aligned} \tag{A.24}$$

This, asymptotics (A.20) and (A.2) yield that there exist positive numbers ε_3 and α_3 such that for all $\varepsilon \in (0,\ \varepsilon_3)$ the following estimate holds:

$$\begin{aligned} I_1 \quad &\le \frac{\alpha_3}{\varepsilon^{1/3}} \int_{h_\varepsilon^-}^{\hat{h}_\varepsilon^c} \frac{dh}{\sqrt{(h-h_\varepsilon^-)(h-\hat{h}_\varepsilon^-)}} \le \\ &\le \frac{\alpha_3}{\varepsilon^{1/3}} \left(-\log\left(h_\varepsilon^- - \hat{h}_\varepsilon^-\right) + 2\log\left(2\left(\hat{h}_\varepsilon^c - \hat{h}_\varepsilon^-\right)\right)\right). \end{aligned} \tag{A.25}$$

Finally, combining the estimates (A.23), (A.25) together with formula (A.21) and using (A.2) one obtains

$$\log(h_\varepsilon^- - \hat{h}_\varepsilon^-) \le -\left(\frac{L - A/P}{2\alpha_3}\right)\frac{1}{\varepsilon^{2/3}} + o\left(\frac{1}{\varepsilon^{2/3}}\right).$$

This together with condition (2.3) imply that the estimate (A.11) holds for sufficiently small $\varepsilon > 0$ with the positive constant

$$\alpha := \frac{L - A/P}{4\alpha_3}.$$

e) The asymptotic form (A.12) follows from (A.11) and (A.2). In turn (A.13) follows from (A.12) and (A.16b), (A.5).

f) Let us finally show the estimate (A.14). Using (1.11a) and Peano's formula one obtains

$$h''_{0,\varepsilon}(\pm L) = \Pi\left(h_\varepsilon^-\right) - \varepsilon P = \Pi'(\hat{h}_\varepsilon^-)(h_\varepsilon^- - \hat{h}_\varepsilon^-) + \Pi''(\theta_\varepsilon)(h_\varepsilon^- - \hat{h}_\varepsilon^-)^2,$$

where $\theta_\varepsilon \in [\hat{h}_\varepsilon^-,\, h_\varepsilon^-]$. The representations (A.2) and (A.12) yield that $\Pi''(\theta_\varepsilon) < 0$ and $\Pi'(\hat{h}_\varepsilon^-) < 1$ for sufficiently small $\varepsilon > 0$. Therefore, applying the estimate (A.11) one obtains (A.14). □

Let us now prove Lemmata 2.2, 3.1

Proof of Lemma 2.2:

Proof. Using the estimates, (A.12)–(A.13) and (1.11c) uniquely define a_ε for each sufficiently small $\varepsilon > 0$ as

$$L > a_\varepsilon > 0: \quad h_{0,\varepsilon}(a_\varepsilon/\varepsilon) = \varepsilon^{-3/4}. \tag{A.26}$$

Then by (1.6) and (A.26) one has

$$\Pi(h_{0,\varepsilon}(x)) = O(\varepsilon^{9/4}) \text{ and } \frac{d^2 h_{0,\varepsilon}(x)}{dx^2} = -P\,\varepsilon + O(\varepsilon^{9/4}), \text{ for } x \in [0,\ a_\varepsilon/\varepsilon].$$

Integrating two times and using $h'_{0,\varepsilon}(0) = 0$ by (1.11c) one obtains

$$\begin{aligned} h'_{0,\varepsilon}(x) &\sim -P\,\varepsilon x, \\ h_{0,\varepsilon}(x) &\sim \frac{P\,\varepsilon}{2}\left(\left(\frac{C}{\varepsilon}\right)^2 - x^2\right) \text{ for all } x \in [0,\ a_\varepsilon/\varepsilon]. \end{aligned} \tag{A.27}$$

Taking $x = 0$ in the last expression and using (A.13) one obtains

$$C = \frac{A}{P}.$$

Taking next $x = a_\varepsilon$ gives

$$a_\varepsilon \sim \frac{A}{P} \text{ and } h'_{0,\varepsilon}(a_\varepsilon/\varepsilon) \sim -A. \tag{A.28}$$

The estimate $\varepsilon^{-3/4} \le h_{0,\varepsilon}(x) = O(1/\varepsilon)$ for all $x \in [0,\ a_\varepsilon/\varepsilon]$ follows from (A.13), definition (A.26) and monotonicity of $h_{0,\varepsilon}(x)$ for $x > 0$ by (1.11c). Therefore, assertion (iii) of the lemma is proved.

Next, for each $\varepsilon > 0$ by definition of $\hat{h}_\varepsilon^c$ there exists a unique $x_\varepsilon^c \in (0,\, L)$ such that

$$h_{0,\varepsilon}(x_\varepsilon^c/\varepsilon) = \hat{h}_\varepsilon^c \quad \text{and} \quad h''_{0,\varepsilon}(x_\varepsilon^c/\varepsilon) = 0. \tag{A.29}$$

For each $\varepsilon > 0$ the function $h_{0,\varepsilon}(x)$ decreases on $[0, L/\varepsilon]$, therefore one can define its inverse function $x_\varepsilon(h)$ decreasing on $[h_\varepsilon^-,\, h_\varepsilon^+]$ as

$$x_\varepsilon\left(h_{0,\varepsilon}(x)\right) := x \quad \text{and} \quad x'_\varepsilon(h) = \frac{1}{h'_{0,\varepsilon}(x_\varepsilon(h))}. \tag{A.30}$$

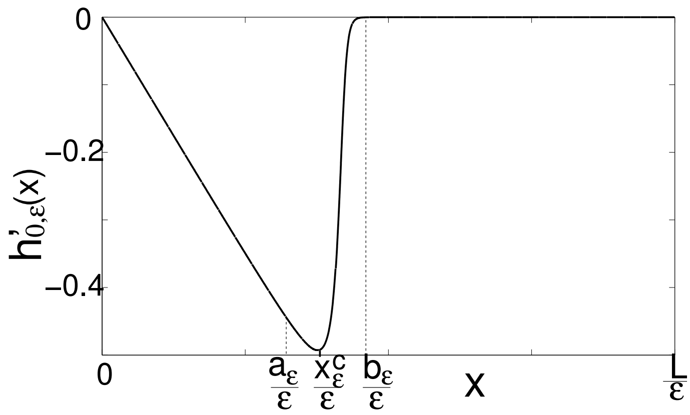

Figure 7: Plot of $h'_{0,\varepsilon}(x)$ obtained numerically for $\varepsilon = 0.1,\ P = 0.1,\ L = 20$ and corresponding to $h_{0,\varepsilon}(x)$ from Figure 1.

By this and (A.2), (A.26) and (A.28) one obtains

$$\frac{\left|x_\varepsilon^c - a_\varepsilon\right|}{\varepsilon} \le \int_0^1 \left|x'_\varepsilon\left(t\,\hat{h}_\varepsilon^c - (1-t)\varepsilon^{-3/4}\right)\right| dt \left|\hat{h}_\varepsilon^c - \varepsilon^{-3/4}\right| \le$$
$$\max_{\hat{h}_\varepsilon^c \le h \le \varepsilon^{-3/4}} \left|x'_\varepsilon(h)\right| \left|\hat{h}_\varepsilon^c - \varepsilon^{-3/4}\right| = \frac{\left|\hat{h}_\varepsilon^c - \varepsilon^{-3/4}\right|}{\left|h'_{0,\,\varepsilon}(a_\varepsilon/\varepsilon)\right|} = O(\varepsilon^{-3/4}), \tag{A.31}$$

where we also use that $\left|h'_{0,\varepsilon}(x)\right|$ increases for $x \in [0,\ x_\varepsilon^c/\varepsilon]$ by (1.11a) and (A.29) (see also Figure 7). Therefore, using (A.28) one obtains

$$\begin{aligned} x_\varepsilon^c &\sim a_\varepsilon \sim \frac{A}{P}, \\ h'_{0,\varepsilon}(x_\varepsilon^c/\varepsilon) &\sim -A, \\ x_\varepsilon^c - a_\varepsilon &= O(\varepsilon^{1/4}). \end{aligned} \tag{A.32}$$

Using this and that $\left|h'_{0,\varepsilon}(x)\right|$ decreases for $x \in [x_\varepsilon^c/\varepsilon,\ L/\varepsilon]$ (see Figure 7), we define for each sufficiently small $\varepsilon > 0$ a unique b_ε as

$$L > b_\varepsilon > x_\varepsilon^c: \quad h'_{0,\varepsilon}(b_\varepsilon/\varepsilon) := -\varepsilon^{1/6}. \tag{A.33}$$

Using this, the first integral (A.10a), definition (A.4a)–(A.4b), Peano's formula and (A.11) yields

$$
\begin{aligned}
-\mathcal{U}_\varepsilon(h_{0,\,\varepsilon}(b_\varepsilon/\varepsilon)) &= \frac{1}{2}\varepsilon^{1/3} - \mathcal{U}_\varepsilon(h_\varepsilon^-) + \mathcal{U}_\varepsilon(\hat{h}_\varepsilon^-) = \\
&= \frac{1}{2}\left(\varepsilon^{1/3} + \Pi'(\theta_\varepsilon)(h_\varepsilon^- - \hat{h}_\varepsilon^-)^2\right) \le \frac{1}{2}\left(\varepsilon^{1/3} + \exp\left(-\frac{\alpha}{\varepsilon^{2/3}}\right)\right) = \\
&= O(\varepsilon^{1/3}),
\end{aligned}
\tag{A.34}
$$

where $\theta_\varepsilon \in [\hat{h}_\varepsilon^-,\ h_\varepsilon^-]$ and by (A.2), (A.12) $\Pi'(\theta_\varepsilon) \le \Pi'(\hat{h}_\varepsilon^-) \le 1$. Applying Proposition A.1 to the last inequality one gets

$$
h_{0,\varepsilon}(b_\varepsilon/\varepsilon) = 1 + O(\varepsilon^{1/6})\,. \tag{A.35}
$$

Now, the assertion (iv) of the lemma follows from (A.12) and (1.11c).

Finally, we show $b_\varepsilon - a_\varepsilon = O(\varepsilon^{1/4})$. Using the inverse function $x_\varepsilon(h)$ defined in (A.30) one writes

$$
\begin{aligned}
\frac{\left|x_\varepsilon^c - b_\varepsilon\right|}{\varepsilon} &\le \int_0^1 \left|x_\varepsilon'\left(t\,\hat{h}_\varepsilon^c - (1-t)h_{0,\varepsilon}(b_\varepsilon/\varepsilon)\right)\right| dt \left|\hat{h}_\varepsilon^c - h_{0,\varepsilon}(b_\varepsilon/\varepsilon)\right| \le \\
&\le \max_{h_{0,\varepsilon}(b_\varepsilon/\varepsilon)\le h\le \hat{h}_\varepsilon^c} \left|x_\varepsilon'(h)\right|\left|\hat{h}_\varepsilon^c - O(1)\right| = \frac{\left|\hat{h}_\varepsilon^c - O(1)\right|}{\varepsilon^{1/6}} = O(\varepsilon^{-1/2}),
\end{aligned}
\tag{A.36}
$$

where we also used definition (A.33) and (A.35), (A.2). From the last estimate one obtains $b_\varepsilon - x_\varepsilon^c = O(\varepsilon^{1/2})$. Combining this with (A.32) yields $b_\varepsilon - a_\varepsilon = O(\varepsilon^{1/4})$, which in turn, noting (A.28), implies assertion (ii) of the lemma. This concludes the proof of the lemma. ☐

Proof of Lemma 3.1:

Proof. By definition (1.15a)

$$
r_\varepsilon(x) = -\frac{4}{(h_{0,\varepsilon}(x))^5} + \frac{3}{(h_{0,\varepsilon}(x))^4} \tag{A.37}
$$

By assumption (iii) of Lemma 2.2 and (A.37), (1.15b) it follows that

$$
O(\varepsilon^4) \le r_\varepsilon(x) = \varepsilon^3 + o(\varepsilon^3), \quad O(\varepsilon^3) \le f_\varepsilon(x) \le \varepsilon^{9/4} \text{ for all } x \in [0,\ a_\varepsilon/\varepsilon]
$$

By assumption (iv) of Lemma 2.2 and (A.37), (1.15b) it follows

$$
-1 \le r_\varepsilon(x) = -1 + O(\varepsilon^6), \quad 1 - O(\varepsilon^6) \le f_\varepsilon(x) \le 1 \text{ for all } x \in [b_\varepsilon/\varepsilon,\ L/\varepsilon].
$$

Therefore, assertion (i) follows. The stationary points of the function $r_\varepsilon(x)$ are given by equation

$$
r_\varepsilon'(x) = \left(\frac{20}{(h_{0,\varepsilon}(x))^6} - \frac{12}{(h_{0,\varepsilon}(x))^5}\right) h_{0,\varepsilon}'(x) = 0
$$

Using (A.12)–(A.13) and that $h'_{0,\varepsilon}(x) < 0$ for all $x \in (0,\ L/\varepsilon)$, one obtains that for each sufficiently small $\varepsilon > 0$ there exists a unique x^m_ε at which $r_\varepsilon(x)$ attains its maximum $k_1 := (3/5)^5 > 0$ such that $h_{0,\varepsilon}(x^m_\varepsilon) = 5/3$. Therefore, by (A.26), (A.35) and again $h'_{0,\varepsilon}(x) < 0$ for all $x \in (0,\ L/\varepsilon)$ it follows that $a_\varepsilon/\varepsilon < x^m_\varepsilon < b_\varepsilon/\varepsilon$, and hence the assertions (ii) and (iii) are proved. ☐

Conclusions and discussion

In this article we considered the asymptotics of the spectrum of the linearized thin-film equation (1.7) with (1.8) at the steady state solution $h_{0,\varepsilon}$ as the singular parameter $\varepsilon \to 0$. It corresponds to the physical sitution of a single droplet that is connected to the boundaries via a thin layer of thickness ε. We constructed the leading order approximations (4.29)-(4.30) for the eigenvalues and eigenfunctions of the corresponding linearized eigenvalue problems (1.14) and (1.19) and confirmed them numerically. In particular, (4.29)-(4.30) show the existence of the spectral gap (4.2), which is a central property for the application of an extension of the center-manifold reduction method of [15] to the derivation of the reduced ODE models describing the dynamics of coarsening droplets as mentioned in the introduction.

The natural question that arises is if these approximations can be justified rigorously, i.e. if one can show the existence of eigenvalues to (1.14) having the corresponding leading orders as $\varepsilon \to 0$. For this purpose in [20] an approach was developed based on a variant of the implicit function theorem that has been developed recently for a special class of singular perturbed problems in [22] and [23]. In particular, in [20] the existence of eigenvalues and eigenfunctions with the leading order asymptotics (4.29) for the approximating problems (3.8) was proved.

Part of our future work will be to extend this approach to show the existence of an exponentially small eigenvalue corresponding to the approximation (4.30) with the leading order asymptotics suggested in (2.5). The existence of an exponentially small eigenvalue is an interesting problem on its own because its smallness prescribes the velocity of the slow motion on the center-manifold, its attraction rate and consequently the time scale for the coarsening process. Besides this, exponentially small eigenvalues, are also believed to be the cause for phenomena such as boundary layer resonance, that has been investigated for convection-diffusion-reaction equations, see for example [24, 25] and references therein.

Returning to the eigenvalue problem (1.14), we note that apart from the proof of existence of its solutions with prescribed asymptotics (4.29) and (4.30), one still needs to show that the set of the corresponding eigenfunctions forms a complete system in order to rigorously establish the existence of the spectral gap property (4.2). In view of the fact that the eigenvalue problem (1.14) is a singular perturbed one and the eigenvalues with asymptotics (4.29)–(4.30) all tend to zero as $\varepsilon \to 0$, this problem seems to be a nontrivial problem on its own and is subject of future research. Also on the other hand, as it was shown in this paper, the approximating eigenvalue problems to (1.14) should posses an exponentially fine approximation in order to catch all the eigenvalues of the original problem.

Finally, we would like to mention that a natural extension of the approach developed above would be the investigation of the asymptotics of the spectrum for equations (1.7) or (1.1) linearized about the steady state solutions that correspond to the physical situation of arrays of droplets connected by a thin layer of thickness ε.

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